

DIOPHANTINE GEOMETRY OVER GROUPS X: THE ELEMENTARY THEORY OF FREE PRODUCTS OF GROUPS

Z. SELA^{1,2}

This paper is the 10th in a sequence on the structure of sets of solutions to systems of equations over groups, projections of such sets (Diophantine sets), and the structure of definable sets over few classes of groups. In the 10th paper we study the first order theory of free products of arbitrary groups. With a given (coefficient-free) predicate, we associate (non-canonically) a finite collection of graded resolutions. These graded resolutions enable us to reduce uniformly an arbitrary coefficient-free sentence over a free product to a finite disjunction of conjunctions of sentences over the factors of the free product. The graded resolutions enable a uniform quantifier elimination over free products, where a coefficient-free predicate (over free products) is shown to be equivalent to a predicate in an extended language, that contains finitely many quantifiers over the factors of the free product, and only 3 quantifiers over the ambient free product.

These uniform reductions allow us to answer affirmatively a question of R. Vaught on the elementary equivalence of free products of pairs of elementarily equivalent groups, and to obtain a generalization of Tarski's problem on the elementary equivalence of non-abelian free groups. Finally, we use the resolutions that are associated with a predicate over free products, to prove that the free product of stable groups is stable, generalizing the main theorem of [Se9] on the stability of free groups.

In the first 6 papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects that are required for the analysis of sentences and elementary sets defined over a free group. The techniques we developed, enabled us to present an iterative procedure that analyzes *EAE* sets defined over a free group (i.e., sets defined using 3 quantifiers), and shows that every such set is in the Boolean algebra generated by *AE* sets ([Se6],41), hence, we obtained a quantifier elimination over a free group.

In the 7th paper in the sequence we generalized the techniques and the results from free groups to torsion-free hyperbolic groups, and in the 8th paper we used the techniques that were developed for quantifier elimination to prove that the elementary theories of free and torsion-free hyperbolic groups are stable. In the 9th paper in the sequence we studied definable equivalence relations over free and hyperbolic groups, proved that 3 basic families of equivalence relations are imaginaries (non-reals), and finally proved a geometric elimination of imaginaries when sorts are added for these 3 basic imaginaries.

In this paper we study the first order theory of free products of arbitrary groups. In a joint work with E. Jaligot [Ja-Se] we started this study, by analyzing the set of

¹Hebrew University, Jerusalem 91904, Israel.

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solutions to systems of equations over an arbitrary free product. For that purpose, we used limit groups over free products, and with each system of equations, or alternatively, with each f.p. group, we have associated (non-canonically) a Makanin-Razborov diagram over free products. This Makanin-Razborov diagram encodes the set of solutions to the finite system of equations over an arbitrary free product, or alternatively describes all the quotients of a given f.p. group that are free products.

We start this paper by studying systems of equations with parameters (section 1). We generalize the notion of a graded limit group from free groups to free products, and then define rigid and (weakly) solid limit groups over free products, generalizing the corresponding notions over free and hyperbolic groups. Unfortunately, the boundedness results that were proved in [Se3] for the number of rigid and strictly solid families of homomorphisms over free and hyperbolic groups, can no longer be valid over free products. However, we manage to prove a combinatorial boundedness for rigid and (weakly) strictly solid families (theorems 1.14 and 1.15), that plays an essential role in studying the first order theory of free products, successfully replacing the strong boundedness results of [Se3].

In section 2 we prove a general form of Merzlyakov theorem (over free groups) on the existence of formal solutions for sentences and predicates over varieties that are defined over free products. In particular, we show how to associate (non-canonically) a formal Makanin-Razborov diagram with a given AE sentence or predicate over free products, generalizing the results of [Se2] over free groups.

In section 3 we start studying sentences and predicates over free products. In section 3 we study AE sentences. The strategy that was used to study such sentences over free groups in [Se4] can not be applied over free products, hence, we use a modification of it that uses the tools that were constructed in the first 2 sections and in [Ja-Se]. In section 4 we study AE sets (predicates), and further apply it to study EAE sets and sentences. In section 5 we study AEAE sets and sentences, and finally, in section 6 we study general definable sets and sentences over arbitrary free products.

In theorem 6.1, we associate (non-canonically) finitely many graded resolutions with a given coefficient-free predicate over free products. This finite collection of resolutions is non-canonical, but it is universal, and it is good for all non-trivial free products apart from the infinite dihedral group, D_∞ . In principle, the finite collection of resolutions enables one to reduce a sentence or a predicate from an ambient free product to its factors. Indeed, in theorem 6.3, we show that any given coefficient-free sentence over free products is equivalent to a finite disjunction of conjunctions of (coefficient-free) sentences over the factors of the free product. Furthermore, any given coefficient-free predicate over free products is equivalent to a coefficient-free predicate in an extended language, that involves finitely many quantifiers over the factors of the free product, and only 3 quantifiers over the ambient free product. Note that since the resolutions that we associated with a coefficient-free predicate are universal, the reduction of sentences and predicates from the ambient free product to its factors is uniform, i.e., it is good for all free products, and it does not depend on any particular given one.

The uniform reduction of sentences and predicates, and the resolutions that are associated with a given (coefficient-free) predicate, enable us to prove some basic results on the first order theory of free products in the next sections. In [Fe-Va] S. Feferman and R. Vaught studied the first order properties of certain products of structures. Their methods, that look at the cartesian product of given structures,

do not cover free products of groups (as they indeed indicated in their paper). This and his work with A. Tarski, led R. L. Vaught to ask the question that we answer in the beginning of section 7:

Theorem 7.1. *Let A_1, B_1, A_2, B_2 be groups. Suppose that A_1 is elementarily equivalent to A_2 , and B_1 is elementarily equivalent to B_2 . Then $A_1 * B_1$ is elementarily equivalent to $A_2 * B_2$.*

The existence of graded resolutions that are associated with a given sentence over free products enables one to prove the following theorem, that generalizes Tarski's problem for free groups.

Theorem 7.2. *Let A, B be non-trivial groups, and suppose that either A or B is not Z_2 . Let F be a (possibly cyclic) free group. Then $A * B$ is elementarily equivalent to $A * B * F$.*

The resolutions that are associated with coefficient-free predicates and sentences over free products, that enable a uniform reduction from the ambient free product to its factors, allow us to prove other uniform properties of sentences over free products.

Theorem 7.3. *Let Φ be a coefficient free sentence over groups. There exists an integer, $k(\Phi)$, so that for every group, H , Φ is a truth sentence over $H_1 * \dots * H_{k(\Phi)}$, $H_i \simeq H$, if and only if Φ is a truth sentence over $H_1 * \dots * H_n$, $H_i \simeq H$, for every $n \geq k(\Phi)$.*

Note that the integer $k(\Phi)$ depends on the coefficient free sentence, Φ , but it does not depend on the group, H . It is easy to see that $k(\Phi)$ can not be chosen to be a universal constant, e.g., we can take Φ_m to be a sentence that specifies if the number of conjugacy classes of involutions in the group is at least m . For such a sentence, Φ_m , $k(\Phi_m) = m$.

Theorem 7.3 can be further strengthened for sequences of groups. Let Φ be a coefficient free sentence over groups. Given any sequence of groups, G_1, G_2, \dots , we set $M_1 = G_1$, $M_2 = G_1 * G_2$, $M_3 = G_1 * G_2 * G_3$, and so on. The sentence Φ may be truth or false on any of the groups (free products) M_i , $i = 1, \dots$. Here one can (clearly) not guarantee that the sentence Φ is constantly truth or constantly false staring at a bounded index (of the M_i 's). However, one can prove the following.

Theorem 7.4. *There exists an integer $c(\Phi)$, so that for every sequence of groups, G_1, G_2, \dots , the sentence Φ over the sequence of groups, $M_1 = G_1, M_2 = G_1 * G_2, \dots$ may change signs (from truth to false or vice versa) at most $c(\Phi)$ times.*

In section 8 we use the resolutions that are associated with a coefficient-free predicate over free products, and combine them with a modification of the strategy that was applied to prove the stability of free groups in [Se9], to prove that free products of stable groups is stable.

Theorem 8.1. *Let A and B be stable groups. Then $A * B$ is stable.*

This question was brought to our attention by E. Jaligot, and was the main reason for this entire work. In fact we prove a slightly stronger result, and show

that a free product of a countable collection of groups that are uniformly stable, is stable.

Theorem 8.7. *Let G_1, G_2, \dots be a sequence of groups. Suppose that every sentence Φ is uniformly stable over the sequence $\{G_i\}$. Then the countable free product, $G_1 * G_2 * \dots$, is stable.*

In section 9, we answer a question that we have learnt from I. Kazachkov [Ca-Ka], and prove that the free product of two equationally Noetherian groups is equationally Noetherian (theorem 9.1).

Finally, it is worth noting that our results for free products of groups, or slight strengthenings of them that are still valid over groups, can be shown to be false for free products of semigroups, using techniques of Quine [Qu] and Durnev [Du]. e.g., a free product of finite semigroups is in general unstable (although it is stable if the finite semigroups happen to be groups). Hence, it seems that model theoretic techniques that handle products of general structures, like the ones that were used by Feferman and Vaught, can not suffice to analyze the elementary theory of free products of groups.

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§1. Graded Limit Groups over Free Products

In [Ja-Se] we have studied systems of equations over free products. To do that we generalized limit groups over free groups to limit groups over free products. Limit groups over free products are equipped with an elliptic structure, i.e., with a finite collection of conjugacy classes of some f.g. subgroups, that are supposed to be mapped into elliptic elements in every homomorphism into a free product under consideration.

With a limit group over free products we have associated a JSJ decomposition over free products (theorem 11 in [Ja-Se]), that encodes all the abelian decompositions of such limit groups over non-elliptic abelian subgroups. Such a JSJ decomposition is non-trivial if the limit group is not (entirely) elliptic, and not abelian nor a 2-orbifold group.

We further proved a d.c.c. for limit groups over free products (theorem 13 in [Ja-Se]), that proves that a strictly decreasing sequence of epimorphisms of limit groups over free products that do not map elliptic elements into the trivial element, terminate after finitely many steps. This d.c.c. which is weaker than the one proved for limit groups over a free group (theorem 5.1 in [Se1]), is still sufficient for constructing a Makanin-Razborov diagram for a limit group, or for a system of equations over free products, and such a diagram is the final conclusion of [Ja-Se].

The Makanin-Razborov diagram over free products encodes all the homomorphism from a given f.p. group into free products, and hence, all the solutions of a given system of equations over arbitrary free products. However, unlike the construction of Makanin-Razborov diagrams over a free or a hyperbolic group, the

construction of the diagram over free products does not give a canonical diagram, but rather a collection of (strict) resolutions, that encode the entire set of homomorphisms into free products.

In this section we combine the construction of the Makanin-Razborov diagram over free products, with the construction of the graded Makanin-Razborov diagram over a free group, to construct a graded (relative) diagram over free products.

Let $\Sigma(x, p) = 1$ be a system of equations with (a tuple of) variables x and (a tuple of) parameters p . With $\Sigma(x, p)$ we naturally associate a f.p. group, $G(x, p)$, that is generated by copies of the variables x and parameters p , and the relations are words that correspond to the equations of $\Sigma(x, p)$. By theorem 21 in [Ja-Se] with the f.p. group $G(x, p)$ we associate (canonically) a finite collection of its maximal limit quotients (over free products), $(L_1(x, p), E_{L_1}), \dots, (L_t(x, p), E_{L_t})$. We continue with each of these limit groups in parallel, and denote such a limit group (over free products), $(L(x, p), E_L)$.

With the limit group $(L(x, p), E_L)$ we associate (canonically) a graded (relative) JSJ decomposition over free products (see theorem 11 in [Ja-Se]), that encodes all the splittings of $(L(x, p), E_L)$ over trivial and non-elliptic abelian subgroups, in which the parameter subgroup $\langle p \rangle$ is elliptic (i.e., contained in a vertex group). Like over a free or a hyperbolic group, and in a similar way to the ungraded case, with the graded JSJ decomposition (over free products), we naturally associate a graded modular group (over free products), $GMod(x, p)$, of the limit group $(L(x, p), E_L)$. Given the graded limit group $(L(x, p), E_L)$ and its graded modular group, $GMod(x, p)$, we naturally associate with $(L(x, p), E_L)$ a collection of shortening quotients.

As over free and hyperbolic groups, and unlike the ungraded (non-relative) case, it may be that the graded virtually-abelian JSJ decomposition of a graded limit group (over free products), $(L(x, p), E_L)$, is trivial. By construction, a shortening quotient of a graded limit group $(L(x, p), E_L)$ is a quotient of $L(x, p)$. In the ungraded case (over free products) a shortening quotient is always a proper quotient, or it is entirely elliptic. As in the free and hyperbolic case a shortening quotient may be isomorphic to the original limit group $L(x, p)$, even when it is not entirely elliptic.

Definition 1.1. *Let $(L(x, p), E_L)$ be a graded limit group over free products. If the graded (virtually abelian) JSJ decomposition of $(L(x, p), E_L)$ over free products is trivial, we say that $L(x, p)$ is rigid. If the following 3 conditions hold:*

- (1) *$(L(x, p), E_L)$ admits no non-trivial free decomposition in which the parameter subgroup $\langle p \rangle$, and the elliptic subgroups in $(L(x, p), E_L)$, are elliptic.*
- (2) *the graded JSJ decomposition of $(L(x, p), E_L)$ (over free products) is non-trivial.*
- (3) *$(L(x, p), E_L)$ has a graded shortening quotient which is isomorphic to $L(x, p)$ (as an abstract group), and this shortening quotient is not entirely elliptic.*

We say that $(L(x, p), E_L)$ is solid.

As in the case of free and hyperbolic groups, and unlike the ungraded case, associating shortening quotients with rigid and solid limit groups (over free products) is not helpful (in order to construct a diagram that encodes all the homomorphisms of a given graded limit group over free products). Like in the cases of free and hyperbolic groups, to analyze the collection of homomorphisms that factor through

a given rigid or solid limit group, we need to associate with such a graded limit group a finite collection of *flexible quotients*.

Definition 1.2. Let $(L(x, p), E_L)$ be either a rigid or a solid graded limit group (over free products). If L is a group of rank at most d , then with L we can associate a sequence of f.p. approximating subgroups $F_d \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$, so that G_m converges into L , and the abelian JSJ decomposition over free products of L lifts to abelian decompositions of the f.p. groups G_m . We fix generating sets of the groups G_m , and of the rigid or solid limit group, (L, E_L) , that is obtained from a generating set of the free group, F_d .

Given these generating sets, we associate Cayley graphs with each of the f.p. groups G_m , and with (L, E_L) . Let $\{h_m : G_m \rightarrow A_m^1 * \dots * A_m^\ell\}$ be a sequence of homomorphisms that converges into a quotient of the limit group over free products, (L, E_L) . With each free product, $A_m^1 * \dots * A_m^\ell$, we naturally associate the pointed Bass-Serre tree, (T_m, t_m) , that is associated with the free product (T_m, t_m) is dual to a finite tree of groups, having one vertex with trivial stabilizer, and ℓ vertices connected to it with stabilizers, A_m^1, \dots, A_m^ℓ . We denote by d_{T_m} the (simplicial) metric on the tree T_m . We say that the sequence of homomorphisms, $\{h_m\}$, is a flexible sequence if one of the following holds:

- (i) each homomorphism h_m can not be shortened (as measured in the trees T_m) by an element from the graded modular group (over free products) of the group G_m , (that lifts the graded modular group of the rigid or solid limit group $L(x, p)$).
- (ii) for each index m , let B_1 be the ball of radius 1 in the Cayley graph X_m of the f.p. G_m . Then:

$$\max_{g \in B_1} d_{T_m}(h_m(g)(t_m), t_m) > m \cdot (1 + \max_{1 \leq j \leq \ell} d_{T_m}(h_m(p_j)(t_m), t_m)).$$

where p_1, \dots, p_ℓ is a fixed generating set of the parameter subgroup $\langle p \rangle$ in G_m , which is the image of this subgroup in F_d .

A graded limit group (over free products) which is the limit of a flexible sequence is called a flexible quotient of the rigid or solid graded limit group $(L(x, p), E_L)$.

As in the free group case, the following are immediate properties of flexible quotients.

Lemma 1.3. Let $(Flx(x, p), E_F)$ be a flexible quotient (over free products) of the rigid or solid graded limit group (over free products) $(L(x, p), E_L)$. Then:

- (i) $(Flx(x, p), E_F)$ is not a rigid limit group (over free products), and is not entirely elliptic.
- (ii) $(Flx(x, p, a), E_F)$ is a proper quotient of $(L(x, p), E_L)$.

Proof: Identical to lemma 10.4 in [Se1].

□

Let $(L(x, p), E_L)$ be a rigid or solid limit group over free products. As over (ungraded) limit groups over free products, on the set of flexible quotients of $(L(x, p), E_L)$ we can naturally define a partial order and an equivalence relation, similar to the ones defined on limit groups (over free products) in [Ja-Se]. By the

same argument that is used to prove proposition 20 in [Ja-Se], the set of flexible quotients of $(L(x,p), E_L)$ contains maximal elements with respect to the partial order. Theorem 1.4 proves that there are at most finitely many maximal flexible quotients of a rigid limit group over free products.

Proposition 1.4. *Let $(L(x,p), E_L)$ be a rigid limit group over free products. Then there exist only finitely many maximal flexible quotients of $(L(x,p), E_L)$ (up to equivalence).*

Furthermore, recall that with the (rigid) limit group over free products, (L, E_L) , we can naturally associate finitely many elliptic subgroups, E_1, \dots, E_r . Each maximal flexible quotient of the rigid limit group, (L, E_L) , can be embedded in an (ungraded) completion, $Comp$, where the completion, $Comp$, is obtained from the elliptic subgroups of the rigid limit group, (L, E_L) , E_1, \dots, E_r , by adding finitely many generators and relations.

Proof: Identical to the proof of theorem 21 in [Ja-Se]. □

Note that in case a rigid limit group, (L, E_L) , is finitely presented, theorem 1.4 implies that each maximal flexible quotient of (L, E_L) can be embedded into a f.p. completion (with f.p. terminal elliptic subgroups).

Theorem 1.4 proves that rigid limit groups over free products have finitely many maximal flexible quotients. For solid limit groups (over free products) we prove a slightly weaker statement. We do not prove the existence of finitely many maximal flexible quotients of a solid limit group, but instead we prove the existence of finitely many covers of flexible quotients, that cover all the flexible quotients of the solid limit group (see theorem 24 in [Ja-Se] for a cover of a limit quotient of a limit group over free products).

Proposition 1.5. *Let $(L(x,p), E_L)$ be a solid limit group over free products. Then there exists a finite collection of covers of maximal flexible quotients of $(L(x,p), E_L)$, so that:*

- (1) *each cover is a proper quotient of $(L(x,p), E_L)$.*
- (2) *each cover has a non-trivial graded JSJ decomposition over free products.*
- (3) *let E_1, \dots, E_r be the elliptic subgroups of (L, E_L) . Then each cover can be embedded in an (ungraded) completion that is obtained from the elliptic subgroups of (L, E_L) , E_1, \dots, E_r , by adding finitely many generators and relations.*
- (4) *every flexible quotient of $(L(x,p), E_L)$ is dominated by at least one of the covers from the finite collection.*
- (5) *let $h_n : L(x,p) \rightarrow A_n^1 * \dots * A_n^\ell$ be a sequence of homomorphisms that converges into a flexible quotient of $(L(x,p), E_L)$. Then there exists a subsequence of the homomorphisms $\{h_n\}$ that do all factor through one of the covers from the fixed finite collection of covers.*

Proof: Identical to the proof of theorem 25 in [Ja-Se]. □

As in the case of free groups, having defined flexible quotients over free products, we are able to define *flexible homomorphisms*, and *rigid* and *solid homomorphisms*. Since a limit group over free products is in general f.g. and not necessarily finitely

presented, for rigid and solid limit groups we define both rigid and solid homomorphisms that factor through the (rigid or solid) limit group, and *asymptotically* rigid or solid sequences of homomorphisms that converge to limit groups that are dominated by the rigid or solid limit group.

Definition 1.6. *Let $(Rgd(x, p), E_R)$ be a rigid limit group over free products, and let:*

$$(Flx_1(x, p), E_{F_1}), \dots, (Flx_v(x, p), E_{F_v})$$

*be the maximal flexible quotients of it. A homomorphism $h : Rgd(x, p) \rightarrow A^1 * \dots * A^\ell$ that does not factor through any of the maximal graded flexible quotients of $(Rgd(x, p), E_R)$ is called a rigid homomorphism (specialization) of the rigid limit group $(Rgd(x, p), E_R)$. A homomorphism of $(Rgd(x, p), E_R)$ into a free product that does factor through one of the maximal flexible quotients is called a flexible homomorphism (specialization).*

*Suppose that the rank of the rigid limit group, (Rgd, E_R) , is at most d . A sequence of homomorphisms, $\{h_n : F_d \rightarrow A_n^1 * \dots * A_n^\ell\}$, that converges to a limit quotient (U, E_U) of (Rgd, E_R) is called asymptotically rigid for (Rgd, E_R) , if the limit group (U, E_U) is not dominated by any of the flexible quotients of (Rgd, E_R) , i.e., if there is no epimorphism $\tau : (Flx, E_F) \rightarrow (U, E_U)$, where (Flx, E_F) is one of the flexible quotients of (Rgd, E_R) . A sequence of homomorphisms $\{h_n : F_d \rightarrow A_n^1 * \dots * A_n^\ell\}$, that converges to a limit quotient (U, E_U) of (Rgd, E_R) that is not asymptotically rigid is called asymptotically flexible.*

Definition 1.7. *Let $(Sld(x, p), E_S)$ be a solid limit group over free products. With $(Sld(x, p), E_S)$ we associate a (fixed) finite collection of covers that satisfies the conclusion of proposition 1.5. A homomorphism $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$ for which $h = h' \circ \varphi$ where h' factors through one of the covers from the fixed finite collection of covers that is associated with $(Sld(x, p), E_S)$, and φ is a graded modular automorphism (over free products) of $(Sld(x, p), E_S)$, is called a flexible homomorphism (specialization) of the solid limit group $(Sld(x, p), E_S)$ (with respect to the given finite collection of covers).*

*A non-flexible homomorphism $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$ is called a solid homomorphism (specialization) of the solid limit group (over free products) $(Sld(x, p), E_S)$ (with respect to the fixed finite collection of covers).*

*Suppose that the rank of the solid limit group, (Sld, E_S) , is at most d . With (L, E_L) we can associate a sequence of f.p. groups, $F_d \rightarrow G_1 \rightarrow G_2 \dots$, that converges into Sld , so that the graded abelian decomposition of Sld lifts to abelian decompositions of the f.p. groups $\{G_n\}$. A sequence of homomorphisms $\{h_n : G_n \rightarrow A_n^1 * \dots * A_n^\ell\}$, that converges to a limit quotient (U, E_U) of (Sld, E_S) is called asymptotically solid for (Sld, E_S) , if there is no subsequence of homomorphisms (still denoted) $\{h_n\}$, and automorphisms $\{\varphi_n\}$ from the graded modular automorphisms of the groups $\{g_n\}$, so that the sequence $\{h_n \circ \varphi_n : G_n \rightarrow A_n^1 * \dots * A_n^\ell\}$ converges into a limit quotient of one of the covers that are associated with (Sld, E_S) according to theorem 1.5.*

*A sequence of homomorphisms $\{h_n : G_n \rightarrow A_n^1 * \dots * A_n^\ell\}$, for which there are automorphisms, $\{\varphi_n\}$, so that $h_n \circ \varphi_n$ converges to a limit quotient of one of the covers that are associated with (Sld, E_S) is called asymptotically flexible.*

As in the case of free groups, flexible quotients of rigid and solid limit groups over free products contain all the "generic infinite families" of homomorphisms (into free products) of these graded limit groups. Rigid solutions of rigid limit groups over free products are the exceptional solutions, and solid solutions of solid limit groups are the exceptional families of solutions.

Proposition 1.8. *Let $(Rgd(x, p), E_R)$ be a rigid limit group, and let $(Sld(x, p), E_S)$ be a solid limit group. With $(Sld(x, p), E_S)$ we associate a (fixed) finite collection of covers that satisfies the conclusion of proposition 1.5. Let p_0 be a specialization of the defining parameters p . Let x_1, \dots, x_f be a fixed generating set of Rgd or Sld , and let p_1, \dots, p_u be a fixed generating set for the parameter subgroup $\langle p \rangle$. Then there exists a constant $c(p_0)$ so that:*

- (i) *let $h : Rgd(x, p) \rightarrow A^1 * \dots * A^\ell$ be a rigid homomorphism for which $h(p) = p_0$. With the free product $A^1 * \dots * A^\ell$ we associate its pointed Bass-Serre tree (T, t) . Then:*

$$\max_{1 \leq i \leq f} d_T(h(x_i)(t), t) < c(p_0) \cdot \max_{1 \leq j \leq u} d_T(h(p_j)(t), t)$$

Note that the constant $c(p_0)$ depends on the parameter p_0 , but not on the rigid homomorphism h .

- (ii) *let $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$ be a solid homomorphism for which $h(p) = p_0$ with respect to the given finite collection of covers that is associated with $(Sld(x, p), E_S)$. With the free product $A^1 * \dots * A^\ell$ we associate its pointed Bass-Serre tree (T, t) . Then there exists a graded modular automorphism (over free products) of the solid limit group $Sld(x, p)$, φ , so that $h = h' \circ \varphi$, and:*

$$\max_{1 \leq i \leq f} d_T(h'(x_i)(t), t) < c(p_0) \cdot \max_{1 \leq j \leq u} d_T(h'(p_j)(t), t)$$

Proof: Similar to the proof of proposition 10.7 in [Se1]. □

In studying the first order theory of free and hyperbolic groups, it was essential to strengthen the finiteness of the number of rigid and solid families of specializations for any given value of the defining parameters, to a global bound on the number of rigid and strictly solid families (these bounds were proved in [Se3]).

Over free products the number of rigid or families of solid solutions for a given value of the defining parameters is not finite in general, hence, we can't expect a strong form of global boundedness. One way to generalize the boundedness results from free and hyperbolic groups would be to prove that the constant, $c(p_0)$, that appears in the formulation of proposition 1.8 can be taken to be uniform, i.e., independent of the specific value of the parameters p_0 . This can indeed be done, but it won't suffice to analyze first order predicates and sentences over free products.

To analyze predicates over free products, we look for a different (stronger) type of generalization of proposition 1.8. We show that in an appropriate sense, the collection of rigid and solid families of homomorphisms, is contained in boundedly many families, where the bound on the number of families does not depend on the specific value of the defining parameters. Furthermore, these families can be defined

using certain (AE like) predicates, that are crucial in proving a form of quantifier elimination over free products.

Our strategy to define these families and the uniform bound on their number generalizes the argument that was used in [Se3] to prove a uniform bound on the number of rigid and strictly solid families (over a free group). We start by proving combinatorial boundedness for rigid and (shortest) solid specializations, which is similar to the one that was proved in section 1 of [Se3], and then use this combinatorial boundedness to prove the existence of a uniform bound on the number of rigid and strictly solid families of homomorphisms.

In principle, the combinatorial bound we are looking for is showing that there exist finitely many (combinatorial) ways to cut the specializations of the defining parameters, and finitely many words in these pieces (*fractions*), so that the specializations of the (given) generating set of a rigid or solid limit group inherited from any rigid or (shortest) solid specialization, can be presented as one of these (finitely many) words in fractions of the specializations of the defining parameters. To prove such a combinatorial bound, we actually prove a stronger result (cf. theorems 1.2 and 1.7 in [Se3]). To state the stronger statement we need the notion of an *R-P-covered* (graded) homomorphism.

Let X be the Cayley graph of a rigid or solid limit group (over free products), $Rgd(x, p)$ or $Sld(x, p)$, with respect to the generating system $Rgd(x, p) = \langle x, p \rangle$ (or $Sld(x, p) = \langle x, p \rangle$). Let $h : Rgd(x, p) \rightarrow A^1 * \dots * A^\ell$ (or $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$) be a homomorphism, and let (T, t) be the pointed Bass-Serre tree that corresponds to the free product $A^1 * \dots * A^\ell$. Clearly, the homomorphism h corresponds to a natural equivariant map $\tau : X \rightarrow T$, where each edge in X is mapped to a (possibly degenerate) path in T .

Definition 1.9. *Let B_R be the ball of radius R in the Cayley graph X of $Rgd(x, p)$ (or $Sld(x, p)$). We say that a homomorphism $h : Rgd(x, p) \rightarrow A^1 * \dots * A^\ell$ (or $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$) is *R-P-covered*, if the image in T of the union of edges labeled by an element of $\{p\}$ in B_R covers the entire image in T of the ball B_1 .*

As over free groups (theorem 1.2 in [Se3]), to control the combinatorial types of rigid solutions over free products, we need the following basic theorem, that bounds combinatorially the structure of rigid solutions (specializations) of a rigid limit group (over free products).

Theorem 1.10. *Let $Rgd(x, p)$ be a rigid limit group over free products. There exists a constant R_0 so that every rigid homomorphism $h : Rgd(x, p) \rightarrow A^1 * \dots * A^\ell$ is R_0 -*P-covered*.*

Proof: Identical to the proof of theorem 1.2 in [Se3]. □

As over free groups, to state a similar theorem for (shortest) solid homomorphisms, we first need to define *strictly solid homomorphisms* of a f.p. solid limit group (over free products) with respect to a given finite collection of covers of its flexible quotients (see proposition 1.5). The definition we present is similar to the definition of strictly solid homomorphisms over free groups that are defined in definition 1.5 in [Se3]. Strictly solid homomorphisms, and their families, are the ones

that needed to be considered in analyzing first order predicates over free products.

Definition 1.11. *Let $(Sld(x, p), E_S)$ be a solid limit group over free products, and let:*

$$(CFlx_1(x, p), E_1), \dots, (CFlx_s(x, p), E_s)$$

be a finite collection of covers that satisfies the conclusion of proposition 1.5.

With each cover, $(CFlx(x, p), E)$, we associate a natural one step resolution given by the quotient map: $Sld(x, p) \rightarrow CFlx(x, p)$, and the graded abelian JSJ decomposition over free products of the solid limit group $(Sld(x, p), E_S)$, Λ_S . Given the one step resolution, we associate with it the entire collection of homomorphisms (into free products) that factor through the one step resolution, and the limit group corresponding to that collection that we denote $(Q(x, p), E_Q)$, which is a quotient of the solid limit group $(Sld(x, p), E_S)$.

The limit group $Q(x, p)$ inherits a graded abelian decomposition over free products, Δ_Q , from the graded abelian decomposition of the solid limit group $(Sld(x, p), E_S)$, which is in general either similar to the abelian decomposition Λ_S (i.e., it has isomorphic QH and abelian vertex and edge groups, and the other (rigid) vertex groups in Δ_Q are quotients of the corresponding ones in Λ_S), or it is a degeneration of the abelian decomposition Λ_S , that is associated with $(Sld(x, p), E_S)$. By construction, the one step resolution given by the quotient map $Q(x, p) \rightarrow CFlx(x, p)$, and the graded abelian decomposition Δ_Q of $Q(x, p)$, is a strict resolution.

With the one step, strict (well-structured) resolution $Q(x, p) \rightarrow CFlx(x, p, a)$, with the abelian decomposition Δ_Q , we associate its completion, according to the construction that appears in definition 1.12 of [Se2]. We denote the obtained completion, $Comp_{CFlx}$.

By the construction of the completion, $Comp_{CFlx}$, the solid limit group $(Sld(x, p), E_S)$ is naturally mapped into it (onto the subgroup that is associated with its top level, that is isomorphic to $(Q(x, p), E_Q)$). Let $\tau : Sld(x, p) \rightarrow Comp_{CFlx}$ be this natural map.

A homomorphism, $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$, of the solid limit group over free products, $(Sld(x, p), E_S)$, is called a strictly solid homomorphism (specialization) if it is non-degenerate (definition 11.1 in [Se1]), and for every index i , $1 \leq i \leq s$, it does not factor through the homomorphisms $\tau_i : Sld(x, p) \rightarrow Comp_{CFlx_i}$. Note that a strictly solid specialization is, in particular, a solid specialization (with respect to the given finite collection of covers).

Solid limit groups over free products may be infinitely presented. For completeness, and since it will be required in the next sections, we further associate with a solid limit group its collection of asymptotic strictly solid sequences of homomorphisms.

Definition 1.12. *Let $(Sld(x, p), E_S)$ be a solid limit group over free products, Suppose that the rank of the solid limit group, (Sld, E_S) , is at most d . With (L, E_L) we can associate a sequence of f.p. groups, $F_d \rightarrow G_1 \rightarrow G_2 \dots$, that converges into Sld , so that the graded abelian decomposition of (Sld, E_S) lifts to abelian decompositions of the f.p. groups $\{G_n\}$. With each f.p. group G_n we can formally associate the completion, $Comp_n$, of the identity map, $G_n \rightarrow G_n$ (note that G_n is not a limit group over free products, but only its abelian decompositions (that lifts the one of (Sld, E_S) is needed for defining the completion).*

A sequence of homomorphisms $\{h_n : G_n \rightarrow A_n^1 * \dots * A_n^\ell\}$, that converges to a limit quotient (U, E_U) of (Sld, E_S) is called *asymptotically strictly solid* for (Sld, E_S) , if there is no subsequence of homomorphisms (still denoted) $\{h_n\}$, and homomorphisms $\{u_n : G_n \rightarrow A_n^1 * \dots * A_n^\ell\}$, so that for each index n the pair of homomorphisms, (h_n, u_n) , extends to a homomorphism, $\tau_n : Comp_n \rightarrow A_n^1 * \dots * A_n^\ell$, where the restrictions of τ_n to the two copies of G_n in $Comp_n$ are the homomorphisms h_n and u_n , and so that the sequence $\{u_n\}$ converges into a limit quotient of one of the covers that are associated with (Sld, E_S) according to theorem 1.5.

Like over free groups, and in a similar way to the combinatorial boundedness of rigid homomorphisms, in theorem 1.13 we state the existence of a combinatorial bound for strictly solid homomorphisms that are the shortest in their solid family.

Theorem 1.13. *Let $(Sld(x, p), E_S)$ be a solid limit group, and let: $(CFlx_1, E_1), \dots, (CFlx_s, E_s)$ be a finite collection of covers of flexible quotients of (Sld, E_S) that satisfies the conclusion of proposition 1.5. Then there exists a constant R_0 so that every strictly solid homomorphism $h : Sld(x, p, a) \rightarrow F_k$ which is among the shortest in its solid family, is R_0 - P -covered.*

Proof: Identical to the proof of theorem 1.7 in [Se3]. □

So far we generalized the combinatorial boundedness of rigid and shortest strictly solid homomorphisms from free and hyperbolic groups to the context of rigid and solid limit groups over free products, i.e., we showed that rigid and (shortest) strictly solid homomorphisms over free products of a given f.p. rigid or solid limit group (over free products) are R_0 - P -covered, for some constant R_0 that depends only on the rigid or solid limit group.

Over free and hyperbolic groups we were able to use this combinatorial boundedness to obtain a global bound on the number of rigid and families of strictly solid homomorphisms for any given value of the defining parameters (theorems 2.5 and 2.9 in [Se3]). Over free products such a global bound can not exist. However, once we define appropriately families of rigid and strictly solid homomorphisms over free products, it is possible to use the combinatorial bounds to obtain a global (uniform) bound on the number of rigid and strictly solid families, a bound that suffices for the analysis of first order predicates over free products in the next sections.

Theorem 1.14. *Let $(Rgd(x, p), E_R)$ be a rigid limit group over free products, generated by x_1, \dots, x_s and so that the parameter subgroup is generated by p_1, \dots, p_u . Let $(Flx_1, E_1), \dots, (Flx_t, E_t)$ be the set of maximal flexible quotients of $(Rgd(x, p), E_R)$ (see proposition 1.4).*

There exist finitely many (combinatorial) systems of fractions of the defining parameters (that depend only on the rigid limit group $(Rgd(x, p), E_R)$):

$$p_1 = v_1 \dots v_{i_1}, p_2 = v_{i_1+1} \dots v_{i_2}, \dots, p_u = v_{i_{u-1}+1} \dots v_{i_u}$$

$$x_j = x_j(v_1, \dots, v_{i_u}, a_1, \dots, a_f) \quad j = 1, \dots, s.$$

(where the indices i_1, \dots, i_u and the words x_1, \dots, x_s may depend on the system of fractions), so that:

- (i) *with each triple consisting of a fixed free product, $A^1 * \dots * A^\ell$, a value of the defining parameters p_0 from the free product $A^1 * \dots * A^\ell$, and a*

(combinatorial) system of fractions (one out of the finitely many possible ones), there is either a unique or no associated values of the fractions, v_1, \dots, v_{i_u} (that are all taken from $A^1 * \dots * A^\ell$).

- (ii) for each rigid homomorphism: $h : \text{Rgd}(x, p) \rightarrow A^1 * \dots * A^\ell$, that satisfies $h(p) = p_0$, there exist at least one of the (finitely many combinatorial) systems of fractions, so that the associated values of the fractions, v_1, \dots, v_{i_u} (that depend only on p_0 and the combinatorial system of fractions), together with some elements, $a_1, \dots, a_f \in A^1 \cup \dots \cup A^\ell$, satisfy:

$$h(x_j) = x_j(v_1, \dots, v_{i_u}, a_1, \dots, a_f) \quad j = 1, \dots, s.$$

Proof: By theorem 1.10 there exists some integer, R_0 , so that every rigid homomorphism is R_0 - P -covered. This implies the existence of finitely many combinatorial systems of fractions that have the properties that are listed in the theorem, i.e., that the values of a fixed generating set, x_1, \dots, x_s , under any given rigid homomorphism into a free product, are specified by fixed words that are associated with the combinatorial systems, and these words are in terms of fractions of values of a generating set of the parameter subgroup, p_1, \dots, p_u , and finite number of elements from the factors of the free product.

To complete the proof of the theorem one is still required to show that for each combinatorial system of fractions, there is a global bound b_R (that does not depend on the parameters value), so that for each free product, $G = A^1 * \dots * A^\ell$, and each given value of the parameters, p_1, \dots, p_u , in G , it possible to "cut" the parameters in finitely many ways to obtain the values of (the fractions), v_1, \dots, v_{i_u} , to obtain all the rigid specializations that are associated with the value of p_1, \dots, p_u , and the number of ways we need to "cut" the parameters to obtain the fractions, is bounded by the global bound b_R .

The existence of such a bounded follows from the proof of theorem 2.5 in [Se3]. Note that the argument that is used to prove theorem 2.5 in [Se3] proves exactly the statement of theorem 1.13 (also in the case of free products), just that in the case of free (or hyperbolic) groups this statement implies a bound on the number of rigid specializations for each value of the defining parameters (theorem 2.5 in [Se3]), and this can not be deduced (in general) over free products. □

The bound on the number of rigid families of homomorphisms of a rigid limit group over free products that is stated in theorem 1.14 has an analogue for strictly solid families of homomorphisms of a solid limit group over free products.

Theorem 1.15. *Let $\text{Sld}(x, p)$ be a solid limit group over free products, generated by x_1, \dots, x_s and so that the parameter subgroup is generated by p_1, \dots, p_u . Let $(\text{CFl}x_1, E_1), \dots, (\text{CFl}x_t, E_t)$ be a finite collection of covers of flexible quotients of $(\text{Sld}(x, p), E_S)$ (see proposition 1.5).*

There exist finitely many (combinatorial) systems of fractions of the defining parameters (that depend only on the solid limit group $(\text{Sld}(x, p), E_S)$ and the fixed collection of covers):

$$p_1 = v_1 \dots v_{i_1}, \quad p_2 = v_{i_1+1} \dots v_{i_2}, \quad \dots, \quad p_u = v_{i_{u-1}+1} \dots v_{i_u}$$

$$x_j = x_j(v_1, \dots, v_{i_u}, a_1, \dots, a_f) \quad j = 1, \dots, s.$$

(where the indices i_1, \dots, i_u and the words x_1, \dots, x_s may depend on the combinatorial system of fractions), so that:

- (i) with a triple that consists of a free product, $A^1 * \dots * A^\ell$, a value of the defining parameters p_0 from the free product $A^1 * \dots * A^\ell$, and a combinatorial system of fractions (one out of finitely many), there is either a unique or no associated values of the fractions, v_1, \dots, v_{i_u} (that are all taken from $A^1 * \dots * A^\ell$).
- (ii) for each (almost) shortest strictly solid homomorphism: $h : Sld(x, p) \rightarrow A^1 * \dots * A^\ell$, that satisfies $h(p) = p_0$, there exist at least one of (the finitely many combinatorial) systems of fractions, so that the associated values of the fractions, v_1, \dots, v_{i_u} (that depend only on p_0 and the combinatorial system of fractions), together with some elements, $a_1, \dots, a_f \in A^1 \cup \dots \cup A^\ell$, satisfy:

$$h(x_j) = x_j(v_1, \dots, v_{i_u}, a_1, \dots, a_f) \quad j = 1, \dots, s.$$

Proof: The existence of finitely many combinatorial systems of fractions that are good for all almost shortest strictly solid specializations follows by theorem 1.11. The existence of a global bound on the number of "cuts" or fractions that are needed in order to cover all the almost shortest specializations follows from the proof of theorem 2.9 in [Se3].

□

Graded limit groups, their graded (relative) abelian JSJ decompositions (over free products), their graded shortening quotients, and rigid and solid limit groups and their flexible quotients, allow us to associate a graded Makanin-Razborov diagram over free products with any given f.p. group $G(x, p)$. As in the ungraded case, the diagram is not canonical, but as over free groups, it encodes all the homomorphisms of the group $G(x, p)$ into free products for all the possible values of the parameters p .

To construct the graded Makanin-Razborov diagram over free products, we need two basic objects, that were constructed and used in the ungraded case. First given a f.p. group $G(x, p)$ we need to associate with $G(x, p)$ (or a quotient of it) a collection of (well-structured) graded resolutions. Then given a well-structured graded resolution we need to associate with it a cover that satisfies similar properties to the cover of an ungraded resolution (see theorem 24 in [Ja-Se]). The entire collection of covers of resolutions of $G(x, p)$ should encode all the homomorphisms of $G(x, p)$ into free products, for every possible value of the defining parameters. Finally, to construct the graded Makanin-Razborov diagram we will show that there exist a finite collection of covers of graded resolutions through which all the homomorphisms from the given graded limit group into free products do factor.

Proposition 1.16. *Let $G(x, p)$ be a f.g. group, and let $L(x, p)$ be a limit quotient of $G(x, p)$ (over free products). Then there exists a finite sequence of limit groups over free products:*

$$L = L_0(x, p) \rightarrow L_1(x, p) \rightarrow L_2(x, p) \rightarrow \dots \rightarrow L_s(x, p)$$

for which:

- (i) non-trivial elliptic elements in L_i are mapped to non-trivial elliptic elements in L_{i+1} , $i = 1, \dots, s - 1$.

- (ii) if L_i is not rigid nor solid, then L_{i+1} is a shortening quotient of L_i , for $i = 1, \dots, s - 1$.
- (iii) in case L_i is a free product of a solid limit group with (possibly) elliptic factors and (possibly) a free group, for some $i < s$, then L_{i+1} is a free product of a quotient of one of the covers that are associated with the solid factor (according to proposition 1.5) with (possibly) the same elliptic factors and (possibly) the same free group, for $i = 1, \dots, s - 1$.
- (iv) The epimorphisms along the sequence are proper epimorphisms.
- (v) L_s is a free product of a rigid or a solid limit group with (possibly) some elliptic factors and (possibly) a free group. With the rigid limit group (which is a factor in L_s) we associate its finite collection of maximal flexible quotients, and with a solid limit group (which is a factor of L_s), we associate a finite collection of covers of its flexible quotients that satisfies the conclusion of proposition 1.5.
- (vi) the resolution: $L_0 \rightarrow L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_s$ is a graded strict resolution ([Se1],5), i.e., in each level non-QH, non-virtually-abelian vertex groups and edge groups in the graded JSJ decomposition (over free products) are mapped monomorphically into the limit group in the next level, and QH vertex groups are mapped into non-virtually-abelian, non-elliptic subgroups).
- (vii) the constructed resolution is well-structured (see definition 1.11 in [Se2] for a well-structured resolution). As a corollary, the graded limit group (over free products) $L = L_0$ is embedded into the completion of the well-structured resolution:

$$L_0 \rightarrow L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_s$$

so that all the elliptic elements in L are mapped into conjugates of the elliptic subgroups of the completion.

Proof: Given a sequence of homomorphisms $\{h_n\}$ of a given group G into free products, $A_n^1 * \dots * A_n^\ell$, that converges to a limit group (L, E_L) over free products, it is shown in theorem 18 of [Ja-Se] how to associate a strict resolution with a subsequence of the sequence of homomorphisms. with the notions of graded limit groups over free products, their graded abelian decompositions, rigid and solid graded limit groups over free products, their flexible quotients, and asymptotically rigid and strictly solid sequences, the construction of an (ungraded) resolution over free products generalizes naturally to the construction of a graded resolution. □

Proposition 1.16 associates graded resolutions with a f.g. group. In the ungraded case, in order to associate a (non-canonical) Makanin-Razborov diagram over free products with a given f.p. group, we have replaced each resolution of some limit quotient (over free products) of the given f.p. group with a *cover* of that limit quotient, and the entire resolution with a cover resolution, so that each cover resolution has a f.p. completion (and terminates in a f.p. group). This existence of cover resolutions enables one to show that it is sufficient to choose finitely many cover resolutions, so that every homomorphism of the original (given) f.p. group into free products factors through at least one of the finite (non-canonical) collection of cover resolutions (see theorems 24-27 in [Ja-Se]).

Our approach to constructing a graded Makanin-Razborov diagram of a given f.p. group $G(x, p)$ is conceptually similar. However, we will need to modify the notion of a cover resolution to the graded case. As our goal is to obtain a finite

collection of (graded) covers so that for each value of the defining parameters the homomorphisms of $G(x, p)$ into free factors will factor through the given finite collection, we need to define cover resolutions in a finite way, i.e., graded covers should be embedded into f.p. completions (over free products). Also, we need to define the covers of flexible quotients that are attached to (terminal) rigid and solid limit groups in such a way that they can be embedded into f.p. completions. To achieve these goals we need to slightly modify the definitions and the constructions that we used, so that covers can be defined in a "finite" way, and all our previous results, especially those for rigid and solid limit groups remain valid.

Theorem 1.17. *Let $(Rgd(x, p), E_R)$ be a rigid limit group over free products. Suppose that $(Rgd(x, p), E_R)$ is a (rigid) limit quotient of a f.p. group, $G(x, p)$, and let $\{h_n : G(x, p) \rightarrow A_n^1 * \dots * A_n^\ell\}$ be an asymptotically rigid sequence of homomorphisms (of Rgd) that converges into $(Rgd(x, p), E_R)$.*

There exists an approximating rigid limit group of $(Rgd(x, p), E_R)$, that we denote $(APRgd, E_{APR})$, with the following properties:

- (1) *(Rgd, E_R) is a quotient limit of $(APRgd, E_{APR})$. $APRgd$ is a rigid limit quotient of $G(x, p)$ (over free products).*
- (2) *there is an ungraded resolution (over free products) $WRes, APRgd \rightarrow W_1 \rightarrow \dots \rightarrow W_s$, so that W_s is f.p. and there exists a subsequence of homomorphisms, (still denoted) $\{h_n\}$, that factor through this resolution. In particular, the completion of the resolution $WRes$ is f.p. and the approximating limit group, $(APRgd, E_{APR})$, is embedded into this f.p. completion.*
- (3) *with $(APRgd, E_{APR})$ we associate finitely many limit quotients (over free products) of $G(x, p)$, CF_1, \dots, CF_g , with ungraded resolutions (over free products), Res_1, \dots, Res_g , that terminate in f.p. limit groups. Hence, the completions of Res_1, \dots, Res_g are f.p. and the limit groups (over free products), CF_1, \dots, CF_g , can be embedded in f.p. completions (that have f.p. terminal limit groups).*
- (4) *the finite collection of limit groups, CF_1, \dots, CF_g , are all limit quotients of $G(x, p)$, and they dominate all the flexible quotients of $(APRgd, E_{APR})$ (although they need not be quotients of the rigid limit group $(APRgd, E_{APR})$). Furthermore, every flexible homomorphism of the rigid approximation, $(APRgd, E_{APR})$, factors through at least one of the resolutions, Res_1, \dots, Res_g .*
- (5) *from the subsequence of homomorphisms, $\{h_n\}$, that factor through the resolution, $WRes$, it is possible to extract a subsequence that does not factor through any of the limit groups over free products, CF_1, \dots, CF_g , and in particular they do not factor through the resolutions, Res_1, \dots, Res_g . This implies that the homomorphisms, $\{h_n\}$, are rigid homomorphisms of the rigid limit group (over free products) $(APRgd, E_{APR})$.*

Proof: Let (Rgd, E_R) be a rigid limit group over free products, and let $\{h_n : G(x, p) \rightarrow A_n^1 * \dots * A_n^\ell\}$ be a sequence of asymptotically rigid homomorphisms that that converges into the rigid limit group over free products, (Rgd, E_R) .

By theorem 18 in [Ja-Se], starting with the sequence of homomorphisms, $\{h_n\}$, we can pass to a subsequence, and associate with the rigid limit group over free products, (Rgd, E_R) , a strict well-structured ungraded resolution, that we denote $Res, Rgd = W_0 \rightarrow W_1 \rightarrow \dots \rightarrow W_t$, so that W_t is the free product of (possibly) finitely many elliptic factors and (possibly) a free group.

We set $Comp_W$ to be the completion of this ungraded resolution. Note that $Comp_W$ is generated by the terminal limit group W_t , in addition to finitely many elements and relations. However, W_t may be not finitely presented.

We modify $Comp_W$ by replacing W_t with a sequence of f.p. approximations of it. Starting with $Comp_W$ we define a sequence of completions, obtained by replacing the terminal limit group (over free products) in $Comp_W$, W_t , with approximating f.p. groups, that we denote W_t^m . These groups, W_t^m , are obtained by replacing each of the elliptic factors of W_t , with a f.p. group that is generated by some fixed generating set of the corresponding elliptic factor of W_t , and imposing only the relations up to length m in this (fixed) generating set that are equal to the identity in the corresponding factor of W_t . For every index m , W_t is a quotient of W_t^m , and for m large enough, the group obtained by replacing W_t with W_t^m in $Comp_W$, is a completion of a strict well-structured resolution (over free products), and we denote the obtained completion, $Comp_W^m$.

Note that the approximating completions, $Comp_W^m$, are all finitely presented. With $Comp_W^m$, we naturally associate an ungraded resolution over free products, that we denote Res^m , $W_0^m \rightarrow W_1^m \rightarrow \dots \rightarrow W_t^m$. The limit groups over free products, W_0^m, \dots, W_t^m , are all embedded in the f.p. completion, $Comp_W^m$. The limit groups (over free products), W_0, \dots, W_t , are quotients of the limit groups over free products, W_0^m, \dots, W_t^m , in correspondence, and the direct limit of the limit groups over free products, W_0^m, \dots, W_t^m , are W_0, \dots, W_t , in correspondence.

By construction, with each of the graded limit groups, W_i , there is an associated virtually abelian decomposition, which is the JSJ decomposition (over free products) of W_i . By the construction of $Comp_W^m$, the JSJ decompositions of the limit groups (over free products), W_i , lift to virtually abelian decompositions of the (approximating) graded limit groups over free products, W_i^m .

By the proof of theorem 27 in [Ja-Se], for large enough index m , these virtually abelian decompositions of the limit groups W_i^m (that are lifted from W_i) are their JSJ decompositions over free products. Furthermore, by the same argument (that appears in detail in the proof of theorem 27 in [Ja-Se]), for large enough m , the graded JSJ decomposition of W_0^m is trivial, which means that for large m , W_0^m is rigid.

So far we have shown that for large index m , the resolutions Res^m and their initial rigid limit group W_0^m , satisfy properties (1) and (2) of the theorem. To get the other parts, we first need to show that for large index m , there exists some index n_m , so that for every $n > n_m$, the homomorphism h_n is a rigid homomorphism of the rigid limit group, W_0^m .

The sequence of homomorphisms, $\{h_n\}$, is an asymptotically rigid sequence of homomorphisms with respect to the original rigid limit group, (Rgd, E_R) , and it converges to (Rgd, E_R) . Since the completions, $Comp_W^m$, are finitely presented, for each index m , there exists some index, a_m , so that for every $n > a_m$, the sequence of homomorphisms, $\{h_n\}$, factor through $Comp_W^m$, and in particular through the rigid limit group, W_0^m .

Suppose that there exists a subsequence of indices, still denoted m , so that for every index m , there is a subsequence of homomorphisms (where the subsequence may depend on m), still denoted, $\{h_n\}$, that are not rigid homomorphisms of W_0^m , i.e., they factor through a flexible quotient of W_0^m . In this case, the sequence of homomorphisms, $\{h_n\}$, converges into a flexible quotient of (Rgd, E_R) , a contradiction to (Rgd, E_R) being rigid. Therefore, for large enough index m , there exists

an index n_m , so that for every $n > n_m$, the homomorphisms $\{h_n\}$ that do all factor through W_0^m , are rigid homomorphisms.

Furthermore, for large enough index m , W_0^m is rigid, hence, we can associate with W_0^m its finite collection of maximal flexible quotients, $(Flx_1^m, E_{F_1}^m), \dots, (Flx_v^m, E_{F_{v(m)}}^m)$ (such a collection of maximal flexible quotients exists by proposition 1.4). If there exists a subsequence of the indices m , so that the original rigid limit group, (Rgd, E_R) , which is a limit quotient of W_0^m , is a limit quotient of at least one of the maximal flexible quotients of W_0^m , then (Rgd, E_R) , can not be a rigid limit group, a contradiction. Therefore, for m large enough, (Rgd, E_R) is not a limit quotient of any of the maximal flexible quotients of W_0^m .

We pick such a large index m . Since (Rgd, E_R) is not a limit quotient of any of the maximal flexible quotients: $(Flx_1^m, E_{F_1}^m), \dots, (Flx_v^m, E_{F_{v(m)}}^m)$, and since $G(x, p)$ maps epimorphically onto its maximal flexible quotients and onto (Rgd, E_R) , there exist elements $r_1, \dots, r_{v(m)} \in G(x, p)$, so that for each j , $1 \leq j \leq v(m)$, r_j is either mapped into non-trivial element in (Rgd, E_R) and to the identity element in Flx_j^m , or it is mapped into a non-elliptic element in (Rgd, E_R) , and to an elliptic element in $(Flx_j^m, E_{F_j^m})$.

With each of the maximal flexible quotients, $(Flx_j^m, E_{F_j^m})$, we associate its (ungraded) Makanin-Razborov diagram according to theorem 27 in [Ja-Se]. Given one of these flexible quotients, $(Flx_j^m, E_{F_j^m})$, and a resolution that appears in its (strict) ungraded Makanin-Razborov diagram, we take the completion of this resolution, into which $G(x, p)$ is mapped. We further take a f.p. approximation of this completion into which $G(x, p)$ is mapped as well. Clearly, we can choose the f.p. approximation of the completion, so that the element $r_j \in G(x, p)$ is mapped either to a trivial or to an elliptic element, depending whether it is mapped to a trivial or to an elliptic element in $(Flx_j^m, E_{F_j^m})$. Finally we set the groups, (CF_1, \dots, CF_g) to be the images of $G(x, p)$ in the corresponding f.p. completions of the resolution in the (strict) ungraded Makanin-Razborov diagrams of the maximal flexible quotients: $(Flx_1^m, E_{F_1^m}), \dots, (Flx_v^m, E_{F_{v(m)}}^m)$. By construction, this finite collection of limit groups with f.p. completions (and corresponding strict well-structured resolution), together with the rigid limit group, W_0^m , and its f.p. completion, satisfy properties (1)-(5) of the theorem. □

Theorem 1.17 associate (non-canonically) a rigid limit group that can be embedded into a f.p. completion with any given rigid limit group over free products. To construct a graded Makanin-Razborov diagram over free products we further need to associate a limit group that can be embedded into a f.p. completion with any given solid limit group. In order to associate similar limit groups with a solid limit group over free products, we need to weaken the notion of a solid limit group, to a *weakly solid* limit group.

Definition 1.18. *Let $(L(x, p), E_L)$ be a graded limit group over free products. Suppose that (L, E_L) does not admit a non-trivial free decomposition in which the parameter subgroup, $\langle p \rangle$, and all the elliptic subgroups of $L(x, p)$ are elliptic (i.e., elliptic subgroups are contained in conjugates of the factors of the free decomposition). Let Λ be the graded virtually abelian JSJ decomposition of (L, E_L) over free products, and suppose that it is non-trivial.*

Let $GMod(L, E_L)$ be the graded modular group of (L, E_L) over free products.

Even though (L, E_L) need not be solid, with the graded limit group, (L, E_L) , we can associate its collection of flexible quotients according to definition 1.2. Hence, following the proof of proposition 1.5, with (L, E_L) we can associate a finite collection of covers that dominate all the flexible quotients of (L, E_L) , and satisfy the properties of covers (that are associated with a solid limit group) that are listed in proposition 1.5.

We say that a graded limit group over free products, $(L(x, p), E_L)$, that does not admit a graded free decomposition over free products, and that has a non-trivial graded JSJ decomposition, is weakly solid, if there exists a finite collection of covers of the flexible quotients of L , (CF_1, \dots, CF_b) , so that there are homomorphisms of (L, E_L) into free products that are weakly strictly solid, i.e., they can not be extended to homomorphisms of any of the completions of the one step resolutions, $L \rightarrow CF_i$ (cf. definition 1.11).

Proposition 1.19. *Let $(L(x, p), E_L)$ be a weakly solid limit group, and let (CF_1, \dots, CF_b) , be a finite collection of covers that dominate all the flexible quotients of (L, E_L) . Then there exists a constant, R_0 , so that every (almost) shortest weakly strictly solid homomorphism of (L, E_L) into free products is R_0 - P -covered. Furthermore, the conclusion of theorem 1.15 (for shortest strictly solid homomorphisms of a solid limit group over free products) hold for weakly strictly solid homomorphisms of (L, E_L) .*

Proof: The same arguments that are used to prove proposition 1.8, and theorems 1.13 and 1.15 for strictly solid homomorphisms of a solid limit group (over free products) remain valid for weakly strictly solid homomorphisms of a weakly solid limit group. □

The notion of weakly solid limit group, and the ability to generalize the bounds on strictly solid homomorphisms of solid limit groups to weakly strictly solid homomorphisms of weakly solid limit groups, enable us to state an analogue of theorem 1.16 for approximations of solid limit groups over free products. Theorems 1.17 and 1.20 finally allow us to generalize the construction of the Makanin-Razborov diagram over free products from the ungraded to the graded case.

Theorem 1.20. *Let $(Sld(x, p), E_S)$ be a solid limit group over free products. Suppose that $(Sld(x, p), E_S)$ is a (solid) limit quotient of a f.p. group, $G(x, p)$, and let $\{h_n : G(x, p) \rightarrow A_n^1 * \dots * A_n^\ell\}$ be an asymptotically strictly solid sequence of homomorphisms (of Sld) that converges into $(Sld(x, p), E_S)$.*

There exists an approximating weakly solid limit group of $(Sld(x, p), E_S)$, (see definition 1.18 for weakly solid) that we denote $(WSld, E_{WS})$, with the following properties:

- (1) *(Sld, E_S) is a quotient limit of $(WSld, E_{WS})$. $WSld$ is a weakly solid limit quotient of $G(x, p)$ (over free products).*
- (2) *there is an ungraded resolution (over free products) $Res, WSld \rightarrow V_1 \rightarrow \dots \rightarrow V_r$, so that V_r is f.p. and there exists a subsequence of homomorphisms, (still denoted) $\{h_n\}$, that factor through this resolution. In particular, the completion of the resolution Res is f.p. and the approximating weakly solid limit group, $(WSld, E_{WS})$, is embedded into this completion.*

- (3) with $(WSld, E_{WS})$ we associate finitely many limit groups (over free products), CF_1, \dots, CF_g , with ungraded resolutions (over free products), Res_1, \dots, Res_g , that terminate in f.p. limit groups. Hence, the completions of Res_1, \dots, Res_g are f.p. and the limit groups (over free products), CF_1, \dots, CF_g , can be embedded in f.p. completions (that have f.p. terminal limit groups).
- (4) the finite collection of limit groups, CF_1, \dots, CF_g , are all limit quotients of $G(x, p)$, and they dominate (as quotients of the f.p. group $G(x, p)$) all the flexible quotients of $(WSld, E_{WS})$ (although they need not be quotients of the weakly solid limit group $(WSld, E_{WS})$). Furthermore, for each homomorphism $f : WSld \rightarrow A^1 * \dots * A^\ell$ that is not weakly strictly solid, there exists a homomorphism, $u : WSld \rightarrow A^1 * \dots * A^\ell$, such that the pair, (f, u) , extends to a homomorphism of the completion of the identity resolution, $WSld \rightarrow WSld$, and the homomorphism u as a homomorphism of the f.p. group, $G(x, p)$, factors through at least one of the resolutions, Res_1, \dots, Res_g .
- (5) from the sequence of homomorphisms, $\{h_n\}$, it is possible to extract a subsequence of weakly solid homomorphisms of the approximating solid limit group, $(WSld, E_{WS})$. Furthermore, for each homomorphism h_n in this subsequence, there does not exist a homomorphism, $u_n : WSld \rightarrow A_n^1 * \dots * A_n^\ell$, so that the pair of homomorphisms, (h_n, u_n) , extends to a homomorphism of the completion of the one step identity resolution, $WSld \rightarrow WSld$, and the homomorphism u_n , as a homomorphism from the f.p. group $G(x, p)$ (into $A_n^1 * \dots * A_n^\ell$), factors through one of the limit groups: CF_1, \dots, CF_g .

Proof: The argument that we use follows the proof of theorem 1.17. Let (Sld, E_S) be a solid limit group, and let $\{h_n : G(x, p) \rightarrow A_n^1 * \dots * A_n^\ell\}$ be a sequence of asymptotically strictly solid homomorphisms that converges into the solid limit group over free products, (Sld, E_S) .

Starting with the sequence of homomorphisms, $\{h_n\}$, we can pass to a subsequence, and associate with the solid limit group over free products, (Sld, E_S) , a strict well-structured ungraded resolution, that we denote $Res, Sld = W_0 \rightarrow W_1 \rightarrow \dots \rightarrow W_t$, so that W_t is the free product of (possibly) finitely many elliptic factors and (possibly) a free group.

We set $Comp_W$ to be the completion of this ungraded resolution. $Comp_W$ is generated by the terminal limit group W_t , in addition to finitely many elements and relations. However, W_t may be not finitely presented. We modify $Comp_W$, by replacing W_t with a sequence of f.p. approximations of it precisely as we did in the rigid case, i.e., for each index m , we replace each factor of W_t by a f.p. group, so that the presentation of each (new) factor includes only relations up to length m from the relations of the corresponding factor of W_t . We denote the obtained f.p. group W_t^m , and the completion that is obtained from $Comp_W$, by replacing W_t with W_t^m , $Comp_W^m$.

For each m , $Comp_W^m$, is finitely presented. With $Comp_W^m$, we naturally associate an ungraded resolution over free products, that we denote $Res^m, W_0^m \rightarrow W_1^m \rightarrow \dots \rightarrow W_t^m$. The limit groups over free products, W_0^m, \dots, W_t^m , are all embedded in the f.p. completion, $Comp_W^m$. The limit groups (over free products), W_0, \dots, W_t , are quotients of the limit groups over free products, W_0^m, \dots, W_t^m , in correspondence, and the direct limit of the limit groups over free products, W_0^m, \dots, W_t^m , are W_0, \dots, W_t , in correspondence.

By construction, with each of the graded limit groups, W_i , there is an associated virtually abelian decomposition, which is the JSJ decomposition (over free products) of W_i . By the construction of $Comp_W^m$, the JSJ decompositions of the limit groups (over free products), W_i , lift to virtually abelian decompositions of the (approximating) graded limit groups over free products, W_i^m . By the proof of theorem 27 in [Ja-Se], for large enough index m , these virtually abelian decompositions of the limit groups W_i^m (that are lifted from W_i) are their JSJ decompositions over free products.

The sequence of homomorphisms, $\{h_n\}$, is an asymptotically strictly solid sequence of homomorphisms with respect to the original rigid limit group, (Sld, E_S) , and it converges to (Sld, E_S) . Since the completions, $Comp_W^m$, are finitely presented, for each index m , there exists some index, a_m , so that for every $n > a_m$, the sequence of homomorphisms, $\{h_n\}$, factor through $Comp_W^m$, and in particular through the limit group, W_0^m .

Suppose that there exists a subsequence of indices, still denoted m , so that for every index m , there is a subsequence of homomorphisms (where the subsequence may depend on m), still denoted, $\{h_n\}$, that are not weakly strictly solid homomorphisms of W_0^m , i.e., they can be extended to a completion of a one step resolution: $W_0^m \rightarrow CFlex_m$, where $CFlex_m$ is a (cover of a) flexible quotient of the graded limit group W_0^m (note that W_0^m need not be solid, still one can define a flexible quotient of it as it appears in definition 1.18). In this case, the original sequence of homomorphisms, $\{h_n\}$, is not an asymptotically strictly solid sequence of homomorphisms with respect to the solid limit group, (Sld, E_S) , a contradiction. Therefore, for large enough m , the approximating limit groups, W_0^m are weakly solid, and for every index m , there exists an index n_m , so that for every $n > n_m$, the homomorphisms, $\{h_n\}$, are weakly strictly solid homomorphisms of W_0^m .

For each index m , we look at some finite collection of covers of its flexible quotients. We further look at a strict (ungraded) Makanin-Razborov resolution of each of the (ungraded) resolutions in this diagram, and with each completion of a resolution that appears in the diagram we associate a f.p. completion, that is obtained by replacing the terminal limit group of such an ungraded resolution with some f.p. approximation of it. By looking at the image of the f.p. group $G(x, p)$ in each of these (finitely many) f.p. completions, we obtained finitely many limit groups (covers), that we denote $CF_1^m, \dots, CF_{v(m)}^m$, that are quotients of $G(x, p)$, and can be embedded into f.p. completions, and cover all the flexible quotients of the weakly solid limit groups (over free products) W_0^m .

The limit groups, $\{W_0^m\}$, do converge into the original solid limit group, (Sld, E_S) . As the flexible quotients of W_0^m are (by definition) quotients of W_0^m , by choosing the approximations properly, i.e., choosing enough of the relations of the terminal limit group of each completion, to obtain the covers (that are not necessarily quotients of W_0^m), $CF_1^m, \dots, CF_{v(m)}^m$, we can guarantee that any sequence of homomorphisms, $s_m : CF_{i(m)}^m \rightarrow A_m^1 * \dots * A_m^\ell$, that converges to a limit group, converges into a quotient of (Sld, E_S) .

Suppose that there exists a subsequence of indices, still denoted m , so that for every index m , there is a subsequence of homomorphisms (where the subsequence may depend on m), still denoted, $\{h_n\}$, so that for every h_n there exists a homomorphism, $u_n : W_0^m \rightarrow A_n^1 * \dots * A_n^\ell$, that as a homomorphism of the f.p. group, $G(x, p)$, factors through one of the finitely many covers, $CF_{i(m)}^m$, and the pair of

homomorphisms of $WSld$, (h_n, u_n) , extend to a homomorphism of the completion of the identity resolution, $W_0^m \rightarrow W_0^m$ (with the graded JSJ decomposition of $WSld$ as the associated virtually abelian decomposition). In this case, it is possible to extract a subsequence of the homomorphisms, $\{h_n\}$, with homomorphisms $\{u_n : W_0^m \rightarrow A_n^1 * \dots * A_n^\ell\}$, so that the pairs, (h_n, u_n) , extend to homomorphisms of the identity resolution, $W_0^m \rightarrow W_0^m$, and the homomorphisms, $\{u_n\}$, converge into a flexible quotient of (Sld, E_S) , a contradiction to our assumption that the sequence, $\{h_n\}$, is weakly strictly solid.

Therefore, for large enough m , the weakly solid limit group, W_0^m , that can be embedded into a f.p. completion, and the covers, $CF_1^m, \dots, CF_{v(m)}^m$, that can be embedded into f.p. completions as well, satisfy properties (1)-(5) of the theorem. \square

Given a sequence of homomorphisms from a fixed graded group into free products, theorem 1.16 enables us to associate a graded resolution with a subsequence of the homomorphisms. Theorems 1.17 and 1.20 finally allow us to generalize the notion of a cover of an ungraded resolution (theorem 24 in [Ja-Se]), to define a cover of a resolution, so that, in particular, given a sequence of homomorphisms of a f.p. group into free products, there is a subsequence of a sequence of homomorphisms that factor through a cover of the resolution that was constructed in proposition 1.16.

Theorem 1.21. *Let $G(x, p)$ be a f.p. group, and let $\{h_n : G(x, p) \rightarrow A_n^1 * \dots * A_n^\ell\}$ be a sequence of homomorphisms of $G(x, p)$ into free products that converges into a limit group (over free products), $(L(x, p), E_L)$. Let $L(x, p) = L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$ be a graded resolution, that we denote $GRes$, of the limit group $(L(x, p), E_L)$, that is obtained from a subsequence of the sequence of homomorphisms, $\{h_n\}$, according to proposition 1.16.*

There exists a f.g. graded limit quotient of $G(x, p)$, CM , with a set of elliptics, E_{CM} , and a well-structured graded resolution of CM , $CGRes$, $CM = CM_0 \rightarrow CM_1 \rightarrow \dots \rightarrow CM_s$, that satisfies similar properties to the ones listed in proposition 1.16. The graded resolution, $CGRes$, satisfies the following properties:

- (1) *for each index i , $i = 1, \dots, s$, there is an epimorphism of limit groups over free products, $\tau_i : CM_i \rightarrow L_i$. The epimorphisms τ_i commute with the quotient maps in the two graded resolutions, $GRes$ and $CGRes$.*
- (2) *non-trivial elliptic elements in CM_i are mapped to non-trivial elliptic elements in CM_{i+1} , $i = 1, \dots, s - 1$.*
- (3) *the epimorphisms along $CGRes$ are proper epimorphisms.*
- (4) *the resolution $CGRes$ is a graded strict resolution ([Se1], 5), and a well-structured resolution. All the graded abelian decompositions that are associated with the various limit groups (over free products), CM_i , $i = 1, \dots, s$, are their graded JSJ decompositions over free products. Furthermore, the graded JSJ decompositions (over free products) of the limit groups CM_i , $i = 1, \dots, s$, have the same structure as the corresponding graded JSJ decompositions (over free products) of the limit groups, L_i , where the difference is only in the rigid vertex groups of the JSJ decompositions and in the elliptic factors.*
- (5) *if no factor of L_i is rigid nor solid, then CM_{i+1} is a cover of a shortening quotient of CM_i , for $i = 1, \dots, s - 1$.*

- (6) if a factor of L_i is rigid then the corresponding factor of CM_i is rigid. This can happen only for $i = s$. If a factor of L_i is solid then the corresponding factor of CM_i is weakly solid (definition 1.18).
- (7) in case L_i is a free product of a solid limit group with (possibly) elliptic factors and (possibly) a free group, for some $i < s$, then by (6) the corresponding factor of CM_i is weakly solid, and CM_{i+1} is a free product of a cover of a flexible quotient of the weakly solid factor in CM_i with (possibly) the same elliptic factors of CM_i and (possibly) the same free group (as in CM_i).
- (8) L_s , the terminal limit group of the graded resolution $GRes$, is a free product of a rigid or a solid limit group with (possibly) some elliptic factors and (possibly) a free group. CM_s is a free product of a rigid or a weakly solid factor, that we denote RSF_s , in correspondence with the factor of L_s , with (possibly) f.p. covers of the elliptic factors of L_s , and a free group (of the same rank as in L_s).

With the rigid or the weakly solid factor of CM_s , RSF_s , we associate finitely many limit groups (over free products), CF_1, \dots, CF_g , with ungraded resolutions (over free products), Res_1, \dots, Res_g , that terminate in f.p. limit groups (cf. theorems 1.17 and 1.20). Hence, the completions of Res_1, \dots, Res_g are f.p. and the limit groups (over free products), CF_1, \dots, CF_g , can be embedded in f.p. completions (that have f.p. terminal limit groups).

- (9) the finite collection of limit groups, CF_1, \dots, CF_g , are all limit quotients of $G(x, p)$, and they dominate all the flexible quotients of RSF_s (although they need not be quotients of RSF_s).

By theorem 1.16, from the sequence of homomorphisms, $\{h_n\}$, it is possible to extract a subsequence, still denote $\{h_n\}$, on which it is possible to perform iterative shortenings, and obtain another sequence of homomorphisms, $\{u_n\}$, which is asymptotically rigid or asymptotically strictly solid and converges into the rigid or solid factor of L_s , the terminal graded limit group in $GRes$.

The homomorphisms $\{u_n\}$ factor through CM_s , and they restrict to rigid or weakly strictly solid homomorphisms of RSF_s (with respect to that cover). Furthermore, in the rigid case, every flexible homomorphism of RSF_s factors through at least one of the resolutions, Res_1, \dots, Res_g , that are associated with the limit groups, CF_1, \dots, CF_g . In the weakly solid case, for every non-weakly strictly solid homomorphism of RSF_s , $f : RSF_s \rightarrow A^1 * \dots * A^\ell$, there exists a homomorphism, $u : RSF_s \rightarrow A^1 * \dots * A^\ell$, such that the pair, (f, u) , extends to a homomorphism of the completion of the identity resolution, $RSF_s \rightarrow RSF_s$, and the homomorphism u as a homomorphism of the f.p. group, $G(x, p)$, factors through at least one of the resolutions, Res_1, \dots, Res_g .

- (10) with the terminal graded limit group CM_s , it is possible to associate an ungraded strict well-structured resolution: $CM_s = V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_t$, so that the terminal (ungraded) limit group (over free products) V_t is a free product of (possibly) elliptic factors and (possibly) a free group. Every non-trivial elliptic element in each of the limit groups (over free products) V_i , $i = 0, \dots, t - 1$, is mapped to a non-trivial element in V_{i+1} .

With the combined (ungraded) resolutions, $CM_0 \rightarrow \dots \rightarrow CM_s \rightarrow V_1 \rightarrow \dots \rightarrow V_t$ it is possible to associate a completion, $Comp_{CM}$, which is f.p.

(and terminates in a f.p. group).

Proof: Given the sequence of homomorphisms, $\{h_n : G(x, p) \rightarrow A_n^1 * \dots * A_n^\ell\}$, we can extract a subsequence from it (still denoted $\{h_n\}$), from which it is possible to obtain a graded strict, well-structured resolution over free products, $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$, according to theorem 1.16. Furthermore, with each homomorphism h_n (from that subsequence) it is possible to associate a homomorphism $u_n : L \rightarrow A_n^1 * \dots * A_n^\ell$, so that the homomorphisms $\{u_n\}$ restrict to an asymptotically rigid or to an asymptotically strictly solid homomorphisms with respect to the rigid or solid factor of the terminal limit group, L_s .

Following theorem 18 in [Ja-Se], starting with the sequence of homomorphisms, $\{u_n\}$, we can pass to a further subsequence, and associate with the limit group over free products, L_s , a strict well-structured ungraded resolution, $L_s = W_0 \rightarrow W_1 \rightarrow \dots \rightarrow W_t$, so that W_t is the free product of (possibly) finitely many elliptic factors and (possibly) a free group.

At this point we look at the combined (ungraded) resolution:

$$L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s = W_0 \rightarrow W_1 \rightarrow \dots \rightarrow W_t$$

We set $Comp_L$ to be the completion of this ungraded combined resolution. Note that all the limit groups over free products, L_0, \dots, L_s and W_1, \dots, W_t , are embedded in the completion $Comp_L$.

Starting with $Comp_L$ we define a sequence of completions, obtained by replacing the terminal limit group (over free products) in $Comp_L$, W_t , with approximating f.p. groups, that are obtained from the terminal limit group W_t , by imposing only relations of length at most m from each factor of W_t (with respect to a fixed set of generators of these factors). We denote the approximating f.p. subgroups, W_t^m , and the group that is obtained from $Comp_L$ by replacing W_t with W_t^m , we denote $Comp_L^m$.

For large enough index m , we do obtain a new completion, hence, for large m , $Comp_L^m$ is a completion of a strict well-structured ungraded resolution over free products. We denote the resolution that is associated with $Comp_L^m$, Res^m , and this is the resolution:

$$L_0^m \rightarrow L_1^m \rightarrow \dots \rightarrow L_s^m = W_0^m \rightarrow W_1^m \rightarrow \dots \rightarrow W_t^m$$

where the groups L_i^m and W_j^m approximate the corresponding subgroups, L_i and W_j .

By construction, with each of the graded limit groups, L_i , there is an associated graded (virtually) abelian decomposition, which is the graded virtually abelian JSJ decomposition of L_i over free products. Also, with each of the ungraded limit group, W_j , there is an associated (virtually) abelian decomposition which is the virtually abelian decomposition of W_j over free products. These graded and ungraded JSJ decompositions lift to virtually abelian decompositions of the approximating limit groups, L_i^m and W_j^m .

Moreover, by the proof of theorem 27 in [Ja-Se], for large enough index m , these graded virtually abelian decompositions of the limit groups L_i^m (that are lifted from L_i) are their graded JSJ decompositions over free products, and the virtually abelian decompositions of the limit groups W_j^m (that are lifted from W_j) are their JSJ decompositions over free products.

By theorems 1.17 and 1.20, if L_i is solid, then for large m , L_i^m is weakly solid. If L_s is rigid (it's the only graded limit group along the resolution Res that can be rigid), then for large m , L_s^m is rigid. Therefore, for large enough m , the resolution, Res^m , satisfy properties (1)-(7) and (10). Properties (8) and (9) hold for Res^m , for large enough m , by the conclusions of theorems 1.17 and 1.20. □

Theorem 1.21 associates a collection of covers of resolutions (over free products) with a given f.p. group. Every homomorphism of a given f.p. group into free products factors through one of these cover resolutions. As in the ungraded case (theorem 26 in [Ja-Se]), if we start with a f.p. group, $G(x, p)$, there exists a finite collection of cover resolutions through which all the homomorphism from the f.p. group into free products do factor. Such a finite collection is not canonical, but as in the ungraded case, we view each such finite collection as the *graded Makanin-Razborov* diagram of the graded f.p. group, $G(x, p)$, over free products.

Theorem 1.22. *Let $G(x, p)$ be a f.p. group. There exist finitely many well-structured resolutions: $CM = CM_0 \rightarrow CM_1 \rightarrow \dots \rightarrow CM_s$, where CM is a graded limit quotient of $G(x, p)$ (over free products), and CM_s is a free product of either a rigid or a weakly solid factor with (possibly) finitely many elliptic factors, and (possibly) a free group, that are all covers of graded resolutions, that satisfy properties (1)-(10) that are listed in theorem 1.21.*

With each of the finitely many cover graded resolutions we associate the collection of homomorphisms that factor through it. Homomorphisms of the terminal graded limit group, CM_s , are obtained from rigid or weakly strictly solid homomorphisms of the rigid or weakly solid factor of CM_s that are combined with homomorphisms of the elliptic factors (into elliptic subgroups), and arbitrary homomorphisms of the (possible) free factor. We further require that the restriction of weakly strictly solid homomorphisms to edge groups and to abelian and QH vertex groups in the graded abelian JSJ decomposition of the weakly solid factor are not (entirely) elliptic.

Homomorphisms of the limit groups, CM_{s-1}, \dots, CM_0 (and hence of the given f.p. group $G(x, p)$), are obtained from homomorphisms of CM_s , in a similar way to homomorphisms that factor through graded resolutions over free and hyperbolic group, i.e., in the following way:

- (1) *in case CM_i , $i = 0, \dots, s - 1$, does not have a factor which is rigid or weakly solid, homomorphisms of CM_i are obtained from homomorphisms of CM_{i+1} , by precomposing the homomorphisms of CM_{i+1} with graded modular automorphisms of CM_i .*
- (2) *in case CM_i , $i = 0, \dots, s - 1$, is weakly solid, we look at the following homomorphisms. With CM_i we associate a one step (well-structured) resolution $CM_i \rightarrow CM_{i+1}$. With this one step resolution we associate its completion, $Comp_i$. Both CM_i and CM_{i+1} are naturally embedded in $Comp_i$. With CM_i we associate all its homomorphisms that can be extended to homomorphisms of the completion, $Comp_i$, so that the restrictions of these homomorphisms of $Comp_i$ to the image of CM_{i+1} in $Comp_i$ is a homomorphism that is associated with CM_{i+1} .*
- (3) *the restriction of the homomorphisms of the various groups, CM_i , $i = 0, \dots, s - 1$, to the edge groups and to abelian and QH vertex groups in the graded abelian JSJ decomposition of theses are not (entirely) elliptic.*

Every homomorphism of $G(x, p)$ into a free product factors through at least one of the finitely many (cover) resolutions that are associated with the f.p. group $G(x, p)$ (i.e., the cover resolutions that appear in its (non-canonical) Makanin-Razborov diagram).

Proof: The cover graded resolutions that appear in theorem 1.21, can be defined by finitely many generators, relations, and generators of subgroups, since these are determined by finitely many f.p. completions and finitely many f.g. subgroups of these f.p. completions. The construction of cover graded resolutions in theorem 1.21 allows one to consider only homomorphisms that satisfy properties (1)-(3) as homomorphisms that factor through a cover graded resolution.

By theorem 1.21, every homomorphism of the f.p. group, $G(x, p)$, factors through at least one of the cover graded resolutions that can be associated with $G(x, p)$. Therefore, by ordering the countable set of cover graded resolutions, and applying the same argument that was used to prove theorem 26 in [Ja-Se], i.e., the argument that was used to show that the set of all homomorphisms of a f.p. group into free products factor through finitely many cover ungraded resolutions, there exists a finite collection of cover graded resolutions, through which all the homomorphisms of the f.p. group, $G(x, p)$, do factor. □

The finite collection of graded resolutions that are associated with a f.p. group, $G(x, p)$, according to theorem 1.22, and through which all the homomorphisms of $G(x, p)$ into free products do factor, form a graded Makanin-Razborov diagram of $G(x, p)$ over free products. Note that the collection of graded resolutions is not canonical, hence, the diagram we constructed is not canonical.

Also, note that over free and hyperbolic groups we needed to study the singular locus of the graded resolutions that appear in the (in these cases canonical) graded Makanin-Razborov diagram. This was crucial in studying sentences and predicates over these groups. The diagram that we constructed over free products, and particularly the conditions on homomorphisms that factor through each graded resolution in the diagram, guarantee that the collection of homomorphisms that are associated with each graded resolution from the diagram lies outside the singular locus, and these collections cover the entire set of homomorphisms from the f.p. group $G(x, p)$ into free products. Hence, there is no need to study the singular locus of the constructed graded resolutions.

§2. Formal Solutions and Formal Limit Groups

So far we have generalized the construction of the Makanin-Razborov diagram from a free group to a free product of groups [Ja-Se], and in section 1 we have generalized the study of systems of equations with parameters (over free products), and associated a graded Makanin-Razborov with such a system.

To analyze the structure of elementary sets over a free product, we first need to generalize *formal solutions*, and their (formal) limit groups ([Se2]), to formal solutions and formal limit groups over free products. As in the construction of the Makanin-Razborov diagrams (graded and ungraded), we study formal solutions over the entire class of free products, and not over a particular free product. This approach will finally lead us to the construction of uniform proofs of a given sentence (or predicate), uniform proofs that lead to a (uniform) reduction of a sentence over

a free product to sentences over the various factors.

We start this section with a generalization of (a special case of) Merzlyakov original theorem, mainly to demonstrate that basic concepts over free groups generalize to free products.

Theorem 2.1 (cf. [Me]). *Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ . Let $y = (y_1, \dots, y_\ell)$ and $x = (x_1, \dots, x_q)$. Let $w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1$ be a system of equations over G , where a is a tuple of elements from the factors, A^1, \dots, A^ℓ . Suppose that the sentence:*

$$\forall y \exists x \quad w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1$$

*is a truth sentence over G . Then there exists a formal solution $x = x(y, a')$, where a' is a tuple of elements from the factors, A^1, \dots, A^ℓ , so that each of the words $w_j(x(y, a'), y, a)$ is the trivial word in the free product $\langle y \rangle * A^1 * \dots * A^\ell = \langle y \rangle * G$.*

Proof: In a non-trivial free product, $G = A^1 * \dots * A^\ell$, which is not isomorphic to D_∞ , there exists a free, purely hyperbolic subgroup, so that its Cayley graph is quasi-isometrically mapped, under the embedding of the free group, into the Bass-Serre tree that corresponds to the free product, $G = A^1 * \dots * A^\ell$. We can clearly write G as a non-trivial free product, $G = C * B$, where C is not isomorphic to Z_2 . If $c_1, c_2 \in C$ are distinct non-trivial elements, and $b \in B$ is non-trivial, then $w_n = (c_1 b)^n$ and $t_n = (c_2 b)^n$, generate such a free group of rank 2, for n large enough (e.g. $n > 5$).

The sentence is a truth sentence over the free product, G , hence, we may assign arbitrary values to the universal variables y , and for every chosen values (in G), there exist values for the existential variables x , so that the equalities, $w_1 = 1, \dots, w_s = 1$, hold in G .

In G there is a copy of the free group F_2 that is quasi-isometrically embedded in the Bass-Serre tree that corresponds to the free product, $G = A^1 * \dots * A^\ell$. We assign a sequence of values in G to the universal variables y . We give the universal variables y , a sequence of values in G that we denote $\{y(n)\}$, so that these values are in (the image of) the (quasi-isometrically embedded) free group F_2 in G , that form a test sequence in F_2 (see the proof of theorem 1.1 and definition 1.20 in [Se2] for the notion of test sequence).

For every index n , there are values in G of the existential values, x , that we denote $x(n)$, so that $w_i(x(n), y(n), a) = 1$, $i = 1, \dots, s$. For each n we choose the shortest possible $x(n)$ (in the Bass-Serre tree that is associated with the free product, $G = A^1 * \dots * A^\ell$, with its simplicial metric) that satisfies these equalities.

By the same (geometric) argument that was used to prove Merzlyakov's theorem over free groups (theorem 1.1 in [Se2]), by possibly iteratively modifying the shortest values $x(n)$ finitely many times, the sequence of elements $\{(x(n), y(n), a)\}$ has a subsequence that converges to the limit group over free products, $L(x, y, a) = \langle y \rangle * A_1 * \dots * A_\ell * E_1 * \dots * E_t * F_r$, where:

- (1) $\langle y \rangle$ is a free group generated by the universal variables: y_1, \dots, y_ℓ .
- (2) A_i , $i = 1, \dots, \ell$, is elliptic and contains the tuple of (fixed) elements a from the factor A^i .
- (3) E_1, \dots, E_r are elliptic. F_r is a (possibly trivial) free group of rank r .

- (4) L contains the elements x , and the elements, $w_1(x, y, a), \dots, w_s(x, y, a)$, represent the trivial element in the limit group L .

We replace L by its (quotient) factor, $R(x, y, a) = \langle y \rangle * A_1 * \dots * A_\ell$, which is also a (limit) quotient of L (R is in fact a retract of L). L , and hence R , are f.g. groups, but in general they need not be finitely presented, i.e., the elliptic factors A_1, \dots, A_ℓ , are f.g. but not necessarily finitely presented. Still, the words w_1, \dots, w_r represent the trivial element in R , as they represent the trivial element in L , and R is a retract of L . Hence, if we look at a free group, F_y , generated by the elements y_1, \dots, y_ℓ and finite (fixed) generating sets for A_1, \dots, A_ℓ , then there is a finite collection of relations, r_1, \dots, r_f , in the (fixed) generating sets of A_1, \dots, A_ℓ , relations that hold in these factors, such that if these relations are imposed on F , then the words, w_1, \dots, w_s , represent the trivial element in the obtained quotient f.p. group.

L is a limit group that is obtained as a limit from a sequence of elements, $\{(x(n), y(n), a)\}$. Hence, for large enough n , the relations r_1, \dots, r_f that hold in A_1, \dots, A_ℓ , hence in L , hold for the specializations, $\{(x(n), y(n), a)\}$, and the fixed set of generators of A_1, \dots, A_ℓ are elliptic, hence, their specializations are in the factors A^1, \dots, A^ℓ correspondingly. Therefore, using the specialization $\{(x(n), y(n), a)\}$, for large n , it is possible to find elements $x \in \langle y \rangle * A^1 * \dots * A^\ell$, so that the words, $w_1(x, y, a), \dots, w_s(x, y, a)$ are the trivial elements in the free product, $\langle y \rangle * G$. \square

To analyze sentences and predicates over free products, we will need a generalization of Merzlyakov theorem to a truth sentence defined over an arbitrary (given) variety. Before generalizing the results that were proved over free group for general varieties, we generalize Merzlyakov theorem to a coefficient free sentence that contains inequalities. For such sentences the conclusion can be stated in a uniform way for all free products for which the sentence is a truth sentence.

Theorem 2.2. *Let $w_1(x, y) = 1, \dots, w_s(x, y) = 1$ be a system of equations (over a group), and let $v_1(x, y), \dots, v_r(x, y)$ be a collection of words in the free group generated by $\{x, y\}$. Let:*

$$\forall y \exists x \quad w_1(x, y) = 1, \dots, w_s(x, y) = 1 \wedge v_1(x, y) \neq 1, \dots, v_r(x, y) \neq 1$$

be a sentence over groups. Then there exist finitely many f.p. limit groups over free products:

$$H_j = \langle y \rangle * S_1 * \dots * S_{m_j} * F_{r_j} \quad j = 1, \dots, t$$

and tuples of elements (formal solutions), $x_j \in H_j$, so that: $w_1(x_j, y) = 1, \dots, w_s(x_j, y) = 1$ in the limit group H_j .

*Furthermore, for every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , and so that the given sentence is a truth sentence over G , for at least one index j , $1 \leq j \leq t$, there exists a homomorphism $\tau : H_j \rightarrow G * \langle y \rangle$, that maps each factor S_i , $1 \leq i \leq m_j$, into an elliptic subgroup in G , the factor $\langle y \rangle$ in H_j isomorphically onto the factor $\langle y \rangle$ in $G * \langle y \rangle$, and the free factor F_{r_j} into G , such that the sentence:*

$$\exists y \quad v_1(\tau(x_j), y) \neq 1, \dots, v_r(\tau(x_j), y) \neq 1$$

is a truth sentence in G .

Proof: The argument that we use is a combination of the proof of theorem 2.1, with the proof of Merzlyakov theorem with inequalities (theorem 1.2 in [Se2]), and the construction of formal limit groups in section 2 of [Se2].

As we have pointed out in the proof of theorem 2.1, in a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , there exists a free, purely hyperbolic subgroup, so that its Cayley graph is quasi-isometrically mapped, under the embedding of the free group, into the Bass-Serre tree that corresponds to the free product, $G = A^1 * \dots * A^\ell$.

We look at all the non-trivial free products, $G = A^1 * \dots * A^\ell$ (for an arbitrary finite ℓ), that are not isomorphic to D_∞ , over which the given sentence is a truth sentence. For each such free product G , we look at all the f.g. free groups that are embedded in G , are purely hyperbolic in G , and the Cayley graph of F is quasi-isometrically mapped into the Bass-Serre tree that is associated with the free product of G , $G = A^1 * \dots * A^\ell$.

We look at all such free products $G = A^1 * \dots * A^\ell$ (over which the given sentence is a truth sentence), the quasi-isometrically embedded free groups, F , all possible test sequence for the universal variables y in F , and for each value of the universal variables y , the shortest possible values for the existential variables x (in the simplicial metric of the Bass-Serre tree that is associated with the free product $G = A^1 * \dots * A^\ell$), for which both the equalities, w_1, \dots, w_s , and the inequalities, v_1, \dots, v_r , hold (in G).

By the construction of formal limit groups that appears in section 2 of [Se2], and by the argument that was used to prove that a f.p. group has only finitely many maximal limit quotients over free products (theorem 21 in [Ja-Se]), with the entire collection of these test sequences it is possible to associate finitely many (maximal) formal limit groups (over free products). Since these limit groups were constructed from sequences that restrict to test sequences for the universal variables, and shortest possible values for the existential variables, each of the finitely many maximal formal limit groups has the structure:

$$H_j = \langle y \rangle * S_1 * \dots * S_{m_j} * F_{r_j} \quad j = 1, \dots, t$$

where the factors, S_1, \dots, S_{m_j} , are elliptic, and F_{r_j} is free. Furthermore, each such (maximal) formal limit group, H_j , is finitely presented, and with each such formal limit group, H_j , there are associated tuples of elements, $x_j \in H_j$, so that the words, $w_1(x_j, y), \dots, w_s(x_j, y)$, are trivial in H_j , and the words, $v_1(x_j, y), \dots, v_r(x_j, y)$, are non-trivial in H_j .

Suppose that $G = A^1 * \dots * A^\ell$ is a non-trivial free product, that is not isomorphic to D_∞ , and the given coefficient free sentence is true over G . Then there exists a free group F_2 , with an embedding into G so that the Cayley graph of F_2 is mapped quasi-isometrically into the Bass-Serre tree that is associated with the free product, $G = A^1 * \dots * A^\ell$. We assign (tuples of) values, $y(n)$, from (the embedding in G of) a test sequence of F_2 to the universal variables y . Given each $y(n)$, we set $x(n)$ to have the shortest possible values in G (with respect to the simplicial metric on the Bass-Serre tree that is associated with $G = A^1 * \dots * A^\ell$).

Since the limit groups over free products, $\{H_j\}$, are finitely presented, the sequence $\{(x(n), y(n))\}$ has a subsequence (still denoted $\{(x(n), y(n))\}$, that factors thorough one of the limit groups, H_j . Furthermore, we may assume that for every index n , the restriction of the specializations, $\{(x(n), y(n))\}$, to the (el-

liptic) factors of H_j, S_1, \dots, S_{m_j} , are elliptic subgroups in G , and the elements, $v_1(x(n), y(n)), \dots, v_r(x(n), y(n))$, are all non-trivial in G .

For every index n , we can define a homomorphism $h_n : H_j \rightarrow G$, given by the specialization $\{(x(n), y(n))\}$. We modify each homomorphism h_n , to a homomorphism: $u_n : H_j \rightarrow \langle y \rangle * G$, by sending the subgroup $\langle y \rangle$ in H_j isomorphically onto the subgroup $\langle y \rangle$ in $\langle y \rangle * G$, and setting u_n restricted to the factor $S_1 * \dots * S_{m_j} * F_{r_j}$ to be identical to h_n restricted to that factor. u_n is clearly a homomorphism, so all the elements, w_1, \dots, w_s , which are the identity element in H_j are mapped to the identity element in $\langle y \rangle * G$. Furthermore, v_1, \dots, v_r are mapped to non-trivial elements in G by the homomorphisms h_n , hence, they must be mapped to non-trivial elements by u_n , as $h_n = \nu_n \circ u_n$, where $\nu_n : \langle y \rangle * G \rightarrow G$, $\nu_n(y) = y(n)$.

Finally, we set $\tau = u_n$ for an arbitrary n . The equalities $w_i(\tau(x), y) = 1$ hold in G (as they hold in $\langle y \rangle * G$), $i = 1, \dots, s$. Furthermore, the sentence:

$$\exists y \quad v_1(\tau(x_j), y) \neq 1, \dots, v_r(\tau(x_j), y) \neq 1$$

is a truth sentence in G , as the inequalities hold for $y = y(n)$. □

To generalize Merzlyakov's original theorem to sentences (over a free group) that hold over some varieties and not only over the entire affine set, we have associated with each well-structured resolution a *completion*, and with a completion we have associated *closures* of it, that are obtained by adding roots to a finite collection of elements in abelian vertex groups that appear along the well-structured resolution (see definitions 1.12 and 1.15 in [Se2]).

Given a completion of a well-structured resolution over a free group, and a (finite) collection of closures of that completion, we call the finite collection of closures, a *covering closure* of the completion (definition 1.16 in [Se2]), if every homomorphism that factors through the well-structured resolution can be extended to a completion of at least one of the closures of the completion (from the given finite collection).

These notions generalize to well-structured resolutions over free products, although the generalizations require some modifications. The construction of a completion generalizes naturally and canonically from well-structured resolutions over a free group (definition 1.12 in [Se2]) to well-structured resolutions over free products. The notion of a closure of the completion over free products, requires some changes in comparison with the similar object over a free group.

Definition 2.3. *Let $Res(y)$ be a well-structured resolution over free products, and let $Comp(Res)(z, y)$ be its completion.*

Let E_1, \dots, E_r be the terminal elliptic subgroups (factors) in the well-structured resolution, $Res(y)$. Note that these are also the terminal elliptic subgroups in the completion, $Comp(Res)(z, y)$. Let Ab_1, \dots, Ab_d be the non-conjugate, non-cyclic, maximal abelian subgroups that appear along the completion, $Comp(Res)(z, y)$, and are mapped onto a non-cyclic, (non-elliptic) abelian factor in a free decomposition associated with one of the levels of the completion.

Let PAb_1, \dots, PAb_{pd} be the non-conjugate, non-cyclic, (non-elliptic) maximal pegged abelian groups that appear along the completed resolution, i.e., maximal non-cyclic abelian subgroups in $Comp(Rlim)(z, y)$, that are mapped onto a non-cyclic, (non-elliptic) abelian vertex group in some abelian decomposition associated with some

level of the completed resolution $\text{Comp}(\text{Res})(z, y)$, and this abelian vertex is connected to the other vertices of the completed decomposition of that level by an edge with (maximal) cyclic (non-elliptic) stabilizer. We call the maximal cyclic subgroup of a pegged abelian group connecting it to the other vertices of the corresponding completed decomposition, the *peg* of the pegged abelian group PAb .

Let S_1, \dots, S_d be free abelian groups so that $Ab_1 < S_1, \dots, Ab_d < S_d$ are subgroups of finite index. Let PS_1, \dots, PS_{pd} be free abelian groups so that $PAb_1 < PS_1, \dots, PAb_{pd} < PS_{pd}$ are subgroups of finite index, and the pegs $\text{peg}_1, \dots, \text{peg}_{pd}$ are primitive elements in the ambient free abelian groups PS_1, \dots, PS_{pd} .

A closure of the completed resolution $\text{Comp}(\text{Res})(z, y)$ is obtained by replacing the free abelian groups Ab_1, \dots, Ab_d by the free abelian groups S_1, \dots, S_d , and the pegged abelian groups PAb_1, \dots, PAb_{pd} by the free abelian groups PS_1, \dots, PS_{pd} in correspondence, along the entire completed resolution, i.e., from the top level through the bottom level in which a subgroup of the pegged abelian group appears along the completed resolution.

We also associate with the closure new elliptic subgroups, D_1, \dots, D_f , where $\ell \leq r$, and B_1, \dots, B_t , with the following properties:

- (1) (up to a change in the order of the elliptic factors E_1, \dots, E_r) E_i is mapped into D_i , $i = 1, \dots, f$.
- (2) for each index i , $f + 1 \leq i \leq r$, we add a new (free) generator, c_i , and map E_i into D_{j_i} .

The closure is obtained from the completion, $\text{Comp}(\text{Res})$, by possibly enlarging the maximal abelian and maximal pegged abelian subgroups (by finite index supergroups), and further replacing each of the terminal elliptic subgroups, E_1, \dots, E_f , with the corresponding subgroup, D_1, \dots, D_f , and replacing each of the subgroups, E_{f+1}, \dots, E_r , by its image in D_{j_i} conjugated by c_i , $i = f + 1, \dots, r$. The terminal limit group of the closure is the free product of the elliptic subgroups, $D_1 * \dots * D_f$, with the free product of the (additional) elliptic subgroups, $B_1 * \dots * B_t$ (possibly) free product with a free group and finitely many abelian (non-elliptic) factors and closed surface groups. The completion, $\text{Comp}(\text{Res})(z, y)$, is mapped naturally into a closure of it.

Having defined closures of a completion, we can generalize the notion of a *covering closure*.

Definition 2.4. Let $\text{Res}(y)$ be a well-structured resolution over free products, let $\text{Comp}(\text{Res})(z, y)$ be its completion, and let $Cl_1(\text{Res}), \dots, Cl_v(\text{Res})$ be a finite set of closures of $\text{Comp}(\text{Res})$.

Let $G = A^1 * \dots * A^\ell$ be a (non-trivial) free product that is not isomorphic to D_∞ . We say that the given finite collection of closures is a *covering closure* of the completion, $\text{Comp}(\text{Res})$, over the free product G , if every tuple of specializations in G of the variables y , that factors through the resolution $\text{Res}(y)$, can be extended to a specialization that factors through at least one of the closures, Cl_1, \dots, Cl_v .

The completion of a resolution $\text{Comp}(\text{Res})(z, y)$, its closures $Cl(\text{Res})(s, z, y)$, and the notion of a *covering closure*, finally allow us to present *formal solutions* associated with a well-structured resolution of a limit group over free products.

Theorem 2.5. Let $u_1(y), \dots, u_m(y)$ be a collection of words in the free group,

$\langle y \rangle$, and let (L, E_L) be a limit group over free products that is a limit quotient (over free products) of the f.p. group: $G(y) = \langle y \mid u_1(y), \dots, u_m(y) \rangle$. Let $Res(y)$ be a well-structured resolution of the limit group (L, E_L) , and let $Comp(Res)(z, y)$ be the completion of the resolution $Res(y)$.

Let $w_1(x, y) = 1, \dots, w_s(x, y) = 1$ be a system of equations, and let $v_1(x, y), \dots, v_t(x, y)$ be a collection of words in the alphabet $\{x, y\}$. Let the sentence:

$$\forall y (u_1(y) = 1, \dots, u_m(y) = 1) \exists x \ w_1(x, y) = 1, \dots, w_s(x, y) = 1 \wedge \\ \wedge v_1(x, y) \neq 1, \dots, v_t(x, y) \neq 1$$

be a sentence over groups.

There exists a finite collection of closures of the resolution $Res(y)$ over free products: $Cl(Res)_1(s, z, y), \dots, Cl(Res)_q(s, z, y)$, so that for each index i , $1 \leq i \leq q$, there exists a limit group over free products, $H_i = Cl_i(Res) * F_{d_i}$, with a tuple of elements $x_i \in H_i$, for which the words, $w_1(x_i, y), \dots, w_s(x_i, y)$, are the trivial elements in the limit groups H_i , and the words, $v_1(x_i, y), \dots, v_t(x_i, y)$, are non-trivial elements in the limit groups H_i , for $i = 1, \dots, q$.

In addition, let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ , and suppose that the given sentence is a truth sentence over G . Let E_1, \dots, E_r be the terminal elliptic subgroups in the completion, $Comp(Res)$, of the resolution, $Res(y)$. Then:

- (1) the closures, Cl_1, \dots, Cl_q , form a covering closure (definition 2.4) of the completion, $Comp(Res)$, over the free product G .
- (2) let (z_0, y_0) be a tuple of specializations from G that factors through the completed resolution, $Comp(Res)(z, y)$. There exists an index i , $1 \leq i \leq q$, for which the specialization (z_0, y_0) extends to a specialization (s_0, z_0, y_0) of the closure, $Cl_i(Res)$.

the specialization (z_0, y_0) restricts to homomorphisms of the elliptic subgroups of the completion, $Comp(Res)$, E_1, \dots, E_r , into the free product G . These homomorphisms extend to a homomorphism $h : H_i \rightarrow G$ for which for every index j , $1 \leq j \leq r$: $v_j(h(x_i), h(y)) \neq 1$ in G .

Proof: To construct the set of closures that is associated with a given sentence, we start with the collection of free products of non-trivial groups, $G = A^1 * \dots * A^\ell$, where $\ell > 1$ is (an arbitrary) positive integer, and G is not isomorphic to $D_\infty = Z_2 * Z_2$, and over which the given sentence is a truth sentence. Given the collection of all these groups, $\{G\}$, we look at the collection of all the test sequences of the given resolution over free products, $Res(y)$. Note that the collection of test sequences is divided into finitely many subsets, where in each subset it is specified which of the terminal elliptic subgroups of $Res(y)$, E_1, \dots, E_r , are mapped into conjugate factors in G . If for some test sequence of $Res(y)$ over a free products $G = A^1 * \dots * A^\ell$, two terminal elliptic factors, E_i and E_j , are mapped into conjugate factors of G , we add new elements to the (generators of the) resolution, $Res(y)$, that conjugate the factors into which E_i and E_j are mapped.

Since the sentence is a truth sentence over the free products G that we consider, for the specializations of each test sequence (in the free products G), it is possible to add specializations to the existential variables x , so that both the equalities, w_1, \dots, w_s , and the inequalities, v_1, \dots, v_t , do hold in G . Given each specialization in a test sequence, we choose the shortest possible specialization of the existential

variables x (in the simplicial metric of the Bass-Serre tree that is associated with the free product, $G = A^1 * \dots * A^\ell$).

Given each test sequence of $Res(y)$, and the extensions to shortest possible existential variables, it is possible to extract a subsequence, that converges into a limit group (over free products), that has the form, $Cl(Res) * F_d$, for some closure of the given resolution, $Res(y)$, and (possibly trivial) free group, F_d . As we did in proving theorem 21 in [Ja-Se], we can define a partial order on the collection of these obtained limit groups. We can replace each of the obtained closures, $Cl(Res)$, with a closure that is obtained from the completion, $Comp(Res)$, by adding finitely many generators and relations, and so that the elements, w_1, \dots, w_s are still trivial, and the elements, v_1, \dots, v_t , are non-trivial in the relatively finitely presented closure. By the ability to perform such a replacement, every maximal element with respect to the partial order on the obtained limit groups, $Cl(Res) * F_d$, is obtained from $Comp(Res)$ by adding finitely many generators and relations. By the same argument that was used to prove the finiteness of maximal limit quotients (over free products) of a f.p. group (theorem 21 in [Ja-Se]), there are only finitely many maximal limit groups of the form, $Cl(Res) * F_d$, that dominate all the limit groups that are obtained from test sequences of the resolution, Res , and its extension to shortest existential variables.

We denote these maximal limit groups, H_1, \dots, H_q , where for each i , $i = 1, \dots, q$, $H_i = Cl_i(Res) * F_{d_i}$. Since the limit groups $\{H_i\}$ dominate all the possible test sequences over groups for which the sentence is a truth sentence (together with their extensions to the existential variables), the closures, Cl_1, \dots, Cl_q , do form a covering closure for the resolution, Res , for every non-trivial free product G , which is not D_∞ , and over which the sentence is a truth sentence. By construction, the elements w_1, \dots, w_s represent the trivial element in each of the groups H_i . The elements v_1, \dots, v_t represent non-trivial elements. Part (2) follows since the limit groups H_i , $i = 1, \dots, q$, dominate (the tails of) all the test sequences of the resolution, $Res(y)$, over free products for which the given sentence is a truth sentence (and their extensions to shortest existential variables). □

Theorem 2.5 proves, in particular, that if a given sentence is a truth sentence over a variety V that is defined over some free product G , then there exist formal solutions that prove the validity of the sentence for generic points in the variety V . As over free and hyperbolic groups, we will need a uniform way to pick these formal solutions. To get the type of uniformity that we need, we still need to collect the entire set of formal solutions. These can be encoded using formal limit groups (over free products) and graded formal limit groups.

As over free groups (definition 2.1 in [Se2]), given a completion of a well-structured resolution over free products, $Comp(Res)(z, y)$, and a system of equations, $\Sigma(x, y) = 1$, we define a *formal limit group* that is associated with the completion and the system of equations, as a limit of a sequence of homomorphisms: $h_n : \langle x, z, y \rangle \rightarrow A_n^1 * \dots * A_n^\ell$, where the restrictions, $h_n : \langle z, y \rangle \rightarrow A_n^1 * \dots * A_n^\ell$, form a test sequence of the completion, $Comp(Res)(z, y)$, and $\Sigma(h_n(x), h_n(y)) = 1$ in $A_n^1 * \dots * A_n^\ell$.

In [Ja-Se] we have associated well-structured resolutions with a limit group over free products. If we combine the results and the techniques of [Ja-Se], with the construction of formal resolutions of a formal limit group (over a free group) in the second chapter of [Se2], then given a formal limit group, $FL(x, z, y)$, we are able to associate with it its collection of formal resolutions. Furthermore, in [Ja-Se] it is

shown that given a f.p group it is possible to associate with it finitely many well-structured resolutions, so that every homomorphism from the given f.p. group into a free product factors through at least one of these finitely many resolutions, and so that the completion of each of the finitely many resolutions is finitely presented (such a finite collection of resolutions is taken to be the (non-canonical) Makanin-Razborov diagram of the given f.p. group over free products).

Similarly, let $Res(y)$ be a well-structured resolution with a f.p. completion, $Comp(Res)(z, y)$ (hence, $comp(Res)(z, y)$ terminates in a free product of finitely many f.p. elliptic factors and (possibly) a f.g. free group). Given a (finite) system of equations, $\Sigma(x, y) = 1$, it is possible to associate with the given system of equations, $\Sigma(x, y) = 1$, and the given resolution over free products, $Res(y)$, a finite collection of (well-structured) formal resolutions, with f.p. completions, so that:

- (1) each resolution terminates in a f.p. group, $Cl_i(Res) * F_{d_i}$, where $Cl_i(Res)$ is a closure of the completion, $Comp(Res)$, and the closure, $Cl_i(Res)$, and its terminating elliptic factors are all finitely presented.
- (2) every formal solution of the given completion and the given system of equations, over any non-trivial free product that is not isomorphic to D_∞ , factors through at least one of the finitely many formal resolutions.

Like the Makanin-Razborov diagram (over free products) that we associated with a f.p. group, the finite collection of formal resolutions that we associated with a given f.p. completion and a finite system of equations is not canonical. However, it encodes all the formal solutions that can be associated with the given completion and the given system of equations, and it suffices to analyze sentences and predicates. Hence, we call such a finite collection of formal resolutions (that satisfy properties (1) and (2)), a *formal Makanin-Razborov* diagram of the completion, $Comp(Res)$, and the finite system of equations, Σ , over free products.

To analyze sentences and predicates over free groups, we needed to collect not only formal solutions that are defined over a given completion and a given system of equations, but rather to collect all the formal solutions that are defined over a given graded completion, $Comp(Res(y, z, p))$, and a system of equations with parameters, $\Sigma(x, y, p) = 1$.

Note that a technical difficulty that exists in the graded case, and does not appear in the ungraded case, is that a (graded) completion of a graded resolution over free products, can not be assumed to be f.p. even if we start with a graded f.p. group. f.p. is essential in our approach to constructing (graded) Makanin-Razborov diagrams over free products. However, this technical difficulty can be overcome by applying the techniques and the results that appear in theorems 1.17, 1.20, and 1.21, while studying graded resolutions and their covers.

We start the construction of a graded formal Makanin-Razborov diagram over free products, by following the construction of formal graded limit groups and their formal graded Makanin-Razborov diagrams over free groups, that appears in section 3 in [Se2]. Let $Res(y, p)$ be a graded well-structured resolution over free products. We assume that $Res(y, p)$ terminates in a free product of a rigid or a solid limit group with (possibly) finitely many elliptic factors and (possibly) a f.g. free group. In case the terminal graded limit group of $Res(y, p)$ contains a solid factor we assume that with the solid factor, there is an associated finite collection of covers of its flexible quotients (see proposition 1.5). Let $Comp(Res)(z, y, p)$ be the graded completion of $Res(y, p)$.

As in the ungraded case (over free products), given the graded completion, $Comp(Res)(z, y, p)$, of the graded well-structured resolution over free products, $Res(y, p)$, and a (finite) system of equations, $\Sigma(x, y, p) = 1$, we define a *graded formal limit group* that is associated with the completion and the system of equations, as a limit of a sequence of homomorphisms $h_n : \langle x, z, y, p \rangle \rightarrow A_n^1 * \dots * A_n^\ell$, where the restrictions, $h_n : \langle z, y, p \rangle \rightarrow A_n^1 * \dots * A_n^\ell$, form a graded test sequence of the graded completion, $Comp(Res)(z, y, p)$, and $\Sigma(h_n(x), h_n(y), h_n(p)) = 1$ in $A_n^1 * \dots * A_n^\ell$. Furthermore, by possibly passing to a subsequence of the homomorphisms, $\{h_n\}$, it is possible to use the subsequence of homomorphisms (still denoted $\{h_n\}$), and associate with them a graded formal resolution, that terminates in a graded limit group of the form, $Cl(Res) * F_d$, where $Cl(Res)$ is a closure of the given completion, $Comp(Res)$, and F_d is a possibly trivial free group. Note that even if the completion, $Comp(Res)$, is f.p. it does not imply that the closure, $Cl(Res)$, is f.p. as well.

Since the closure, $Cl(Res)$, may be infinitely presented, to construct finitely many formal graded resolutions that will cover all the graded formal resolutions that are associated with $Comp(Res)$, we need to replace each graded formal resolution by a *cover*, in a similar way to the construction of covers of graded resolutions over free products that appear in theorem 1.21.

Theorem 2.6. *Let $G(y, p)$ be a f.p. group, let $Res(y, p)$ be a well-structured resolution over free products of a (graded) limit quotient of $G(y, p)$ that terminates in a free product of a rigid or solid factor and possibly finitely many elliptic factors and a (possibly trivial) free group. Let $Comp(Res)(z, y, p)$ be the completion of $Res(y, p)$, and let $\Sigma(x, y, p) = 1$ be a (finite) system of equations.*

*Let $\{h_n : \langle x, z, y, p \rangle \rightarrow A_n^1 * \dots * A_n^\ell\}$ (where $A_n^1 * \dots * A_n^\ell$ are non-trivial free products that are not isomorphic to D_∞) be a sequence of homomorphisms, where the restrictions, $h_n : \langle z, y, p \rangle \rightarrow A_n^1 * \dots * A_n^\ell$, form a graded test sequence of the graded completion, $Comp(Res)(z, y, p)$, and $\Sigma(h_n(x), h_n(y), h_n(p)) = 1$ in $A_n^1 * \dots * A_n^\ell$. Furthermore, we assume that the sequence $\{h_n\}$ converges into a formal limit group over free products, $FL(x, z, y, p)$, and that from the sequence, $\{h_n\}$, it is possible to construct a formal resolution (over free products), $FGRes$:*

$$FL(x, z, y, p) = FL_0 \rightarrow FL_1 \rightarrow \dots \rightarrow FL_s = Cl(Res) * F_d$$

where $Cl(Res)$ is a graded closure of the completion, $Comp(Res)(z, y, p)$, and F_d is a possibly trivial free group.

There exists a f.g. graded formal limit quotient of $G(y, p)$, CF , and a well-structured graded formal resolution of CF , $CFGRes$, $CF = CF_0 \rightarrow CF_1 \rightarrow \dots \rightarrow CF_s$, that covers the graded formal resolution that is constructed from the homomorphisms, $\{h_n\}$, i.e., the formal graded resolution, $CFGRes$, has the following properties (cf. theorem 1.21):

- (1) the (cover) graded formal resolution, $CFGRes$, is a strict well-structured resolution over free products. In particular it maps non-trivial elliptic elements to non-trivial elliptic elements. Furthermore, for each index i , $i = 1, \dots, s$, there is an epimorphism of limit groups over free products, $\tau_i : CF_i \rightarrow FL_i$. The epimorphisms τ_i commute with the quotient maps in the two graded resolutions, $FGRes$ and $CFGRes$.
- (2) the epimorphisms along $CFGRes$ are proper epimorphisms.

- (3) all the graded formal abelian decompositions that are associated with the various formal limit groups (over free products), CF_i , $i = 1, \dots, s$, are their graded formal JSJ decompositions over free products. Furthermore, the graded formal JSJ decompositions (over free products) of the formal limit groups, CF_i , $i = 1, \dots, s$, have the same structure as the corresponding graded formal JSJ decompositions (over free products) of the formal limit groups, FL_i , where the difference is only in the rigid vertex groups of the JSJ decompositions and in the elliptic factors.
- (4) if no factor of FL_i is (formal) rigid nor solid, then CF_{i+1} is a cover of a (formal) shortening quotient of CF_i , for $i = 1, \dots, s - 1$.
- (5) for $i < s$ no factor of FL_i can be (formal) rigid. If for $i < s$ a factor of FL_i is (formal) solid then the corresponding factor of CF_i is formal weakly solid (see definition 1.18 for a weakly solid group). In this (solid) case, when $i < s$, CF_{i+1} is a free product of a cover of a (formal) flexible quotient of the (formal) weakly solid factor in CF_i with (possibly) the same elliptic factors and (possibly) the same free group as in CF_i .
- (6) FL_s , the terminal formal graded limit group of the formal graded resolution, $FGRes$, is of the form $Cl(Res) * F_d$, where $Cl(Res)$ is a closure of the completion, $Comp(Res)$, and F_d is a possibly trivial free group. $Cl(Res)$ terminates in a free product of either a (non-formal) rigid or solid factor with (possibly) finitely many elliptic factors and (possibly) a free factor.

CF_s , the terminal limit group of the cover formal graded resolution, $CFGRes$, is of the form $CF_s = CCl(Res) * F_d$, where $CCl(Res)$ is a cover of the closure, $Cl(Res)$, which is a factor of FL_s . The cover closure, $CCl(Res)$, has the same structure as the closure, $Cl(Res)$, and it differs from the closure $Cl(Res)$ only in the terminating limit group. The cover closure $CCl(Res)$ terminates in a free product of a (non-formal) rigid or a weakly solid factor, that we denote RSF_s , in correspondence with the factor of the terminal limit group of $Cl(Res)$, with (possibly) f.p. covers of the elliptic factors in the terminal limit group of $Cl(Res)$, and a free group (of the same rank as in the terminal limit group of $Cl(Res)$).

With the rigid or the weakly solid factor of the terminal limit group of $CCl(Res)$, RSF_s , we associate finitely many limit groups (over free products), CFl_1, \dots, CFl_g , with ungraded resolutions (over free products), Res_1, \dots, Res_g , that terminate in f.p. limit groups (cf. theorems 1.17 and 1.20). Hence, the completions of Res_1, \dots, Res_g are f.p. and the limit groups (over free products), CFl_1, \dots, CFl_g , can be embedded in f.p. completions (that have f.p. terminal limit groups).

- (7) the finite collection of limit groups, CFl_1, \dots, CFl_g , are all limit quotients of the f.p. group $G(y, p) * \langle x \rangle$, and they dominate all the flexible quotients of RSF_s (although they need not be quotients of RSF_s).

From the sequence of homomorphisms, $\{h_n\}$, it is possible to perform iterative shortenings, and obtain another sequence of homomorphisms, $\{u_n\}$, that restrict to asymptotically rigid or asymptotically strictly solid homomorphisms that converge into the rigid or solid factor of the terminal limit group of the closure, $Cl(Res)$, that appears as a factor in a free product, $FL_s = Cl(Res) * F_d$, where FL_s is the terminal graded limit group in $FGRes$.

The homomorphisms $\{u_n\}$ factor through CFL_s , and they restrict to

rigid or weakly strictly solid homomorphisms of RSF_s (with respect to the cover, CFl_1, \dots, CFl_g). Furthermore, in the rigid case, every flexible homomorphism of RSF_s factors through at least one of the resolutions, Res_1, \dots, Res_g , that are associated with the limit groups, CFl_1, \dots, CFl_g . In the weakly solid case, for every non-weakly strictly solid homomorphism of RSF_s , $f : RSF_s \rightarrow A^1 * \dots * A^\ell$, there exists a homomorphism, $u : RSF_s \rightarrow A^1 * \dots * A^\ell$, such that the pair, (f, u) , extends to a homomorphism of the completion of the identity resolution, $RSF_s \rightarrow RSF_s$, and the homomorphism u as a homomorphism of the f.p. group, $G(y, p) * \langle x \rangle$, factors through at least one of the resolutions, Res_1, \dots, Res_g .

- (8) with the terminal graded limit group of the cover closure, $CCL(Res)$, that we denote, $TCCL(Res)$, it is possible to associate an ungraded strict well-structured resolution: $TCCl = V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_t$, so that the terminal (ungraded) limit group (over free products) V_t is a free product of (possibly) f.p. elliptic factors and (possibly) a free group. Every non-trivial elliptic element in each of the limit groups (over free products) V_i , $i = 0, \dots, t-1$, is mapped to a non-trivial element in V_{i+1} .

It is clearly possible to combine the cover formal graded resolution, $CFGRes$, with the ungraded resolution, $TCl = V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_t$, and obtain a combined ungraded resolution over free products. With this resolution we can naturally associate an (ungraded) completion, that we denote, $Comp_{CF}$. Then the completion, $Comp_{CF}$, and its terminal limit group, V_t , are finitely presented. This implies that all the formal limit groups that appear along the cover formal graded resolution, $CFGRes$, and in particular the terminal cover closure, $CCL(Res)$, can all be embedded in a f.p. completion.

Proof: With the techniques for the construction of graded formal limit groups (over free groups) that appear in section 3 of [Se2], the proof is identical to the proof of theorem 1.21. □

Given a graded well-structured resolution, $Res(y, p)$, of a graded limit quotient of a f.p. group, $G(y, p)$, its completion, $Comp(Res)(z, y, p)$, a system of equations, $\Sigma(x, y, p) = 1$, and a sequence of homomorphisms, $\{h_n : \langle x, z, y, p \rangle \rightarrow A_n^1 * \dots * A_n^\ell\}$ that restricts to a test sequence of the completion, $Comp(Res)$, we first passed to a subsequence of the homomorphisms, $\{h_n\}$, from which we constructed a formal graded resolution over a closure of the completion, $Comp(Res)$, and then by theorem 2.6 we have associated (non-canonically) a cover formal graded resolution with the constructed formal graded resolution. By the properties of the cover formal resolution, all the graded limit groups that are involved in its construction can be embedded in f.p. graded completions (see theorem 2.6).

Therefore, with the graded resolution, $Res(y, p)$, and the system of equations, $\Sigma(x, y, p) = 1$, theorem 2.6 associates a collection of cover formal graded resolutions, so that every formal solution, that is associated with the resolution and the system of equations, factors through at least one of these cover formal graded resolutions. As in the ungraded case (theorem 26 in [Ja-Se]), and as in the graded case (theorem 1.22), from this collection of cover formal graded resolutions it is possible to find a finite subcollection, so that every formal solution that is associated with the graded resolution, $Res(y, p)$, and the system of equations, $\Sigma(x, y, p) = 1$, factors through at least one of the cover formal graded resolutions that belong to the finite

subcollection. Hence, we can view this (non-canonical) finite subcollection as a *formal graded Makanin-Razborov diagram* of the resolution, $Res(y, p)$, and the system of equations, $\Sigma(x, y, p) = 1$.

Theorem 2.7. *Let $G(y, p)$ be a f.p. group, let $Res(y, p)$ be a well-structured resolution over free products of a (graded) limit quotient of $G(y, p)$ that terminates in a free product of a rigid or solid factor, and possibly finitely many elliptic factors, and a (possibly trivial) free group. Let $Comp(z, y, p)$ be the completion of $Res(y, p)$, and let $\Sigma(x, y, p) = 1$ be a (finite) system of equations.*

*There exist finitely many well-structured formal graded resolutions that satisfy properties (1)-(8) of theorem 2.6: $CF = CF_0 \rightarrow CF_1 \rightarrow \dots \rightarrow CF_s$, where CF is a graded formal limit group over free products that is associated with the f.p. group $G(y, p)$ and the system of equations, $\Sigma(x, y, p) = 1$, and CF_s is a free product $CF_s = CCl(Res) * F_d$, i.e., a free product of a cover of a closure of $Comp(Res)(z, y, p)$ with a (possibly trivial) free group.*

With each of the finitely many cover formal graded resolutions we associate the collection of formal solutions that factor through it. These formal solutions, are obtained from homomorphisms of the terminal graded limit group, $TCCl(Res)$ of the cover closure, $CCl(Res)$, which is a factor of CF_s . These homomorphisms of $TCCl(Res)$ are obtained from rigid or weakly strictly solid homomorphisms of the rigid or weakly solid factor of $TCCl(Res)$, that are combined with homomorphisms of the f.p. elliptic factors (into elliptic subgroups), and arbitrary homomorphisms of the (possible) free factor (of $TCCl(Res)$). We further require that the restriction of weakly strictly solid homomorphisms to edge groups and to abelian and QH vertex groups in the graded abelian JSJ decomposition of the weakly solid factor are not (entirely) elliptic.

Formal solutions that are associated with the given cover formal graded resolution, and with the formal limit groups, $CF_{s-1}, \dots, CF_0 = CF$, are obtained from homomorphisms of CF_s , in a similar way to homomorphisms that factor through graded resolutions over free and hyperbolic group, i.e., according to parts (1)-(3) that are listed in theorem 1.22.

Every formal solution that is associated with the given resolution, $Res(y, p)$, and the system of equations, $\Sigma(x, y, p) = 1$, factors through at least one of the finitely many (cover) formal graded resolutions that are associated with the f.p. group $G(y, p)$, the resolution, $Res(y, p)$, and the system of equations, $\Sigma(x, y, p) = 1$.

Proof: Given the construction of a cover formal graded resolution that appears in theorem 2.6, the proof of theorem 2.7 is identical to the proof of theorem 1.22 (and to theorem 26 in [Ja-Se]).

□

The finite collection of formal graded resolutions that are associated (non-canonically) with a f.p. group, $G(y, p)$, a well-structured resolution of a limit quotient (over free products) of it, $Res(y, p)$, and a system of equations, $\Sigma(x, y, p) = 1$, according to theorem 2.7, and through which all the formal solutions that are associated with this triple do factor, form a *formal graded Makanin-Razborov diagram* over free products. Note that the collection of formal graded resolutions is not canonical, hence, the diagram we constructed is not canonical.

As we have already indicated in constructing the graded Makanin-Razborov diagram, the construction of the formal graded Makanin-Razborov diagram does not

require a separate study of the singular locus of the constructed formal graded resolutions, as by construction, this singular locus is covered by other formal graded resolutions from the constructed finite collection.

§3. AE Sentences

In [Ja-Se] and the first 2 sections in this paper we have generalized the results and notions that were presented in [Se1], [Se2] and [Se3], for studying varieties, sentences and predicates defined over a free group, to free products. In this section we show how to use these notions and constructions to reduce an AE sentence over free products to a sentence in the Boolean algebra of AE sentences in the factors. Like all our constructions in the previous sections and in [Ja-Se], the reduction of an AE sentence over free products to a sentence in the factors is uniform, and does not depend on a specific free product (although it does depend on the number of factors in the free product).

In [Se4] we have associated a *complexity* with each well-structured resolution, or an induced resolution (see section 3 of [Se4] for an induced resolution). This complexity is slightly modified in studying AE sentences over hyperbolic groups (definition 4.2 in [Se7]). Over free products we use the same complexity of well-structured resolutions (and their completions), as the one that is used over hyperbolic groups.

Definition 3.1 ([Se4],4.2). *Let $Comp(Res)(t, y)$ be a completion of a well-structured resolution, $Res(y)$, over free products, with (possibly) reduced modular groups associated with each of its various QH subgroups. Let Q_1, \dots, Q_m be the QH subgroups that appear in the completion, $Comp(Res)(t, y)$, and let S_1, \dots, S_m be the (punctured) surfaces associated with the reduced modular group associated with each of the QH vertex group. To each (punctured) surface S_j we may associate an ordered couple (genus(S_j), $|\chi(S_j)|$). We will assume that the QH subgroups Q_1, \dots, Q_m are ordered according to the lexicographical (decreasing) order of the ordered couples associated with their corresponding surfaces. Let $rk(Res(y))$ be the rank of the free group that is dropped along the resolution $Res(y)$, let $fact(Res(y))$ be the number of elliptic terminal factors of the resolution $Res(y)$, and let $Abrk(Res(y))$ be the sum of the ranks of the kernels of the mappings of (free) abelian groups that appear as vertex groups along the resolution $Res(y)$ (see definition 1.15 in [Se4]). We set the complexity of the resolution, $Res(y)$, and the completion, $Comp(Res)$, denoted $Cmplx(Res(y))$, to be:*

$$Cmplx(Res(y)) = (fact(Res(y)) + rk(Res(y)), (genus(S_1), |\chi(S_1)|), \dots \\ \dots, (genus(S_m), |\chi(S_m)|), Abrk(Res(y))).$$

On the set of complexities of completed resolutions with (possibly) reduced modular groups we can define a linear order. Let $Res_1(y)$ and $Res_2(y)$ be two completed resolutions with (possibly) reduced modular groups. We say that $Cmplx(Res_1(y)) = Cmplx(Res_2(y))$ if the tuples defining the two complexities are identical. We say that $Cmplx(Res_1(y)) < Cmplx(Res_2(y))$ if:

- (1) *the "Kurosh" rank, $fact(Res_1(y)) + rk(Res_1(y))$ is smaller than the Kurosh rank $fact(Res_2(y)) + rk(Res_2(y))$.*
- (2) *the above ranks are equal and the tuple:*

$$((genus(S_1^1), |\chi(S_1^1)|), \dots, (genus(S_{m_1}^1), |\chi(S_{m_1}^1)|))$$

is smaller in the lexicographical order than the tuple:

$$((\text{genus}(S_1^2), |\chi(S_1^2)|), \dots, (\text{genus}(S_{m_2}^2), |\chi(S_{m_2}^2)|)).$$

(3) the above ranks and tuples are equal and $\text{Abrk}(\text{Res}_1(y)) < \text{Abrk}(\text{Res}_2(y))$.

Given a complete, well-separated resolution, $\text{Res}(t, y)$, over a free group, and a subgroup $\langle y \rangle$ of its associated limit group, we have constructed in section 3 of [Se4], the induced resolution, $\text{Ind}(\text{Res}(t, y))(u, y)$. The construction of the induced resolution generalizes directly to well-separated resolutions over free products, hence, we omit its detailed description.

In section 4 of [Se4] a procedure for a validation of an AE sentence over a free group is presented. This procedure is generalized to torsion-free hyperbolic groups in [Se8]. In this section our goal is to analyze uniformly an AE sentence over all non-trivial free products. Hence, we are not aiming at validating an AE sentence, but rather our goal is to find a uniform way to reduce an AE sentence over a free product to a sentence over its factors.

In the procedure that is presented in [Se4] we used a single formal solution (at times particular families of formal solutions), at each step of the procedure. As we look for a uniform construction, we will need to collect all the formal solutions that are defined over a completion (or rather on closures of it) in each step of the procedure. Also, as we will see in the sequel, to obtain the reduction of an AE sentence from a free product to its factors, it is easier to analyze the places in which the iterative procedure of the type that is presented in [Se4] fails, rather than the places it succeeds (these possible "failures" are provided by the iterative procedure for validation of an AE sentence over free and hyperbolic groups (see section 4 in [Se4]), and they are later used in the quantifier elimination procedure over these groups).

Let:

$$\forall y \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1$$

be a sentence over groups. Let $F_y = \langle y \rangle$, be a free group that is freely generated by (copies of) the (universal) variables y . Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ . If the given sentence is a false sentence over G , then there is a homomorphism $h : F_y \rightarrow G$, so that for the corresponding values of the universal variables y (i.e., the image of the variables y under the homomorphism h), there exist no values for the existential variables x , for which both the equalities $\Sigma(x, y) = 1$ and the inequalities $\Psi(x, y) \neq 1$ hold.

We look at all the possible sequences of homomorphisms, $\{h_n : F_y \rightarrow A_n^1 * \dots * A_n^\ell\}$, where $\ell \geq 2$ is an arbitrary positive integer, and the free products, $A_n^1 * \dots * A_n^\ell$, are non-trivial and not isomorphic to D_∞ . We further assume that each of the homomorphisms h_n fails the given AE sentence for $A_n^1 * \dots * A_n^\ell$, i.e., for each h_n there are no values for the existential variables x (in $A_n^1 * \dots * A_n^\ell$), so that for the tuple, $(x, h_n(y))$, both the equalities Σ and the inequalities Ψ hold (in $A_n^1 * \dots * A_n^\ell$).

By theorem 18 in [Ja-Se] given such a sequence, $\{h_n\}$, we can pass to a subsequence that converges into a well-structured (even well-separated) resolution: $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$, where L_s is a free product of (possibly) a free group and (possibly) finitely many elliptic factors. We denote this resolution $B\text{Res}$. Note

that the terminal elliptic subgroups in $BRes$ are f.g. but they may be infinitely presented.

As the the terminal elliptic factors of the resolution, $BRes$, and its completion, may be infinitely presented, we start by iteratively approximating it by resolutions with the same structure that have f.p. completions and terminal elliptic factors, approximations (or covers) that we used in constructing the ungraded Makanin-Razborov diagram over free products (see theorem 25 in [Ja-Se]).

Let E_1, \dots, E_r be the elliptic factors in the free decomposition of the terminal limit group of the resolution, $BRes$, L_s . E_1, \dots, E_r are all f.g. but they may be infinitely presented. Hence, we fix a system of f.p. approximations of E_1, \dots, E_r , that we denote, E_1^m, \dots, E_r^m , that are obtained from E_1, \dots, E_r by fixing a generating set of E_1, \dots, E_r , and keeping only the relations of length up to m in each of the elliptic factors, E_1, \dots, E_r .

For sufficiently large index m , we set Res^m to be the resolution (over free products) that is obtained from $BRes$, by replacing the elliptic factors, E_1, \dots, E_r , by the f.p. factors, E_1^m, \dots, E_r^m (note that for sufficiently large m it is guaranteed that the all the retractions in the resolution, $BRes$, lift to corresponding retractions in the resolutions, Res^m). Since the resolution $BRes$ is well-structured (and well-separated), the resolutions, Res^m , are well-structured and well-separated for large m . Since the elliptic factors, E_1^m, \dots, E_r^m , are all finitely presented, for each index m , there exists some index n_m , so that for all $n > n_m$, the homomorphisms $\{h_n\}$ factor through the resolution Res^m .

In section 2 we have shown that given a resolution over free products, $Res(y)$, with a f.p. completion, and a finite system of equations, $\Sigma(x, y) = 1$, it is possible to associate with them (non-canonically) a formal Makanin-Razborov diagram that encodes all the formal solutions that are defined over (a closure of) $Res(y)$, and over every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ (see theorem 2.5). Furthermore, each formal resolution in the formal Makanin-Razborov diagram terminates in a closure of $Res(y)$, and this closure, as well as its terminating elliptic factors are all finitely presented.

Therefore, with each approximating resolution of the constructed resolution $BRes$, Res^m , we associate (non-canonically) a formal Makanin-Razborov diagram (over free products). Note that the completions of the formal resolutions in these formal Makanin-Razborov diagrams are all finitely presented.

For each index m , there exists an index n_m , so that for all $n > n_m$, the homomorphisms $\{h_n : A_n^1 * \dots * A_n^\ell\}$ factor through the resolution, Res^m . Hence, with each such homomorphism, h_n , we can associate values (specializations) with each of the elliptic subgroups, E_1^m, \dots, E_r^m .

If there exists an index m , for which there is an infinite subsequence of homomorphisms (still denoted), $\{h_n\}$, so that for each of the specializations of the elliptic subgroups, E_1^m, \dots, E_r^m , that are associated with the homomorphisms, $\{h_n\}$, there exists a test sequence of Res^m that extends the values of the elliptic factors, that does not extend to formal solutions over (a closure of) Res^m , or it does extend to formal solutions over (a closure of) Res^m , but for each such formal solution at least one of the inequalities in the system, $\Psi(x, y) \neq 1$, does not hold (i.e., it is an equality and not an inequality), we reached a terminal point of the iterative procedure. In this case, the associated output is the (approximating) resolution, Res^m , and its associated formal Makanin-Razborov diagram.

Suppose that for every index m , there is no subsequence of homomorphisms

$\{h_n\}$, for which for the values of the elliptic factors of Res^m that are associated with each homomorphism from the subsequence, there exists a test sequence that extends these values, and either there exists no formal solution (over a closure of Res^m) that extends the specializations of the test sequence, or there exist such formal solutions but for all of them the system of inequalities, $\Psi(x, y) \neq 1$ does not hold. In this case for each index m , there exists an index $k_m > n_m > m$, so that h_{k_m} factors through Res^m , and with the specializations of the elliptic subgroups, E_1^m, \dots, E_r^m , that are associated with h_{k_m} , it is possible to associate a formal solution x_m that does satisfy $\Sigma(x_m, y) = 1$ and $\Psi(x_m, y) \neq 1$ for generic y (generic for the fiber that is associated with the resolution, $Res(y)$, and the corresponding specializations of the elliptic factors, E_1^m, \dots, E_r^m).

By construction, the elliptic factors in the resolutions, $\{Res^m\}$, E_1^m, \dots, E_r^m , converge into the elliptic factors in the resolution, $BRes$, E_1, \dots, E_r (as they are f.p. approximations of E_1, \dots, E_r). The formal solutions, $\{x_m\}$, are defined over closures of the resolutions, $\{Res^m\}$. Using the techniques to construct formal limit groups over free products, that were presented in the previous section, from the sequence of formal solutions, $\{x_m\}$, it is possible to extract a subsequence (still denoted) $\{x_m\}$, that converges into a formal limit group over a closure of the (limit) resolution $BRes$, $FL(x, z, y)$. By the construction of the formal limit group $FL(x, z, y)$, the equations from the system, $\Sigma(x, y) = 1$, represent the trivial word in $FL(x, z, y)$, whereas each of the inequations, $\Psi(x, y) \neq 1$, represent a non-trivial element in $FL(x, z, y)$ (as these elements are non-trivial for the formal solutions, $\{x_m\}$, and generic values of the variables y).

At this point we look at the sequences of specializations, $\{(x_m, y_m, z_1(m), \dots, z_t(m))\}$, of the formal solution x_m , the universal variables y , and its successive shortenings (z_1, \dots, z_t) , that take their values in the free products, $\{A_{k_m}^1 * \dots * A_{k_m}^\ell\}$, and each of the specializations, $(y_m, z_1(m), \dots, z_t(m))$, factors through the resolution, Res^m . The elements, $\{y_m\}$, are precisely the subsequence of values of the universal variables for which the sentence fail to hold for the free products, $\{A_{k_m}^1 * \dots * A_{k_m}^\ell\}$.

Given this sequence of specializations, we apply the first step of the procedure for validation of an AE sentence, that is presented in section 4 of [Se4], and extract a subsequence, that converges into a quotient resolution of the one that is associated with the formal limit group, $FL(x, z, y)$ (where the last one is a closure of the original resolution, $BRes$). Since the formal solutions, $\{x_m\}$, were assumed to satisfy both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, the obtained quotient resolution is not a closure of the resolution, $BRes$, that we have started with, but rather a resolution of "reduced complexity" (in the sense of the iterative procedure that is presented in section 4 in [Se4]).

We continue iteratively. At each step we start with a quotient resolution, $QRes$, that was constructed in the previous step of the procedure, using the general step of the iterative procedure that is presented in section 4 in [Se4], and a sequence of homomorphisms into free products, $\{h_n\}$, that converges into the (completion of the) quotient resolution, $QRes$. The homomorphisms, $\{h_n\}$, that are associated with $QRes$, are constructed from a subsequence of the sequence of homomorphisms that are associated with the quotient resolution that was constructed in the previous step, in addition to elements that are associated with the formal solution that is imposed on that quotient resolution. Furthermore, the sequence $\{h_n\}$, restricts to specializations of the universal variables y , that demonstrate that the given AE sentence is a false sentence for the corresponding free products (i.e., for these values

of the universal variables there exist no specializations of the existential variables for which both the equalities, Σ , and the inequalities, Ψ , do hold).

We start the current (general) step, by fixing a sequence of finitely presented approximations, $QRes_m$, of the quotient resolution, $QRes$. Note that for each index m , there exists an index n_m , so that for every $n > n_m$, the homomorphisms $\{h_n\}$ factor through $QRes_m$, and for each homomorphism h_n there are associated specializations of the elliptic factors of $QRes_m$ in the free product that is associated with h_n .

With each f.p. completion (of $QRes_m$) we associate (not in a canonical way) a formal Makanin-Razborov diagram. Note that each formal resolution in such a formal diagram has a f.p. completion. If there exists an index m , and a subsequence of the given sequence of homomorphisms, $\{h_n\}$, that we still denote $\{h_n\}$, so that over the specializations of the elliptic factors of $QRes_m$ that are associated with the subsequence of homomorphisms, $\{h_n\}$, there exists a test sequence that does not extend to any formal solutions over $QRes_m$, so that both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, hold for generic y 's (over the corresponding sequence of free products, $\{A_n^1 * \dots * A_n^\ell\}$), we reached a terminal point of our iterative procedure. In this case the final output of the iterative procedure is the quotient resolution, $QRes_m$, (that has a f.p. completion), and its (non-canonical) formal Makanin-Razborov diagram, where each formal resolution in this diagram has a f.p. completion as well, and this formal Makanin-Razborov diagram encodes all the formal solutions over $QRes_m$ that satisfy the equalities, $\Sigma(x, y) = 1$.

Suppose that there is no such index m , and no such subsequence of the homomorphisms, $\{h_n\}$. In this case for each index m , there exists some index $k_m > m$, so that there is a formal solution x_m that is defined over the resolution, $QRes_m$, with values of its elliptic factors that are associated with the homomorphism, h_{k_m} , so that both the equalities, $\Sigma(x_m, y) = 1$, and the inequalities, $\Psi(x_m, y) \neq 1$, hold for generic values of the universal variables y (generic in the corresponding fibers of the resolution $QRes_m$, and the values of its terminal elliptic factors that are associated with the homomorphisms, $\{h_{k_m}\}$). In this case we look at the sequence, $\{(x_m, h_{n_m})\}$, that has a subsequence that converges into a quotient resolution, that is obtained using the general step of the iterative procedure for the analysis of an AE sentence that is presented in section 4 of [Se4]. Since the formal solutions are guaranteed to satisfy both the equalities, $\Sigma(x_m, y) = 1$, and the inequalities, $\Psi(x_m, y) \neq 1$, the complexity of the obtained quotient resolution is strictly smaller than the complexity of the resolution, $QRes$, that we have started the current step with.

By theorem 4.12 in [Se4], this iterative procedure terminates after finitely many steps. If we reached a terminal quotient resolution along the process, we found a f.p. resolution, $QRes_m$, and a subsequence of the original sequence of homomorphisms, for which for the specializations of the elliptic factors of $QRes_m$ that are associated with the subsequence of homomorphisms (these elliptic factors take their values in the free products, $\{A_n^1 * \dots * A_n^\ell\}$, that are associated with the homomorphisms, $\{h_n\}$), there exist test sequences that do not extend to formal solutions that are defined over (closures of) $QRes_m$, and for which both the equalities and the inequalities hold for generic value of the universal variables y (generic in the fibers that are associated with $QRes_m$ and the corresponding values of the elliptic factors of $QRes_m$ in $A_n^1 * \dots * A_n^\ell$).

If we didn't find such a f.p. resolution along the iterative procedure, it continued

until we reached a quotient resolution which is a free product of elliptic factors (as by theorem 4.12 in [Se4] the iterative procedure terminates after finitely many steps). Once again, we look at a sequence of f.p. approximations of these elliptic factors. These f.p. approximations are free products of elliptic factors as well.

Since the given sequence of homomorphisms $\{h_n\}$, testifies that the given sentence is not valid for the free products, $\{A_n^1 * \dots * A_n^\ell\}$, there must exist a f.p. approximation of the terminal elliptic factors, and a subsequence of the given sequence of homomorphisms, $\{h_n\}$, that we still denote $\{h_n\}$, so that over the specializations of the elliptic factors of the f.p. approximation that are associated with the subsequence of homomorphisms, $\{h_n\}$, no formal solution over the f.p. approximating free product can be constructed so that both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, hold for the corresponding value of the universal variables y .

Therefore, whatever terminal point of the iterative procedure we have reached, we found a resolution (over free products) with f.p. completion, through which a subsequence of the given sequence of homomorphisms do factor, and for which for some test sequences of this resolution that are associated with those values of the elliptic factors of the terminal limit group of the resolution, that are associated with the given subsequence of homomorphisms, the specializations in these test sequences can not be extended to formal solutions (over closures of the resolution) that satisfy both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, for generic values of the variables y .

This procedure that starts with a given AE sentence, and with a sequence of values of the universal variables y of that sentence, that testifies that the given AE sentence fails (over the free products in which the universal variables take their values), and extracts a subsequence that factors through a resolution (over free products) with a f.p. completion, is what we need in order to reduce a sentence uniformly from free products to their factors. It enables one to associate with a given AE sentence, a finite collection of resolutions with f.p. completions, so that if the given sentence is false for a non-trivial free product, $A^1 * \dots * A^\ell$ (which is not D_∞), it must be false for a generic point of one of the constructed resolutions, where the elliptic factors take some value in $A^1 * \dots * A^\ell$. This is the key for reducing an AE sentence over a free product to a sentence over the factors of the free product, and a key for our approach to the analysis of general sentences and predicates over free products.

Theorem 3.2. *Let:*

$$\forall y \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1$$

be a sentence over groups. Then there exist finitely many resolutions over free products: $Res_1(z, y), \dots, Res_d(z, y)$ with the following properties:

- (1) *the completion of each of the resolutions, $Res_i(z, y)$, is finitely presented.*
- (2) *with each resolution, $Res_i(z, y)$, we associate (non-canonically) its formal Makanin-Razborov diagram over free products with respect to the system of equations, $\Sigma(x, y) = 1$. Every resolution in these formal Makanin-Razborov diagrams has a f.p. completion.*
- (3) *for every non-trivial free product, $A^1 * \dots * A^\ell$, which is not D_∞ , and for which the given sentence is false over $A^1 * \dots * A^\ell$, there exists an index*

i , $1 \leq i \leq d$, and a specialization of each of the terminal elliptic factors of $\text{Res}_i(z, y)$ in elliptic factors in the free products, $A^1 * \dots * A^\ell$, so that for these specializations of the elliptic factors of $\text{Res}_i(z, y)$, there exists a test sequence that is associated with these values of the elliptic factors, that does not extend to formal solutions over (closures of) $\text{Res}_i(z, y)$, for which both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, hold for generic values of y .

In other words, there exists a finite collection of resolutions (with f.p. completion) over free products, so that the failure of an AE sentence over general free products can be demonstrated by the lack of the existence of a formal solution over a (generic point in a) fiber of at least one of these resolutions.

Proof: Let A_n^1, \dots, A_n^ℓ , for some $\ell > 1$, be a sequence of non-trivial free products, that are not isomorphic to D_∞ , over which the given AE sentence is false. Let $\{y_n\}$ be a sequence of specializations of the universal variables y , in the free products, A_n^1, \dots, A_n^ℓ , that fail the given AE sentence over these free products, i.e., the existential sentences (with coefficients) over the free products, $A_n^1 * \dots * A_n^\ell$:

$$\exists x \Sigma(x, y_n) = 1 \wedge \Psi(x, y_n) \neq 1$$

are false.

Starting with the sequence, $\{y_n\}$, the terminating iterative procedure that we have presented constructs a resolution, $VRes$, with the following properties:

- (i) the resolution $VRes$ has a f.p. completion and terminal limit group.
- (ii) there exists a subsequence of the sequence of specializations, $\{y_n\}$, that extend to specializations that factor through the resolution, $VRes$. Hence, with each specialization from this subsequence of the sequence, $\{y_n\}$, specializations of the f.p. terminal limit group of the resolution, $VRes$, can be associated. For each such specialization of the terminal limit group of $VRes$, there exists an associated subsequence, that does not extend to any formal solutions (over closures of $VRes$) that satisfy both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, for generic values of the variables y .
- (iii) with the resolution $VRes$, and the system of equations, $\Sigma(x, y) = 1$, we associate (non-canonically) a formal Makanin-Razborov diagram (see section 2). Every (formal) resolution in this diagram has a f.p. completion and terminal limit groups as well.

At this point we are able to apply the argument that we used in constructing the ungraded and graded Makanin-Razborov diagrams (theorems 26 in [Ja-Se] and 1.22 in this paper). We look at all the non-trivial free products, $A^1 * \dots * A^\ell$ (possibly for varying $\ell > 1$), that are not isomorphic to D_∞ , over which the given AE sentence is false. We further look at sequences of specializations of the universal variables y , $\{y_n\}$, that testifies that the given AE sentence fail over the corresponding sequence of non-trivial free products (that are not D_∞), $A_n^1 * \dots * A_n^\ell$. From every such sequence we use our terminating iterative procedure, and extract a subsequence of the specializations, $\{y_n\}$, and a resolution, $VRes$, that has the properties (i)-(iii), and in particular, the subsequence of specializations, (still denoted) $\{y_n\}$, extend to specializations that factor through the resolution $VRes$, and no formal solutions that satisfy both the equalities, $\Sigma(x, y) = 1$, and the inequalities, $\Psi(x, y) \neq 1$, for

generic values (i.e., some test sequences) of the variables y can be constructed, for those values of the elliptic factors in the terminal limit group of $VRes$, that are associated with the subsequence of specializations of the sequence, $\{y_n\}$.

The completion of each of the constructed resolutions, $VRes$, is finitely presented, and so are the resolutions in its associated formal Makanin-Razborov diagram. Hence, we can define a linear order on this (countable) collection of resolutions ($VRes$), and their (non-canonically) associated formal Makanin-Razborov diagrams. By the same argument that was used in constructing the Makanin-Razborov diagram (theorem 26 in [Ja-Se]), there exists a finite subcollection of these resolutions that satisfy properties (1)-(3) of the theorem. □

Theorem 3.2 constructs a finite collection of resolutions over free products, with f.p. completions, that demonstrate the failure of an AE sentence over general free products. This is precisely what is required in order to reduce an AE sentence over a free product to a sentence which is in the Boolean algebra of AE sentences over its factors, which is a special case of our general goal. Before stating this reduction of a sentence from a free product to its factors, we need to study the *singular locus* of a resolution over free products.

Recall that in constructing the graded and ungraded Makanin-Razborov diagrams over free products, and the formal Makanin-Razborov diagram, we have considered the set of specializations that factor through a resolution in one of these diagrams, as those homomorphisms that factor through the resolutions (in the sense that they are obtained from specializations of the terminal limit group of the resolution by iteratively precomposing these homomorphisms with automorphisms from the modular groups (over free products) that are associated with the various levels), and in addition we required that the associated specializations of the various limit groups along the resolution restrict to non-elliptic specializations of all the abelian edge (and vertex) groups, and all the QH vertex groups along the resolution.

This assumption on the non-ellipticity of abelian edge groups, and QH vertex groups, allows us to ignore the singular locus in studying (graded) resolutions and the formal resolutions that are defined over them. However, it is essential to determine the singular locus of the resolutions, Res_1, \dots, Res_d , that were constructed in theorem 3.2, as specializations that do factor through the singular locus are not being considered in further analyzing a given resolution, in order to reduce the AE sentence over the free product G to an AE sentence over the factors of G , and this singular locus needs to be defined uniformly (for all free products that are not isomorphic to D_∞).

Definition 3.3. *Let $Res(y)$ be a well-structured, coefficient free resolution over free products. Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ . A specialization of the terminal limit group of $Res(y)$ in G is said to be in the singular locus of $Res(y)$ over the free product G if it does not extend to a test sequence over G , or equivalently, if for every specialization of $Res(y)$ in G that is obtained from the given specialization of the terminal limit group of $Res(y)$, by a finite sequence of automorphisms in the modular groups that are associated with the various levels of $Res(y)$, the associated specialization of at least one of the edge groups in the virtually abelian decompositions that are associated with the various levels of $Res(y)$ is elliptic or trivial, or the specialization of at least one of the QH vertex groups in these abelian decompositions is elliptic or trivial or virtually*

abelian.

To obtain a (uniform) reduction of AE sentences, we need a uniform description of the singular locus of a given resolution (over all free products).

Proposition 3.4. *Let $Res(y)$ be a well-structured, coefficient free resolution over free products with a f.p. completion (and terminal limit group). Then there exist finitely many graded resolutions with f.p. completions and terminal limit groups, $SLRes_1(y), \dots, SLRes_u(y)$, so that for every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , a specialization of the terminal limit group of $Res(y)$ is in the singular locus of $Res(y)$, if and only if it extends to a specialization of (at least) one of the terminal limit groups of the resolutions, $SLRes_1(y), \dots, SLRes_u(y)$.*

Proof: Let the resolution, $Res(y)$, be given by the sequence of epimorphisms: $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$. Given an edge or a QH vertex in the virtually abelian decomposition that is associated with level i of the resolution $Res(y)$, $0 \leq i \leq s-1$, we look at the partial resolution, Res_{i+1} , that is given by: $L_{i+1} \rightarrow L_{i+2} \rightarrow \dots \rightarrow L_s$. Since $Res(y)$ is well-structured and coefficient-free, so is Res_{i+1} . Since L_s , the terminal limit group of $Res(y)$, is finitely presented, so is the completion of Res_{i+1} .

An edge group or a QH vertex group in the virtually abelian decomposition that is associated with L_i , is mapped into L_{i+1} . Given the edge or a QH vertex group in the abelian decomposition that is associated with L_i , $0 \leq i \leq s-1$, we look at all the test sequences of the completion of the resolution, Res_{i+1} , over arbitrary non-trivial free products that are not D_∞ , for which the image of the given edge group is elliptic or trivial or the image of the given QH vertex group in L_{i+1} is either elliptic or trivial or virtually abelian.

Every such test sequence subconverges into a limit quotient U (over free products) of the completion, $Comp(Res_{i+1})$, in which the image of the given edge group is elliptic or trivial, or the image of the given QH vertex group in L_{i+1} is elliptic or trivial or virtually abelian. As the limit quotient U is obtained as a limit of a test sequence of specializations of $Comp(Res_{i+1})$, U has the same structure as the completion, $Comp(Res_{i+1})$, but its terminal limit group is a quotient of the terminal limit group of $Res(y)$, L_s . Note that the terminal limit group of L_s is f.g. but it may be infinitely presented.

On the set of limit quotients that are obtained from convergent sequences of test sequences of Res_{i+1} in which the image of the given edge group is trivial or elliptic, or the the image of the given QH vertex group in L_{i+1} is either elliptic or trivial or virtually abelian, we can naturally define a partial order, that is identical to the partial order that is defined on limit quotients (over free products) of a given f.g. group in definition 12 in [Ja-Se]. Since the completion, $Comp(Res_{i+1})$, is finitely presented, and we require that finitely many more elements (the image of a generating set of an edge group or a QH vertex group in L_{i+1}) are elliptic or trivial, or that the image of a QH vertex group is virtually abelian (which means that it has an abelian subgroup of index 2 in our case), every limit quotient U that is obtained as a limit of a test sequence of Res_{i+1} in which the image of the given edge group is trivial or elliptic, or the image of the given QH vertex group in L_{i+1} is either elliptic or trivial or virtually abelian, is dominated (under the natural partial order) by such a limit quotient which is finitely presented.

Therefore, by the same argument that was used to prove the finiteness of maximal

limit quotients (over free products) of a f.p. group (theorem 21 in [Ja-Se]), with a given edge or a QH vertex in the abelian decomposition that is associated with L_i in $Res(y)$, $0 \leq i \leq s-1$, we can associate finitely many limit quotients U_1, \dots, U_d , that are all f.p. and do all have the same structure as Res_{i+1} , such that every test sequence of Res_{i+1} for which the image of the given edge group is trivial or elliptic, or the image of the given QH vertex group in L_{i+1} is either elliptic or trivial or virtually abelian, has a subsequence that factors through one of the limit groups, U_1, \dots, U_d .

Hence, by taking the finite union of these limit quotients for all the (finitely many) edge and QH vertex groups in the abelian decompositions that are associated with all the limit groups, L_i , $0 \leq i \leq s-1$, we constructed a finite set of limit quotients, that are all completions, and together they satisfy the conclusion of the proposition. □

To reduce an AE sentence from the ambient free group to its factors, we also need to construct auxiliary resolutions, that will enable one to decide uniformly if over a given free product G , the set of specializations that do factor through a given resolution are covered by a given finite collection of closures. Following the construction of the *Root* resolutions (over free groups) in sections 1 and 3 of [Se5], we associate a finite collection of *Root resolutions* with a given resolution and a finite collection of its closures (over free products).

Proposition 3.5. *Let $Res(y)$ be a well-structured, coefficient free resolution over free products with a f.p. completion (and terminal limit group), and let $Cl_1(Res), \dots, Cl_f(Res)$ be a given finite set of closures of $Res(y)$ (see definition 2.3).*

Each abelian vertex group that appears in the abelian decompositions that are associated with the various levels of $Res(y)$ is contained as a finite index subgroup in an abelian vertex group in each of the closures, $Cl_1(Res), \dots, Cl_f(Res)$. We set $Pind$ to be the product of all the indices of the abelian vertex groups of $Res(y)$ in the corresponding abelian vertex groups in the closures, $Cl_1(Res), \dots, Cl_f(Res)$.

*Let pg_1, \dots, pg_t be the pegs of pegged abelian groups in $Res(y)$. With each collection of positive integers, d_1, \dots, d_t , that do all divide $Pind$, we associate a finite (possibly empty) collection of closures of $Res(y)$ with f.p. completions and terminal limit groups. We denote the entire finite collection of such closures, $\{RootRes_r\}$, and call them *Root resolutions*.*

*Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ . We fix a specialization of the terminal limit group of $Res(y)$ in G . The pegs of abelian vertex groups in $Res(y)$, pg_1, \dots, pg_t , have roots of orders d_1, \dots, d_t , for every test sequence (in G) over the given specialization of the terminal limit group of $Res(y)$, if and only if the specialization of the terminal limit group of G can be extended to at least one of the *Root resolutions*, $RootRes_r$, that are associated with the sequence, d_1, \dots, d_t .*

Proof: Let d_1, \dots, d_t be a tuple of integers that divide $Pind$. We look at the collection of all the test sequences of $Res(y)$, over all possible non-trivial free products that are not isomorphic to D_∞ , for which the pegs, pg_1, \dots, pg_t , have roots of orders d_1, \dots, d_t . By the construction of the formal Makanin-Razborov diagram over free products in section 2, with this collection of test sequences, it is possible to associate (non-canonically) a finite collection of closures of $Res(y)$, that do all

have f.p. completions. This finite collection of closures has the property, that every test sequence of $Res(y)$ (over an arbitrary non-trivial free product that is not isomorphic to D_∞), for which the pegs in $Res(y)$, pg_1, \dots, pg_t , have roots of orders, d_1, \dots, d_t , in correspondence, has a subsequence that factors through one of the closures from the finite collection.

By taking the finite union of the finite collections of closures for every tuple of integers, d_1, \dots, d_t , for which each d_i divides $Pind$, we get a (non-canonical) finite collection that satisfies the conclusion of the theorem. □

Proposition 3.5 associates Root resolutions with a given coefficient-free resolution over free products, and a finite collection of closures of this resolution. Using these Root resolutions, it is possible to write a sentence, which is a disjunction of conjunctions of sentences over the factors of a free product, that determines (uniformly) if for a given free product the finite collection of closures forms a covering closure of the given resolution.

Corollary 3.6. *With the notation of proposition 3.5, and for every positive integer $\ell > 1$, the set of closures, $Cl_1(Res), \dots, Cl_f(Res)$, is a covering closure of the resolution, $Res(y)$, over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ (see definition 2.4 for a covering closure), if and only if a (finite) conjunction of disjunctions of coefficient free AE sentences over the factors, A^1, \dots, A^ℓ is a truth sentence.*

Proof: The given closures, $Cl_1(Res), \dots, Cl_f(Res)$, do not form a covering closure of the resolution, $Res(y)$, over a free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , if and only if there exist specializations of the elliptic factors in the terminal limit group of $Res(y)$, for which:

- (i) the specializations extend to specializations of the elliptic factors in the terminal limit groups of (one of finitely many possible) prescribed subset of the associated Root resolutions (see proposition 3.5) and to any of the terminal limit groups of the Root resolutions in the complement of the prescribed set.
- (ii) the specializations do not extend to specializations of the elliptic factors in the terminal limit groups of (one of finitely many possible) prescribed subset of the closures, $Cl_1(Res), \dots, Cl_f(Res)$, that given condition (i) on the Root resolutions, cover the corresponding fiber of $Res(y)$ that is associated with the specializations of the elliptic factors.

By counting all the possible finite subsets of Root resolutions, and all the possible finite subsets of closures, $Cl_1(Res), \dots, Cl_f(Res)$, that satisfy part (ii), given the chosen finite subset of Root resolutions, the set of closures, $Cl_1(Res), \dots, Cl_f(Res)$, do not form a covering closure of $Res(y)$ over G , if and only if a (finite) disjunction of conjunctions of EA sentences over the factors A^1, \dots, A^ℓ is a truth sentence. Hence, the converse is a finite conjunction of disjunctions of AE sentences over the factors, A^1, \dots, A^ℓ . □

Proposition 3.4 on the structure of the singular locus of a resolution, the construction of Root resolutions in proposition 3.5, together with theorem 3.2 that associates with a given AE sentence over free products, finitely many resolutions, and their formal Makanin-Razborov diagrams, enable us to reduce (uniformly) a

given AE sentence from free products to their factors.

Theorem 3.7. *Let:*

$$\forall y \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1$$

be a sentence over groups.

Then for every positive integer, $\ell > 1$, there exists a coefficient-free sentence over free products, which is a disjunction of conjunctions of AE sentences, where each of these last AE sentences involves elliptic elements from the same factor, A^1, \dots, A^ℓ , such that for every non-trivial free product, $G = A^1 * \dots * A^\ell$, which is not D_∞ , the original AE sentence over the free product $G = A^1 * \dots * A^\ell$ is a truth sentence, if and only if the sentence which is a disjunction of conjunctions of AE sentences over the factors A^1, \dots, A^ℓ is a truth sentence.

Proof: By theorem 3.2, with the given AE sentence, it is possible to associate finitely many well-structured resolutions, Res_1, \dots, Res_d , that do all have f.p. completion, and terminal limit groups. With each resolution, Res_i , and the system of equations, $\Sigma(x, y) = 1$, it is possible to associate (non-canonically) its formal Makanin-Razborov diagram. We denote each resolution in the formal Makanin-Razborov diagram of Res_i , $FRes_i^j$. Since the completion of Res_i is f.p. each formal resolution, $FRes_i^j$, has a f.p. completion and terminal limit group.

By the construction of the formal resolutions, the elements that correspond to the equations in the system, $\Sigma(x, y) = 1$, represent the identity element in the completion, $Comp(FRes_i^j)$. We further look at all the test sequences of each of the formal resolutions, $FRes_i^j$, for which at least one of the words in the system of inequalities, $\Psi(x, y) \neq 1$, holds as equality. By the construction of the formal Makanin-Razborov diagram, this associates a finite collection of resolutions with each of the formal resolutions, $FRes_i^j$, that do all have f.p. completion and terminal limit groups. In each of these resolutions, at least one of the words in the system of inequalities, $\Psi(x, y) \neq 1$ is the trivial element. We denote each of these resolutions, $\Psi FRes_i^{j,k}$.

By proposition 3.4, with the resolutions, Res_1, \dots, Res_d , that are associated with the given AE sentence, it is possible to associate finitely many resolutions, $SLRes_i^v$, that have f.p. completions and terminal limit groups. A specialization of the terminal limit group of Res_i is in the singular locus, if and only if this specialization extends to a specialization of the terminal limit group of one of the resolutions, $SLRes_i^v$. Similarly we associate with each formal resolution, $FRes_i^j$, finitely many formal resolutions that are associated with its singular locus, and we denote of these resolutions, $SLFRes_i^{j,v}$.

Each of the formal resolutions, $FRes_i^j$, terminates with a closure of the resolution Res_i . Hence with the resolution, Res_i , we can associate finitely many closures, $Cl(Res_i)^j$. By proposition 3.5, with the resolution, Res_i , and its closures, $Cl(Res_i)^j$, we can associate finitely many Root resolutions, that we denote $RootRes_i^r$, that collect all the specializations of the terminal limit group of Res_i , for which the pegs in the resolution, Res_i , have roots of prescribed orders for every test sequence that extends the given specialization of the terminal limit group of Res_i .

Therefore, by theorem 3.2 and propositions 3.4 and 3.5, the given AE sentence is false over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic

to D_∞ , if and only if there exists a specialization in G of the terminal limit group of a resolution Res_i (one of the resolutions, Res_1, \dots, Res_d , that were associated with the sentence by theorem 3.2), i.e., specializations of the elliptic factors of the terminal limit group of Res_i in the factors, A^1, \dots, A^ℓ , for which:

- (1) the specialization does not extend to a specialization of the terminal limit group of one of the resolutions, $SLRes_i^v$ (the singular locus of Res_i).
- (2) the specialization extends to the terminal limit group of some prescribed Root resolutions (possibly only the trivial roots), $RootRes_i^r$, and not to other ones.
- (3) the specialization does not extend to a (finite) collection of specializations of the formal resolutions, $FRes_i^j$, that are associated with the resolution Res_i , that satisfy:
 - (i) these specializations are not in the singular locus of $FRes_i$, i.e., they don't extend to specializations of the terminal limit group of one of the resolutions, $SLRes_i^{j,v}$.
 - (ii) these specializations do not extend to specializations of the terminal limit group of any of the resolutions, $\Psi FRes_i^{j,k}$.
 - (iii) the fibers that are associated with these specializations and the corresponding formal resolutions, $FRes_i^j$, form a covering closure of the fiber that is associated with Res_i and the given specialization of its terminal limit group.

Finally, the existence of such a specialization of the terminal limit group of one of the resolutions, Res_i , that satisfies properties (1)-(3), is clearly a finite disjunction of finite conjunctions of EA sentences over the factors of G , A^1, \dots, A^ℓ . Hence, the AE sentence, which is exactly the negation of this sentence is a finite disjunction of finite conjunctions of AE sentences over the factors, A^1, \dots, A^ℓ . □

§4. EAE sentences and predicates

In the previous section we used the iterative procedure that is presented in section 4 of [Se4] to reduce AE sentences from the ambient free product to its factors. In this section we use the procedures that were used to prove quantifier elimination over free groups, to analyze EAE predicates and sentences over free products.

Let:

$$EAE(p) = \exists w \forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1$$

be a predicate over groups. Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product (for some $\ell > 1$) that is not isomorphic to D_∞ . If $p_0 \in EAE(p)$ over the free product G , then there exists some specializations w_0 of the existential variables w , so that the AE sentence:

$$\forall y \exists x \Sigma(x, y, w_0, p_0) = 1 \wedge \Psi(x, y, w_0, p_0) \neq 1$$

is a truth sentence over G .

In the previous section we studied AE sentence over free products, and we have finally shown that an AE sentence over free products is a truth sentence over a non-trivial free product $G = A^1 * \dots * A^\ell$ (that is not isomorphic to D_∞) if and

only if a disjunction of conjunctions of sentences over the factors A^1, \dots, A^ℓ does hold.

To prove a similar reduction from a given EAE sentence over free products to a (finite) disjunction of (finite) conjunctions of sentences over the factors of the free product, we use the procedure for quantifier elimination over free groups that was presented in [Se5] and [Se6]. Recall that the analysis of an EAE set over a free group is divided into two steps. In the first step one analyzes uniformly the AE sentences:

$$\forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1$$

(uniformly in the pair (w, p) , which is equivalent to the analysis of the corresponding AE set, $AE(w, p)$). Then one uses this uniform analysis of AE sentences, and the (iterative) sieve procedure that is presented in [Se6], to analyze an EAE set over a free group.

We start with the uniform analysis of AE sentences, which is equivalent to the analysis of an AE set, that combines what we did in the previous section with the iterative procedure that is presented in section 2 in [Se5]. Let:

$$AE(w, p) = \forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1$$

be an AE set. Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ , and suppose that (w_0, p_0) is a specialization of the free variables, (w, p) , in the free product G , that is not in the definable set $AE(w, p)$ over the free product G . Then there exists a specialization y_0 of the universal variables y in G , for which for all the possible values of the existential variables x in G either the equalities, $\Sigma(x, y_0, w_0, p_0) = 1$, do not hold, or at least one of the inequalities, $\Psi(x, y_0, w_0, p_0) \neq 1$, does not hold in G .

We continue by combining the construction of the diagram that is associated with an AE sentence over free products, that was constructed in the previous section, with the procedure for the analysis of AE sets (over free groups) that is presented in section 2 of [Se5]. We start by looking at all the sequences of specializations of tuples, $\{(p_n, w_n)\}$, that take their values in non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$, that are not isomorphic to D_∞ , specializations of the tuple (p, w) that are not in the definable set, $AE(w, p)$. This implies that for every pair, (w_n, p_n) , in the sequence, there exists a specialization of the universal variables, y_n , in the free product, G_n , for which there are no values for the existential variables x (in G_n) so that both the equalities $\Sigma(x, y, w, p) = 1$ and the inequalities $\Psi(x, y, w, p) \neq 1$ hold.

By proposition 1.16, given such a sequence of triples, $\{(p_n, w_n, y_n)\}$, we can pass to a subsequence that converges into a well-structured (even well-separated) graded resolution with respect to the parameter subgroup $\langle w, p \rangle$, that we denote $GRes$:

$$L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$$

where L_s is a free product of a rigid or a solid limit group over free products (that contains the subgroup, $\langle w, p \rangle$), and (possibly) a free group and (possibly) finitely many elliptic factors. Note that the terminal limit group L_s of $GRes$ is f.g. but it may be infinitely presented.

As the terminal limit group L_s of $GRes$ is f.g. but perhaps not f.p. we fix a sequence of approximating covers of the resolution $GRes$, that we denote, $\{CGRes_m\}$.

These approximating covers are constructed according to the construction of a cover approximating resolution that appears in theorem 1.21, hence, the cover approximating resolutions, $\{CGRes_m\}$, satisfy all the properties that are listed in theorem 1.21. In particular, the graded completions of the covers, $CGRes_m$, can be embedded into f.p. completions, and if the terminal limit group L_s of $GRes$ is a free product of a rigid (solid) factor with elliptic and free factors, then the terminal limit group of the covers, $CGRes_m$, is a free product of a rigid (weakly solid) factor with (f.p. approximating) elliptic and free factors. With each of the graded covers, $CGRes_m$, there is an associated cover of its flexible quotients, and each cover (of flexible quotients) from this finite collection can be embedded into a f.p. completion. Also, for each cover, $CGRes_m$, there exists a subsequence of the triples, $\{(p_n, w_n, y_n)\}$, that factor through it, and restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $CGRes_m$. By the construction of approximating cover resolutions, we further assume that the sequence of cover approximations, $\{CGRes_m\}$, converges into the original graded resolution, $GRes$, and in particular the terminal limit groups of the covers, $\{CGRes_m\}$, that we denote, L_s^m , converge into the terminal limit group of $GRes$, L_s .

With each of the approximating covers, $CGRes_m$, and the system of equations, $\Sigma(x, y, w, p) = 1$, we associate (non-canonically) a graded formal Makanin-Razborov diagram. Note that by theorem 2.7, the (formal graded) completion of each of the resolutions in such a formal graded Makanin-Razborov diagram can be embedded into a f.p. (formal graded) completion, and that with the terminal limit group of the cover closure of $CGRes_m$, that is associated with a resolution in the graded formal Makanin-Razborov diagram, there is an associated finite collection of covers of the flexible quotients of the rigid or weakly solid factor of the terminal limit group, and these covers are all embedded in f.p. completions (see theorem 2.7).

By the construction of the approximating covers, $\{CGRes_m\}$, for each index m , there is a subsequence of the sequence of tuples, $\{(p_n, w_n, y_n)\}$, that factor through $CGRes_m$. In particular, for each index m , with the subsequence of tuples, (still denoted) $\{(p_n, w_n, y_n)\}$, that factor through $CGRes_m$, we can associate a sequence of specializations of the terminal limit group of $CGRes_m$, L_s^m , that restricts to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of L_s^m , and to specializations of the elliptic factors of L_s^m .

If there exists an index m , for which there is an infinite subsequence of tuples (still denoted), $\{(p_n, w_n, y_n)\}$, that factor through $CGRes_m$, and so that the rigid or weakly strictly solid specializations of the rigid or weakly solid factor in L_s^m , and the specializations of the elliptic factors of L_s^m , that are associated with the tuples, $\{(p_n, w_n, y_n)\}$, extend to test sequences (of $CGRes_m$) that do not extend to formal solutions, or they do extend to formal solutions (over closures of $CGRes_m$), but for each such formal solution, at least one of the inequalities in the system, $\Psi(x, y, w, p) \neq 1$, does not hold (for generic value of the universal variables y), we reached a terminal point of the iterative procedure. In this case, the associated output is the approximating cover, $CGRes_m$, its associated (finite) collection of covers of the flexible quotients of the rigid or weakly solid factor of L_s^m , and its formal graded Makanin-Razborov diagram.

Suppose that for every index m , there is no such subsequence of tuples $\{(p_n, w_n, y_n)\}$. In this case for each index m , there exists an index $n_m > m$, so that the tuple $(p_{n_m}, w_{n_m}, y_{n_m})$ factors through $CGRes_m$, and with the associated rigid or weakly

strictly solid specialization of the rigid or weakly solid factor of L_s^m , and the associated specializations of the elliptic factors of L_s^m , it is possible to associate a test sequence and a formal solution x_{n_m} that does satisfy $\Sigma(x_{n_m}, y, w_{n_m}, p_{n_m}) = 1$ and $\Psi(x_{n_m}, y, w_{n_m}, p_{n_m}) \neq 1$ for generic y .

By construction, the factors of the terminal limit groups of the graded resolutions, $\{CGRes_m\}$, L_s^m , converge into the corresponding factors of the terminal limit group of the graded resolution, $GRes$, L_s . The formal solutions, $\{x_{n_m}\}$, are defined over the approximating cover resolutions, $\{CGRes_m\}$. Using the techniques to construct graded formal limit groups over free products, that were presented in section 2, from the sequence of formal solutions, $\{x_{n_m}\}$, and the specializations of the terminal limit groups, L_s^m , that are associated with the tuples, $\{(p_{n_m}, w_{n_m}, y_{n_m})\}$, it is possible to extract a subsequence that converges into a graded formal limit group over the original (limit) graded resolution Res , $FL(x, z, y, w, p)$. By the construction of the graded formal limit group $FL(x, z, y, w, p)$, the equations from the system, $\Sigma(x, y, w, p) = 1$, represent the trivial word in $FL(x, z, y, w, p)$, whereas each of the inequations in the system, $\Psi(x, y, w, p) \neq 1$, represent a non-trivial element in $FL(x, z, y, w, p)$.

At this point we look at the sequences of specializations:

$$\{(x_{n_m}, y_{n_m}, z_1(n_m), \dots, z_t(n_m), w_{n_m}, p_{n_m})\}$$

of the formal solution x_{n_m} , the universal variables y_{n_m} , and its successive shortenings (z_1, \dots, z_t) , and the parameters, (p_{n_m}, w_{n_m}) , that take their values in the free products, $\{A_{n_m}^1 * \dots * A_{n_m}^\ell\}$, and each of the tuples, $(p_{n_m}, w_{n_m}, y_{n_m})$, factors through the cover graded resolution, $CGRes_m$. The elements, $\{y_m\}$, are precisely the subsequence of values of the universal variables for which the sentence fail to hold for the free products, $\{A_{n_m}^1 * \dots * A_{n_m}^\ell\}$.

Given this sequence of specializations, we apply the first step of the procedure for the analysis of an AE set, that is presented in section 2 of [Se5], and extract a subsequence, that converges into a quotient graded resolution of the one that is associated with the formal limit group, $FL(x, z, y, w, p)$. Since the formal solutions, $\{x_{n_m}\}$, were assumed to satisfy both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, the obtained quotient resolution is not a graded closure of the original graded resolution, $GRes$, that we have started with, but rather a resolution of "reduced complexity" (in the sense of the iterative procedure that is presented in section 2 in [Se5]).

We continue iteratively. At each step we start with a quotient graded resolution (with respect to the parameter subgroup), $\langle w, p \rangle$, $QRes$, that was constructed in the previous step of the procedure, using the general step of the iterative procedure that is presented in section 2 in [Se5], and a sequence of homomorphisms into free products, that extends a subsequence of the original sequence of tuples, $\{(p_n, w_n, y_n)\}$, that converges into the (completion of the) quotient resolution, $QRes$. Note that the completion of such a quotient resolution need not be finitely presented.

We start the current (general) step, by fixing a sequence of approximating cover graded resolutions, $QRes_m$, of the quotient resolution, $QRes$, that satisfy the properties that are listed in theorem 1.21. By the construction of cover resolutions (theorem 1.21), we may further assume that the sequence of cover resolutions, $QRes_m$, converges into the original quotient resolution, $QRes$. Note that for each index

m , there exists a subsequence of the original sequence of specializations (that converges into the completion of the quotient resolution, $QRes$) that factor through the cover (graded) resolution, $QRes_m$. In particular, this subsequence restricts to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of the cover resolution, $QRes_m$.

With each cover (approximating) graded resolution, $QRes_m$, we associate (not in a canonical way) its graded formal Makanin-Razborov diagram (see theorem 2.7). Note that the completion of each resolution in the formal Makanin-Razborov diagram can be embedded in a f.p. completion, and that with each such formal resolution there is an associated finite collection of covers of the flexible quotients of the rigid or weakly solid factor of its terminal limit group, and each cover from this finite collection can be embedded in a f.p. completion (see theorem 2.7). Also, recall that a formal Makanin-Razborov diagram of $QRes_m$ encodes all the formal solutions that can be defined over it for all possible free products, $A^1 * \dots * A^\ell$.

If there exists an index m , and a subsequence of the original sequence of specializations that factor through the cover approximating resolution, $QRes_m$, so that over the rigid or the weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $QRes_m$, and the specializations of the elliptic factors of the terminal limit group of $QRes_m$, that are associated with the subsequence, there exist test sequences that can not be extended to formal solutions over closures of $QRes_m$, so that both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, hold for generic y 's (over the corresponding subsequence of free products, $\{A_n^1 * \dots * A_n^\ell\}$), we reached a terminal point of our iterative procedure. In this case the final output of the iterative procedure is the graded quotient resolution, $QRes_m$, and its (non-canonical) graded formal Makanin-Razborov diagram.

Suppose that there is no such index m , and no such subsequence of the specializations that factor through $QRes_m$. In this case for each index m , there exists a specialization from the sequence that converges to $QRes$ (of index bigger than m in that sequence) that factors through $QRes_m$, and a formal solution x_m that is defined over (a closure of) the graded resolution, $QRes_m$, so that for the specialization of the terminal limit group of $QRes_m$ that is associated with this specialization, and for the formal solution, x_m , both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, hold for generic values of the universal variables y (generic values that are associated with the specialization of the terminal limit group of $QRes_m$, and the specialization that factors through it from the sequence that converges to $QRes$).

In this case we look at the sequence of these specializations (that factor through $QRes_m$), and their associated formal solutions, $\{x_m\}$, that are defined over $QRes_m$. We analyze the combined sequence (of pairs of a specialization that factors through $QRes_m$ and the formal solution x_m), according to the analysis of quotient graded resolutions, which is part of the general step of the iterative procedure for the analysis of AE sets (over free groups), that is presented in section 2 of [Se5], and finally extract a subsequence that converges into a quotient graded resolution of the resolutions that we have this step with, $QRes$ (see section 2 of [Se5]). Since the formal solutions are guaranteed to satisfy both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, for generic values of y , the complexity of the obtained quotient graded resolution is strictly smaller than the complexity of the graded resolution, $QRes$, that we have started the current step with.

By theorem 2.10 in [Se5], this iterative procedure terminates after finitely many steps (the proof of the termination over free products is identical to the one over free groups). If we reached a terminal quotient graded resolution along the process, we found a cover graded resolution, $QRes_m$, that satisfies the properties of theorem 1.21. Furthermore, there exists a subsequence of the original sequence of tuples, $\{(p_n, w_n, y_n)\}$, that extend to specializations that factor through this cover resolution, $QRes_m$, and these extended specializations restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $QRes_m$. Each extended specialization from this subsequence restricts to a specialization of the terminal graded limit group of $QRes_m$, and there exists a test sequence that is associated with it, that can not be extended to a formal solution that is defined over (a closure of) $QRes_m$, for which both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, hold for generic values of the universal variables y , that are associated with $QRes_m$, and the associated specialization of the terminal limit group of $QRes_m$.

If we didn't find such an approximating cover resolution, $QRes_m$, along the iterative procedure, it continued until we reached a quotient graded resolution which is a free product of a rigid limit group over free products (with respect to the parameter subgroup $\langle w, p \rangle$) and elliptic factors. Once again we look at a sequence of approximating covers of the rigid limit group (over free products) and the elliptic factors, and we construct these covers according to the construction that appears in theorem 1.21. With each such approximating cover, and the system of equations, $\Sigma(x, y, w, p) = 1$, we associate (non-canonically) its graded formal Makanin-Razborov diagram, according to theorem 2.7.

Hence, whatever terminal point of the iterative procedure we have reached, we found an approximating cover (graded) resolution (over free products) that satisfies the properties of approximating covers that appear in theorem 1.21. A subsequence of the given sequence of specializations, $\{(p_n, w_n, y_n)\}$, extend to specializations that do factor through this approximating cover. Each extended specialization restricts to a specialization of the terminal limit group of the approximating cover resolution. For each associated specialization of the terminal limit group of the cover resolution, there exists a test sequence for which there is no formal solution that is defined over the approximating cover resolution, and for which the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, do hold for generic values of the universal variables y , in the the fiber that is associated with the approximating cover resolution, and the specialization of its terminal graded limit group.

As in the previous section, the existence of such a cover resolution, that satisfies the properties that are listed in theorem 1.21, is sufficient to enable one to associate with a given AE set, a finite collection of (cover) graded resolutions that satisfy the properties that are listed in theorem 1.21, and their (non-canonical) formal graded Makanin-Razborov diagrams, so that if a specialization of the tuple (w, p) in some free product, $A^1 * \dots * A^\ell$ (for some $\ell > 1$), is not in the definable set $AE(w, p)$, then the corresponding sentence must be false for a generic point of one of the constructed (cover) graded resolutions, where the terminal factors of the graded resolutions take some value in the free product, $A^1 * \dots * A^\ell$. This is the key for analyzing an AE set over free products, i.e., the key for obtaining quantifier elimination from a predicate over the ambient free product to a predicate over the

factors.

Theorem 4.1. *Let:*

$$AE(w, p) = \forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1$$

be an AE set over groups. Then there exist finitely many (cover) graded resolutions over free products (with respect to the parameter subgroup $\langle w, p \rangle$) that satisfy the properties that are listed in theorem 1.21: $GRes_1(z, y, w, p), \dots, GRes_d(z, y, w, p)$, with the following properties:

- (1) with each graded resolution, $GRes_i(z, y)$, we associate (non-canonically) its graded formal Makanin-Razborov diagram over free products with respect to the system of equations, $\Sigma(x, y, w, p) = 1$.
- (2) each of the graded resolutions, $GRes_i$, can be extended to an ungraded resolution with a f.p. completion, $Comp_i$ (see theorem 1.21).
- (3) let $G = A^1 * \dots * A^\ell$ be a non-trivial free product (for some $\ell > 1$) that is not isomorphic to D_∞ . Let (w_0, p_0) be a specialization of the parameters (w, p) in the free product $A^1 * \dots * A^\ell$, for which $(w_0, p_0) \notin AE(w, p)$ over the free product $G = A^1 * \dots * A^\ell$. Then there exists an index i , $1 \leq i \leq d$, a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of $GRes_i$ in $G = A^1 * \dots * A^\ell$, that extends (w_0, p_0) , and specializations of the elliptic factors of that terminal limit group in elliptic subgroups in $G = A^1 * \dots * A^\ell$, so that the combined specialization of the terminal limit group of the graded resolution, $GRes_i$, can be extended to a specialization of the f.p. completion, $Comp_i$, and with this combined specialization of the terminal limit group of $GRes_i$ there exists an associated test sequence for which there is no formal solution over (a closure of) $GRes_i(z, y, w, p)$ for which both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, do hold for generic values of the universal variables y .

In other words, given an AE set, there exists a finite collection of cover graded resolutions over free products (with the properties that are listed in theorem 1.21), so that the failure of a specialization of the parameters (free variables) to be in the AE set over any given free product, can be demonstrated by the lack of the existence of a formal solution over a test sequence of at least one of these (finitely many cover) graded resolutions.

Proof: Let $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), be a sequence of non-trivial free products that are not isomorphic to D_∞ . Let $\{(p_n, w_n, y_n)\}$ be a sequence of tuples in G_n , so that $(p_n, w_n) \notin AE(p, w)$ over G_n , and y_n testifies for the failure of the pair (p_n, w_n) to be in $AE(p, w)$, i.e. the following existential sentence (with coefficients):

$$\exists x \Sigma(x, y_n, w_n, p_n) = 1 \wedge \Psi(x, y_n, w_n, p_n) \neq 1$$

is false over G_n .

Starting with the sequence, $\{(p_n, w_n, y_n)\}$, the terminating iterative procedure that we have presented constructs a graded resolution, $GRes$, with the following properties:

- (i) the graded resolution $GRes$ satisfies the properties of a cover graded resolution that are listed in theorem 1.21. In particular, its completion can

be extended to (an ungraded) f.p. completion. With the rigid or weakly solid factor of the terminal limit group of $GRes$, there is a finite collection of covers of its flexible quotients that can all be embedded into f.p. completions.

- (ii) there exists a subsequence of the sequence of specializations, $\{(p_n, w_n, y_n)\}$, that extend to specializations that factor through the resolution, $GRes$, and to specializations of the f.p. completion of the ungraded resolution that extends the graded resolution, $GRes$ (see theorem 1.21). Hence, with each specialization from this subsequence of the tuples, $\{(p_n, w_n, y_n)\}$, specializations of the terminal limit group of the graded resolution, $GRes$, can be associated. Furthermore, these specializations of the terminal limit group of $GRes$ can be extended to specializations of the f.p. completion of some ungraded resolution of the terminal limit group of $GRes$. With this subsequence of specializations of the terminal limit group of the graded resolution, $GRes$, there are associated test sequences for which no formal solutions that satisfy both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, for generic values of the universal variables y , can be constructed.
- (iii) with the resolution $GRes$, and the system of equations, $\Sigma(x, y, w, p) = 1$, we associate (non-canonically) a formal graded Makanin-Razborov diagram (see theorem 2.7). The completion of every (formal) resolution in this diagram can be embedded into a f.p. completion, and the finite collection of covers that is associated with the rigid or weakly solid factor of the terminal limit group of each resolution in these formal graded Makanin-Razborov diagrams, can be embedded in a f.p. completion as well.

At this point we are able to apply the argument that was used to prove theorem 3.2 (for AE sentences). We look at all the sequences of non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), that are not isomorphic to D_∞ , an associated sequence of the tuples, $(p_n, w_n) \notin AE(p, w)$, over G_n , and specializations y_n (in G) of the universal variables y that testify for that. Given every such sequence we use our terminating iterative procedure, and extract a subsequence of the tuples, (still denoted) $\{(p_n, w_n, y_n)\}$, and a graded resolution, $GRes$, that has the properties (i)-(iii), and in particular, the subsequence of tuples, (still denoted) $\{(p_n, w_n, y_n)\}$, extend to specializations that factor through the resolution $GRes$, and to specializations of the f.p. completion of the ungraded resolutions that extends the graded resolution $GRes$. Furthermore, no formal solution that satisfies both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$, for generic values of the variables y can be constructed, for some test sequences that are associated with those values of the terminal limit group of $GRes$, that are associated with the subsequence of tuples, $\{(p_n, w_n, y_n)\}$.

The completion of each of the constructed resolutions, $GRes$, can be extended to an ungraded f.p. completion, and so are the resolutions in its associated formal graded Makanin-Razborov diagram. Furthermore, the (finite collection of) covers of flexible quotients of the rigid or weakly solid factors of the terminal limit groups of the resolutions, $GRes$, and of the formal graded resolutions that are associated with the resolutions $GRes$, can all be embedded into (ungraded) resolutions with f.p. completions. Hence, we can define a linear order on this (countable) collection of resolutions ($GRes$), and their (non-canonically) associated formal Makanin-

Razborov diagrams, and (finite collections of) covers of flexible quotients. By the same argument that was used in constructing the Makanin-Razborov diagram (theorem 26 in [Ja-Se]), there exists a finite subcollection of these graded resolutions that satisfy properties (1)-(3) of the theorem. □

Given an AE set, theorem 4.1 constructs a finite collection of (cover) graded resolutions over free products, where the completions of all these graded resolutions can be embedded into (ungraded) f.p. completions (cf. theorem 1.21), that demonstrates the failure of specializations of the parameters (free variables) to be in the given AE set over general free products. This is an analogue of the uniformization of proofs (the construction of the tree of stratified sets) that was proved in section 2 of [Se5] for AE sets over free groups.

To use these (finitely many) constructed resolutions for the analysis of EAE sets, we still need to associate with the graded resolutions that are constructed in theorem 4.1, a (finite) collection of resolutions that contain their singular locus, and a finite collection of resolutions that classifies those values of their terminal limit groups for which the pegs of these graded resolutions have roots of prescribed orders (cf. propositions 3.4 and 3.5).

As in the ungraded case, in constructing the graded and formal graded Makanin-Razborov diagrams over free products, we have considered the set of specializations that factor through a resolution in one of these diagrams, as those homomorphisms that factor through the graded resolutions, and in addition we required that the associated specializations of the various limit groups along the resolution restrict to non-elliptic specializations of all the abelian edge (and vertex) groups, and all the QH vertex groups along the resolution.

In applying theorem 4.1 to analyze EAE sets, and general definable sets, over free products, it is essential to determine the singular locus of the graded resolutions that are associated with an AE set by the theorem, as we need to ignore specializations that do factor through the singular locus.

The singular locus of a graded resolution, $GRes(y, p)$, is defined precisely in the same way as in the ungraded case (definition 3.3). To analyze the singular locus of a graded resolution we prove a graded version of proposition 3.4. Given the techniques of the first two sections, especially the construction of the formal graded Makanin-Razborov diagram (theorem 2.7), the generalization of proposition 3.4 to the graded case is rather straightforward.

Proposition 4.2. *Let $GRes(y, p)$ be a well-structured, coefficient free, graded resolution over free products. There exist finitely many graded approximating cover resolutions, that satisfy the properties of the resolutions in the formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7), $SLRes_1(y, p), \dots, SLRes_u(y, p)$, and in particular the (graded) completion of each of these graded resolutions embeds into an (ungraded) f.p. completion.*

*For every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , a specialization of the terminal limit group of $GRes(y)$ is in the singular locus of $GRes(y, p)$, if and only if it extends to a specialization of (at least) one of the terminal limit groups of the resolutions, $SLGRes_1(y, p), \dots, SLGRes_u(y, p)$, and further extends to a specialization of the f.p. (ungraded) completion that contains the (graded) completion of the corresponding resolution, $SLGRes_i(y, p)$.*

Proof: The statement of the theorem follows by exactly the same argument that was used to prove the analogous statement in the ungraded case (proposition 3.4), where instead of using the construction of the (ungraded) formal Makanin-Razborov diagram, we apply the construction of the graded formal Makanin-Razborov diagram (theorem 2.7). □

As in the ungraded case, to analyze an EAE and general definable sets over free products, we also need to construct auxiliary resolutions, that will enable one to decide uniformly if over a given free product G , the set of specializations that do factor through a given resolution are covered by a given finite collection of closures. The construction of these *graded Root* resolutions essentially follows the construction of ungraded Root resolutions in proposition 3.5.

Proposition 4.3. *Let $GRes(y, p)$ be a well-structured, coefficient free resolution over free products with a f.p. completion (and terminal limit group), and let:*

$$Cl_1(Res), \dots, Cl_f(Res)$$

be a given finite set of graded closures of $GRes(y, p)$ (see definition 2.3).

Each of the abelian vertex groups that appear in the abelian decompositions that are associated with the various levels of $GRes(y, p)$ is contained as a finite index subgroup in an abelian vertex group in each of the closures, $Cl_1(Res), \dots, Cl_f(Res)$. We set $Pind$ to be the product of these indices (where the product is over all the abelian vertex groups in $GRes(y, p)$, and all the closures, $Cl_1(Res), \dots, Cl_f(Res)$).

Let pg_1, \dots, pg_t be a generating set for the pegs of pegged abelian groups in $GRes(y, p)$. Note that in the graded case, the peg of a pegged abelian vertex group is in general a f.g. free abelian group, hence, its generating set may consist of more than a single element). With each collection of positive integers, d_1, \dots, d_t , that do all divide $Pind$, we associate a finite (possibly empty) collection of graded closures of $GRes(y, p)$ with f.p. completions and terminal limit groups. We denote the entire finite collection of such graded closures, $\{GRootRes_r\}$, and call them (graded) Root resolutions.

*Let $G = A^1 * \dots * A^\ell$ be a non-trivial free product that is not isomorphic to D_∞ . We fix a specialization of the terminal limit group of $GRes(y, p)$ in G . The pegs of abelian vertex groups in $GRes(y, p)$, pg_1, \dots, pg_t , have roots of orders d_1, \dots, d_t , for every test sequence (in G) over the given specialization of the terminal limit group of $GRes(y, p)$, if and only if the specialization of the terminal limit group of G can be extended to at least one of the graded Root resolutions, $GRootRes_r$, that are associated with the sequence, d_1, \dots, d_t .*

Proof: Identical to the proof of the analogous statement in the ungraded case (proposition 3.5), were as in the construction of the graded resolutions that are associated with the singular locus of a graded resolution (proposition 4.2), we apply the construction of the graded formal Makanin-Razborov diagram (theorem 2.7), instead of the construction of the (ungraded) formal Makanin-Razborov diagram that was used in proving proposition 3.5. □

In a similar way to the ungraded case, the graded Root resolutions that are constructed in proposition 4.3, enable one to associate a predicate with the set of

values of the parameters p , for which a given set of (graded) closures is a covering closure.

The finite collection of graded resolutions that are associated with an AE set in theorem 4.1, their formal graded Makanin-Razborov diagrams, the singular loci of all these resolutions, and their associated graded Root resolutions, enable us to analyze EAE sets over free products. For such analysis we need a variation of the sieve procedure over free groups that is presented in [Se6] for free products.

Let:

$$EAE(p) = \exists w \forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1$$

be a predicate over groups. With the given EAE set we naturally look at the AE set:

$$AE(p, w) = \forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1.$$

By theorem 4.1 with this AE set it is possible to associate finitely many (cover) graded resolutions that satisfy the properties that are listed in theorem 1.21. With each of these (finitely many) graded resolutions, we have associated (non-canonically) a graded formal Makanin-Razborov diagram, a finite collection of graded resolutions that are associated with the singular loci of all these resolutions, and a finite collection of graded Root resolutions, that are associated with each of the graded resolutions that are associated with the AE set by theorem 4.1, and the closures that are associated with the resolutions in the formal Makanin-Razborov diagrams of these graded resolutions. These finite collection of graded (and graded formal) resolutions, demonstrate the failure of a specialization of a pair (p, w) to be in the definable set $AE(p, w)$ over an arbitrary non-trivial free product, $G = A^1 * \dots * A^\ell$ (for an arbitrary $\ell > 1$), where G is not isomorphic to D_∞ .

Let: $GRes_1(z, y, w, p), \dots, GRes_d(z, y, w, p)$ be the (cover) graded resolutions that we have associated with the set $AE(p, w)$. Each of these graded resolutions terminates in a graded limit group which is a free product of a rigid or a weakly solid limit group with (possibly) a finite collection of f.p. elliptic factors, and the completion of each of these cover graded resolutions can be embedded into a f.p. (ungraded) completion, that we denote $Comp_1, \dots, Comp_d$.

With each of the graded resolutions, $GRes_i$, and the system of equations, $\Sigma(x, y, w, p) = 1$, we have associated (non-canonically) its graded formal Makanin-Razborov diagram, and the completion of each formal resolution in one of these diagrams can be embedded into a f.p. completion as well. We denote each of the (finitely many) resolutions in the formal graded Makanin-Razborov diagram of a graded resolution, $GRes_i, FGRes_i^j$.

With each of the formal graded resolutions, $FGRes_i^j$, we further associate a finite collection of graded resolutions. For each formal graded resolution, $FGRes_i^j$, we look at all of its test sequences, for which at least one of the inequalities in the system, $\Psi(x, y, w, p) \neq 1$, fails to be an inequality and is in fact an equality for the entire test sequence. With the collection of all such test sequences it is possible to associate a finite collection of graded resolutions, that do all have the same structure as the formal graded resolution, $FGRes_i^j$, just that each factor in the terminal limit group of $FGRes_i^j$ is replaced by a quotient. We denote each of the (finitely many) obtained resolutions, that are associated with $FGRes_i^j, CollFGRes_i^{j,k}$.

With each of the graded resolutions, $GRes_i$, we have associated a finite collection of graded resolutions that are associated with its singular locus. We denote these

graded resolutions, $SLGRes_i^v$. We also associate finitely many resolutions with each graded formal resolution, $FGRes_i^j$, that we denote, $SLFGRes_i^{j,v}$.

We further associate graded Root resolutions with each of the graded resolutions, $GRes_i$, and the collection of closures that are associated with the graded resolutions in its formal graded Makanin-Razborov diagram. We denote these graded Root resolutions, $GRootRes_i^r$.

We start the analysis of the set $EAE(p)$ with all the sequences of specializations of the tuple, (p, w) , $\{(p_n, w_n)\}$, that take values in non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$ (for some $\ell > 1$), which is not isomorphic to D_∞ . We further assume that for every index n , $p_n \in EAE(p)$ over the free product, G_n , and that w_n is a witness for p_n , i.e., a specialization of the existential variables w in G_n , so that the AE sentence:

$$\forall y \exists x \Sigma(x, y, w_n, p_n) = 1 \wedge \Psi(x, y, w_n, p_n) \neq 1$$

is a truth sentence over G_n .

By theorem 1.16, given such a sequence of pairs, $\{(p_n, w_n)\}$, we can pass to a subsequence that converges into a well-structured (even well-separated) graded resolution with respect to the parameter subgroup $\langle p \rangle$: $M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_s$, where M_s is a free product of a rigid or a solid group over free products (that contains the subgroup, $\langle p \rangle$), and (possibly) a free group and (possibly) finitely many elliptic factors. We denote this graded resolution $WRes$. Note that the terminal limit group M_s in $WRes$ is f.g. but it may be infinitely presented.

As we did in analyzing the set, $AE(p, w)$, we fix a sequence of approximating cover resolutions of $WRes$, that we denote, $\{WRes_m\}$, that are constructed following the construction that appears in theorem 1.21, and hence each of them satisfies the properties of cover graded resolutions that are listed in theorem 1.21. By construction we may further assume that the sequence of approximating cover resolutions, $\{WRes_m\}$, converge into the original graded resolution, $WRes$.

By the construction of the approximating cover graded resolutions that appears in theorem 1.21, for every index m , there is a subsequence of the sequence of specializations, $\{(p_n, w_n)\}$, that factor through the approximating cover resolution, $WRes_m$, and they further extend to specializations of the f.p. completion of the ungraded resolution that extends the approximating cover, $WRes_m$ (see theorem 1.21). In particular, with this subsequence it is possible to associate specializations of the terminal limit group of $WRes_m$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal (graded) limit group of $WRes_m$.

First, suppose that there exists an approximating resolution, $WRes_m$, for which there exists a subsequence of pairs, (still denoted) $\{(p_n, w_n)\}$, so that for each specialization of the terminating limit group of $WRes_m$ that is associated with a pair (p_n, w_n) from the subsequence, there exists a test sequence in the fiber that is associated with such a specialization of the terminal graded limit group (which takes its values in G_n) that satisfies $p_n \in EAE(p)$ (over G_n), and the values of the variables w from the test sequence are witnesses for p_n (over G_n).

In this case with the approximating graded resolution, $WRes_m$, we associate finitely many graded resolutions, that have similar properties to the resolutions in the formal graded Makanin-Razborov diagram (see theorem 2.7).

First, we look at all the test sequences of $WRes_m$, that can be extended to specializations of one of the terminal limit groups of the graded resolutions, $GRes_1, \dots, GRes_d$,

that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, and so that the specializations of the terminal limit group, $GRes_i$, extends to a specializations of the f.p. completion, $Comp_i$, into which the completion of $GRes_i$ is embedded.

By the construction the formal graded Makanin-Razborov diagram (theorem 2.7), with the approximating (cover) graded resolution, $WRes_m$, and the collection of test sequences of it, that can be extended to specializations of the terminal limit groups of the graded resolutions, $GRes_1, \dots, GRes_d$, it is possible to associate (non-canonically) a finite collection of (formal like) graded resolutions, that do all terminate in graded cover closures of the graded resolution, $WRes_m$, $CCI(WRes_m)$, possibly free product with a free group. Each resolution from this finite collection has the same properties as the resolutions in the formal Makanin-Razborov diagram (see theorems 2.6 and 2.7). We denote each of the constructed (finitely many, graded, cover) resolutions, $ExWRes_m$.

In a similar way to the formal Makanin-Razborov diagram (cf. theorem 2.7), given a test sequence of $WRes_m$ that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, and its associated f.p. completion $Comp_i$, it is possible to extract a subsequence of the extended specializations that factor through one of the resolution, $ExWRes_m$.

Given this finite collection of closures, $ExWRes_m$, that collect all the test sequences of the graded resolution, $WRes_m$, that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, we further associate with each of the graded cover resolutions, $ExWRes_m$, finitely many graded resolutions, that terminate in (cover) closures of the graded resolutions, $ExWRes_m$. These include 6 types of cover graded resolutions, each for collecting test sequences of $ExWRes_m$ with different properties. With each collection of test sequences it is possible to associate a finite collection of cover graded resolutions, in a similar way, and with similar properties to those of the resolutions in the formal Makanin-Razborov diagram (see theorems 2.6 and 2.7):

- (1) first we look at all the test sequences of $ExWRes_m$ for which the restrictions to specializations of the rigid or weakly solid factor of the terminal limit group of one of the resolutions, $GRes_1, \dots, GRes_d$, is not rigid or not weakly strictly solid.
- (2) second, we look at all the test sequences of $ExWRes_m$ for which the restrictions to specializations of the terminal limit group of one of the resolutions, $GRes_1, \dots, GRes_d$, can be extended to specializations of the terminal limit group of one the graded resolutions, $SLGRes_i^v$, that are associated with the singular locus of the corresponding graded resolution, $GRes_i$.
- (3) third, we look at all the test sequences of $ExWRes_m$ for which the restrictions to specializations of the terminal limit group of one of the resolutions, $GRes_1, \dots, GRes_d$, can be extended to specializations of the terminal limit group of one the graded Root resolutions, $GRootRes_i^r$, that are associated with the corresponding graded resolution, $GRes_i$.
- (4) we look at all the test sequences of $ExWRes_m$, for which to the restriction to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, it is possible to add a specialization of the terminal limit group of one of the associated graded formal (cover) resolution, $FGRes_i^j$.
- (5) given the finite collection of graded cover resolutions that are associated

with the test sequences that are described in part (4), we look at those test sequences of these graded cover resolutions, for which the restrictions to the specializations of the rigid or weakly solid factor of the terminal limit group of $FGRes_i$ are not rigid or not weakly strictly solid.

- (6) given the finite collection of graded cover resolutions that are associated with part (4), we look at those test sequences of these (finitely many) graded resolutions, for which the restrictions to the terminal limit groups of one of the resolutions, $FGRes_i^j$, either factor through the terminal limit group of one of the associated collapse formal graded resolutions, $CollFGRes_i^{j,k}$, or they can be extended to specializations of the terminal limit group of one of the resolutions, $SLFGRes_i^{j,v}$, that are associated with the singular locus of the corresponding resolution, $FGRes_i^j$.

Finally, with the resolution, $WRes_m$, itself, we associate finitely many graded resolutions, that are associated with its singular locus, according to proposition 4.2. We further associate with $WRes_m$, and the finite collection of closures of it, that include the resolutions, $ExWRes_m$, and the resolutions that are associated with it according to parts (1)-(6), a finite collection of graded Root resolutions, according to proposition 4.3.

Suppose that there is no approximating resolution, $WRes_m$, with a subsequence of (the original sequence of) tuples $\{(p_n, w_n)\}$ that factor through it, and that can be extended to specializations of the f.p. completion that extends $WRes_m$, for which for the fiber of $WRes_m$ that contains the tuple, (p_n, w_n) , the restrictions of a test sequence in the fiber to the existential variables w testify that $p_n \in EAE(p)$ over G_n , i.e., for these w 's, the tuple (p_n, w) is not in the associated set $AE(p, w)$ (over G_n).

In this case we can associate with the original graded resolution, $WRes$, a graded closure that is constructed from a limit of test sequences of a sequence of approximating resolutions, $WRes_m$, that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal group, where the specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, can not be extended to formal solutions that are defined over these graded resolutions, and these formal solutions form a covering closure (definition 2.4), and satisfy both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$. We denote such a closure, $ExWRes$.

Now, we continue in one of two possible ways, depending on the sequence of specializations, $\{(p_n, w_n)\}$, and the fibers that contain them in the approximations, $WRes_m$. The first possibility is to look at a sequence of cover approximations of the closure, $ExWRes$, (see theorem 1.21 for the construction of such approximations), and force one of 3 possible collapse conditions on it:

- (i) we require that the additional rigid or weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of the associated graded resolution, $GRes_1, \dots, GRes_d$, that was added to $ExWRes$, will be either non-rigid or non weakly strictly solid.
- (ii) we require that the specialization of the terminal limit group of the associated graded resolution, $GRes_i$, extends to a specialization of the terminal limit group of one of the graded resolutions, $SLGRes_i^v$, that are associated with the graded resolution, $GRes_i$.

- (iii) we require that the specialization of the terminal limit group of the associated graded resolution, $GRes_i$, extends to a specialization of the terminal limit group of one of the graded Root resolutions, $GRootRes_i^r$, that are associated with the graded resolution, $GRes_i$ (and the restrictions of the test sequences from which the resolution, $ExWRes$, was constructed, do not extend to specializations of this graded Root resolution).

By passing to a further subsequence, and applying the first step of the sieve procedure, that is presented in [Se6], we obtain a quotient resolution of the closure, $ExWRes$.

The second possibility is to look at a sequence of cover approximations of the closure, $ExWRes$, and force the existence of a formal solution that is associated with the specialization of the terminal limit group of one of the resolutions, $GRes_1, \dots, GRes_d$, that was added to specializations of $ExWRes$. i.e., to each specialization in a test sequence of a cover approximation of $ExWRes$ that we consider, we add specializations of the terminal limit group of one of the formal resolutions, $FGRes_i^j$, that are associated with the graded resolutions, $GRes_1, \dots, GRes_d$. From the collection of test sequences that factor through cover approximations of $ExWRes$, and the additional specializations of the terminal limit groups of one of the formal resolutions that are associated with one of the resolutions, $GRes_1, \dots, GRes_d$, we extract a subsequence that converges into a quotient resolution of the closure, $ExWRes$. The structure of such a quotient resolution is determined by the first step of the sieve procedure [Se6], in the same way that it was used in proving the equationality of Diophantine sets in section 2 of [Se9].

We continue iteratively, at each step we first look at a sequence of approximating (cover) resolutions of a quotient resolution that was constructed in the previous step, where these approximating (cover) resolutions satisfy the properties that are listed in theorem 1.21, and in particular they can be extended to ungraded resolutions with f.p. completions. If there exists an approximating cover resolution for which there exists a subsequence of pairs, (still denoted) $\{(p_n, w_n)\}$, so that for each specialization of the terminating limit group of the approximating resolution that is associated with a pair (p_n, w_n) from the subsequence, there exists a test sequence in the fiber that is associated with such a specialization (which takes its values in G_n) that satisfies $p_n \in EAE(p)$ (over G_n), and the restrictions of the test sequence to the existential variables w are witnesses for p_n , we do what we did in this case in the first step.

This means that we associate with the approximate resolution finitely many auxiliary resolutions. First, we look at all the test sequences of the approximating cover resolution that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, and use a construction similar to the construction of the formal graded Makanin-Razborov diagram to associate with the cover approximating resolution finitely many resolutions that are similar to the resolutions, $ExWRes_m$, that were constructed in the first step. Then we look at all the test sequences of these constructed resolutions that satisfy properties (1)-(6), where for each such collection of test sequences, we use a construction similar to the one that was used to construct a formal Makanin-Razborov diagram (theorems 2.6 and 2.7). Finally, with the approximating resolution and the auxiliary resolutions that are associated with it, we associate finitely many graded resolutions that are associated with the singular locus of the approximating resolution, and with it and the

finite set of closures that are associated with its auxiliary resolutions, according to propositions 4.2 and 4.32. Note that each of the constructed (auxiliary) resolutions has the same properties of a resolution in the formal Makanin-Razborov diagram (see theorem 2.6), so in particular it terminates in a cover closure of the approximating cover resolution, and each of the constructed resolution can be extended to an ungraded resolution with a f.p. completion.

If there is no such approximating resolutions for a constructed quotient resolution, we associate with it a graded resolution that is constructed from test sequences that can be extended to specializations of the terminal limit groups of one of the graded resolutions: $GRes_1, \dots, GRes_d$ (that can not be extended to formal solutions that are defined over these resolutions and satisfy both the equalities, $\Sigma(x, y, w, p) = 1$, and the inequalities, $\Psi(x, y, w, p) \neq 1$). We further force one of the 3 possible collapse conditions over the constructed closure of the quotient resolution, or the existence of a formal solution that is associated with the additional specialization of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, and apply the general step of the sieve procedure [Se6], and associate a quotient resolution of the quotient resolution that we have started this (general) step with.

By an argument which is similar to the argument that guarantees the termination of the sieve procedure in [Se6], and the argument that was used to prove the equationality of Diophantine sets over free and hyperbolic groups (proposition 2.2 in [Se9]), we obtain a termination of the this procedure.

Theorem 4.4. *The procedure for the analysis of an EAE set over free products terminates after finitely many steps.*

Proof: At each step of the procedure, on the quotient resolution, $WRes$, that is analyzed in that step, we impose one of two types of restrictions. The first adds to generic points of fibers (test sequences) of the resolution, $WRes$, specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, and associate with $WRes$ a (cover) closure. Then we impose one of 3 possible Diophantine conditions on the obtained (cover) closure, that either forces the additional rigid or weakly strictly solid specialization to be either non-rigid or non weakly strictly solid, or forces the specialization of the terminal limit group of one of the graded resolution, $GRes_1, \dots, GRes_d$, to be extended to a specialization of the terminal limit group of one of the resolutions that are associated with the singular locus, $SLGRes_i^v$, or to the terminal limit group of one of the graded Root resolutions, $GRootRes_i^r$.

The second type adds specializations of the terminal limit groups of one of the associated formal graded (cover) resolutions, $FGRes_i^j$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor in the terminal limit group of that formal graded resolution, and these specializations do not extend to a closure of the quotient resolution from the previous step.

These (3 possible) Diophantine conditions are similar in nature to the Diophantine condition that is imposed on quotient resolutions in the general step of the sieve procedure (over free groups) in [Se6]. The additional specializations of the second type (that do not extend to a closure of the previous quotient resolution), are similar to the rigid or strictly solid specializations that are added in each step in

the iterative procedure that is used to prove the equationality of Diophantine sets over free and hyperbolic groups (section 2 in [Se9]). However, we need to modify the argument for the termination of the sieve procedure over free groups, to guarantee the termination of the procedure for the analysis of an EAE set over free products.

The sieve procedure over free groups [Se6], analyzes a sequence of quotient resolutions, where each quotient resolution is obtained from the previous one by imposing one of finitely many Diophantine conditions on a closure of the quotient resolution that was obtained in the previous step. The complexity of such a quotient resolution is measured by some finite collection of associated graded limit groups (over free groups), a finite collection of core resolutions, their associated induced resolutions, and possibly a finite collection of sculpted resolutions and carriers. Since the analysis and the definitions of these objects are rather involved, we won't elaborate on them, and the interested reader is referred to [Se6] for the complete details.

The analysis of quotient resolutions over free products, along the steps of the iterative procedure for the analysis of an EAE set, is done according to the general step of the sieve procedure. Hence, with each quotient resolution we also associate (finitely many) graded limit groups (over free products), core resolutions, induced resolutions, and possibly carriers and sculpted resolutions, precisely as they appear in the sieve procedure [Se6].

As the graded limit groups, the core resolutions, and the induced resolutions, that are constructed along the iterative procedure for the analysis of an EAE set over free products, are similar to the ones that are constructed along the sieve procedure over free groups in [Se6], the proof of theorem 22 in [Se6] that guarantees the termination of the sieve procedure over free groups, implies that the limit groups, the core resolutions, and the induced resolutions, that appear along the procedure for the analysis of an EAE set over free products, can change only finitely many times along the procedure. Hence, by the same proof of theorem 22 in [Se6], if the procedure does not terminate, the number of sculpted resolutions that are associated with the quotient resolutions that are constructed in the various steps can not be bounded.

Suppose that the procedure for the analysis of an EAE set over free products does not terminate. By the argument that is used in proving theorem 22 in [Se6], the larger and larger numbers of sculpted resolutions that are constructed along the iterative procedure, in combination with the argument that was used to prove the combinatorial bounds on rigid and (weakly) strictly solid specializations of rigid and (weakly) solid limit groups over free products (theorems 1.10 and 1.13), together with the argument that was used to prove the existence of finitely many systems of fractions for rigid and (weakly) strictly solid specializations in theorems 1.14 and 1.15, guarantee the existence of a subsequence of steps along the iterative procedure, with an unbounded number of associated sculpted resolutions, for which the additional specializations of the rigid or weakly solid factors of the terminal limit groups of one of the graded resolutions, $GRes_1, \dots, GRes_d$, satisfy the conclusions of theorems 1.10, 1.13, 1.14 and 1.15 (even though these specializations are not rigid nor weakly strictly solid in general).

Over free groups, such a subsequence of steps gives an immediate contradiction, as theorems 1.14 and 1.15 in the case of a free group imply a global bound on the number of (families of) the specializations of the rigid or weakly solid factors of the terminal limit groups of $GRes_1, \dots, GRes_d$, that contradicts the lack of a bound on the number of sculpted resolutions along the subsequence of steps of the

procedure. Over free products, theorems 1.14 and 1.15 do not imply similar bounds on the number of specializations of the rigid or weakly solid factors, hence, we need to slightly modify the argument.

The quotient resolutions that are constructed along the procedure, are obtained by either imposing one of 3 possible Diophantine conditions on a closure of the quotient resolution that was constructed in the previous step, or by requiring the existence of an additional specialization of the terminal limit group of one of the formal graded resolutions, $FGRes_i^j$, that are associated with the graded resolutions, $GRes_1, \dots, GRes_d$, a specialization that does not extend to a closure of the previously constructed quotient resolution. The second type of requirement, the existence of an additional specialization of one of the terminal limit groups of the associated graded formal resolutions, can occur for only boundedly many sculpted resolutions, by the same argument that was used in the proof of theorem 22 in [Se6] (over free groups), or alternatively by the argument that was applied in proving the equationality of Diophantine sets over free and hyperbolic groups (theorem 2.2 in [Se9]).

Hence, to prove the termination of the iterative procedure for the analysis of an EAE set, we need to prove that the requirement that the additional specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, satisfy one of 3 possible Diophantine conditions, can occur for boundedly many sculpted resolutions as well.

Each of the 3 possible Diophantine requirements, is expressed by an extension of the specialization of the additional terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, to a specialization of one of finitely many possible graded limit groups. With each such graded limit group we associate a (non-canonical) graded Makanin-Razborov diagram (see theorem 1.22) with respect to the parameter subgroup that is generated by the fixed generating set of the (original) terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$. Each such graded resolution in the graded Makanin-Razborov diagram terminates in a free product of a rigid or a weakly solid limit group with finitely many elliptic factors and possibly a free group.

Therefore, the Diophantine conditions that are imposed in each step of the iterative procedure on the additional specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, can be expressed by the possibility to extend this specialization to a specialization of one of the terminal limit groups in the finitely many graded Makanin-Razborov diagrams that are associated with the 3 possible Diophantine conditions, that restrict to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the corresponding terminal graded limit group.

We have already pointed out that in case the iterative procedure doesn't terminate, there exists a subsequence of steps for which the additional specializations of the terminal limit groups of the graded resolutions, $GRes_1, \dots, GRes_d$, that are associated with unbounded collections of sculpted resolutions, satisfy the conclusions of theorems 1.10, 1.13, 1.14 and 1.15. Precisely the same argument guarantees that we can pass to a further subsequence for which the extended specializations to terminal limit groups of the resolutions in the graded Makanin-Razborov diagrams that are associated with the 3 possible Diophantine conditions, satisfy the conclusions of these (1.10, 1.13, 1.14 and 1.15) theorems as well.

Now we can apply the argument that was used to prove the termination of

the sieve procedure over free groups (theorem 22 in [Se6]), and conclude that by the conclusions of theorems 1.14 and 1.15 for the extended specializations of the terminal limit groups in the graded Makanin-Razborov diagrams that are associated with the Diophantine conditions, if there is no bound on the number of sculpted resolutions that are associated with the quotient resolutions along the iterative procedure, then there must exist a specialization of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, that already satisfies one of the associated Diophantine conditions within the sculpted (or developing) resolution in which it was constructed, which is a contradiction to the way sculpted and developing resolutions are constructed in the general step of the iterative procedure.

Therefore, there exists a global bound on the number of possible sculpted resolutions that can be associated with a quotient resolution along the iterative procedure. This global bound implies the termination of the iterative procedure for the analysis of an EAE set over free products, since as we have already pointed out, in case it doesn't terminate, there is no bound on the number of associated sculpted resolutions. Therefore, the iterative procedure terminates after finitely many steps. \square

By theorem 4.4 the procedure for the analysis of an EAE set terminates after finitely many steps. When it terminates we are left with a quotient resolution, that has all the properties of a cover approximating (graded) resolution that are listed in theorem 1.21, and a finite collection of auxiliary resolutions, that do all satisfy the properties of the resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7)

This terminating iterative procedure that allows one to associate an approximating cover graded resolution, and its finite collection of auxiliary resolutions (that can all be extended to ungraded resolutions with f.p. completions), with a subsequence of any given sequence of pairs, $\{(p_n, w_n)\}$, of specializations in free products G_n for which $p_n \in EAE(p)$ over G_n , and w_n are witnesses for p_n , enables one to construct finitely many such approximating cover resolutions with their (finitely many) auxiliary resolutions, that can be associated with the set $EAE(p)$. In the case of an EAE sentence, the construction of these (finitely many) approximating and auxiliary resolutions, allows one to reduce (uniformly) the given EAE sentence over free products to a (finite) disjunction of conjunctions of sentences over the factors of the free product.

Theorem 4.5. *Let:*

$$EAE(p) = \exists w \forall y \exists x \Sigma(x, y, w, p) = 1 \wedge \Psi(x, y, w, p) \neq 1$$

be an EAE set over groups. Then there exist finitely many (coefficient-free) graded resolutions over free products (with respect to the parameter subgroup $\langle p \rangle$): $WRes_1(z, y, w, p), \dots, WRes_e(z, y, w, p)$, with the following properties:

- (1) *each of the graded resolutions, $WRes_i$, is a graded cover resolution that satisfies the properties of cover graded resolutions that are listed in theorem 1.21. In particular, each of the graded resolutions, $WRes_i$, can be continued to an ungraded resolution with a f.p. completion. With the rigid or weakly solid factor of the terminal limit group of $WRes_i$ there are finitely many associated covers of the flexible quotients of that rigid or weakly solid factor, and each of these covers can be embedded in an ungraded resolution with a f.p. completion (see theorem 1.21).*

- (2) *with each graded resolution, $WRes_i(z, y, w, p)$, we associate (non-canonically) finitely many graded (auxiliary) resolutions, precisely in the same way that we have associated such auxiliary resolutions with an approximating cover resolution, $WRes_m$, in the first step of the iterative procedure, in case the iterative procedure terminates in the first step. These auxiliary graded resolutions have the same properties as the resolutions in a formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7), and they are constructed from test sequences of each of the graded resolutions, $WRes_i$, $i = 1, \dots, e$, that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, and test sequences that can be extended to specializations that satisfy one of the properties (1)-(6), that are listed in constructing the auxiliary resolutions that are associated with the resolution, $WRes_m$, in the first step of the iterative procedure (in case this procedure terminates in the first step). We further associate auxiliary resolutions that are associated with the singular loci of the resolutions, $WRes_i$, (see proposition 4.2) and auxiliary resolutions that are graded Root resolutions that are associated with the graded resolutions, $WRes_i$, and the (finite) set of closures that are associated with all the previously constructed auxiliary resolutions (see proposition 4.3).*
- (3) *let $G = A^1 * \dots * A^e$ be a non-trivial free product, which is not isomorphic to D_∞ . Let p_0 be a specialization of the parameters p in the free product G , for which $p_0 \in EAE(p)$ over G . Then there exists an index i , $1 \leq i \leq e$, a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of $WRes_i$ in G (that restricts to p_0), and specializations of the elliptic factors of that terminal limit group in elliptic factors in G , so that for the combined specialization of the terminal limit group of the graded resolution, $WRes_i$, there is a test sequence in the fiber that is associated with this specialization of the terminal limit group (of $WRes_i$), that either can not be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, or if it can be extended to such specializations, then either their restrictions to the rigid or weakly solid factor in the terminal limit group of $GRes_1, \dots, GRes_d$, is not rigid or not weakly solid, or the specializations of the terminal limit group of $GRes_1, \dots, GRes_d$, can be extended to specializations of the terminal limit group of one of the graded resolutions, $SLGRes_i^v$, that are associated with the singular locus of the graded resolutions, $GRes_1, \dots, GRes_d$, or the specializations of the terminal limit group of $GRes_1, \dots, GRes_d$, can be further extended to specializations of the terminal limit groups of one or more formal graded resolutions, $FGRes_i^j$, that are associated with, $GRes_1, \dots, GRes_d$, so that the associated fibers of the formal resolutions, $FGRes_i^j$, form a covering closure of the corresponding fiber of the graded resolution $GRes_i$, and these specializations of the terminal limit groups of the formal resolutions, $FGRes_i$, can not be further extended to the terminal limit groups of the collapse resolutions, $CollFRes_i^{j,k}$, that are associated with the formal graded resolutions, $FRes_i^j$, and not to specializations of the terminal limit group of one of the graded resolutions, $SLFGRes_i^{j,v}$, that are associated with the singular locus of the formal resolutions $FGRes_i^j$ (see the construction of the auxiliary resolutions that are associated with $WRes_m$ in the first step*

of the iterative procedure for the analysis of an EAE set, and in particular properties (1)-(6)).

In other words, given an EAE set, there exists a finite collection of graded resolutions (that can be extended to ungraded resolutions with f.p. completions) over free products, and finitely many associated auxiliary resolutions of these graded resolutions, so that the inclusion of a specialization of the parameters (free variables) in the EAE set over any given free product, can be demonstrated by a generic point in at least one of these resolutions, and (possibly) their associated auxiliary resolutions.

Proof: The argument that we use is similar to the one that was used to prove theorems 4.1 and 3.2, that is based on the arguments that were used in constructing the ungraded and graded Makanin-Razborov diagrams (theorem 26 in [Ja-Se] and theorem 1.22).

Let $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), be a sequence of non-trivial free products that are not isomorphic to D_∞ . Let $\{(p_n, w_n)\}$ be a sequence of tuples in G_n , so that $p_n \in EAE(p)$ over G_n , and w_n is a witness for p_n , i.e., (p_n, w_n) is in the associated set $AE(p, w)$ over G_n , which means that the following sentence:

$$\forall y \exists x \Sigma(x, y, w_n, p_n) = 1 \wedge \Psi(x, y, w_n, p_n) \neq 1$$

is true over G_n .

Starting with the sequence, $\{(p_n, w_n)\}$, the terminating iterative procedure for the analysis of an EAE set that we have presented constructs a graded resolution, $WRes$, with the following properties:

- (i) the graded resolution $WRes$ satisfies the properties of a cover graded resolution that are listed in theorem 1.21. In particular, its completion can be extended to (an ungraded) f.p. completion. With the rigid or weakly solid factor of the terminal limit group of $WRes$, there is a finite collection of covers of its flexible quotients that can all be embedded into f.p. completions.
- (ii) there exists a subsequence of the sequence of specializations, $\{(p_n, w_n)\}$, that extend to specializations that factor through the resolution, $WRes$, and to specializations of the f.p. completion of the ungraded resolution that extends the graded resolution, $WRes$ (see theorem 1.21). Hence, with each specialization from this subsequence of the tuples, $\{(p_n, w_n)\}$, specializations of the terminal limit group of the graded resolution, $WRes$, can be associated. Furthermore, these specializations of the terminal limit group of $WRes$ can be extended to specializations of the f.p. completion of some ungraded resolution of the terminal limit group of $WRes$. Restrictions of generic points in the fibers that are associated with these specializations of the terminal limit group of $WRes$, to the variables (p, w) , that we denote, (p_n, w) , are in the set $AE(p, w)$ that is associated with the given EAE set, $EAE(p)$. i.e., for every index n , and for generic w in the fiber that is associated with (p_n, w_n) (i.e., for the restriction of some test sequence in this fiber to the existential variables w), the sentence:

$$\forall y \exists x \Sigma(x, y, w, p_n) = 1 \wedge \Psi(x, y, w, p_n) \neq 1$$

is true over G_n .

- (iii) with the resolution $WRes$, we associate (non-canonically) a finite collection of graded auxiliary resolutions, that are part of the output of the terminating iterative procedure for the analysis of an EAE set, i.e., auxiliary resolutions of the same type as those that were associated with an approximating cover resolution, $WRes_m$, in the first step of the procedure, in case it terminates in the first step. These auxiliary resolutions have the same properties of the resolutions in a formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7), and in particular they can be extended to ungraded resolutions with f.p. completions.

Now we can apply the argument that was used to prove theorems 3.2 and 4.1. We look at all the sequences of non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), that are not isomorphic to D_∞ , an associated sequence of tuples, $p_n \in EAE(p)$ over G_n , and witnesses w_n for p_n , i.e., a sequence of pairs $(p_n, w_n) \in AE(p, w)$ over G_n . From every such sequence we use our terminating iterative procedure, and extract a subsequence of the tuples, (still denoted) $\{(p_n, w_n)\}$, and a graded resolution, $WRes$, that has the properties (i)-(iii), and in particular, the subsequence of tuples, (still denoted) $\{(p_n, w_n)\}$, extend to specializations that factor through the resolution $WRes$, and to specializations of the f.p. completion of the ungraded resolutions that extends the graded resolution $WRes$. Furthermore, the restrictions of generic points in the fibers that are associated with the pairs, $\{(p_n, w_n)\}$, are in the set $AE(p, w)$ (i.e., generic values of w in these fibers are witnesses for $p_n \in EAE(p)$ over G_n).

The completion of each of the constructed resolutions, $WRes$, can be extended to an ungraded f.p. completion, and so are its associated (finitely many) auxiliary resolutions. Furthermore, the (finite collection of) covers of flexible quotients of the rigid or weakly solid factors of the terminal limit groups of the resolutions, $WRes$, and of the auxiliary resolutions that are associated with $WRes$, can all be embedded into (ungraded) resolutions with f.p. completions. Hence, we can define a linear order on this (countable) collection of graded resolutions ($WRes$), and their (non-canonically) associated auxiliary resolutions, and (finite collections of) covers of flexible quotients. By the same argument that was used in constructing the Makanin-Razborov diagram (theorem 26 in [Ja-Se]), there exists a finite sub-collection of these graded resolutions that satisfy properties (1)-(3) of the theorem. \square

As we have already pointed out, the existence of finitely many graded resolutions that satisfy the properties that are listed in theorem 4.5, and their (finitely many) auxiliary resolutions, allows one to reduce an EAE sentence from free products to a sentence over its factors in a uniform way.

Theorem 4.6. *Let:*

$$\exists w \forall y \exists x \Sigma(x, y, w) = 1 \wedge \Psi(x, y, w) \neq 1$$

be an EAE sentence over groups.

*Then there exists a coefficient-free sentence over free products, which is a (finite) disjunction of conjunctions of EAE sentences, where each of these last EAE sentences involves elliptic elements from the same factor, such that for every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , the original EAE sentence over the free product $G = A^1 * \dots * A^\ell$ is a truth sentence, if and*

only if the sentence which is a (finite) disjunction of conjunctions of EAE sentences over the factors A^1, \dots, A^ℓ is a truth sentence.

Proof: By theorem 4.5, with a given EAE set, it is possible to associate finitely many graded resolutions, $WRes_1, \dots, WRes_e$, that do all satisfy the properties that are listed in theorem 1.21, and in particular they can all be embedded into f.p. completions. With each graded resolution, $WRes_i$, we have associated finitely many auxiliary resolutions, according to parts (1)-(6) that appear in the first step of the iterative procedure for the analysis of an EAE set, as well as finitely many graded resolutions that are associated with the singular locus of $WRes_i$ (according to proposition 4.2), and finitely many graded Root resolutions that are associated with $WRes_i$ and its collection of auxiliary resolutions according to proposition 4.3. Note that all the (finitely many) graded resolutions that are associated with $WRes_i$ have the properties of the resolutions in the formal graded Makanin-Razborov diagram as listed in theorems 2.6 and 2.7, and in particular they can all be embedded into f.p. completions.

In case of an EAE sentence, the same constructions that enable one to associate graded resolutions and their auxiliary resolutions with an EAE set (i.e., the iterative procedure for the analysis of an EAE set, and the proof of theorem 4.5), enable one to associate with a given EAE sentence a (non-canonical) finite collection of (ungraded) resolutions (over free products) with f.p. completions, and with each resolution finitely many (ungraded) auxiliary resolutions that do all have f.p. completions, and these auxiliary resolutions have the same properties and they are constructed in the same way as the (graded) auxiliary resolutions that are constructed in the case of an EAE set.

We (still) denote the (ungraded) resolutions that are associated with the given EAE sentence, $WRes_1, \dots, WRes_e$. By theorems 4.5 and 3.2, and propositions 4.2 and 4.3, the given EAE sentence is true over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , if and only if there exists a specialization in G of the terminal limit group of a resolution $WRes_i$ (one of the resolutions, $WRes_1, \dots, WRes_e$, i.e., specializations of the elliptic factors of the terminal limit group of $WRes_i$ in the factors, A^1, \dots, A^ℓ , for which:

- (1) the specialization does not extend to a specialization of the terminal limit group of one of the auxiliary resolutions, that are associated with the singular locus of $WRes_i$ according to proposition 4.2. The specialization extends to the terminal limit group of some prescribed Root resolutions (possibly only the trivial roots), that are associated with $WRes_i$ according to theorem 4.3, and does not extend to the other Root resolutions.
- (2) the specialization does not extend to a (finite) collection of specializations of the terminal limit groups of the auxiliary resolutions that are constructed from test sequences of $WRes_i$ that extend to specializations of the terminal limit groups of the graded resolutions, $GRes_1, \dots, GRes_d$, that are associated with the AE set, $AE(w)$:

$$\forall y \exists x \Sigma(x, y, w) = 1 \wedge \Psi(x, y, w) \neq 1$$

that satisfy:

- (i) these specializations extend to specializations of the terminal limit groups of a prescribed finite collection (possibly empty) of auxiliary

resolutions (that are associated with $WRes_i$) that are constructed from test sequences for which the additional specializations of $GRes_1, \dots, GRes_d$, are either collapsed, i.e., the restriction of the additional specializations to the rigid or weakly solid factor are not rigid nor weakly strictly solid, or they can be extended to either the the terminal limit groups of resolutions that are associated with the singular locus of the corresponding resolution, $GRes_1, \dots, GRes_d$, or to the terminal limit groups of resolutions that are associated with the graded Root resolutions that are associated with the resolutions, $GRes_1, \dots, GRes_d$. The specializations extend only to specializations of the terminal limit groups of such auxiliary resolutions from the prescribed set, and not to specializations of terminal limit groups of auxiliary resolutions that do not belong to the prescribed set.

- (ii) these specializations extend to specializations of the terminal limit groups of a prescribed finite collection (possibly empty) of auxiliary resolutions (that are associated with $WRes_i$) that are constructed from test sequences for which the additional specializations of $GRes_1, \dots, GRes_d$, extend to specializations of the terminal limit groups of formal graded resolution, $FGRes_i^j$, that are associated with the graded resolutions, $GRes_i$. The specializations extend only to specializations of the terminal limit groups of such auxiliary resolutions from the prescribed set, and not to specializations of terminal limit groups of auxiliary resolutions that do not belong to the prescribed set.
- (iii) the extended specializations of the terminal limit groups of the auxiliary resolutions that are associated with the formal graded resolutions, $FGRes_i^j$, extends further to specializations of the terminal limit groups of a prescribed finite collection (possibly empty) of auxiliary resolutions (that are associated with $WRes_i$) that are constructed from test sequences for which the additional specializations of the terminal limit groups of the formal graded resolutions, $FGRes_i$, are either collapsed, i.e., the restriction of the additional specializations to the rigid or weakly solid factor are not rigid nor weakly strictly solid, or they can be extended to either the the terminal limit groups of resolutions that are associated with the singular locus of the corresponding graded formal resolution, $FGRes_i^j$, or to the terminal limit groups of one of the graded resolutions, $\Psi FGRes_i^{j,k}$. Again, the specializations extend only to specializations of the terminal limit groups of such auxiliary resolutions from the prescribed set, and not to specializations of terminal limit groups of auxiliary resolutions that do not belong to the prescribed set.
- (iv) the fibers that are associated with the specializations of the terminal limit groups of auxiliary resolutions from the prescribed (finite) sets, form a covering closure of the fiber that is associated with the specialization of the terminal limit group of $WRes_i$.

Finally, by going over the finitely many possibilities for prescribed sets of auxiliary resolutions that satisfy part (iv), the existence of a specialization of the terminal limit group of one of the resolutions, $WRes_i$, that satisfies properties (1)-(2), is clearly a finite disjunction of finite conjunctions of EAE sentences over the factors

of G, A^1, \dots, A^ℓ , i.e., conditions (i)-(iv) can be easily written as a disjunction of conjunctions of such sentences in the factors by using all the constructed auxiliary resolutions. □

§5. AEAE sentences and predicates

In the previous two sections we used the iterative procedure for the analysis of an AE sentence over free groups [Se4], and the sieve procedure that was used for quantifier elimination over free groups [Se6], to analyze AE and EAE predicates and sentences. In particular we showed that AE and EAE sentences over free products can be reduced to (finite) disjunction of conjunctions of sentences over the factors of the free product. In this section we further continue towards similar statements for general predicates and sentences. We use what we proved for AE and EAE predicates, to analyze AEAE predicates and sentences. We expand the structure that we are working with, by adding new quantifiers over elliptic factors, and prove that an AEAE set can be defined using a predicate that requires only 3 quantifiers over the ambient free product, and additional quantifiers over elements that are in elliptic factors. The tools that are used in this reduction enable us to reduce an AEAE sentence over the ambient free product to a disjunction of conjunctions of sentences over the elliptic factors. These results and the tools and techniques that are used in their proof, are the basis for our general results for sentences and predicates (with an arbitrary (finite) number of quantifiers) over free products, and therefore they are the key for all our results on the first order theory of free products of groups.

Let:

$$AEAE(p) = \forall t \exists w \forall y \exists x \Sigma(x, y, w, t, p) = 1 \wedge \Psi(x, y, w, t, p) \neq 1$$

be a predicate over groups. With the set $AEAE(p)$ we naturally associate an EAE set:

$$EAE(p, t) = \exists w \forall y \exists x \Sigma(x, y, w, t, p) = 1 \wedge \Psi(x, y, w, t, p) \neq 1$$

and with the sets, $AEAE(p)$ and $EAE(p, t)$, we naturally associate an AE set:

$$AE(p, t, w) = \forall y \exists x \Sigma(x, y, w, t, p) = 1 \wedge \Psi(x, y, w, t, p) \neq 1.$$

Recall that with the set $AE(p, t, w)$ we have associated (in theorem 4.1) finitely many graded resolutions, $GRes_1, \dots, GRes_d$, with respect to the parameter subgroup $\langle p, t, w \rangle$, that do all satisfy the properties of graded cover resolutions that are listed in theorem 1.21. With these graded resolutions we have associated graded resolutions that are associated with their singular locus (proposition 4.2), graded Root resolutions (proposition 4.3), and graded formal resolutions, $FGRes_i^j$, with which we associated graded resolutions that are associated with their singular locus, and collapse graded formal resolutions, $CollFGRes_i^{j,k}$. A specialization $(p_0, t_0, w_0) \notin AE(p, t, w)$, over some non-trivial free product $G = A^1 * \dots * A^\ell$, which is not isomorphic to D_∞ , if and only if there is a specialization of the terminal graded limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, that extends the specialization (p_0, t_0, w_0) , a specialization that is composed from a rigid or a weakly strictly solid specialization of the rigid or solid factor of that graded

limit group, and specializations of the f.p. elliptic factors of the terminal graded limit group, so that the restrictions to the universal variables y of generic elements (i.e., a test sequence) in the fiber that is associated with the specialization of the terminal limit group, are witnesses that $(p_0, t_0, w_0) \notin AE(p, t, w)$.

With the EAE set, $EAE(p, t)$, we associated finitely many graded resolution (in proposition 4.3) that we denoted, $WRes_1, \dots, WRes_e$, that satisfy the properties of theorem 1.21. With each of these graded resolutions we have associated finitely many auxiliary resolutions that satisfy the properties of the resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7). Both the graded resolutions, $GRes_i$, and their associated auxiliary resolutions can be extended to ungraded resolutions with f.p. completions. By part (3) of theorem 4.5, a specialization $(p_0, t_0) \in EAE(p, t)$ over some non-trivial free product $G = A^1 * \dots * A^\ell$ which is not isomorphic to D_∞ , if and only if there is a specialization of the terminal graded limit group of one of the graded resolutions, $WRes_1, \dots, WRes_e$, that extends (p_0, t_0) , so that a generic point (i.e., a test sequence) in the fiber that is associated with this specialization of the terminal limit group, restricts to specializations of the existential variables w that are witnesses that $(p_0, t_0) \in EAE(p, t)$.

Both in the analysis of AE sets, and in the analysis of EAE sets, the existence of finitely many graded resolutions with the properties listed above enabled us to reduce a 2 quantifier and a 3 quantifier sentence to a finite disjunction of conjunctions of sentences over the factors of the free product (theorems 3.7 and 4.6).

In this section we use our results on AE and EAE sets over free products to analyze AEAE sets over free products, and in particular we associate finitely many graded resolutions, that satisfy the properties of theorem 1.21, and finitely many auxiliary resolutions with each of these graded resolutions (that satisfy the properties of formal graded resolutions that are listed in theorem 2.6), with any given AEAE set. The construction of these graded and auxiliary resolutions uses the structural results that were proved in sections 1 and 2, and is once again a variation of the sieve procedure that was presented in [Se6].

We start the analysis of the set $AEAE(p)$ with all the sequences of specializations of the tuple, $(p, t), \{(p_n, t_n)\}$, that take values in non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$ (for possibly varying $\ell > 1$), which are not isomorphic to D_∞ . We further assume that for every index n , $p_n \notin AEAE(p)$ over the free product, G_n , and that t_n is a witness for p_n , i.e., a specialization of the universal variables t in G_n , so that the EAE sentence:

$$\exists w \forall y \exists x \Sigma(x, y, w, t_n, p_n) = 1 \wedge \Psi(x, y, w, t_n, p_n) \neq 1$$

is a false sentence over G_n .

By theorem 1.16, given such a sequence of pairs, $\{(p_n, t_n)\}$, we can pass to a subsequence that converges into a well-structured (even well-separated) graded resolution with respect to the parameter subgroup $\langle p \rangle$: $T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_s$, where T_s is a free product of a rigid or a solid group over free products (that contains the subgroup, $\langle p \rangle$), and (possibly) a free group and (possibly) finitely many elliptic factors. We denote this graded resolution $TRes$. As we noted in similar constructions in the previous sections, the terminal limit group T_s in $TRes$ is f.g. but it may be not finitely presented.

We apply the techniques of theorem 1.21, and fix a sequence of cover approximating graded resolutions of the resolution, $TRes$, that we denote, $\{TRes_m\}$, that

satisfy the properties of cover approximating resolutions that are listed in theorem 1.21, and converge into the graded resolution, $TRes$. For each index m , there is a subsequence of the sequence of pairs, $\{(p_n, t_n)\}$, that factors through the cover approximating (graded) resolution, $TRes_m$.

For each index m , and each pair, (p_n, t_n) , that factors through $TRes_m$, we can associate a specialization of the terminating limit groups of the (approximating) resolution, $TRes_m$ (in the free product G_n). Such specialization is composed from a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of $TRes_m$, and specializations of the f.p. elliptic factors of that terminal graded limit group.

As in the analysis of an EAE set in section 4, we first assume that there exists an approximating resolution, $TRes_m$, for which there exists a subsequence of pairs, (still denoted) $\{(p_n, t_n)\}$, so that for each specialization of the terminating limit group of $TRes_m$ that is associated with a pair (p_n, t_n) from the subsequence, a generic pair in the fiber (i.e., a test sequence) that is associated with such a specialization of the terminal graded limit group of $TRes_m$ (which values in G_n) satisfies $p_n \notin AEAE(p)$ (over G_n), and a generic t in the fiber is a witness for that. i.e., for generic t in the fiber, the pairs $(p_n, t) \notin EAE(p, t)$ over G_n .

In this case with the approximating (cover) graded resolution, $TRes_m$, that has the properties that are listed in theorem 1.21, we associate finitely many auxiliary resolutions, that have similar properties to the resolutions in the formal graded Makanin-Razborov resolution (theorems 2.6 and 2.7). These auxiliary resolutions are similar in nature to the auxiliary resolutions that were associated in section 4 with an EAE set, and with a cover approximating resolution, $WRes_m$ (see the first step of the iterative procedure for the analysis of an EAE set in section 4). The auxiliary resolutions that are associated with the approximating cover, $TRes_m$, are constructed from test sequences of $TRes_m$ that have the following properties:

- (1) we start by associating (non-canonically) finitely many graded resolutions with the singular locus of $TRes_m$ (according to proposition 4.2), that we denote $SLTRes$, and finitely many graded Root resolutions (according to proposition 4.3), that we denote $GRootTRes$.
- (2) we look at all the test sequences of $TRes_m$, that can be extended to specializations of the terminal limit group of one of the cover graded resolutions, $WRes_i$, that are associated with the EAE set, $EAE(p, t)$, and to the f.p. completion into which this terminal limit group embeds. By the construction of the formal graded Makanin-Razborov diagram, with this collection of test sequences it is possible to associate finitely many graded resolutions that have the same properties as the resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7). In particular, the completion of each of the constructed resolutions embeds into a f.p. completion. We denote the constructed resolutions, $WTRes$.
- (3) we look at test sequences of the graded resolutions that are constructed in part (2), for which the restriction of the additional specializations of the terminal limit group of the resolution, $WRes_i$, to a specialization of the rigid or weakly solid factor of that terminal limit group, is non-rigid or non weakly solid. By the construction of the formal Makanin-Razborov diagram, it is possible to associate finitely many graded (cover) resolutions with these test sequences that have the properties of the resolutions in the

formal Makanin-Razborov diagram. We denote the constructed resolutions, $CollWTRes$.

Similarly, we look at test sequences of the graded resolutions that are constructed in part (2), for which the restrictions of the additional specializations of the terminal limit group of the resolution, $WRes_i$, can be extended to specializations of the terminal limit group of one the auxiliary resolutions that are associated with the singular locus of $WRes_i$, or of one of the graded Root resolutions that are associated with $WRes_i$. We denote the obtained auxiliary resolutions, $SLWTRes$ and $GRootWTRes$, in correspondence.

- (4) we look at test sequences that are constructed in part (2), that can be further extended to specializations of the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, $WRes_i$, and was constructed from test sequences that can be extended to the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, that are associated with the AE set, $AE(p, t, w)$. We further require that the specialization of the terminal limit group of this auxiliary resolution can be extended to the f.p. completion that is associated with the auxiliary resolution, and the terminal limit group (of the auxiliary resolution) is embedded into it. Once again with this collection of sequences it is possible to associate finitely many graded resolutions that have the properties of the resolutions in a formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7). We denote the constructed resolutions, $YWTRes$.

- (5) we look at test sequences that were considered in part (4) for which the additional specialization of the terminal limit group of the associated auxiliary resolution, restricts to non-rigid non-weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of the auxiliary resolution. With this collection of sequences we associate finitely many graded resolutions following the construction of the formal graded Makanin-Razborov diagram. We denote the constructed resolutions, $CollYWTRes$.

Similarly, we look at test sequences of the graded resolutions that are constructed in part (4), for which the restrictions of the additional specializations of the associated auxiliary resolution of $WRes_i$, can be extended to specializations of the terminal limit group of one the auxiliary resolutions that are associated with $WRes_i$ and with the singular locus of $GRes_i$, or with one of the graded Root resolutions that are associated with $GRes_i$. We denote the obtained auxiliary resolutions, $SLYWTRes$ and $GRootYWTRes$, in correspondence.

- (6) we look at test sequences that are constructed in part (4), that can be further extended to specializations of the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, $WRes_i$, and were constructed from test sequences that can be extended to the terminal limit group of one of the graded formal resolutions, $FGRes_i^j$, that are associated with the graded resolution $GRes_i$, and both are associated with the AE set, $AE(p, t, w)$. We further require that the specialization of the terminal limit group of this auxiliary resolution can be extended to the f.p. completion that is associated with the auxiliary resolution, and the terminal limit group (of the auxiliary resolution) is embedded into it. With this collection of sequences we, once again, associate finitely many resolutions, in

a construction that follows the construction of the formal graded Makanin-Razborov diagram (theorem 2.7). We denote the constructed resolutions, $XYWTRes$.

- (7) we look at test sequences that were considered in part (6) for which the additional specialization of the terminal limit group of the associated auxiliary resolution (that is associated with $FGRes_i^j$), restricts to non-rigid non-weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of the auxiliary resolution. With this collection of sequences we associate finitely many graded resolutions, following the construction of the formal graded Makanin-Razborov diagram. We denote the constructed resolutions, $CollXYWTRes$.

Similarly, we look at test sequences of the graded resolutions that are constructed in part (6), for which the restrictions of the additional specializations of the terminal limit group of one the auxiliary resolutions that are associated with $WRes_i$ and one of the formal graded resolution, $FGRes_i^j$, can be extended to specializations of the terminal limit group of one the auxiliary resolutions that are associated with $WRes_i$ and with the singular locus of $FGRes_i^j$. We denote the obtained auxiliary resolutions $SLXYWTRes$.

- (8) we look at test sequences that are constructed in part (6), that can be further extended to specializations of the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, $WRes_i$, and were constructed from test sequences that can be extended to the terminal limit group of one of the collapse graded formal resolutions, $CollFGRes_i^{j,k}$, that are associated with the graded formal resolution, $FGRes_i^j$, that is associated with the graded resolution, $GRes_i$, and all the three are associated with the AE set, $AE(p, t, w)$. We further require that the specialization of the terminal limit group of this auxiliary resolution can be extended to the f.p. completion that is associated with the auxiliary resolution, and the terminal limit group (of the auxiliary resolution) is embedded into it. With this collection of sequences we associate finitely many resolutions, in a construction that follows the construction of the formal graded Makanin-Razborov diagram (theorem 2.7). We denote the constructed resolutions, $\Psi XYWTRes$, as well.
- (9) we look at test sequences that were considered in part (8) for which the additional specialization of the terminal limit group of the associated auxiliary resolution (that is associated with $CollFGRes_i^{j,k}$), restricts to non-rigid non-weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of the auxiliary resolution. With this collection of sequences we associate finitely many graded resolutions, following the construction of the formal graded Makanin-Razborov diagram. We denote the constructed resolutions, $Coll\Psi XYWTRes$.

Hence, with the cover approximating graded resolution, $TRes_m$, that satisfies the properties of theorem 1.21, we associated finitely many auxiliary resolutions that do all satisfy the properties of resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7).

Suppose that there is no approximating cover resolution, $TRes_m$, with a subsequence of the original sequence, $\{(p_n, t_n)\}$ (for which $p_n \notin AEAE(p)$ (over G_n), and t_n (a specialization of the universal variables t in G_n) is a witness for that (in

G_n)), that factor through $TRes_m$, so that for each specialization of the terminal limit group of $TRes_m$ that is associated with a pair (p_n, t_n) from the subsequence, a generic pair (i.e., a test sequence) in the fiber that is associated with such a specialization of the terminal graded limit group (which takes its values in G_n), satisfies $p_n \notin AEAE(p)$ (over G_n), and t (a generic value of t in the associated fiber) is a witness for that.

As in analyzing EAE sets in sections 4, in this case (in which there is no approximating cover $TRes_m$ with the desired properties) we can associate with the original graded resolution, $TRes$, another graded resolution, which is obtained by applying the first step of the sieve procedure for the analysis of quotient resolutions [Se6]. The new graded resolution has a smaller complexity than the original graded resolution $TRes$, and the original sequence of specializations, $\{(p_n, t_n)\}$, is guaranteed to have a subsequence that extends to specializations that converge into the obtained quotient resolution (which is of smaller complexity). This enables us to continue iteratively, in a similar way to the sieve procedure [Se6], and to the iterative procedure that was used to prove the equationality of Diophantine sets over free and hyperbolic groups in section 2 of [Se8].

With the resolution, $TRes$, we have associated a sequence of approximating cover resolutions, $TRes_m$, that satisfy the properties that are listed in theorem 1.21. In particular, for each index m , there is a subsequence of the original sequence of pairs, $\{(p_n, t_n)\}$, that factor through $TRes_m$. Recall that by our assumptions, for every index m , there is no subsequence of the pairs, $\{(p_n, t_n)\}$, that do factor through the approximating cover, $TRes_m$, for which for generic pair (p_n, t) (i.e., for a restriction of a test sequence to the universal variables t) in the fiber that contains the pair (p_n, t_n) (in the graded variety that is associated with $TRes_m$), $p_n \notin AEAE(p)$ over G_n , and the generic t (in the fiber) is a witness for that, i.e., the generic pairs, $(p_n, t) \notin EAE(p, t)$ over G_n .

We construct a sequence of specializations over G_n as follows. We go over the indices m , and given an index m we pick an index $n_m > m$, and a tuple, $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m}, \tilde{t}_{n_m})$, that takes its values in G_{n_m} , with the following properties:

- (1) (p_{n_m}, t_{n_m}) factors through the graded resolution, $TRes_m$, and \hat{t}_{n_m} is the specialization of the terminal limit group of the graded resolution, $TRes_m$, that contains the pair, (p_{n_m}, t_{n_m}) .
- (2) \tilde{t}_{n_m} is a specialization of the variables t , from the fiber that is associated with \hat{t}_{n_m} , i.e., the fiber that contains the pair (p_{n_m}, t_{n_m}) in the graded variety that is associated with $TRes_m$. Furthermore, for every index m , $(p_{n_m}, \tilde{t}_{n_m}) \in EAE(p, t)$ over G_n .
- (3) the sequence $\{(p_{n_m}, \tilde{t}_{n_m})\}$ is a graded test sequence that converges into the graded resolution, $TRes$.
- (4) the tuples $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m})$, are specializations of the terminal limit group of one of the finitely many graded resolutions, $WRes$ (that are associated with the set $EAE(p, t)$), that do not extend to specializations of the terminal limit groups of the graded resolutions that are associated with the singular locus of $WRes_m$, and these specializations restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group (with respect to the parameter group $\langle p, t \rangle$). Furthermore, a generic value of the existential variables w in the fiber of $WRes$ that is associated with the tuple, $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m})$, is a witness that: $(p_{n_m}, \tilde{t}_{n_m}) \in$

$EAE(p, t)$ over G_{n_m} .

From the sequence of tuples, $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m}, \tilde{t}_{n_m})$, we can extract a subsequence that converges into a closure of the graded resolution, $TRes$, that we have started with. We denote this closure, $ExWTRes$. Recall that we have assumed that the original sequence of pairs, $\{(p_n, t_n)\}$, satisfies $p_n \notin EAE(p)$, and t_n is a witness for that (i.e., $(p_n, t_n) \notin EAE(p, t)$). On the other hand, in the sequence of tuples that we chose (in the fibers that are associated with a subsequence of the original sequence of specializations), a generic value of the existential variables w in the fiber that is associated with the tuple, $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m})$, is a witness that the pair $(p_{n_m}, \tilde{t}_{n_m}) \in EAE(p, t)$.

Given the closure, $ExWTRes$, we continue in a similar way to what we did in analyzing EAE sets in section 4, a way that is adapted to sets with 4 quantifiers. The closure, $ExWTRes$, contains an (additional) rigid or weakly strictly solid specialization of the terminal existential variables w , that for generic value of the universal variables t (i.e., for restrictions of a test sequence to the variables t) are witnesses that the pairs, $(p_{n_m}, t) \in EAE(p, t)$. Hence, we apply the construction that was used in theorem 1.21, and associate a sequence of approximating covers of $ExWTRes$ that satisfy the properties that are listed in theorem 1.21. We denote this sequence, $\{ExWTRes_r\}$.

By construction, for each index r , it is possible to extract a subsequence of the original sequence of tuples: $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m}, \tilde{t}_{n_m})$, that factor through $ExWTRes_r$. The tuples $(p_{n_m}, t_{n_m}) \notin EAE(p, t)$, whereas the tuples $(p_{n_m}, \tilde{t}_{n_m}) \in EAE(p, t)$, and generic values (a test sequence) of the existential variables w in the fiber that is associated with the tuple, $(p_{n_m}, \hat{t}_{n_m}, \tilde{w}_{n_m})$, are witnesses for this last inclusion. Hence, on the tuples of variables (p, t, w) , that are associated with the additional tuples, $(p_{n_m}, \tilde{t}_{n_m}, \tilde{w}_{n_m})$, in the approximating cover $ExWTRes_r$, at least one of the following collapse forms can be imposed (cf. the list of test sequences that are associated with the approximating cover $TRes_m$ in the first step):

- (1) the additional specialization of the variables (p, t, w) in $ExWTRes_r$, that is assumed to restrict to rigid or weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of a resolution, $WRes$ (that is associated with $EAE(p, t)$) restricts to non-rigid or non weakly strictly solid specialization of that factor. This forces a Diophantine condition on the specializations of $ExWTRes_r$, similar to the ones that are forced along the sieve procedure in [Se6].
- (2) the additional specialization of the variables (p, t, w) in $ExWTRes_r$, extends to a specialization of the terminal limit group of one of the graded resolutions that are associated with the singular locus of the corresponding resolution, $WRes_i$. This forces a Diophantine condition on the specializations of $ExWTRes_r$.
- (3) we look at all the test sequences of $ExTWRes_r$, for which the specialization of the variables (p, t, w) in $ExWTRes_r$, can be extended to a specialization of the terminal limit group of an auxiliary resolution that is associated with graded Root resolution of $WRes$ (and with the set $EAE(p, t)$), and not to a specializations of an auxiliary resolutions that is associated with graded Root resolutions of $WRes$ of higher order roots, and for which the specializations of the variables, (p, t, w) , can be extended to specializations of the terminal limit groups of a prescribed set of auxiliary resolutions that

are associated with the resolution $WRes$ (and with the set $EAE(p, t)$), auxiliary resolutions that were constructed from test sequences of $WRes$ that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, or with the formal graded resolutions, $FGRes_j^i$, or with their associated graded Root resolutions, that are associated with $AE(p, t, w)$. It is further required that the prescribed set of auxiliary resolutions that are associated with the graded resolutions, $GRes_1, \dots, GRes_d$, form a covering closure of the graded resolution $WRes$, but that they do not form a covering closure if we take out the auxiliary resolutions that are associated with the graded formal resolutions, $FGRes_i^j$.

By the techniques for the construction of the graded formal Makanin-Razborov diagram (theorem 2.7), with this collection of test sequences and their extended specializations (for all possible graded root resolutions and prescribed sets of auxiliary resolutions), it is possible to associate finitely many graded resolutions that have the properties of resolutions in the formal Makanin-Razborov diagrams (theorems 2.6 and 2.7), and they are all closures of $ExWTRes_r$. We denote each of these finitely many graded resolutions, $XYExWTRes_r^s$.

On each of the closures, $XYExWTRes_r^s$, we further impose Diophantine conditions (in parallel). We require that the additional specializations of the terminal limit group of one of the auxiliary resolutions of $WRes$ that is associated with $FGRes_i^j$ in $XYExWTRes_r^s$, extend to specializations of the terminal limit group of either an auxiliary resolution that is associated with the singular locus of the corresponding resolution, $FGRes_i^j$, or to the terminal limit group of an auxiliary resolution that is associated with a resolution, $\Psi FGRes_i^{j,k}$, or to an auxiliary resolution that is associated with specializations of the terminal limit group of a corresponding graded resolution, $FGRes_j^i$, that do not restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $FGRes_i^j$, or that the specializations of the terminal limit group of such an auxiliary resolution of $WRes$ do not restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of their terminal limit group.

Another possible Diophantine restrictions on $XYExWTRes$ is that the specializations of the terminal limit group of the graded resolution $WRes$, extend to specializations of the terminal limit group of an auxiliary resolution of $WRes$, that is associated with a graded Root resolution of $WRes$ or with a graded Root resolution of one of the graded resolutions, $GRes_1, \dots, GRes_d$, from the prescribed set, with roots of higher orders than the prescribed ones.

- (4) we further look at all the specializations of the graded resolution, $ExWTRes_r$, for which to the specialization of the the variables (p, t, w) in $ExWTRes_r$, it is possible to add additional specialization of the terminal limit group of one of the finitely many auxiliary resolutions that are associated with $WRes$ (and with $EAE(p, t)$), auxiliary resolutions that were constructed from test sequences of $WRes$ that can be extended to specializations of the terminal limit group of one of the graded resolutions, $GRes_1, \dots, GRes_d$, that are associated with $AE(p, t, w)$, and the combined specialization does not extend to a specialization that factors through (or in the same weakly strictly

solid family of a specialization in) one of the closures, $XYExWTRes_r^s$, that were constructed in part (3). We further assume that these additional specializations of the terminal limit group of an auxiliary resolution of $WRes$ (that is associated with $GRes_i$), restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of the auxiliary resolution, and they can not be extended to specializations of the terminal limit groups of auxiliary resolutions that are associated with the singular locus of $GRes_i$, or of auxiliary resolutions that are associated with specializations of the terminal limit group of $GRes_i$ that do not restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $GRes_i$.

For each index r , we can find an index $n_r > r$, so that the specialization, (p_{n_r}, t_{n_r}) (in G_{n_r}), extends to a specialization that satisfies the conditions that are specified by one the parts, (1)-(4) (for $ExWTRes_r$). We further apply the analysis of quotient resolutions that appear in the first step of the sieve procedure [Se6], and pass to a subsequence of the extended specializations (of the sequence $\{(p_{n_r}, t_{n_r})\}$, that converges into a quotient resolution of smaller complexity (in terms of the sieve procedure) than the complexity of the original resolution, $TRes$ (note that the obtained quotient resolution may be a proper closure of $TRes$, but there is a global bound (depending on $TRes$) on the number of steps for which the obtained quotient resolution is a closure of $TRes$. After a number of steps that is bigger than the bound, the obtained quotient resolution is no longer a closure, and hence, there is a reduction in other parts of the complexity of the obtained quotient resolution).

We continue iteratively. At each step we first look at a sequence of approximating resolutions of a quotient resolution that was constructed in the previous step, where these approximating resolutions satisfy the properties that are listed in theorem 1.21. If there exists an approximating resolution for which there exists a subsequence of pairs, (still denoted) $\{(p_n, t_n)\}$, so that for each specialization of the terminating limit group of the approximating resolution that is associated with a pair (p_n, t_n) from the subsequence, a generic pair (i.e. a restriction of a test sequence to the pair (p, t)) in the fiber that is associated with such a specialization of the terminal graded limit group (which takes its values in G_n) forms a witness for $p_n \notin AEAE(p)$ (over G_n), we do what we did in this case in the first step (i.e., we look at all the test sequences of the approximating resolutions that satisfy properties (1)-(9) and associate finitely many auxiliary resolutions with it).

If there is no such approximating resolutions for a constructed quotient resolution, we associate with it a closure that is constructed from test sequences that can be extended to additional (new) specialization of the terminal limit group of one of the resolutions, $WRes$ (that are associated with $EAE(p, t)$), in a similar way to the construction of the graded closure, $ExWTRes$, in the first step of the procedure. We further force a collapse condition over the constructed closure of the quotient resolution, according to parts (1)-(4), and apply the general step of the sieve procedure [Se6], extract a subsequence of the original sequence of tuples, $\{(p_n, t_n)\}$, and construct a quotient resolution that has smaller complexity than the quotient resolution that was constructed in the previous step (smaller complexity in light of the sieve procedure).

By an argument which is similar to the argument that guarantees the termination of the sieve procedure in [Se6], and to the termination of the procedure for the

analysis of EAE sets, we obtain a termination of the this procedure.

Theorem 5.1. *The procedure for the analysis of an AEAE set over free products terminates after finitely many steps.*

Proof: The argument that we use is similar to the proof of theorem 4.4 (termination of the procure for the analysis of an EAE set). At each step of the procedure, on the quotient resolution, $TRes$, that is analyzed in that step we impose one of finitely many restrictions. The first ones impose a non-trivial Diophantine condition on specializations of a closure of the resolution, $TRes$. This is similar to the Diophantine conditions that are imposed on quotient resolutions in the general step of the iterative procedure for the analysis of an EAE set. Hence, by the termination of this iterative procedure (theorem 4.4), these type of restrictions can occur only at finitely many steps along the iterative procedure for the analysis of an AEAE set.

The second type of restrictions, adds specializations of the terminal limit groups of one of the associated auxiliary resolutions that are associated with the graded resolution, $WRes$, where these specializations restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group. We further require that the additional specializations do not factor through (or are not in the same (weak) strictly solid families as specializations of) closures that are built from test sequences of $TRes$ that can be extended to specializations of such terminal limit groups of auxiliary resolutions.

Therefore, the argument that was used to prove the termination of the iterative procedure for the analysis of an EAE set (theorem 4.4), proves that restrictions of the second type can also occur in only finitely many steps (this type of restrictions are precisely the ones that are analyzed in proving the equationality of Diophantine sets (over free and hyperbolic) groups in [Se9]). Since the restrictions of both the first and the second type can occur at only finitely many steps, the iterative procedure for the analysis of an AEAE set terminates after finitely many steps. \square

As we argued for AE and EAE sets, the termination of the procedure for the analysis of AEAE sets, enables one to associate with the given AEAE set a finite collection of graded resolutions, that do all satisfy the properties that are listed in theorem 1.21, and for each of these graded resolutions an associated finite collection of auxiliary resolutions, that do all satisfy the properties of resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7).

This finite collection of graded resolutions, enable us to obtain a form of quantifier elimination for AEAE sets over free products, i.e., it enables one to show that an AEAE set can be defined by a predicate that uses only 3 quantifiers over the free product, and additional quantifiers over the various factors. In the case of an AEAE sentence, the construction of these (finitely many) graded resolutions (and their auxiliary resolutions), allows one to reduce the given AEAE sentence over free products to a (finite) disjunction of conjunctions of sentences over the factors of the free product. These statements are generalized to arbitrary definable sets and predicates in the next section, and they are the key for all the results in this paper.

Theorem 5.2. *Let:*

$$AEAE(p) = \forall t \exists w \forall y \exists x \Sigma(x, y, w, t, p) = 1 \wedge \Psi(x, y, w, t, p) \neq 1$$

be an AEAE set over groups. Then there exist finitely many graded resolutions over free products (with respect to the parameter subgroup $\langle p \rangle$):

$$TRes_1(z, y, w, t, p), \dots, TRes_g(z, y, w, t, p)$$

with the following properties:

- (1) The graded resolutions, $TRes_i(z, y, w, t, p)$, satisfy the properties that are listed in theorem 1.21. In particular, with each of them there is an associated f.p. completion, into which the completion of them embeds.
- (2) with each graded resolution, $TRes_i(z, y, w, t, p)$, we associate (non-canonically) a finite collection of auxiliary resolutions, that do all have the properties of resolutions in the formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7). These auxiliary resolutions are constructed from test sequences of $TRes_i$, precisely in the way we associated auxiliary resolutions with an approximating cover, $TRes_m$, in the first part of the procedure, i.e., according to parts (1)-(9) in the first step. We denote these auxiliary resolutions: $WTRes$, $CollWTRes$, $YWTRes$, $CollYWTRes$, $XYWTRes$, $CollXYWTRes$, $\Psi XYWTRes$, and $Coll\Psi XYWTRes$, and their associated graded Root resolutions and resolutions that are associated with the corresponding singular locus (see parts (1)-(9) in the first step of the procedure).
- (3) let $G = A^1 * \dots * A^\ell$ be a non-trivial free product, which is not isomorphic to D_∞ . Let p_0 be a specialization of the parameters p in the free product G , for which $p_0 \notin AEAE(p)$ over G . Then there exists an index i , $1 \leq i \leq g$, a specialization of the terminal limit group of $TRes_i$ (in G) that restricts to p_0 , that extends to a specialization of the f.p. completion into which the completion of $TRes_i$ embeds, and restricts to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of $TRes_i$, such that one of the following holds:
 - (i) the specialization of the terminal limit group of $TRes_i$, does not extend to a specialization of the terminal limit group of any graded resolutions, $SLTRes$, that are associated with the singular locus of $TRes$. It may extend to specializations of the terminal limit groups of some of the auxiliary resolutions, $WTRes$, but the corresponding fibers do not cover the fiber of $TRes$ that is associated with the given specialization.
 - (ii) the specialization of the terminal limit group of $TRes_i$ does not extend to the terminal limit group of any of the resolutions, $SLTRes$, but it extends to specializations of the terminal limit groups of some graded resolutions, $WTRes$, that do not extend to specializations of the terminal limit groups of the auxiliary resolutions, $SLWTRes$ and $CollWTRes$, and the corresponding fibers do cover the fiber of $TRes$ that is associated with the given specialization.

In that case, for each extension of the specialization of the terminal limit group of $TRes_i$, to a specialization of the terminal limit group of one of the auxiliary resolutions, $WTRes$, there can be further extensions to specializations of the terminal limit groups of auxiliary resolutions, $YWTRes$. Furthermore, these specializations of the terminal limit group of $YWTRes$ may extend to specializations of the terminal limit groups of auxiliary resolutions, $XYWTRes$. For every finite collection of extensions of the specialization of the terminal limit group of $TRes_i$ to the terminal limit groups

of (finitely many) auxiliary resolutions $WTRes$, the fibers that are associated with these extensions (and with the auxiliary resolutions $WTRes$), minus the fibers that are associated with further extensions to specializations of the terminal limit groups of auxiliary resolutions $YWTRes$, that are not covered by fibers that are associated with further extensions to the terminal limit groups of auxiliary resolutions $XYWTRes$ (that can not be extended to fibers of auxiliary resolutions $CollXYWTRes$, $\Psi XYWTRes$, and $SLXYWTRes$), do not cover the fiber that is associated with the original specialization of the terminal limit group of $TRes_i$.

In other words, given an AEAE set, there exists a finite collection of graded resolutions over free products (that satisfy the properties that are listed in theorem 1.21), and a finite collection of auxiliary resolutions (that satisfy the properties that are listed in theorem 2.6), so that the inclusion of a specialization of the parameters (free variables) in the complement of the given AEAE set over any given free product, which is not D_∞ , can be demonstrated by a generic point (test sequences) in a disjunction of conjunctions of (fibers of) these resolutions.

Proof: The argument that we use is similar to the one that was used to prove theorems 4.5, 4.1 and 3.2, that is based on the arguments that were used in constructing the ungraded and graded Makanin-Razborov diagrams (theorem 26 in [Ja-Se] and theorem 1.22).

Let $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), be a sequence of non-trivial free products that are not isomorphic to D_∞ . Let $\{(p_n, t_n)\}$ be a sequence of tuples in G_n , so that $p_n \notin AEAE(p)$ over G_n , and t_n is a witness for p_n , i.e., $(p_n, t_n) \notin EAE(p, t)$ over G_n .

Starting with the sequence, $\{(p_n, t_n)\}$, the terminating iterative procedure for the analysis of an AEAE set that we have presented, constructs a graded resolution, $TRes$, with the following properties:

- (i) the graded resolution $TRes$ satisfies the properties of a cover graded resolution that are listed in theorem 1.21. In particular, its completion can be extended to (an ungraded) f.p. completion. With the rigid or weakly solid factor of the terminal limit group of $WRes$, there is a finite collection of covers of its flexible quotients that can all be embedded into f.p. completions.
- (ii) there exists a subsequence of the sequence of specializations, $\{(p_n, t_n)\}$, that extend to specializations that factor through the resolution, $TRes$, and to specializations of the f.p. completion of the ungraded resolution that extends the graded resolution, $TRes$ (see theorem 1.21). Hence, with each specialization from this subsequence of the tuples, $\{(p_n, t_n)\}$, specializations of the terminal limit group of the graded resolution, $TRes$, can be associated. Furthermore, these specializations of the terminal limit group of $TRes$ can be extended to specializations of the f.p. completion of some ungraded resolution of the terminal limit group of $TRes$. Restrictions of generic points in the fibers that are associated with these specializations of the terminal limit group of $TRes$, to the variables (p, t) , that we denote, (p_n, t) , are not in the set $EAE(p, t)$ that is associated with the given AEAE set, $AEAE(p)$.
- (iii) with the resolution $TRes$, we associate (non-canonically) a finite collection of graded auxiliary resolutions, according to parts (1)-(9) of the first step of

the iterative procedure for the analysis of an AEAE set. These auxiliary resolutions are part of the output of this terminating procedure. The auxiliary resolutions have the same properties of the resolutions in a formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7), and in particular they can be extended to ungraded resolutions with f.p. completions.

Now we can apply the argument that was used to prove theorems 3.2, 4.1 and 4.5. We look at all the sequences of non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), that are not isomorphic to D_∞ , an associated sequence of tuples, $p_n \notin AEAE(p)$ over G_n , and witnesses t_n for p_n , i.e., a sequence of pairs $(p_n, t_n) \notin EAE(p, t)$ over G_n . From every such sequence we use our terminating iterative procedure, and extract a subsequence of the tuples, (still denoted) $\{(p_n, t_n)\}$, and a graded resolution, $TRes$, that has the properties (i)-(iii), and in particular, the subsequence of tuples, (still denoted) $\{(p_n, t_n)\}$, extend to specializations that factor through the resolution $TRes$, and to specializations of the f.p. completion of the ungraded resolutions that extends the graded resolution $TRes$. Furthermore, the restrictions of generic points in the fibers that are associated with the pairs, $\{(p_n, t_n)\}$, are not in the set $EAE(p, t)$ (i.e., generic values of t in these fibers are witnesses for $p_n \notin AEAE(p)$ over G_n).

The completion of each of the constructed resolutions, $TRes$, can be extended to an ungraded f.p. completion, and so are its associated (finitely many) auxiliary resolutions. Furthermore, the (finite collection of) covers of flexible quotients of the rigid or weakly solid factors of the terminal limit groups of the resolutions, $TRes$, and of the auxiliary resolutions that are associated with $TRes$, can all be embedded into (ungraded) resolutions with f.p. completions. Hence, we can define a linear order on this (countable) collection of graded resolutions ($TRes$), and their (non-canonically) associated auxiliary resolutions, and (finite collections of) covers of flexible quotients. By the same argument that was used in constructing the Makanin-Razborov diagram (theorem 26 in [Ja-Se]), there exists a finite subcollection of these graded resolutions that satisfy properties (1)-(3) of the theorem. \square

As we argued for AE and EAE sets, the existence of a finite collection of graded resolutions and their auxiliary resolutions, with the properties that are listed in theorem 5.2, allows one to reduce an AEAE sentence from free products to a sentence over its factors in a uniform way.

Theorem 5.3. *Let:*

$$\forall t \exists w \forall y \exists x \Sigma(x, y, w, t) = 1 \wedge \Psi(x, y, w, t) \neq 1$$

be an AEAE sentence over groups.

*Then there exists a coefficient-free sentence over free products, which is a (finite) disjunction of conjunctions of AEAE sentences, where each of these last AEAE sentences involves elliptic elements from the same factor, such that for every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , the original AEAE sentence over the free product G is a truth sentence, if and only if the sentence which is a (finite) disjunction of conjunctions of AEAE sentences over the factors, A^1, \dots, A^ℓ is a truth sentence.*

Proof: The argument that we use, that is based on theorem 5.2, is similar to the proof of theorem 4.6 (which is based on theorem 4.5). By theorem 5.2, with

a given AEAE set, it is possible to associate finitely many graded resolutions, $TRes_1, \dots, TRes_g$, that do all satisfy the properties that are listed in theorem 1.21, and in particular they can all be embedded into f.p. completions. With each graded resolution, $TRes_i$, we have associated finitely many auxiliary resolutions, according to parts (1)-(9) that appear in the first step of the iterative procedure for the analysis of an AEAE set. Note that all the (finitely many) graded resolutions that are associated with $TRes_i$ have the properties of the resolutions in the formal graded Makanin-Razborov diagram as listed in theorems 2.6 and 2.7, and in particular they can all be embedded into f.p. completions.

In case of an AEAE sentence, the same constructions that enable one to associate graded resolutions and their auxiliary resolutions with an AEAE set (i.e., the iterative procedure for the analysis of an AEAE set, and the proof of theorem 5.2), enables one to associate with a given AEAE sentence a (non-canonical) finite collection of (ungraded) resolutions (over free products) with f.p. completions, and with each resolution finitely many (ungraded) auxiliary resolutions that do all have f.p. completions, and these auxiliary resolutions have the same properties and they are constructed in the same way as the (graded) auxiliary resolutions that are constructed in the case of an AEAE set.

We (still) denote the (ungraded) resolutions that are associated with the given AEAE sentence, $TRes_1, \dots, TRes_g$. By theorems 5.2, 4.5 and 3.2, and propositions 4.2 and 4.3, the given AEAE sentence is false over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , if and only if there exists a specialization in G of the terminal limit group of a resolution $TRes_i$ (one of the resolutions, $TRes_1, \dots, TRes_g$), i.e., specializations of the elliptic factors of the terminal limit group of $TRes_i$ in the factors, A^1, \dots, A^ℓ , so that the specialization does not extend to a specialization of the terminal limit group of one of the auxiliary resolutions that are associated with the singular locus of $TRes_i$ (see proposition 4.2). This specialization of the terminal limit group of $TRes_i$ also satisfies:

- (1) it extends to the terminal limit group of an auxiliary resolution that is associated with a Root resolution of $TRes_i$ according to proposition 4.3 (possibly only the trivial roots), and does not extend to specializations of the terminal limit group of other Root resolutions, $RootTRes$, that are associated with higher order roots.
- (2) it may extend to specializations of the terminal limit groups of some of the auxiliary resolutions $WTRes$, that do not extend to specializations of the terminal limit groups that are associated with the singular locus of the associated graded resolution, $WRes$. Either the collection of the fibers that are associated with these specializations of the terminal limit groups of $WTRes$ do not form a covering closure of the fiber that is associated with the specialization of the terminal limit group of $TRes_i$, or (some of) these specializations can be extended to the specializations of the terminal limit groups of the following auxiliary resolutions (see pars (1)-(9) in the first step for the construction and the properties of the auxiliary resolutions that we refer to):
 - (i) each of the specializations of the terminal limit groups of $WTRes$ extends to a specialization of the terminal limit group of one of the auxiliary resolutions, $GRootWTRes$, and not to similar auxiliary resolutions that are associated with roots of higher orders.

- (ii) some of these specializations extend to specializations of the terminal limit groups of auxiliary resolutions, $YWTRes$, that do not extend to specializations of the terminal limit groups of auxiliary resolutions, $CollYWTRes$ and $SLYWTRes$.

Each such specialization extends to a specialization of the terminal limit group of an auxiliary resolution, $GRootYWTRes$, and not to the terminal limit groups of such resolutions that are associated with higher order roots.

- (iii) some of the specializations in (ii) can further extend to specializations of the terminal limit groups of auxiliary resolutions, $XYWTRes$, that do not further extend to specializations of the terminal limit groups of auxiliary resolutions, $SLXWTRes$, $CollXYWTRes$, or $\Psi XYWTRes$.
- (iv) the fibers that are associated with the specializations of the terminal limit groups of auxiliary resolutions $WTRes$, minus the fibers that are associated with their extensions to specializations of the terminal limit groups of auxiliary resolutions $YWTRes$, from which we take out fibers that are covered by fibers that are associated with further extensions to specializations of the terminal limit groups of auxiliary resolutions $XYWTRes$, that do not extend to terminal limit groups of auxiliary resolutions: $SLXWTRes$, $CollXYWTRes$, or $\Psi XYWTRes$, do not form a cover of the fiber that is associated with the original specialization of the graded resolution $TRes_i$.

Finally, by going over the finitely many possibilities for prescribed sets of auxiliary resolutions that satisfy part (iv), the existence of a specialization of the terminal limit group of one of the resolutions, $TRes_i$, that satisfies properties (1)-(2), is clearly a finite disjunction of finite conjunctions of EAEA sentences over the factors of G , A^1, \dots, A^ℓ , i.e., conditions (i)-(iv) can be easily written as a disjunction of conjunctions of such sentences in the factors by using all the constructed auxiliary resolutions. Hence, the given AEAE set is a disjunction of conjunctions of AEAE sentences over the factors A^1, \dots, A^ℓ . □

In analyzing AE and EAE sets, we have associated finitely many graded resolutions with these sets, and using them we were able to reduce an AE or an EAE sentence over free products to disjunctions of conjunctions of sentences over the factors. The graded resolutions that we constructed for analyzing AEAE sets, enable us to reduce not only AEAE (and EAEA) sentences over free products, but also AEAE (and EAEA) predicates. As we will see in the next section, this kind of reduction, or quantifier elimination, hold for arbitrary predicates (or definable sets) over free products, and can be viewed as a form of quantifier elimination over free products.

Theorem 5.4. *Let:*

$$AEAE(p) = \forall t \exists w \forall y \exists x \Sigma(x, y, w, t, p) = 1 \wedge \Psi(x, y, w, t, p) \neq 1$$

be an AEAE predicate over groups.

Then there exists a coefficient-free predicate over groups that are free products, that is composed from only 3 quantifiers over variables that take values in the ambient free product, and additional quantifiers over variables that take their values in the various factors of the free product, so that for every non-trivial free product,

$G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , the set $AEAE(p)$ over G can be defined by the following predicate over free products:

$$AEAE(p) = \exists u (\forall t \exists w \forall y \exists x) \forall v \exists s (\exists e)$$

$$(\Sigma_1(x, y, w, t, u, v, s, p) = 1 \wedge \Psi_1(x, y, w, t, u, v, s, p) \neq 1) \vee \dots \vee$$

$$(\Sigma_k(x, y, w, t, u, v, s, p) = 1 \wedge \Psi_k(x, y, w, t, u, v, s, p) \neq 1)$$

where the variables u, v, s take values in the ambient free product G , and the variables t, w, y, x, e take values in the factors A^1, \dots, A^ℓ .

Furthermore, an EAEA set, $EAEA(p)$, can be defined over every non-trivial free product $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , by the following predicate:

$$EAEA(p) = \exists u (\exists t \forall w \exists y \forall x) \forall v \exists s (\exists e)$$

$$(\Sigma_1(x, y, w, t, u, v, s, p) = 1 \wedge \Psi_1(x, y, w, t, u, v, s, p) \neq 1) \vee \dots \vee$$

$$(\Sigma_k(x, y, w, t, u, v, s, p) = 1 \wedge \Psi_k(x, y, w, t, u, v, s, p) \neq 1)$$

where (as in the AEAE case) the variables u, v, s take values in the ambient free product G , and the variables t, w, y, x, e take values in the factors A^1, \dots, A^ℓ .

Proof: Our argument for the analysis of AEAE and EAEA predicates combines theorem 5.3, that proves the reduction of sentences from the ambient free product to the factors, with theorem 5.2 that associates with an AEAE or EAEA predicates a finite collection of graded resolutions and their auxiliary resolutions that enable one to perform the reduction of sentences uniformly.

By theorem 5.2, with a given AEAE set, it is possible to associate finitely many graded resolutions, $TRes_1, \dots, TRes_g$, that do all satisfy the properties that are listed in theorem 1.21, and in particular they can all be embedded into (ungraded) f.p. completions. Furthermore, with the rigid or weakly strictly solid factor of the terminal limit group of each of the graded resolutions, $TRes_i$, we have associated finitely many (graded limit group) covers of its flexible quotient, and each of these covers embeds into an (ungraded) f.p. completion as well (see theorem 1.21).

With each graded resolution, $TRes_i$, we have associated finitely many auxiliary resolutions, according to parts (1)-(9) that appear in the first step of the iterative procedure for the analysis of an AEAE set. Note that all the (finitely many) graded resolutions that are associated with $TRes_i$ have the properties of the resolutions in the formal graded Makanin-Razborov diagram as listed in theorems 2.6 and 2.7, and in particular they can all be embedded into f.p. completions, and with the rigid or weakly solid factor of their terminal limit groups, there are associated finitely many covers of its flexible quotients, and these covers are embedded into (ungraded) f.p. completions.

Theorem 5.3 analyzes logically an AEAE sentence over free products, and shows how the graded resolutions, $TRes_i$, can be used to reduce (uniformly) such an AEAE sentence from the ambient free product to a (finite) disjunction of conjunctions of AEAE sentences over the factors of the free product.

For the analysis of sentences, we have used ungraded resolution over free products, that terminate in a free product of elliptic factors and possibly a free group. To analyze predicates we have to use graded resolutions, that terminate in a free product of a rigid or a weakly solid factor with (possibly) finitely many elliptic

factors and possibly a free factor. Hence, to apply the same analysis that was used in the analysis of AEAE sentences to analyze AEAE predicates, we need to further find a way to encode all the rigid or all the weakly strictly solid specializations of the rigid or weakly solid factors, and to encode the fact that two specializations belong to the same rigid or weakly strictly solid family of a rigid or a weakly solid limit group (over free products).

By theorem 1.14, for any given rigid limit group, $Rgd(x, p)$, and any given (finite) covers of its flexible quotients, there exist finitely many combinatorial systems of fractions (that depend only on the rigid limit group and its generating set), so that for every value p_0 of the parameter subgroup $\langle p \rangle$, we can associate at most one value of the fractions for each combinatorial system (these values depend on the value of the defining parameters p_0), so that every rigid specialization that is associated with p_0 can be expressed as a fixed word (that depends only on the combinatorial system, i.e., depends only on the group and its generating set), in the fractions (that take their values in the ambient free product), and in finitely many elliptic elements (see theorem 1.14 for the exact statements). By theorem 1.15 the same holds for almost shortest weakly strictly solid homomorphisms that are associated with the parameters value p_0 .

These finite collection of combinatorial systems that describe the structure of all the rigid or almost shortest weakly strictly solid homomorphisms of rigid and weakly solid limit groups, together with theorems 5.2 and 5.3, allow us to replace an AEAE (or EAEA) predicate with a predicate that uses only 3 quantifiers on elements from the ambient free product, as described in the statement of the theorem.

Given an AEAE set, $AEAE(p)$, we can write a predicate of the form:

$$AEAE(p) = \exists u (\forall t \exists w \forall y \exists x) \forall v \exists s (\exists e)$$

$$(\Sigma_1(x, y, w, t, u, v, s, p) = 1 \wedge \Psi_1(x, y, w, t, u, v, s, p) \neq 1) \vee \dots \vee$$

$$(\Sigma_k(x, y, w, t, u, v, s, p) = 1 \wedge \Psi_k(x, y, w, t, u, v, s, p) \neq 1)$$

where the existential variables u represent the fractions in all the possible system of fractions that are associated (by theorems 1.14 and 1.15) with the rigid or weakly solid factors of the terminal limit groups of the graded resolutions, $TRes_1, \dots, TRes_g$, and their auxiliary resolutions, that are associated with the set $AEAE(p)$ by theorem 5.2. The universal variables v and the existential variables s and e (the variables e are contained in a factor A^1, \dots, A^ℓ of the free product), enable us to guarantee that the values of the fractions u satisfy the conclusions of theorems 1.14 and 1.15, i.e., enable the covering of all the families of rigid or weakly strictly solid families that are associated with any given value of the defining parameters.

The variables t, w, y, x take their values in the various elliptic factors, A^1, \dots, A^ℓ , of the given free product G , and this part of the predicate is given by the description that appears in theorem 5.3, only that the elements t, w, y, x are either specializations of the elliptic factors of the terminal limit groups of the graded resolution, $TRes_1, \dots, TRes_g$, and of their auxiliary resolutions, or they are elliptic elements that appear in the combinatorial description of rigid and almost shortest weakly strictly solid specializations (given by theorems 1.14 and 1.15) of the rigid or weakly solid factors of these terminal limit groups. The universal variables v (that get values in the ambient free product) enable us to guarantee that the existential variables

w and x represent rigid or weakly strictly solid specializations. The existential variables s (that get values in the ambient free product) enable us to guarantee that the universal variables t and y represent rigid or weakly strictly solid specializations, and that two given specializations of a weakly solid limit group belong to the same family.

Once we have used the variables u, v, s (that take their values in the ambient free product) and the elliptic variables e , to go over all the rigid and weakly strictly solid specializations, the theorem for AEAE sets follows by the proof of theorem 5.3. The proof in the EAEA case, is naturally identical. □

§6. Definable sets and sentences over free products

In the previous 3 sections we used the iterative procedure for the analysis of an AE sentence over free groups [Se4], and the sieve procedure that was used for quantifier elimination over free groups [Se6], to analyze AE, EAE, and AEAE predicates and sentences. In particular, we showed that AEAE and EAEA sentences over free products can be reduced to a (finite) disjunction of conjunctions of sentences over the factors of the free product (theorem 5.3), and that every AEAE or EAEA set over free products can be defined by a predicate that contains only 3 quantifiers over the ambient free product, and additional quantifiers over elements in the various factors (theorem 5.4). These were conclusions of the existence of finitely many graded resolutions that satisfy the properties that are listed in theorem 1.21, and their (finitely many) auxiliary resolutions (that satisfy the properties of resolutions in the formal graded Makanin-Razborov diagram), so that a specialization p_0 of the defining parameters (free variables) is in the AEAE (EAEA) set if and only if a generic element in a fiber of one of these resolutions is a witness for that, and the existence of such a generic witness can be reduced to the terminal limit groups of these graded resolutions, and their auxiliary resolutions, and the condition on the terminal limit groups can be expressed by a predicate of the indicated form.

In this section we use the same techniques that were used to analyze AEAE and EAEA sets in section 5, to analyze general definable sets and sentences over free products. First we apply a finite induction process and associate (non-canonically) a finite collection of graded resolutions with any given definable set over free products, graded resolutions that have similar properties to the ones that are associated with an AEAE and EAEA sets (see theorem 5.2). The existence of finitely many graded resolutions with such properties allows one to deduce that every definable set over free products can be defined by a predicate that contains only 3 quantifiers over the ambient free product and finitely many quantifiers over elements in the factors of the free product. These resolutions also enable one to deduce that every sentence over free products can be reduced to a finite disjunction of finite conjunctions of sentences over the factors over the free product. As we show in the next section, these reductions have somewhat surprising (uniformity) corollaries for sentences over free products, and we are sure that they will find quite a few generalizations and further applications in the near future.

Let:

$$E(AE)^k(p) = \exists t \forall y_1 \exists x_1 \dots \forall y_k \exists x_k \Sigma(t, y_1, x_1, \dots, y_k, x_k, p) = 1 \wedge \\ \wedge \Psi(t, y_1, x_1, y_k, x_k, p) \neq 1$$

be a predicate (with $2k+1$ quantifiers) over groups. Our goal is to associate with this set a finite collection of graded resolutions that have the properties of the graded resolutions that are associated with an AEAE set according to theorem 5.2.

Theorem 6.1. *Let $E(AE)^k$ be a definable set over free products. Then there exist finitely many graded resolutions over free products (with respect to the parameter subgroup $\langle p \rangle$):*

$$DRes_1(z, x_k, y_k, \dots, x_1, y_1, t, p), \dots, DRes_g(z, x_k, y_k, \dots, x_1, y_1, t, p)$$

with the following properties:

- (1) *the resolutions, $DRes_i$, $i = 1, \dots, g$, satisfy the properties that are listed in theorem 1.21. In particular, with each of them there is an associated f.p. completion, into which the completion of them embeds.*
- (2) *with each of the graded resolutions, $DRes_i$, $i = 1, \dots, g$, we associate (non-canonically) a finite collection of auxiliary resolutions, that do all have the properties of resolutions in the formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7). These auxiliary resolutions are constructed from test sequences of $DRes_i$, precisely in the way we associated auxiliary resolutions with the graded resolutions, $TRes_i$, that are associated with an AEAE set (part (2) of theorem 5.2). These include auxiliary resolutions that are associated with the singular locus of $DRes_i$ according to proposition 4.2, that we denote $SLDRes$, and graded resolutions that are associated with possible roots of pegs in $DRes_i$, according to proposition 4.3, that we denote $GRootDRes$. Besides these resolutions, the auxiliary resolutions that are associated with $DRes_i$ are similar to the ones that are constructed in that case of an AEAE set in part (2) of theorem 5.2.*

We denote these auxiliary resolutions, Y_1DRes , X_1Y_1DRes , \dots , $X_1Y_1 \dots X_kY_kDRes$, $\Psi X_1Y_1 \dots X_kY_kDRes$, and their associated collapsed resolutions, $CollY_1DRes$, $CollX_1Y_1DRes$, \dots , $CollX_1Y_1 \dots X_kY_kDRes$, $Coll\Psi X_1Y_1 \dots X_kY_kDRes$, that indicate that certain specializations that are assumed to be rigid or weakly strictly solid, are non-rigid or non weakly strictly solid, and similar auxiliary resolutions that are associated with the singular locus of the corresponding resolutions, SLY_1DRes, \dots , and auxiliary resolutions that are associated with possible roots of pegs in the associated graded resolutions, $GRootY_1DRes, \dots$.

- (3) *let $G = A^1 * \dots * A^\ell$ be a non-trivial free product, which is not isomorphic to D_∞ . Let p_0 be a specialization of the parameters p in the free product G , for which $p_0 \in E(AE)^k(p)$ over the free product G . Then there exists an index i , $1 \leq i \leq g$, a specialization of the terminal limit group of $DRes_i$ (in G) that restricts to p_0 , that extends to a specialization of the f.p. completion into which the completion of $DRes_i$ embeds, does not extend to a specialization of an auxiliary resolution that is associated with the singular locus, and restricts to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of $DRes_i$, such that one of the following holds:*
 - (i) *the specialization of the terminal limit group of $DRes_i$ does not extend to specializations of the terminal limit group of auxiliary resolutions, Y_1DRes , that are associated with $DRes_i$.*

(ii) suppose that the specialization of the terminal limit group of $DRes_i$ extends to specializations of the terminal limit groups of some of the auxiliary resolutions, Y_1DRes . Then each such specialization of the terminal limit group of an auxiliary resolution, Y_1DRes , either extends to specializations of the terminal limit groups of auxiliary resolutions, $CollY_1DRes$ and SLY_1DRes , so that the corresponding fibers of these resolutions cover the fiber of Y_1DRes , or the specialization of the terminal limit group of Y_1DRes extends to a specialization of the terminal limit group of an auxiliary resolution, X_1Y_1DRes , that does not extend to specializations of the terminal limit groups of any auxiliary resolutions, $CollX_1Y_1DRes$, SLX_1Y_1DRes , and $Y_2X_1Y_1DRes$.

In this case, for each extension of the original specialization (of the terminal limit group of $DRes$) to the terminal limit group of Y_1DRes , the fiber that is associated with such extended specialization (and with Y_1DRes), is covered by the fibers that are associated with extended specializations to the terminal limit groups of auxiliary resolutions: $CollY_1DRes$, SLY_1DRes , and X_1Y_1DRes (that do not extend to specializations of the terminal limit groups of auxiliary resolutions: $CollX_1Y_1DRes$, SLX_1Y_1DRes , and $Y_2X_1Y_1DRes$).

(iii) suppose that the specialization of the terminal limit group of $DRes_i$ extends to specializations of the terminal limit groups of some of the auxiliary resolutions, Y_1DRes , and this extension does not satisfy part (ii) (i.e., the corresponding fibers are not covered by fibers of the auxiliary resolutions: $CollY_1DRes$, SLY_1DRes , and fibers of an auxiliary resolution X_1Y_1DRes that satisfy the properties that are listed in part (ii)).

In this case we continue the conditions on extensions of specializations iteratively. If we got to an extension of the specialization of the terminal limit group of $DRes_i$, to a specialization of the terminal limit group of the terminal limit group of an auxiliary resolution, $Y_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$, $j < k$, then every such specialization extends to specializations of the terminal limit groups of $CollY_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$, or of $SLY_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$, or to specializations of the terminal limit groups of auxiliary resolutions, $X_jY_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$, that do not extend to a specialization of the terminal limit group of $CollX_jY_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$ nor of

$$SLX_jY_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$$

nor of $Y_{j+1}X_jY_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$. Furthermore, the fibers that are associated with these extended specializations (and their associated auxiliary resolutions) cover the fiber that is associated with the specialization of the terminal limit group of $Y_jX_{j-1}Y_{j-1}\dots X_1Y_1DRes$.

In case we got to an extension of the specialization of the terminal limit group of $DRes_i$, to a specialization of the terminal limit group of the terminal limit group of an auxiliary resolution, $Y_kX_{k-1}Y_{k-1}\dots X_1Y_1DRes$, then every such specialization extends to a specialization of the terminal limit group of $CollY_kX_{k-1}Y_{k-1}\dots X_1Y_1DRes$ or $SLY_kX_{k-1}Y_{k-1}\dots X_1Y_1DRes$, or to a specialization of the terminal limit group of an auxiliary resolution, $X_kY_kX_{k-1}Y_{k-1}\dots X_1Y_1DRes$, that does not extend to a specialization of the terminal limit group of $CollX_kY_kX_{k-1}Y_{k-1}\dots X_1Y_1DRes$ nor

of $SLX_k Y_k X_{k-1} Y_{k-1} \dots X_1 Y_1 DRes$, nor to a specialization of the terminal limit group of an auxiliary resolution, $\Psi X_k Y_k X_{k-1} Y_{k-1} \dots X_1 Y_1 DRes$, that does not extend to a specialization of the terminal limit group of

$$Coll \Psi X_k Y_k X_{k-1} Y_{k-1} \dots X_1 Y_1 DRes$$

Furthermore, the fibers that are associated with these extended specializations (and their associated auxiliary resolutions) cover the fiber that is associated with the specialization of the terminal limit group of $Y_k X_{k-1} Y_{k-1} \dots X_1 Y_1 DRes$.

In other words, given an $E(AE)^k$ set, there exists a finite collection of graded resolutions (with completions that can be embedded into f.p. completions) over free products, with finitely many auxiliary resolutions (that have the properties of resolutions in the formal Makanin-Razborov diagram), such that the inclusion of a specialization of the parameters (free variables) in the $E(AE)^k$ set over any given non-trivial free product (which is not D_∞), can be demonstrated by a generic point in a disjunction of conjunctions of (fibers of) these resolutions and their auxiliary resolutions.

Proof: We prove theorem 6.1 by induction on the index k , that determines the number of quantifiers that are used in defining the set, $E(AE)^k(p)$. Our induction hypothesis is therefore that theorem 6.1 holds for $(AE)^k$ sets. Note that in section 3 and 5 we proved theorem 6.1 for AE and AEAE sets, hence, the induction hypothesis holds for $k = 1, 2$.

Having the induction hypothesis, we prove theorem 6.1 following our analysis of EAE and AEAE sets. We start with all the sequences of specializations of the tuple, (p, t) , $\{(p_n, t_n)\}$, that take values in non-trivial free products, $G_n = A^1 * \dots * A^\ell$ (possibly for varying ℓ), that are not isomorphic to D_∞ . We further assume that for every index n , $p_n \in E(AE)^k(p)$ over the free product, G_n , and that t_n is a witness for p_n , i.e., $(p_n, t_n) \in (AE)^k(p, t)$ over G_n .

Given such a sequence of pairs, $\{(p_n, t_n)\}$, we can pass to a subsequence that converges into a well-structured (even well-separated) graded resolution with respect to the parameter subgroup $\langle p \rangle$: $T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_s$, where T_s is a free product of a rigid or a solid group over free products (that contains the subgroup, $\langle p \rangle$), and (possibly) a free group and (possibly) finitely many elliptic factors. We denote this graded resolution $DRes$. The terminal limit group T_s in $DRes$ is f.g. but it may be infinitely presented.

Following our arguments from previous sections, we apply the construction that appears in theorem 1.21, and fix a sequence of approximating covers of T_s , that we denote, $DRes_m$. The approximating covers, $DRes_m$, satisfy the properties that are listed in theorem 1.21, and they converge to the resolution T_s , precisely like the approximating covers that we chose in analyzing EAE and AEAE sets in sections 4 and 5.

By the construction of approximating covers, with each approximating cover, $DRes_m$, there exists a subsequence of the original sequence of tuples, $\{(p_n, t_n)\}$, that factor through it. As in the analysis of an AEAE set in section 5, we first assume that there exists an approximating cover, $DRes_m$, for which there exists a subsequence of tuples, (still denoted) $\{(p_n, t_n)\}$, so that for each specialization of the terminal limit group of $DRes_m$ (in G_n), that is associated with a tuple (p_n, t_n) from the subsequence, a generic pair (i.e., a test sequence) in the fiber that

is associated with such a specialization of the terminal graded limit group satisfies $p_n \in E(AE)^k(p)$ (over G_n), and (a generic) t is a witness for that.

In this case with the approximating graded resolution, $DRes_m$, we associate finitely many auxiliary resolutions, that do all have the properties of resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7), in a similar way to what we did in the procedure for the analysis of an AEAE set in section 5. The auxiliary resolutions that are associated with the approximating cover, $DRes_m$, are constructed from test sequences of $DRes_m$ that have the following properties:

- (1) With $DRes_m$ we associate finitely many graded resolutions that are associated with its singular locus (according to proposition 4.2), that we denote $SLDRes$, and given all the auxiliary resolutions that are constructed below, a finite collection of graded Root resolutions (proposition 4.3), that we denote $GRootDRes$.
- (2) we start by looking at all the test sequences of $DRes_m$, that can be extended to specializations of the terminal limit group of one of the cover graded resolutions, Y_1Res , that are associated with the set, $(AE)^k(p, t)$, and to the f.p. completion into which this terminal limit group embeds. By the construction of the formal graded Makanin-Razborov diagram, with this collection of test sequences it is possible to associate finitely many graded resolutions that have the same properties as the resolutions in the formal Makanin-Razborov diagram (theorems 2.6 and 2.7). In particular, the completion of each of the constructed resolutions embeds into a f.p. completion. We denote the constructed resolutions, Y_1DRes .

We further associate with each of these resolutions, (collapse) auxiliary resolutions, $CollY_1DRes$, auxiliary resolutions that are associated with the singular locus of the resolutions Y_1Res , that we denote SLY_1DRes , and auxiliary resolutions that are associated with the graded Root resolutions of Y_1Res , that we denote $GRootY_1DRes$.

- (3) we look at test sequences that are constructed in part (2), that can be further extended to specializations of the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, Y_1Res (that are associated with the set $(AE)^k(p, t)$), and was constructed from test sequences that can be extended to the terminal limit group of one of the graded resolutions, X_1Y_1Res , that are associated with the set $E(AE)^{k-1}(p, t, y_1)$. We further require that the specialization of the terminal limit group of this auxiliary resolution can be extended to the f.p. completion that is associated with the auxiliary resolution, and the terminal limit group (of the auxiliary resolution) is embedded into it. Once again with this collection of sequences it is possible to associate finitely many graded resolutions that have the properties of the resolutions in a formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7). We denote the constructed resolutions, X_1Y_1DRes .

We further associate with each of these resolutions, (collapse) auxiliary resolutions, $CollX_1Y_1DRes$, auxiliary resolutions that are associated with the singular locus of the resolutions X_1Y_1Res , that we denote SLX_1Y_1DRes , and auxiliary resolutions that are associated with the graded Root resolutions of X_1Y_1Res , that we denote $GRootX_1Y_1DRes$.

- (4) we continue iteratively. For each index $j \leq k$, we look at all the test se-

quences that were looked at while constructing the auxiliary resolutions, $X_{j-1}Y_{j-1} \dots X_1Y_1DRes$, that can be extended to the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, Y_1Res , and one of the resolutions, $Y_jX_{j-1}Y_{j-1} \dots X_1Y_1Res$, that are associated with the set $(AE)^{k+1-j}$. We further require that the specialization of the terminal limit group of this auxiliary resolution can be extended to the f.p. completion that is associated with the auxiliary resolution, and the terminal limit group (of the auxiliary resolution) is embedded into it. With this collection of sequences we, once again, associate finitely many resolutions, in a construction that follows the construction of the formal graded Makanin-Razborov diagram (theorem 2.7). We denote the constructed resolutions, $Y_jX_{j-1}Y_{j-1} \dots X_1Y_1DRes$. With each such resolution we also associate finitely many collapse auxiliary resolutions, $CollY_jX_{j-1}Y_{j-1} \dots X_1Y_1DRes$, finitely many auxiliary resolutions that are associated with the singular locus of the corresponding resolutions, $SLY_jX_{j-1}Y_{j-1} \dots X_1Y_1DRes$, and finitely many auxiliary resolutions that are associated with all the corresponding graded Root resolutions, $GRootY_jX_{j-1}Y_{j-1} \dots X_1Y_1DRes$.

Similarly, we look at all the test sequences that were looked while constructing these last auxiliary resolutions that can be extended to the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, Y_1Res , and one of the auxiliary resolutions, $X_jY_j \dots X_1Y_1Res$, that are associated with the set $E(AE)^{k-j}$. We denote the constructed resolutions, $X_jY_j \dots X_1Y_1DRes$. With such a resolution we associate finitely many collapse auxiliary resolutions, $CollX_jY_j \dots X_1Y_1DRes$, auxiliary resolutions that are associated with the singular loci of the constructed resolutions, $SLX_jY_j \dots X_1Y_1DRes$, and auxiliary resolutions that are associated with graded Root resolutions, $GRootX_jY_j \dots X_1Y_1DRes$.

- (5) Finally, we look at all the test sequences that were looked at while constructing the auxiliary resolutions, $X_kY_k \dots X_1Y_1DRes$, that can be extended to the terminal limit group of one of the auxiliary resolutions that are associated with the graded resolution, Y_1Res , and one of the resolutions, $\Psi X_kY_k \dots X_1Y_1Res$, that are associated with the set (AE) . We denote the constructed resolutions, $\Psi X_kY_k \dots X_1Y_1DRes$. With each such resolution we also associate finitely many collapse auxiliary resolutions, $Coll\Psi X_kY_k \dots X_1Y_1DRes$.

Suppose that there is no approximating cover resolution, $DRes_m$, with a subsequence of the original sequence, $\{(p_n, t_n)\}$ (for which $p_n \in E(AE)^k(p)$ (over G_n), and t_n (a specialization of the universal variables t in G_n) is a witness for that (in G_n)), that factor through $DRes_m$, so that for each specialization of the terminal limit group of $DRes_m$ that is associated with a pair (p_n, t_n) from the subsequence, a generic pair in the fiber that is associated with such a specialization of the terminal graded limit group (which takes its values in G_n), satisfies $p_n \in E(AE)^k(p)$ (over G_n), and t (a generic value of t in the associated fiber) is a witness for that.

As in analyzing EAE and AEAE sets in sections 4 and 5, in this case we can associate with the original graded resolution, $DRes$, another graded resolution, of smaller complexity, which is obtained by applying the first step of the sieve procedure for the analysis of quotient resolutions [Se6]. Furthermore, a subsequence

of the original sequence of specializations, $\{(p_n, t_n)\}$, is guaranteed to have a subsequence that extends to specializations that converge into the obtained quotient resolution. This enables us to continue iteratively, in a similar way to the sieve procedure [Se6], and to the iterative procedure that was used to prove the equationality of Diophantine sets over free and hyperbolic groups in section 2 of [Se8].

With the resolution, $DRes$, we have associated a sequence of approximating cover resolutions, $DRes_m$, that satisfy the properties that are listed in theorem 1.21. In particular, for each index m , there is a subsequence of the original sequence of pairs, $\{(p_n, t_n)\}$, that factor through $DRes_m$. By our assumptions, for every index m , there is no subsequence of the pairs, $\{(p_n, t_n)\}$, that do factor through the approximating cover, $DRes_m$, for which for generic pair (p_n, t) in the fiber that contains the pair (p_n, t_n) (in the graded variety that is associated with $DRes_m$), $p_n \in E(AE)^k(p)$ over G_n , and the generic t (a test sequence in the fiber) is a witness for that, i.e., the generic pairs, $(p_n, t) \in (AE)^k(p, t)$ over G_n .

We construct a sequence of specializations over G_n , in the same way as what we did in analyzing an AEAE set in section 5 as follows. We go over the indices m , and given an index m we pick an index $n_m > m$, and a tuple, $(p_{n_m}, \hat{t}_{n_m}, (\tilde{y}_1)_{n_m}, \tilde{t}_{n_m})$, that takes its values in G_{n_m} , with the following properties:

- (1) (p_{n_m}, t_{n_m}) factors through the graded resolution, $DRes_m$, and \hat{t}_{n_m} is the specialization of the terminal limit group of the graded resolution, $DRes_m$, that contains the pair, (p_{n_m}, t_{n_m}) .
- (2) \tilde{t}_{n_m} is a specialization of the variables t , from the fiber that is associated with \hat{t}_{n_m} , i.e., the fiber that contains the pair (p_{n_m}, t_{n_m}) in the graded variety that is associated with $DRes_m$. Furthermore, for every index m , $(p_{n_m}, \tilde{t}_{n_m}) \notin (AE)^k(p, t)$ over G_n .
- (3) the sequence $\{(p_{n_m}, \tilde{t}_{n_m})\}$ is a graded test sequence that converges into the graded resolution, $DRes$.
- (4) the tuples $(p_{n_m}, \tilde{t}_{n_m}, (\tilde{y}_1)_{n_m})$, are specializations of the terminal limit group of one of the finitely many graded resolutions, Y_1Res (that are associated with the set $(AE)^k(p, t)$), that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group of Y_1Res , with respect to the parameter group $\langle p, t \rangle$. Furthermore, a generic value of the existential variables y_1 in the fiber of Y_1Res that is associated with the tuple, $(p_{n_m}, \tilde{t}_{n_m}, (\tilde{y}_1)_{n_m})$, is a witness that: $(p_{n_m}, \tilde{t}_{n_m}) \notin (AE)^k(p, t)$ over G_{n_m} .

From the sequence of tuples, $(p_{n_m}, \hat{t}_{n_m}, (\tilde{y}_1)_{n_m}, \tilde{t}_{n_m})$, we can extract a subsequence that converges into a closure of the graded resolution, $DRes$, that we have started with. We denote this closure, ExY_1DRes . Recall that we have assumed that the original sequence of pairs, $\{(p_n, t_n)\}$, satisfies $p_n \in E(AE)^k(p)$, and t_n is a witness for that (i.e., $(p_n, t_n) \in (AE)^k(p, t)$). On the other hand, in the sequence of tuples that we chose (in the fibers that are associated with a subsequence of the original sequence of specializations), a generic value of the existential variables y_1 in the fiber that is associated with the tuple, $(p_{n_m}, \tilde{t}_{n_m}, (\tilde{y}_1)_{n_m})$, is a witness that the pair $(p_{n_m}, \tilde{t}_{n_m}) \notin (AE)^k(p, t)$.

Given the closure, ExY_1DRes , we continue in a similar way to what we did in analyzing AEAE sets in section 4, a way that is adapted to sets with arbitrarily many quantifiers. The closure, ExY_1DRes , contains an (additional) specialization

of the variables y_1 , so that for a generic value of the variables t and y_1 , the value of the existential variables y_1 are witnesses that the pairs, $(p_n, t) \notin (AE)^k(p, t)$. Hence, we look at a sequence of approximating covers of the closure, ExY_1DRes , that we denote $\{ExY_1DRes_r\}$, that satisfy the properties that are listed in theorem 1.21. On the additional (existential) variables y_1 in the approximating covers, ExY_1DRes , we impose one of the (finitely many) possible collapse forms that are defined iteratively. These possible collapse forms are:

- (1) the additional specialization of the terminal limit group of the graded resolutions, Y_1Res , in ExY_1DRes_r , that is assumed to restrict to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of the terminal limit group of the graded resolution Y_1Res , is non-rigid or non weakly strictly solid. This forces a Diophantine condition on the specializations of ExY_1DRes_r , similar to the ones that are forced along the sieve procedure in [Se6].

Similarly, we impose the Diophantine condition that forces the additional specialization of the terminal limit group of Y_1Res to extend to a specialization of the terminal limit group of one of the resolutions, SLY_1Res , that are associated with the singular locus of Y_1Res .

- (2) to the specialization of the closure, ExY_1DRes_r , it is possible to add an additional specialization of the terminal graded limit group of one of the graded resolutions, X_1Y_1Res , that is associated by our induction hypothesis with the set, $(AE)^k(p, t)$, that restricts to a rigid or a weakly strictly solid specialization of the rigid or solid factor of that terminal graded limit group. If the extended specializations can not be extend to a specialization of a closure of ExY_1DRes_r , then we collect these extended specializations (for a subsequence of the original sequence of specializations, $\{(p_n, t_n)\}$), in a graded resolution that has smaller complexity than the graded resolution, $DRes$, according to the sieve procedure [Se6], or rather according to the iterative procedure that was used to prove the equationality of Diophantine sets in [Se9].
- (3) suppose that there is a test sequence of the graded resolution, ExY_1DRes , for which for each additional specialization of the terminal limit group of the resolution Y_1Res (that is associated with the set $(AE)^h(p, t)$), there is an extension to specializations of the terminal limit group of some auxiliary resolutions X_1Y_1Res (that are associated with the resolution Y_1Res and the set $(AE)^k(p, t)$), so that the associated fibers covers the corresponding fiber of Y_1Res .

In this case, we first look at all the test sequences of ExY_1DRes_r , for which the specialization of the variables (p, t, y_1) in ExY_1DRes_r , can be extended to a specialization of the terminal limit group of an auxiliary resolution that is associated with graded Root resolution of Y_1Res (and with the set $(AE)^k(p, t)$), and not to a specialization of an auxiliary resolution that is associated with graded Root resolutions of Y_1Res of higher order roots, and for which the specializations of the variables, (p, t, y_1) , can be extended to specializations of the terminal limit groups of a prescribed set of auxiliary resolutions, X_1Y_1Res and $Colly_1Res$, that are associated with the resolution Y_1Res (and with the set $(AE)^k(p, t)$), so that the corresponding fibers of these auxiliary resolutions cover the fiber of the resolution Y_1Res .

We further repeat the construction of the graded resolution, ExY_1DRes_r , and construct a closure of, ExY_1DRes_r , from test sequences that can be further extended to specializations of the terminal limit group of one of the auxiliary resolutions, $Y_2X_1Y_1Res$, that are associated with the set, $(AE)^k(p, t, y_1, x_1)$, that restrict to rigid or weakly strictly solid specializations of the rigid or solid factor in that graded limit group. We denote this constructed graded resolution, which is a closure of ExY_1DRes_r , $ExY_2X_1Y_1DRes_r$.

First, it may be that for a subsequence of the original sequence of specializations, the additional specialization (that is associated with the variables Y_2) does not restrict to a rigid or a weakly strictly solid specialization, or that this specialization extends to a specialization of the terminal limit group of one of the auxiliary resolutions that are associated with the singular locus of the associated auxiliary resolution $Y_2X_1Y_1Res$ (in a similar to what we did in part (1)), or that the specialization of the terminal limit group of the graded resolution Y_1Res extends to an associated graded Root resolution with higher order roots. In this case we impose a Diophantine condition similar to the ones that are imposed in part (1) on the constructed resolution, $ExY_2X_1Y_1DRes_r$, and as in part (1) we get a graded resolution with a smaller complexity according to the sieve procedure [Se6].

If we can not impose these Diophantine conditions, we further collect specializations that extend to specializations of the terminal limit groups of auxiliary resolutions, $X_2Y_2X_1Y_1Res$ (that restrict to rigid or weakly strictly solid specializations, and do not extend to the associated singular locus). If the extended specializations do not factor through a closure of ExY_1DRes_r , we obtained a graded resolution with smaller complexity than $DRes$.

If for a subsequence of the original specializations, the extended specializations do factor through a closure of $ExY_2X_1Y_1DRes_r$, we continue as in part (3), and collect test sequences that can be extended to specializations that are associated with the variables Y_3 .

- (4) we continue iteratively. The original sequence of specializations, $\{(p_n, t_n)\}$, are in the set, $(AE)^k(p, t)$, but generic points in the fibers of ExY_1DRes_r that are associated with these specializations are not in this set. Hence, after at most k steps we must end up with a graded resolution, that has a smaller complexity than the original graded resolution, $DRes$, and this graded resolution is obtained by either forcing an additional specialization that is associated with variables Y_j to be non-rigid or non weakly strictly solid or extendable to the specialization of the terminal limit group of an auxiliary resolution that is associated with the corresponding singular locus (a Diophantine condition), or it is obtained by collecting additional specializations that are associated with variables X_j that restrict to rigid or weakly strictly solid specializations, and do not factor through a closure of ExY_1DRes_r .

In all these cases we constructed a quotient resolution according to the first step of the sieve procedure [Se6], and pass to a subsequence of the specializations, $\{(p_n, t_n)\}$, that extend to specializations that converge into that quotient resolution. Note that by the construction of these quotient resolutions that appear in [Se6], the complexity of this quotient resolution, in light of the sieve procedure, is smaller

than the complexity of the graded resolution that we have started the first step with, $DRes$ (note that the obtained quotient resolution may be a proper closure of the resolution that we have started the first step with, i.e. the resolution $DRes$). However, there is a global bound on the orders of roots that are needed to be added to such closures, hence, the obtained quotient resolution can be a closure of the quotient resolution that we have started the first step with only finitely (boundedly) many steps. After these finitely many steps, the complexity of the obtained quotient resolution is strictly smaller).

We continue iteratively. At each step we first look at a sequence of approximating (cover) resolutions of the quotient resolution that was constructed in the previous step, where these approximating cover resolutions satisfy the properties that are listed in theorem 1.21, and in particular their completion can be embedded in a f.p. completion. If there exists an approximating cover resolution for which there exists a subsequence of pairs, (still denoted) $\{(p_n, t_n)\}$, so that for each specialization of the terminating limit group of the approximating cover resolution, that is associated with a pair (p_n, t_n) from the subsequence, a generic pair (i.e., a test sequence) in the fiber that is associated with such a specialization of the terminal graded limit group (which takes its values in G_n) forms a witness for $p_n \in E(AE)^k(p)$ (over G_n), we do what we did in this case in the first step (and associate with this approximating cover finitely many auxiliary resolutions).

If there is no such approximating cover resolution for the constructed quotient resolution, we associate with it a closure that is constructed from test sequences that can be extended to additional (new) rigid or weakly strictly solid specializations of the terminal limit groups of one of the graded resolutions Y_1Res , that are associated with the set $(AE)^k(p, t)$. We further associate a sequence of approximating covers with that closure, and force a collapse condition over the constructed approximating covers, according to parts (1)-(4). Finally, we apply the general step of the sieve procedure [Se6], and obtain a new quotient resolution, that has smaller complexity than the quotient resolution that we have started the current step with.

By an argument which combines the argument that guarantees the termination of the sieve procedure in [Se6], with the argument that implies the equationality of Diophantine sets over free groups, we obtain a termination of the this procedure.

Theorem 6.2. *The procedure for the analysis of an $E(AE)^k$ set over free products terminates after finitely many steps.*

Proof: The argument that we use is similar to the proofs of theorems 4.4 and 5.1 (termination of the procedures for the analysis of AEAE and EAE sets). At each step of the procedure, on the quotient resolution, $DRes$, that is analyzed in that step we impose one of finitely many restrictions. The first ones impose a non-trivial Diophantine condition on specializations of a closure of the resolution, $DRes$. The second type of restrictions, adds specializations of the terminal limit groups of one of the associated auxiliary resolutions that are associated with the graded resolution, Y_1Res (and with the set $(AE)^k(p, t)$), where these specializations restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group. We further require that the additional specializations do not factor through (or are not in the same (weak) strictly solid families as specializations of) closures that are built from test sequences of $DRes$ that can be extended to specializations of such terminal limit groups of auxiliary resolutions.

The argument that was used to prove the termination of the sieve procedure (theorem 22 in [Se6]), that was modified to work over free products in the proof of theorem 4.4, proves that restrictions of both types can occur in only finitely many steps. Hence, the iterative procedure for the analysis of an $E(AE)^k$ set terminates after finitely many steps. \square

So far we have shown that given a sequence, $\{(p_n, t_n)\}$, for which $p_n \in E(AE)^k(p)$ over G_n , and t_n is a witness for p_n , i.e., $(p_n, t_n) \in (AE)^k(p, t)$ over G_n , it is possible to extract a subsequence, (still denoted) $\{(p_n, t_n)\}$, and to construct a graded resolution, $DRes$, with the following properties:

- (i) the graded resolution $DRes$ satisfies the properties of a cover graded resolution that are listed in theorem 1.21.
- (ii) the specializations of the subsequence, $\{(p_n, t_n)\}$, extend to specializations that factor through the resolution, $DRes$, and to specializations of the f.p. completion of the ungraded resolution that extends the graded resolution, $DRes$ (see theorem 1.21). Hence, with each specialization from this subsequence of the tuples, $\{(p_n, t_n)\}$, specializations of the terminal limit group of the graded resolution, $DRes$, can be associated. Furthermore, these specializations of the terminal limit group of $DRes$ can be extended to specializations of the f.p. completion of some ungraded resolution of the terminal limit group of $DRes$. Restrictions of generic points (i.e., test sequences) in the fibers that are associated with these specializations of the terminal limit group of $DRes$, to the variables (p, t) , that we denote, (p_n, t) , are in the set $(AE)^k(p, t)$ that is associated with the given set, $E(AE)^k(p)$.
- (iii) with the resolution $DRes$, we associate (non-canonically) a finite collection of graded auxiliary resolutions, that are part of the output of the terminating iterative procedure for the analysis of a definable set, i.e., auxiliary resolutions of the same type as those that were associated with an approximating cover resolution, $DRes_m$, in the first step of the procedure, in case it terminates in the first step. These auxiliary resolutions have the same properties of the resolutions in a formal graded Makanin-Razborov diagram (theorems 2.6 and 2.7), and in particular they can be extended to ungraded resolutions with f.p. completions.

As in the proof of theorems 3.2, 4.1, 4.5 and 5.1, we look at all the sequences of non-trivial free products, $G_n = A_n^1 * \dots * A_n^\ell$ (possibly for varying $\ell > 1$), that are not isomorphic to D_∞ , an associated sequence of tuples, $p_n \in E(AE)^k(p)$ over G_n , and witnesses t_n for p_n , i.e., a sequence of pairs $(p_n, t_n) \in (AE)^k(p, t)$ over G_n . From every such sequence we use our terminating iterative procedure, and extract a subsequence of the tuples, (still denoted) $\{(p_n, t_n)\}$, and a graded resolution, $DRes$, that has the properties (i)-(iii), and in particular, the subsequence of tuples, (still denoted) $\{(p_n, t_n)\}$, extend to specializations that factor through the resolution $DRes$, and to specializations of the f.p. completion of the ungraded resolutions that extends the graded resolution $DRes$. Furthermore, the restrictions of generic points (test sequences) in the fibers that are associated with the pairs, $\{(p_n, t_n)\}$, are in the set $(AE)^k(p, t)$ (i.e., generic values of t in these fibers are witnesses for $p_n \in E(AE)^k(p)$ over G_n).

The completion of each of the constructed resolutions, $DRes$, can be extended

to an ungraded f.p. completion, and so are its associated (finitely many) auxiliary resolutions. Furthermore, the (finite collection of) covers of flexible quotients of the rigid or weakly solid factors of the terminal limit groups of the resolutions, $DRes$, and of the auxiliary resolutions that are associated with $DRes$, can all be embedded into (ungraded) resolutions with f.p. completions. Hence, we can define a linear order on this (countable) collection of graded resolutions ($DRes$), and their (non-canonically) associated auxiliary resolutions, and (finite collections of) covers of flexible quotients. By the same argument that was used in constructing the Makanin-Razborov diagram (theorem 26 in [Ja-Se]), there exists a finite subcollection of these graded resolutions that satisfy properties (1)-(3) of the theorem.

So far we have argued that given our induction hypothesis (the conclusion of theorem 6.1) on sets of the form, $(AE)^k$, the conclusion of theorem 6.1 follows for sets of the form $E(AE)^k$. Clearly, exactly the same argument proves the conclusion of theorem 6.1 for sets of the form $(AE)^{k+1}$, assuming what we already proved, i.e., the conclusion of theorem 6.1 for sets of the form $E(AE)^k$. Hence, we have completed the proof by induction, and theorem 6.1 follows for every definable set over free products. □

Theorem 6.1 associates finitely many graded resolutions with any given definable set over free products. As we argued for AE, EAE and AEAE sets, the existence of a finite collection of graded resolutions with the properties that are listed in theorem 6.1, allows one to reduce a general sentence in the language of groups from free products to a sentence over its factors in a uniform way.

Theorem 6.3. *Let Φ be a coefficient-free sentence over groups. Then there exists a coefficient-free sentence over free products, which is a (finite) disjunction of conjunctions of sentences over the factors of the free product, such that for every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , the original sentence over the free product G is a truth sentence, if and only if the sentence which is a (finite) disjunction of conjunctions of sentences over the factors A^1, \dots, A^ℓ is a truth sentence.*

Proof: The argument that we use, that is based on theorem 6.1, is similar to the proof of theorem 5.3 (which is based on theorem 5.2). By theorem 6.1, with a given coefficient-free definable set, it is possible to associate finitely many graded resolutions, $DRes_1, \dots, DRes_g$, that do all satisfy the properties that are listed in theorem 1.21, and with them we have associated finitely many auxiliary resolutions, according to parts (1)-(5) that appear in the first step of the iterative procedure for the analysis of a definable set. Note that all these auxiliary resolutions have the properties of the resolutions in the formal graded Makanin-Razborov diagram as listed in theorems 2.6 and 2.7, and in particular they can all be embedded into f.p. completions.

As we remarked in the EAE and AEAE cases in sections 4 and 5, in analyzing general sentences (rather than predicates), the same constructions that enable one to associate graded resolutions and their auxiliary resolutions with a definable set (i.e., the iterative procedure for the analysis of a definable set, and the proof of theorem 6.1), enable one to associate with a given sentence a (non-canonical) finite collection of (ungraded) resolutions (over free products) with f.p. completions, and with each resolution finitely many (ungraded) auxiliary resolutions that do all have

f.p. completions, and these auxiliary resolutions have the same properties and they are constructed in the same way as the (graded) auxiliary resolutions that are constructed in the case of a definable set.

We (still) denote the (ungraded) resolutions that are associated with the given AEAE sentence, $DRes_1, \dots, DRes_g$. By theorem 6.2, the given sentence is true over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , if and only if there exists a specialization in G of the terminal limit group of a resolution $DRes_i$ (one of the resolutions, $DRes_1, \dots, DRes_g$), i.e., specializations of the elliptic factors of the terminal limit group of $DRes_i$ in the factors, A^1, \dots, A^ℓ , so that the specialization does not extend to a specialization of the terminal limit group of one of the auxiliary resolutions that are associated with the singular locus of $DRes_i$ (see proposition 4.2). This specialization of the terminal limit group of $DRes_i$ also satisfies:

- (1) it extends to the terminal limit group of an auxiliary resolution that is associated with a Root resolution of $DRes_i$ according to proposition 4.3 (possibly only the trivial roots), and does not extend to specializations of the terminal limit group of other Root resolutions, $RootDRes$, that are associated with higher order roots.
- (2) it may extend to specializations of the terminal limit groups of some of the auxiliary resolutions Y_1DRes , that do not extend to specializations of the terminal limit groups of auxiliary resolutions, SLY_1DRes , that are associated with the singular locus of the associated graded resolution, Y_1Res , nor to the terminal of a collapse auxiliary resolution, $CollY_1DRes$. Either the collection of the fibers that are associated with these specializations of the terminal limit groups of Y_1DRes do not form a covering closure of the fiber that is associated with the specialization of the terminal limit group of $DRes_i$, or (some of) these specializations can be extended to the specializations of the terminal limit groups of the following auxiliary resolutions (see pars (1)-(5) in the first step of the iterative procedure for the analysis of a definable set for the construction and the properties of the auxiliary resolutions that we refer to).
- (3) each of the specializations of the terminal limit groups of Y_1DRes extends to a specialization of the terminal limit group of one of the auxiliary resolutions, $RootY_1DRes$, and not to similar auxiliary resolutions that are associated with roots of higher orders.
- (4) some of the specializations that extend to specializations of the terminal limit groups of the auxiliary resolutions, Y_1DRes , extend further to specializations of the terminal limit groups of auxiliary resolutions, X_1Y_1DRes , that do not extend to specializations of the terminal limit groups of auxiliary resolutions, $CollX_1Y_1DRes$ and SLX_1Y_1DRes .

The fibers that are associated with the specializations of the terminal limit groups of the auxiliary resolutions, Y_1DRes (and not to SLY_1DRes nor $CollY_1DRes$), minus the fibers that are associated with specializations that can be further extended to specializations of the terminal limit groups of auxiliary resolutions, X_1Y_1DRes (and not to SLX_1Y_1DRes nor $CollX_1Y_1DRes$), do not form a covering closure of the fiber that is associated with the specialization of the terminal limit group of $DRes$.

- (5) if none of the specializations of the terminal limit groups of the auxiliary

resolutions, X_1Y_1DRes , that can not be extended to specializations of the terminal limit groups of SLX_1Y_1DRes nor $CollX_1Y_1DRes$, can be further extended to specializations of the terminal limit groups of auxiliary resolutions, $Y_2X_1Y_1DRes$, that can not be further extended to $Sly_2X_1Y_1DRes$ nor $Colly_2X_1Y_1DRes$, we are done.

If some of these specializations of X_1Y_1DRes do extend to $Y_2X_1Y_1DRes$ we do the following. Each such specialization of the terminal limit group of X_1Y_1DRes extends to a specialization of the terminal limit group of an auxiliary resolution, $GRootX_1Y_1DRes$, and not to the terminal limit groups of such resolutions that are associated with higher order roots.

If the fibers that are associated with specializations of (the terminal limit groups of) the auxiliary resolutions, Y_1DRes , minus the fibers that are associated with specializations of X_1Y_1DRes from which we take out fibers that are associated with specializations of $Y_2X_1Y_1DRes$, do not form a covering closure of the fiber that is associated with the specialization of the terminal limit group of $DRes$ we are done.

- (6) otherwise, we continue iteratively. At each step, we first require that the specialization of the corresponding terminal limit groups can be extended to some associated graded Root auxiliary resolutions, and to such auxiliary resolutions that are associated with higher order roots.

Then we check if the inclusion exclusion combination of fibers that are associated with the auxiliary resolutions that are associated with existential and universal variables do cover the fiber that is associated with the (original) specialization of the terminal limit group of $DRes$. If they do not cover the fiber of $DRes$ we are done. Otherwise we continue to the next auxiliary resolutions. After finitely many such steps (that depend only on the number of quantifiers in the original sentence) we have exhausted all the possibilities for the sentence to be a true sentence.

Finally, the given sentence is a true sentence over a non-trivial free group, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , if and only if there exists a specialization of the terminal limit group of $DRes$, that satisfies one of the possibilities that are described in parts (1)-(6). The union of these finitely many possibilities can be expressed as a finite disjunction of finite conjunctions of sentences over the factors, A^1, \dots, A^ℓ .

□

In analyzing AEAE sets in section 5, we were able to reduce not only AEAE (and EAEA) sentences over free products, but also AEAE (and EAEA) predicates. The existence of graded resolutions with the properties that are listed in theorem 6.1, enables us to get a similar reduction for arbitrary predicates over free products.

Theorem 6.4. *Let Φ be an arbitrary coefficient-free predicate over groups. Then there exists a coefficient-free predicate over groups that are free products, that is composed from only 3 quantifiers over variables that take values in the ambient free product, and additional quantifiers over variables that take their values in the various factors of the free product, so that for every non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not isomorphic to D_∞ , the set $\Phi(p)$ over G can be defined by the following predicate over free products:*

$$\Phi(p) = \exists u (\Theta(e_1)) \quad \forall v \exists s (\exists e_2)$$

$$(\Sigma_1(e_1, e_2, u, v, s, p) = 1 \wedge \Psi_1(e_1, e_2, u, v, s, p) \neq 1) \vee \dots \vee \\ (\Sigma_\ell(e_1, e_2, u, v, s, p) = 1 \wedge \Psi_\ell(e_1, e_2, u, v, s, p) \neq 1)$$

where the variables u, v, s take values in the ambient free product G , and the variables e_1, e_2 take values in the factors A^1, \dots, A^ℓ , and Θ is a predicate over elements from the factors A^1, \dots, A^ℓ .

Proof: Given theorems 6.1 and 6.3, the statement of the theorem follows by the same argument that was used to prove theorem 5.4 (that is based on theorems 5.2 and 5.3).

The existential variables u represent the fractions in all the possible systems of fractions that are associated (by theorems 1.14 and 1.15) with the rigid or weakly solid factors of the terminal limit groups of the graded resolutions, $DRes_1, \dots, DRes_g$, and their auxiliary resolutions, that are associated with the given definable set by theorem 6.1. The universal variables v and the existential variables s and e_2 (the variables e_2 are contained in the various factor sA^1, \dots, A^ℓ of the free product), enable us to guarantee that the values of the fractions u satisfy the conclusions of theorems 1.14 and 1.15, i.e., enable the covering of all the families of rigid or weakly strictly solid families that are associated with any given value of the defining parameters.

The variables e_1 take their values in the various elliptic factors, A^1, \dots, A^ℓ , of the given free product G , and this part of the predicate is given by the description that appears in theorem 6.3, only that the elements e_1 are either specializations of the elliptic factors of the terminal limit groups of the graded resolution, $DRes_1, \dots, DRes_g$, and of their auxiliary resolutions, or they are elliptic elements that appear in the combinatorial description of rigid and almost shortest weakly strictly solid specializations (given by theorems 1.14 and 1.15) of the rigid or weakly solid factors of these terminal limit groups. The universal variables v (that get values in the ambient free product) enable us to guarantee that the existential variables among the variables e_1 that are associated with rigid or weakly strictly solid specializations are indeed part of such specializations. The existential variables s (that get values in the ambient free product) enable us to guarantee that the universal variables among the elliptic variables e_1 that represent rigid or weakly strictly solid specializations are indeed part of such specializations, and that two given specializations of a weakly solid limit group belong to the same family.

Once we have used the variables u, v, s (that take their values in the ambient free product) and the elliptic variables e_1, e_2 , to go over all the rigid and weakly strictly solid specializations, the theorem follows by the proof of theorem 6.3. □

§7. Basic properties of the first order theory of free products

In the previous section we associated a finite collection of graded resolutions with any given predicate (definable set) over free products (theorem 6.1). We used these graded resolutions to reduce a general sentence over free products to a (finite) disjunction of conjunctions of sentences over the factors of the free product (theorem 6.3), and to get a form of "quantifier elimination", i.e. a reduction of a predicate over free products to a predicate that contains only 3 quantifiers over the ambient free product and finitely many additional quantifiers over the factors of the free product (theorem 6.4).

The (uniform) reduction of sentences from a free product to its factors has fairly immediate corollaries for the first order theory of free product of groups. We start with a positive answer to a well known problem of R. Vaught (cf. [Fe-Va]).

Theorem 7.1. *Let A_1, B_1, A_2, B_2 be groups. Suppose that A_1 is elementarily equivalent to A_2 , and B_1 is elementarily equivalent to B_2 . Then $A_1 * B_1$ is elementarily equivalent to $A_2 * B_2$.*

Proof: If $A_1 * B_1$ is isomorphic to D_∞ , and the free product is non-trivial, then both A_1 and B_1 are isomorphic to Z_2 . Hence, A_2 and B_2 are isomorphic to Z_2 as well, so if $A_1 * B_1$ is a non-trivial free product that is isomorphic to D_∞ then $A_2 * B_2$ is isomorphic to D_∞ as well, and the theorem follows in this case.

Suppose that $A_1 * B_1$ is a non-trivial free product that is not isomorphic to D_∞ . Let Φ be a coefficient free sentence over groups. By theorem 6.3, the sentence Φ is a truth sentence over a non-trivial free product $U * V$ that is not isomorphic to D_∞ , if and only if a (finite) disjunction of conjunctions of sentences over the factors U and V , that we denote α , is a truth sentence.

Since A_1 is elementarily equivalent to A_2 , and B_1 is elementarily equivalent to B_2 , the sentence α is a truth sentence over the factors, A_1, B_1 , if and only if it is a truth sentence over the factors, A_2, B_2 . Therefore, Φ , is truth over $A_1 * B_1$, if and only if it is truth over $A_2 * B_2$. □

The existence of graded resolutions that are associated with a given sentence over free products enables one to prove the following theorem, that implies Tarski's problem for free groups.

Theorem 7.2. *Let A, B be non-trivial groups, and suppose that either A or B is not Z_2 . Let F be a (possibly cyclic) free group. Then $A * B$ is elementarily equivalent to $A * B * F$.*

Proof: Since by Tarski's problem [Se7], all the non-abelian free groups are elementarily equivalent, by theorem 7.1 we may assume that F is a f.g. free group. With $A * B$ we have associated a Bass-Serre tree, T_1 , that corresponds to the graph of groups that contains two vertex groups, A and B , and an edge (with trivial stabilizer) that connects them. With $A * B * F$ we associate a Bass-Serre tree, T_2 , that corresponds to a graph of groups that contains 3 vertices, one with trivial stabilizer and two with stabilizers A and B that are connected by edges with trivial stabilizer to the vertex group with trivial stabilizer. On the vertex with trivial stabilizer we further place $m \geq 1$ loops, if the free group F is isomorphic to F_m . In particular, in both free products, $A * B$ and $A * B * F$, the elliptic elements are (only) those that can be conjugated into A or B .

Let $\Sigma(x) = 1$ be a (finite, coefficient-free) system of equations. Then, by construction, every (non-canonical) Makanin-Razborov diagram of Σ over the collection of all free products, $A * B$, is a Makanin-Razborov diagram of Σ over the collection of all the free products, $A * B * F$. If $\Theta(x, p) = 1$ is a coefficient-free system of equations with parameters, p , then by construction, every (non-canonical) graded Makanin-Razborov diagram of Θ (that satisfy the properties that are listed in theorem 1.22) over the collection of free products, $A * B$, is a graded Makanin-Razborov diagram of Θ over the entire collection of free products $A * B * F$. This means, in particular, that a graded limit group over the collection, $A * B$, is rigid or weakly

solid, if and only if it is rigid or weakly solid over the collection, $A * B * F$, and the same holds for flexible quotients of a rigid or a (weakly) solid limit group.

$A * B$ naturally embeds (as a factor) in $A * B * F$, and hence each element $y_0 \in A * B$ can be naturally viewed as an element in $A * B * F$. Let:

$$\forall y \exists x \Sigma(x, y) = 1 \wedge \Psi(x, y) \neq 1$$

be a sentence over groups, and let $A * B$ be a non-trivial free product that is not isomorphic to D_∞ . Suppose that $y_0 \in A * B$ is a specialization of the universal variables y , for which there is no specialization $x_0 \in A * B$ of the existential variables x , so that $\Sigma(x_0, y_0) = 1$ and $\Psi(x_0, y_0) \neq 1$ (in $A * B$). Then for y_0 viewed as an element of $A * B * F$, there is no specialization $x_1 \in A * B * F$ of the existential variables x , so that $\Sigma(x_1, y_0) = 1$ and $\Psi(x_1, y_0) \neq 1$ (in $A * B * F$).

Let $y_1 \in A * B * F$ be a specialization of the universal variables y , for which there is no $x_1 \in A * B * F$, such that $\Sigma(x_1, y_1) = 1$ and $\Psi(x_1, y_1) \neq 1$ (in $A * B * F$). Then there exists a sequence of retractions: $\tau_n : A * B * F \rightarrow A * B$, that maps the subgroup $A * B$ of $A * B * F$ identically onto $A * B$, and maps a fixed generators of F into tuples of elements that form a test sequence in $A * B$ (when n grows to infinity), so that for every index n , there is no element $x_n \in A * B$, for which: $\Sigma(x_n, \tau_n(y_1)) = 1$ and $\Psi(x_n, \tau_n(y_1)) \neq 1$ (in $A * B$).

Hence, given a coefficient free AE sentence, a specialization of the universal variables y in $A * B$ that is a witness for the failure of the sentence over $A * B$, is also a witness for the failure of the sentence over $A * B * F$, and with a specialization of the universal variables y in $A * B * F$ that is a witness for the failure of the sentence over $A * B * F$, it is possible to associate a (test) sequence of specializations in $A * B$, that are all witnesses for the failure of the sentence over $A * B$, and these specializations (in $A * B$) "converge" into the specialization of the variables y in $A * B * F$.

This fact, and the identity between ungraded, graded, and (graded) formal Makanin-Razborov diagrams over the collection of non-trivial free products, $A * B$ and $A * B * F$, imply that the (non-canonical) finite collection of (ungraded) resolutions, $Res_1(z, y), \dots, Res_d(z, y)$, and their finite collection of auxiliary resolutions, that are associated in theorem 3.1 with a given coefficient-free sentence, and the entire collection of non-trivial free products $A * B$ (that are not isomorphic to D_∞), can be taken to be the collection of ungraded resolutions and their auxiliary resolutions that is associated (by theorem 3.1) with the same coefficient free sentence, and all the non-trivial free products $A * B * F$. This implies that a coefficient-free AE sentence is a truth sentence over a non-trivial free product, $A * B$, that is not isomorphic to D_∞ , if and only if it is a truth sentence over $A * B * F$.

By the same arguments, given a coefficient-free AE set, $AE(p)$, the (non-canonical) finite collection of graded resolutions and their auxiliary resolutions, that is associated with the set $AE(p)$ and the entire collection of free products $A * B$, in theorem 4.1, can be taken to be the finite collection of graded resolutions and their auxiliary resolutions that is associated (by theorem 4.1) with the given AE set $AE(p)$ and the collection of free products, $A * B * F$.

We continue by induction, similar to the induction that was used in proving theorem 6.1 for general definable sets over free products. We argued that the finite collection of graded resolutions and their auxiliary resolutions that are associated with a given coefficient-free AE set can be taken to be the same over the collection

of non-trivial free products $A * B$ (that are not isomorphic to D_∞), and over the collection of non-trivial free products, $A * B * F$. Hence, given a coefficient-free EAE set, the finite collection of graded resolutions and their auxiliary resolutions that are associated with it (by theorem 4.5) over the entire collection of non-trivial free products, $A * B$, that are not isomorphic to D_∞ , can be taken to be the finite collection of such graded resolutions and auxiliary resolutions that are associated by the same theorem with the collection of non-trivial free products, $A * B * F$. Continuing inductively, we obtain the same conclusion for the finite collection of graded resolutions and their associated auxiliary resolutions that is associated with a given coefficient-free definable set (by theorem 6.1) over the collection of non-trivial free products, $A * B$, and over the collection of non-trivial free products, $A * B * F$.

Theorem 6.3 uses the finite collection of graded resolutions and their auxiliary resolutions that is associated with a coefficient-free definable set, to reduce a given coefficient free sentence from a non-trivial free product to a disjunction of conjunctions of sentences over its factors.

The finite collection of graded resolutions and their auxiliary resolutions that is associated with a given coefficient-free definable set, can be taken to be identical over the entire collection of non-trivial free products, $A * B$, and over the entire collection of free products, $A * B * F$. Hence, we may apply the proof of theorem 6.3, and deduce that a given coefficient-free sentence over the entire collection of non-trivial free products, $A * B$, that are not isomorphic to D_∞ , and over the entire collection of non-trivial free products, $A * B * F$, reduces to the same (finite) disjunction of conjunctions of sentences over the factors A and B . Hence, a given coefficient-free sentence is a truth sentence over a given non-trivial free product, $A * B$, that is not isomorphic to D_∞ , if and only if it is a truth sentence over $A * B * F$. Therefore, every such non-trivial free product $A * B$ is elementarily equivalent to $A * B * F$. □

Note that Tarski's problem for free groups follows if we take A and B to be isomorphic to Z in the statement of theorem 7.2. Also, note that exactly the same argument proves that a non-trivial free product, $A * B$, that is not isomorphic to D_∞ , is elementarily equivalent to a tower over $A * B$, i.e., a completion that has in its bottom level a free product of the form, $A * B * F$, where F is a (possibly trivial) free group (cf. theorem 7 in [Se7] in the free group case, and theorems 7.6 and 7.10 in [Se8] in the torsion-free hyperbolic analogue).

Theorem 7.1 implies that for every group G that is a non-trivial free product, and is not the infinite dihedral group, G is elementarily equivalent to $G * F$ (where F is a free group). In [Se8] it is proved that this holds for every non-elementary (torsion-free) hyperbolic group. By the combination of theorems 7.1 and 7.2, the collection of groups G for which G is elementarily equivalent to $G * F$ is an elementary class, i.e., if G is elementarily equivalent to $G * F$, and H is elementarily equivalent to G , then H is elementarily equivalent to $H * F$. It is then natural to ask what are the properties of groups in this elementary class.

Question. *What are the (algebraic, first order) properties of groups G for which G is elementarily equivalent to $G * F$?*

Other rather straightforward corollaries of theorems 6.1 and 6.3, are uniform

properties of sentences over free products.

Theorem 7.3. *Let Φ be a coefficient free sentence over groups. There exists an integer, $k(\Phi)$, so that for every group, H , Φ is a truth sentence over $H_1 * \dots * H_{k(\Phi)}$, $H_i \simeq H$, if and only if Φ is a truth sentence over $H_1 * \dots * H_n$, $H_i \simeq H$, for every $n \geq k(\Phi)$.*

Proof: By theorem 6.1, with the coefficient free sentence, Φ , it is possible to associate finitely many (ungraded) resolutions (over free products), $DRes_1, \dots, DRes_g$, and with each of them finitely many auxiliary resolutions, so that the sentence Φ is a truth sentence over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not D_∞ , if and only if there exist (or there do not exist) specializations of the terminal limit groups of the resolutions, $DRes_1, \dots, DRes_g$, with properties that are listed in the statement of the theorem.

Note that the finite collection of resolutions, $DRes_1, \dots, DRes_\ell$, is universal (although it is not canonical), i.e., it does not depend on the free product G , nor on the number of factors in the free product G . In theorem 6.3, we used the existence of this finite collection of resolutions, to reduce the coefficient-free sentence Φ to a disjunction of conjunctions of sentences in the factors of the free product G , A^1, \dots, A^ℓ . This disjunction of conjunctions of sentences does depend on the number of factors, even though the resolutions and the auxiliary resolutions that are associated with Φ by theorem 6.1, do not (depend on the number of factors).

For each subset I that contains a resolution, $DRes_i$, and a subset of its (finitely many) associated auxiliary resolutions, we set t_I to be the sum of the numbers of elliptic factors in the terminal limit groups of all the resolutions in the subset I ($DRes_i$ and the subset of its associated auxiliary resolutions). Clearly, there are finitely many such subsets I . We set $k(\Phi)$ to be the maximum between the sum of the numbers t_I over all the possible subsets I , and the number 3 (to guarantee that the corresponding free product is not D_∞ , in case G is non-trivial).

Since the free products we are looking at are free products of the same group G , theorem 6.1 implies that Φ is a truth sentence over an iterated free product of G with itself $k(\Phi)$ times, if and only if it is a truth sentence over an iterated free product of G with itself a larger number of times. □

Note that the integer $k(\Phi)$ depends on the coefficient free sentence, Φ , but it does not depend on the group, G . It is easy to see that $k(\Phi)$ can not be chosen to be a universal constant, e.g., we can take Φ_m to be a sentence that specifies if the number of conjugacy classes of involutions in the group is at least m . For such a sentence, Φ_m , $k(\Phi_m) = m$.

Theorem 7.3 can be further strengthened for sequences of groups. Let Φ be a coefficient free sentence over groups. Given any sequence of groups, G_1, G_2, \dots , we set $M_1 = G_1$, $M_2 = G_1 * G_2$, $M_3 = G_1 * G_2 * G_3$, and so on. The sentence Φ may be truth or false on any of the groups (free products) M_i , $i = 1, \dots$. Here one can (clearly) not guarantee that the sentence Φ is constantly truth or constantly false staring at a bounded index (of the M_i 's). However, one can prove the following.

Theorem 7.4. *There exists an integer $c(\Phi)$, so that for every sequence of groups, G_1, G_2, \dots , the sentence Φ over the sequence of groups, $M_1 = G_1, M_2 = G_1 * G_2, \dots$ may change signs (from truth to false or vice versa) at most $c(\Phi)$ times.*

Proof: The argument that we use is similar to the proof of theorem 7.3. By theorem 6.1, with the coefficient free sentence, Φ , it is possible to associate finitely many (ungraded) resolutions, and with each of them finitely many auxiliary resolutions, so that the sentence Φ is a truth sentence over a non-trivial free product, $G = A^1 * \dots * A^\ell$, that is not D_∞ , if and only if there exist (or there do not exist) specializations of the terminal limit groups of the given list of resolutions, with properties that are listed in the statement of the theorem.

Once again, we note that the collection of resolutions that is associated (by theorem 6.1) with the given coefficient free sentence Φ , and their associated finite collections of auxiliary resolutions, are universal, which means that they are good for all the non-trivial free factors that are not isomorphic to D_∞ , and they do not depend on the number of factors in such a free product. In theorem 6.3, this finite collection of resolutions is used to reduce the coefficient-free sentence Φ to a disjunction of conjunctions of sentences in the factors of the free product. This disjunction of conjunctions of sentences does depend on the number of factors, even though the resolutions and the auxiliary resolutions that are associated with Φ by theorem 6.1, do not (depend on the number of factors).

The (finitely many) resolutions and their auxiliary resolutions, that are associated with the coefficient-free sentence Φ by theorem 6.1, allows one to reduce the sentence Φ into boundedly many sentences on the various factors of a given non-trivial free product, that is not isomorphic to D_∞ , where the given sentence Φ is true or false (over the ambient free product) if and only if a (finite) disjunction of conjunctions of these sentences over the factors is true or false (see theorem 6.3). The length of this disjunction of conjunctions depend on the number of factors in the given free product, but the number of distinct (coefficient-free) sentences over the various factors is uniformly bounded, and depend only on the resolutions that are associated with Φ by theorem 6.1, and on their auxiliary resolutions.

We set $selp$ to be the sum of the numbers of elliptic factors in the terminal limit groups of all the (finitely many) resolutions (and auxiliary resolutions) that are associated with the coefficient free sentence Φ by theorem 6.1. By the proof of theorem 6.3, the number of distinct coefficient-free sentences that are defined over the factors of a given non-trivial free product, such that the given sentence Φ is equivalent to a disjunction of conjunctions of these sentences over the various factors of the given non-trivial free product, is bounded by the number of subsets of a set of size $selp$, i.e, it is bounded by 2^{selp} .

We define the *state* of a given factor of a free product, to be the subcollection of distinct coefficient-free sentences, that are defined over the factors of a free product, and are associated with Φ by theorems 6.1 and 6.3, that are truth sentences over this given factor. Clearly, there are at most $2^{2^{selp}}$ possible states for factors of a free product.

Let G_1, G_2, \dots be a sequence of groups, with the corresponding free products: $M_1 = G_1, M_2 = G_1 * G_2, \dots$. By the proof of theorem 6.3, the question whether the given coefficient-free sentence Φ is true or false depends only on the states of the factors, G_1, \dots, G_n . Furthermore, to determine if Φ is true or false, it suffices to know for every possible factor how many factors do have this state, and for every possible state it suffices to know the number of factors that have this state only up to $selp$, as a larger number will make no difference for the question of Φ being true or false (i.e., for each state it suffices to know the minimum between the actual

number of factors that have this state and the number $selp$).

Hence, with each free product M_n from the sequence, we can associate a tuple of at most $2^{2^{selp}}$ non-negative integers that are all bounded by $selp$. The sequence of tuples that are associated with the free products, $\{M_n\}$, can only increase (in the lexicographical order), and Φ may change from true to false or vice versa, only when the tuple changes (increases). Hence, Φ may change from true to false or vice versa over the sequence of free factors, $\{M_n\}$, at most at $selp \cdot 2^{2^{selp}}$ indices, and the theorem follows. □

§8. Stability

In the previous section we deduced several basic first order properties of free products from the reduction of sentences from a free product to its factors (theorem 6.3), and from the association of finitely many resolutions and their auxiliary resolutions with any given sentence, that enables this reduction (theorem 6.1). This reduction, and the associated resolutions, enable one to prove that a free product inherits certain first order properties from its factors. In this section we use the scheme of argument that was used to prove the stability of free and hyperbolic groups [Se10], to prove that stability is inherited by a free product from its factors. This question about the possibility to lift stability from the factors to a free product was brought to our attention by Eric Jaligot, and was the motivation for our entire work on free products.

Theorem 8.1. *Let A, B be stable groups. Then $A * B$ is stable.*

Proof: We may assume that the free product $A * B$ is non-trivial and not isomorphic to D_∞ . As in the proof of stability of free and (torsion-free) hyperbolic groups, we prove the stability of a free product gradually. We start with the stability of coefficient-free varieties (that can be deduced from complex algebraic geometry in the case of free groups), (coefficient-free) Diophantine sets, (coefficient-free) rigid and weakly solid sets (i.e., the set of specializations of the defining parameters for which there exists a rigid or a strictly solid specialization of a given rigid or a solid limit group), coefficient-free definable sets, and finally the stability of every definable set.

Theorem 8.2. *Let A, B be stable groups. Then every coefficient-free variety over $A * B$ is stable.*

Proof: The approach that we used for proving stability of free and hyperbolic groups, associates certain finite diagrams with some families (Diophantine, rigid and solid) of definable sets, and using these diagrams we further associate (finite) canonical collections of *Duo limit groups* with these definable sets. These finite diagrams and their associated Duo limit groups, together with the bounds on the number of rigid and strictly solid families of specializations of rigid and solid limit groups over free and (torsion-free) hyperbolic groups, are the main tools that enable one to prove stability of free and hyperbolic groups.

Unfortunately, we do not know how to imitate this approach in the free product case, and we won't construct canonical and universal diagrams as in the free group

case. Instead we argue by contradiction. We suppose that there exists a coefficient-free system of equations, $\Sigma(p, q) = 1$, such that the corresponding variety, $V(p, q)$, is not stable over a non-trivial free product $G = A * B$, where both A and B are stable. With such unstable variety $V(p, q)$, we associate (non-canonically) two diagrams that are similar to the (canonical and universal) diagram that was constructed in the case of free and hyperbolic groups, and from the existence of the constructed diagrams, and the stability of the factors, A and B , we deduce a contradiction, that finally proves theorem 8.2.

Let $\Sigma(p, q) = 1$ be a coefficient-free system of equations. Let $G = A * B$ be a non-trivial free product in which both A and B are stable, and suppose that the variety $V(p, q)$, that corresponds to the system of equations $\Sigma(p, q) = 1$, is unstable over G . Since $V(p, q)$ is unstable over G , for every positive integer m , there exists two sequences of tuples with elements in G , $\{p_i^m\}_{i=1}^m$ and $\{q_j^m\}_{j=1}^m$, such that $(p_i^m, q_j^m) \in V(p, q)$ if and only if $j \leq i$.

Given the triangle of pairs, $\{(p_i^m, q_i^m) \mid 1 \leq i \leq m\}$, we can pass to a subtriangle (still denoted with the same indices), such that every sequence of pairs, $\{(p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, converges into the same limit group (over free products), $L(p, q)$. By passing to a further subtriangle, we may assume that every such sequence, $\{(p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, converges into the same graded resolution, with respect to the parameter subgroup $\langle q \rangle$: $L = L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$, where L_s is a free product of a rigid or a solid factor with (possibly) finitely many elliptic factors, and (possibly) a free group. We denote this graded resolution, $GRes_1$.

At this stage we pass to a further subtriangle (still denoted with the same indices). In this subtriangle, we may further assume that every sequence of triples, $\{(p_{i_m}^m, q_1^m, q_2^m)\}_{m=2}^\infty$, where for every index m , $i_m \geq 2$, converges into the same limit group, $U(p, q_1, q_2)$. We apply the construction of quotient resolutions, as it appears in the first step of the sieve procedure [Se6], and pass to a further subtriangle, so that we may assume that every sequence of triples, $\{(p_{i_m}^m, q_1^m, q_2^m)\}_{m=2}^\infty$, where for every index m , $i_m \geq 2$, converges into the same quotient resolution with respect to the parameter subgroup $\langle q_1, q_2 \rangle$ (as constructed according to the first step of the sieve procedure).

We continue iteratively. At each step we pass to a subtriangle, and apply the general step of the construction of a quotient resolution, as it appears in the sieve procedure [Se6]. Note that the resolution that is constructed in step n of the procedure, is graded with respect to the parameter subgroup, $\langle q_1, \dots, q_n \rangle$. Also note that each quotient resolution is either a closure of the quotient resolution that was constructed in the previous step, or it is a proper quotient resolution, which means that its complexity (as it appears in the sieve procedure) is strictly smaller than the complexity of the resolution that was constructed in the previous step.

To conclude the first part of the argument, and continue with it to prove stability of varieties, we need to prove that this (first) iterative procedure that is associated with an unstable variety terminates after finitely many steps.

Proposition 8.3. *There are only finitely many steps along the iterative procedure that is associated with an unstable variety, $V(p, q)$, over a given non-trivial free product, $G = A * B$, in which the constructed graded resolution is not a closure of the graded resolution that was constructed in the previous step of the procedure.*

Proof: Suppose that there are infinitely many steps in which the constructed

graded resolution is not a closure of the graded resolution that was constructed in the previous step of the procedure. This contradicts the termination of the sieve procedure after finitely many steps (theorem 22 in [Se6], see also the proof of theorem 4.4).

□

By proposition 8.3 there exists some step n_0 of the iterative procedure, such that starting at this step, the constructed quotient resolutions along the iterative procedure, are closures of the quotient resolution that was constructed in step n_0 . First, we replace the quotient resolution that was constructed in step n_0 , by a cover resolution, according to theorem 1.21. We denote this cover resolution, $CGRes$. By the construction of the cover resolution, $CGRes$, we can pass to a further subtriangle of specializations, $\{p_i^m\}_{i=1}^m$, and $\{q_j^m\}_{j=1}^m$, such that:

- (1) $(p_i^m, q_j^m) \in V(p, q)$ if and only if $j \leq i$.
- (2) the pairs (p_i^m, q_j^m) extend to specializations that factor through the (cover) graded resolutions, $CGRes$, for $j \leq i$, and $1 \leq j \leq n_0$.

The (cover) graded resolution, $CGRes$, terminates in a free product of a rigid or weakly solid factor with (possibly) finitely many elliptic factors. By theorems 1.14 and 1.15, with the rigid or weakly solid factor, one can associate finitely many (combinatorial) configurations, so that each configuration contains finitely many fractions, and finitely many elliptic elements, and a rigid or almost shortest (weakly) strictly solid specialization is given by fixed words in the fractions and the elliptic elements. The value of these fractions depend only on the specialization of the defining parameters, and not on the (specific) rigid or almost shortest (weakly) strictly solid specialization, whereas the elliptic elements do depend on the specific rigid or weakly strictly solid specialization (and not only on the specialization of the defining parameters). See theorems 1.14 and 1.15 for the precise details.

Once again, we can pass to a further subtriangle of specializations, $\{p_i^m\}_{i=1}^m$ and $\{q_j^m\}_{j=1}^m$, that satisfy properties (1) and (2), and assume that the specializations of the rigid or weakly solid factor of the terminal limit group of $CGRes$ that are associated with the specializations, $\{(p_i^m, q_j^m)\}$, from the subtriangle, are all associated with one (fixed) combinatorial configuration (out of the finitely many combinatorial configurations) that is presented in theorems 1.14 and 1.15, and is associated with the rigid or weakly solid factor of the terminal limit group of $CGRes$.

We denote the fractions that appear in the (fixed) combinatorial configuration (that get the same values for all the rigid or almost shortest strictly solid specializations that are associated with the same specializations of the parameters $(q_1^m, \dots, q_{n_0}^m)$, and with the fixed combinatorial configuration), v_1^m, \dots, v_f^m . We denote the elliptic elements that are associated with the values of the parameters $(q_1^m, \dots, q_{n_0}^m)$, eq_1^m, \dots, eq_d^m , and with the rigid or almost shortest (weakly) strictly solid specializations: ep_1^m, \dots, ep_g^m (see the statements of theorems 1.14 and 1.15 for these notions). For brevity we denote the tuple v_1^m, \dots, v_f^m by v^m , the tuple eq_1^m, \dots, eq_d^m by eq^m , and the tuple ep_1^m, \dots, ep_g^m by ep^m .

With the variety V , the (cover) graded resolution, $CGRes$, the triangle of specializations that factor through $CGRes$, the terminal limit group of $CGRes$, and the (fixed) combinatorial configuration that is associated with the rigid or weakly solid factor of its terminal limit group, we associate a triangle of specializations, and a second iterative procedure, in which each quotient resolution is ungraded, hence, terminates in a free product of f.g. elliptic factors.

First, with the specializations, $q_1^m, \dots, q_{n_0}^m$, and the fixed combinatorial configuration that is associated with the rigid or weakly solid factor of the terminal limit group of $CGRes$, we associate the fixed specializations of the elements, v_1^m, \dots, v_f^m , and of the elliptic elements, eq^m . With each element, p_i^m , $n_0 < i \leq m$, from the triangle of specializations that is associated with $CGRes$, we associate a tuple of elliptic specializations, ep_i^m , that is associated with it and with the fixed combinatorial configuration that is associated with the rigid or weakly solid factor of the terminal limit group of $CGRes$, and with the fixed values of the elements, v^m .

We continue with the triangle of specializations, $\{ep_i^m\}$ and $\{q_j^m\}$, $n_0 < i, j \leq m$. We swap each line of the triangle, i.e., we replace a pair, (ep_i^m, q_i^m) , with the pair, $(ep_{m-i+n_0+1}^m, q_{m-i+n_0+1}^m)$, for $n_0 < i \leq m$. After this swap, the corresponding pair, $(p_i^m, q_j^m) \in V(p, q)$, $n_0 < i, j \leq m$, if and only if $i \leq j$.

Given the triangle of pairs, and the specialization, v^m and eq^m , we can pass to a subtriangle (still denoted with the same indices), such that every sequence of pairs, $\{(ep_{n_0+1}^m, q_{j_m}^m, v^m, eq^m)\}_{m=1}^\infty$, converges into the same limit group (over free products), E . By passing to a further subtriangle, we may assume that every such sequence, converges into the same ungraded resolution (that can be viewed as multi-graded with respect to the elliptic tuples, ep^m and eq^m): $E = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_s$, where E_s is a free product of finitely many f.g. elliptic factors and possibly a free group. We denote this ungraded resolution, $ERes_1$.

As we did in the first iterative procedure, at this stage we pass to a further subtriangle (still denoted with the same indices), and apply the construction of quotient resolutions, as it appears in the first step of the sieve procedure [Se6], and pass to a further subtriangle, so that we may assume that every sequence of tuples, $\{(ep_{n_0+1}^m, ep_{n_0+2}^m, q_{j_m}^m, v^m, eq^m)\}_{m=1}^\infty$, $n_0 + 2 < j_m \leq m$, converges into the same quotient resolution (which is an ungraded resolution), as it is constructed according to the first step of the sieve procedure.

We continue iteratively. At each step we pass to a subtriangle, and apply the general step of the construction of a quotient resolution, as it appears in the sieve procedure [Se6]. As in the first iterative procedure that we associated with $V(p, q)$, each quotient resolution is either a closure of the quotient resolution that was constructed in the previous step, or it is a proper quotient resolution, which means that its complexity (as it appears in the sieve procedure) is strictly smaller than the complexity of the resolution that was constructed in the previous step. The second iterative procedure, satisfies a similar termination property as the first iterative procedure.

Proposition 8.4. *There are only finitely many steps along the (second) iterative procedure that is associated with an unstable variety, $V(p, q)$, over a given non-trivial free product, $G = A * B$, in which the constructed ungraded resolution is not a closure of the ungraded resolution that was constructed in the previous step of the procedure.*

Proof: Identical to the proof of proposition 8.3. □

By proposition 8.4 starting at some step, $n_1 > n_0$, all the quotient resolutions that are constructed along the various steps of the (second) iterative procedure, are closures of the quotient (ungraded) resolution that was constructed in step n_1 .

We denote the ungraded resolution that is constructed in step n_1 , $ERes$. With

$ERes$ we associate a cover (ungraded) resolution, $CERes$, according to theorem 1.21. As (by theorem 1.21) the completion and the terminal limit group of the cover resolution, $CERes$, are finitely presented, we may pass to a further subtriangle of the tuples specializations that were used to construct the resolution $ERes$, such that all the specializations in this subtriangle factor through $CERes$.

The cover resolution, $CERes$, the subtriangle of specializations that are associated with it, together with the stability of the factors A and B of the (non-trivial) free factor, $G = A * B$, contradict the instability of the variety, $V(p, q)$, over G . Indeed, since the constructed quotient resolutions are only replaced by closures along the second iterative (starting with step n_1), after passing to a further subtriangle, with each specialization, q_j^m , $n_1 < j \leq m$, we can associate a tuple of elliptic specializations eq_j^m . By the construction of the (ungraded) Makanin-Razborov diagram of a f.p. group (theorem 26 in [Ja-Se]), the pairs of specializations, $(p_i^m, q_j^m) \in V(p, q)$, for $n_1 \leq i, j < m$, if and only if the tuples of elliptic elements, ep_i^m, eq_j^m , satisfy one out of finitely many system of equations. Since, $(p_i^m, q_j^m) \in V(p, q)$ if and only if $i \leq j$, from the given subtriangle of specializations, we can extract larger and larger sequences of specializations, $\{ep_i^m\}$ and $\{eq_j^m\}$, $n_1 \leq i, j \leq m$, such that the (elliptic) tuples, (ep_i^m, eq_j^m) satisfy a fixed coefficient-free system of equations (independent of the indices, i, j, m), if and only if $i \leq j$. This clearly contradicts the stability of varieties over the factors A and B . Hence, every coefficient-free variety over an arbitrary free product of stable groups is stable, and we get theorem 8.2. □

Theorem 8.2 proves the stability of varieties over non-trivial free products, $G = A * B$, in case both A and B are stable. A similar argument implies the stability of Diophantine sets over such non-trivial free products.

Theorem 8.5. *Let A, B be stable groups that are not both isomorphic to Z_2 . Then every coefficient-free Diophantine set over $A * B$ is stable.*

Proof: The argument that we used to prove stability of varieties, essentially applies to prove stability of Diophantine sets as well. We assume that there exists an unstable coefficient-free Diophantine set over a non-trivial free product, $G = A * B$, that is not isomorphic to D_∞ , in which both A and B are stable. We apply the same iterative procedures that we applied in the case of varieties. After finitely many steps the quotient resolutions that are produced in each of the two iterative procedures are closures of the quotient resolution that was constructed in the previous step. This follows by the argument that proves termination of the iterative procedure for the analysis of an EAE set (theorem 4.4), that is based on the termination of the sieve procedure (theorem 22 in [Se6]), and the proof of equationality of Diophantine sets over free and (torsion-free) hyperbolic groups in theorem 2.2 in [Se9].

Once one proves that after finitely many steps, quotient resolutions that are constructed along the second iterative procedure are closures of the quotient resolutions that were constructed in the previous step, we get a contradiction to the instability of the original Diophantine set, in the same way a contradiction was extracted for unstable varieties in the proof of theorem 8.2. □

Theorem 8.5 proves that coefficient free Diophantine sets are stable over free

products of stable groups. As over free and hyperbolic groups, our next step towards stability of the theory of free products of stable groups, is the analysis of rigid specializations of rigid limit groups, and (weakly) strictly solid specializations of solid limit groups (cf. section 4 in [Se9] for the analysis of the corresponding sets over free and hyperbolic groups).

Over free products, we looked at the set of parameters for which there are at least s rigid or s families of strictly solid specializations. Unfortunately, over free products there are infinitely many such (rigid and strictly solid) families. In theorems 1.14 and 1.15 we proved combinatorial bounds on the collection of rigid and (weakly) strictly solid families, that give bounds on the number of rigid and (weakly) strictly solid families over free products (that are different than the families over free groups). However, these families and their number is not definable over free products. They are definable if we enrich the language and allow quantifiers over elements in the factors of the free product. Hence, instead of looking at sets (of parameters) with at least s families of rigid or weakly strictly solid families, we look at sets of the defining parameters for which there exists a rigid or a weakly strictly solid specialization.

As in the case of a free group, the stability of Diophantine sets (theorem 8.5) plays an essential role in proving the stability of these sets that are naturally associated with rigid and weakly solid limit groups. However, the iterative procedures that served us in proving stability of varieties and Diophantine sets need to be further refined.

Theorem 8.6. *Let $Sld(x, p, q)$ be a coefficient-free (weakly) solid limit group over free products. Suppose that Sld embeds into a f.p. (ungraded) completion, and that with Sld there is an associated (finite) cover of its flexible quotients, so that every graded limit group in this cover embeds into an (ungraded) f.p. completion (see theorems 1.20 and 1.21 for the precise details on such weakly solid limit groups). Note that we say that a homomorphism of $Sld(x, p, q)$ into a free product is (weakly) strictly solid, if it can be extended to a specialization of the f.p. completion that is associated with Sld , and it does not extend to a specialization of any of the f.p. completions that are associated with the (finite) cover of the flexible quotients of Sld .*

We set $ES(p, q)$ to be the (coefficient-free definable) set of specializations of the (free variables) parameters p, q , for which there exists a (weakly) strictly solid homomorphism of $Sld(x, p, q)$. If A and B are non-trivial stable groups that are not both isomorphic to Z_2 , then $ES(p, q)$ is stable.

Proof: The argument that we use is a modification of the argument that was used to prove stability of varieties and Diophantine sets (theorems 8.2 and 8.5). Let $Sld(p, q)$ be a weakly solid limit group, and let $ES(p, q)$ be the (coefficient-free) definable set that contains those values of the parameters (free variables) for which Sld has a weakly strictly solid specialization. Let $G = A * B$ be a non-trivial free product, that is not isomorphic to D_∞ , in which both A and B are stable, and suppose that the definable set, $ES(p, q)$, is unstable over G . Since $ES(p, q)$ is unstable over G , for every positive integer m , there exists two sequences of tuples with elements in G , $\{p_i^m\}_{i=1}^m$ and $\{q_j^m\}_{j=1}^m$, such that $(p_i, q_j) \in ES(p, q)$ if and only if $j \leq i$.

Given the triangle of pairs, $\{(p_i^m, q_i^m) \mid 1 \leq i \leq m\}$, we can pass to a subtriangle

(still denoted with the same indices), and with each pair, (p_i^m, q_1^m) , we associate a weakly strictly solid specialization of the weakly solid limit group $Sld(x, p, q)$, $(x_{i,1}^m, p_i^m, q_1^m)$, such that every sequence of weakly strictly solid specializations, $\{(x_{i_m,1}^m, p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, converges into the same limit group (over free products), $L(x, p, q)$. By passing to a further subtriangle, we may assume that every such sequence, $\{(x_{i_m,1}^m, p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, converges into the same graded resolution, with respect to the parameter subgroup $\langle q \rangle$: $L = L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$, where L_s is a free product of a rigid or a solid factor with (possibly) finitely many elliptic factors and (possibly) a free group. We denote this graded resolution, $GRes_1$.

At this stage we look for a further subtriangle (still denoted with the same indices), for which:

- (1) for each pair (p_i^m, q_2^m) , there exists a weakly strictly solid specialization of $Sld(x, p, q)$, $(x_{i,2}^m, p_i^m, q_2^m)$, $2 \leq i \leq m$.
- (2) every sequence of weakly strictly solid specializations (of $Sld(x, p, q)$):

$$\{(x_{i_m,2}^m, p_{i_m}^m, q_2^m)\}_{m=1}^\infty$$

$2 \leq i_m \leq m$, converges into the same limit group, $U(x, p, q)$.

- (3) every sequence of pairs of weakly strictly solid specializations:

$$\{(x_{i_m,1}^m, x_{i_m,2}^m, p_{i_m}^m, q_1^m, q_2^m)\}_{m=1}^\infty$$

$2 \leq i_m \leq m$, converges into the same quotient resolution, that is constructed according to the first step of the sieve procedure [Se6].

- (4) the quotient resolution that is constructed from the convergent sequences of pairs of weakly strictly solid specializations, is a proper quotient resolution of the graded resolutions that was constructed from the convergent sequences, $\{(x_{i_m,1}^m, p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, in the first step of the iterative procedure, i.e., the quotient resolution that is constructed in the second step of the iterative procedure is not a graded closure of the graded resolution that was constructed in the first step, but rather it is a quotient resolution of strictly smaller complexity according to the sieve procedure [Se6].

We continue iteratively. At each step n we look for a further subtriangle, for which the corresponding sequences of weakly strictly solid specializations (of $Sld(x, p, q)$), $\{(x_{i_m,n}^m, p_{i_m}^m, q_n^m)\}_{m=1}^\infty$, $n \leq i_m \leq m$, converge into the same limit group, and the combined weakly strictly solid specializations converge into the same quotient resolution, that is constructed according to the general step of the sieve procedure. Furthermore, the quotient resolution that is constructed in the n -th step of the procedure is not a closure of the quotient resolution that was constructed in step $n - 1$, but rather it is a quotient resolution of strictly smaller complexity according to the sieve procedure [Se6].

By proposition 8.3 this iterative procedure terminates after finitely many steps. When it terminates we are left with a triangle of specializations, (still denoted) $\{p_i^m\}_{i=1}^m$ and $\{q_j^m\}_{j=1}^m$, which is a subtriangle of the original triangle of specializations in $G = A * B$, for which $(p_i, q_j) \in ES(p, q)$ if and only if $j \leq i$. Furthermore, suppose that the iterative procedure terminated at step n_1 . Then:

- (1) for each pair (p_i^m, q_j^m) , there exists a weakly strictly solid specialization of $Sld(x, p, q)$, $(x_{i,j}^m, p_i^m, q_j^m)$, where $1 \leq j \leq n_1$ and $n_1 + 1 \leq i \leq m$.

(2) every sequence of tuples of weakly strictly solid specializations:

$$\{(x_{i_m,1}^m, \dots, x_{i_m,n_1}^m, p_{i_m}^m, q_1^m, \dots, q_{n_1}^m)\}_{m=n_1+1}^\infty$$

$n_1 + 1 \leq i_m \leq m$, converges into the same quotient resolution, that was constructed according to the general step of the sieve procedure [Se6].

We denote the graded resolution that is constructed in the n_1 -th step of the iterative procedure, $GRes_{n_1}$. By theorem 1.21 we can associate a cover graded resolution with $GRes_{n_1}$, that we denote, $CGRes_{n_1}$, that satisfies all the properties that are listed in theorem 1.21, and after possibly passing to a further subtriangle, we may assume that all the tuples of specializations,

$$\{(x_{i_m,1}^m, \dots, x_{i_m,n_1}^m, p_{i_m}^m, q_1^m, \dots, q_{n_1}^m)\}_{m=n_1+1}^\infty$$

$n_1 + 1 \leq i_m \leq m$, factor through the cover graded resolution, $CGRes_{n_1}$.

At this point we swap the rows in the subtriangle of specializations. For each i , $n_1 + 1 \leq i \leq m$, we replace p_i^m with $p_{m+n_1+1-i}^m$, and for each j , $n_1 + 1 \leq j \leq m$, we replace q_j^m with $q_{m+n_1+1-j}^m$. Note that after this swap, for each pair of indices, $n_1 + 1 \leq i, j \leq m$, $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $i \leq j$.

Each specialization p_i^m , $n_1 + 1 \leq i \leq m$, belongs to some fiber of the graded resolution, $CGRes_{n_1}$. If there is a subtriangle of our given triangle, (still denoted) $\{p_i^m\}$ and $\{q_j^m\}$, $n_1 + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $n_1 + 1 \leq j < i \leq m$, there is no sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m , we reached a terminal point of the first procedure (for proving the stability of $ES(p, q)$), and we continue with the subtriangle, and the graded resolutions, $CGRes_{n_1}$, and the graded resolution, $GRes_{n_1}$, to the second part of the proof, i.e., to a procedure that allows us to analyze ungraded resolutions and not graded ones, as we did along the second iterative procedure in proving the stability of varieties and Diophantine sets (theorems 8.2 and 8.5).

Otherwise, if there is no subtriangle with this property, there exists a subtriangle of the given triangle, $\{p_i^m\}$ and $\{q_j^m\}$, $n_1 + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $n_1 + 1 \leq j < i \leq m$, there exists a sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m .

In this case there exists a further subtriangle of the triangle that is associated with $CGRes_{n_1}$, (that we still denote) $\{p_i^m\}$ and $\{q_j^m\}$, $n_1 + 1 \leq i, j \leq m$, for which:

- (1) for every pair (i, m) , $n_1 + 1 < i \leq m$, there exists a sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_{n_1+1}^m(s))\}$, and an associated test sequence of the cover graded resolution, $CGRes_{n_1}$:

$$(z_i^m(s), p_i^m(s), q_1^m, \dots, q_{n_1}^m)$$

that is all in the fiber that contains p_i^m .

- (2) every combined sequence, $\{(z_{i_m}^m(s_m), x_{i_m, n_1+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_1}^m, q_{n_1+1}^m)\}$, where $n_1 + 1 < i_m \leq m$, and $m < s_m$, converges into the same graded resolution, $FGRes_{n_1+1}$ (it is graded with respect to the parameter subgroup $\langle q_1, \dots, q_{n_1+1}, q_{n_1+1} \rangle$). In particular, the sequence:

$$\{(z_{i_m}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_1}^m)\}$$

converges into the graded resolution, $GRes_{n_1}$.

The graded resolution, $FGRes_{n_1+1}$, can be viewed as a formal graded resolution, and it has the same structure as the graded resolution, $GRes_{n_1}$. With the graded resolution, $FGRes_{n_1+1}$, we associate a cover graded resolution according to theorem 1.21, that we denote, $CFGRes_{n_1+1}$. We can pass to a further subtriangle, so that the sequences of specializations: $\{(z_{i_m}^m(s_m), x_{i_m, n_1+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_1}^m, q_{n_1+1}^m)\}$, where $n_1 + 1 < i_m \leq m$, and $m < s_m$, actually factor through the graded cover resolution, $CFGRes_{n_1+1}$.

As for i, j , $n_1 < i, j \leq m$, $(p_i^m, q_j^m) \in E(p, q)$ if and only if $i \leq j$, the tuple, $\{(x_{i, n_1+1}^m, p_i^m, q_{n_1+1}^m)\}$, that extends to a specialization of $CFGRes_{n_1+1}$, can not be weakly strictly solid for $n_1 + 1 < i \leq m$. Hence, on the specializations that factor through $CFGRes_{n_1+1}$ we may further impose one of finitely many Diophantine conditions that force the specializations, $\{(x_{i, n_1+1}^m, p_i^m, q_{n_1+1}^m)\}$, that extends to a specialization of $CFGRes_{n_1+1}$, to be non weakly strictly solid for $n_1 + 1 < i \leq m$.

We further pass to a subtriangle, so that every sequence of specializations, $(x_{i_m, n_1+1}^m, p_{i_m}^m, q_{n_1+1}^m)$, $n_1 + 1 < i_m \leq m$, together with the specialization of the Diophantine condition that forces it to be non weakly strictly solid, converges into a quotient resolution of $FGRes_{n_1+1}$, that we denote, $GRes_{n_1+1}$. Since generic points in $FGRes_{n_1+1}$ restrict to weakly strictly solid specializations, $GRes_{n_1+1}$, is a proper quotient resolution of $FGRes_{n_1+1}$, i.e., it is a quotient resolution of strictly smaller complexity than $GRes_{n_1}$ (in light of the sieve procedure).

We continue iteratively. At each step we first look for a subtriangle, (still denoted) $\{p_i^m\}$ and $\{q_j^m\}$, $i, j \leq m$, for which for every triple (i, j, m) , $j < i \leq m$, there is no sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m)\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m in the cover of the graded resolution that was constructed in the previous step. If there is such a subtriangle, we reached a terminal point of this part of the procedure.

Otherwise, if there is no subtriangle with this property, there exists a subtriangle of the given triangle, $\{p_i^m\}$ and q_j^m , $i, j \leq m$, for which for every triple (i, j, m) , $j < i \leq m$, there exists a sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m)\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m in the cover of the graded resolution that was constructed in the previous step.

In this case we (possibly) pass to a further subtriangle, and collect test sequences that extend to weakly strictly solid specializations. The combined specializations converge into a graded resolution that is similar to $FGRes_{n_1+1}$. On the specializations in a further subtriangle, that factor through a cover graded resolution, we impose one of finitely many possible Diophantine conditions, and the combined specializations converge into a quotient resolution which is a proper quotient resolution (of strictly smaller complexity) of the resolution that was constructed in the previous step.

By the termination of the sieve procedure (theorem 22 in [Se6], see also the proof of theorem 4.4), this iterative procedure terminates after finitely many steps. We denote its terminating step, n_2 , and clearly $n_2 > n_1$. When it terminates there exists a subtriangle for which for every triple (i, j, m) , $j < i \leq m$, there is no sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m)\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m in the cover of the graded resolution that was constructed

in step n_2 .

At this point we once again swap the rows in the subtriangle of specializations. For each i , $n_2 + 1 \leq i \leq m$, we replace p_i^m with $p_{m+n_2+1-i}^m$, and for each j , $n_2 + 1 \leq j \leq m$, we replace q_j^m with $q_{m+n_2+1-j}^m$. After this swap, for each pair of indices, $n_2 + 1 \leq i, j \leq m$, $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $j \leq i$.

We continue according to the first iterative procedure, i.e., at each step we collect weakly strictly solid specializations, that converge into a proper quotient resolution (a resolution of strictly smaller complexity) of the graded resolution that was constructed in the previous step. This iterative procedure terminates after finitely many steps. After it terminates we swap the rows in the triangle once again, and continue iteratively along the steps of the second iterative procedure. This (second) iterative procedure terminates after finitely many steps, and we continue iteratively. At each step we first swap the rows, and then continue along the first or second iterative procedure until they terminate (after finitely many steps), swap, and continue along the other procedure.

This iterative procedure of swapping and continuing along the two iterative procedures until they terminate, terminates after finitely many steps, by the proof of theorem 4.4. Once it terminates, we are left with a graded resolution, $GRes_{n_k}$, which is graded with respect to the parameter subgroup, $\langle q_1, \dots, q_{n_k} \rangle$, a cover of this graded resolution that satisfies the properties of cover graded resolutions that are listed in theorem 1.21, $CGRes_{n_k}$, and a subtriangle of specializations, (still denoted) $\{p_i^m\}$ and $\{q_j^m\}$, $n_k + 1 \leq i, j \leq m$ that extend to specializations that factor through the cover graded resolution, $CGRes_{n_k}$, for which $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $n_k + 1 \leq j \leq i \leq m$. Furthermore, each specialization p_i^m extends to a specialization that factors through $CGRes_{n_k}$. A test sequence of specializations in the fiber of $CGRes_{n_k}$ that contains p_i^m , restricts to specializations, $\{p_i^m(s)\}$, that extend to weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, if and only if $n_k + 1 \leq j \leq i \leq m$.

Now, we can continue to the second part as we did in proving the stability of varieties and Diophantine sets (theorems 8.2 and 8.5). The (cover) graded resolution, $CGRes_{n_k}$, terminates in a free product of a rigid or a weakly solid factor with (possibly) finitely many elliptic factors and (possibly) a free group. By theorems 1.14 and 1.15, with the rigid or solid factor, one can associate finitely many (combinatorial) configurations, so that each configuration contains finitely many fractions, and finitely many elliptic elements. A rigid or almost shortest (weakly) strictly solid specialization is given by fixed words in the fractions and the elliptic elements. The value of these fractions depend only on the specialization of the defining parameters, and not on the (specific) rigid or almost shortest (weakly) strictly solid specialization, whereas the elliptic elements do depend on the specific rigid or weakly strictly solid specialization (and not only on the specialization of the defining parameters). See theorems 1.14 and 1.15 for the precise details.

We can pass to a further subtriangle of specializations, $\{p_i^m\}_{i=n_k+1}^m$ and $\{q_j^m\}_{j=n_k+1}^m$, so that $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $j \leq i$, and assume that the specializations of the rigid or weakly solid factor of the terminal limit group of $CGRes_{n_k}$ that are associated with the specializations, $\{(p_i^m, q_j^m)\}$, from the subtriangle, are all associated with one (fixed) combinatorial configuration (out of the finitely many combinatorial configurations) that is presented in theorems 1.14 and 1.15, and is associated with the rigid or weakly solid factor of the terminal limit group of

$CGRes_{n_k}$.

We denote the fractions that appear in the (fixed) combinatorial configuration (that get the same values for all the rigid or almost shortest strictly solid specializations that are associated with the same specializations of the parameters $(q_1^m, \dots, q_{n_k}^m)$, and with the fixed combinatorial configuration), v_1^m, \dots, v_f^m , and for brevity v^m . We denote the elliptic elements that are associated with the values of the parameters $(q_1^m, \dots, q_{n_k}^m)$, eq_1^m, \dots, eq_d^m , and for brevity eq^m , and with the rigid or almost shortest (weakly) strictly solid specializations: ep_1^m, \dots, ep_g^m , and for brevity ep^m (see the statements of theorems 1.14 and 1.15 for these notions).

At this point we start the second part of our argument that involves two interchanging iterative procedures like the first part, and in which all the resolutions are ungraded, i.e., they terminate in elliptic subgroups, in a similar way to the second part of the argument that was used to prove stability of varieties and Diophantine sets.

First, With each element, p_i^m , $n_k < i \leq m$, from the triangle of specializations that is associated with the cover graded resolution, $CGRes_{n_k}$, we associate a tuple of elliptic specializations, ep_i^m , that is associated with it and with the fixed combinatorial configuration that is associated with the rigid or weakly solid factor of the terminal limit group of $CGRes_{n_k}$, and with the fixed values of the elements, v^m .

We continue with the triangle of specializations, $\{ep_i^m\}$ and $\{q_j^m\}$, $n_k < i, j \leq m$. We swap each line of the triangle, i.e., we replace a pair, (ep_i^m, q_i^m) , with the pair, $(ep_{m-i+n_k+1}^m, q_{m-i+n_k+1}^m)$, for $n_k < i \leq m$. After this swap, the corresponding pair, $(p_i^m, q_j^m) \in ES(p, q)$, $n_k < i, j \leq m$, if and only if $i \leq j$.

We apply the first iterative procedure for this triangle of specializations. Given the triangle of pairs, and the specialization, v^m and eq^m , we can pass to a subtriangle (still denoted with the same indices), such that every sequence of pairs, $\{(ep_{n_k+1}^m, q_{j_m}^m, v^m, eq^m)\}_{m=1}^\infty$, $n_k + 1 \leq j_m \leq m$, converges into the same limit group (over free products), E . By passing to a further subtriangle, we may assume that every such sequence, converges into the same ungraded resolution (that can be viewed as multi-graded with respect to the elliptic tuples, $ep_{n_k+1}^m$ and eq^m): $E = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_s$, where E_s is a free product of finitely many f.g. elliptic factors and possibly a free group. We denote this ungraded resolution, $ERes_1$.

We continue iteratively. At each step we look for a further subtriangle, for which the corresponding sequences of weakly strictly solid specializations (of $Sld(x, p, q)$), $\{(x_{n, i_m}^m, p_n^m, q_{j_m}^m)\}_{m=1}^\infty$, $n \leq j_m \leq m$, converge into the same limit group, and the combined weakly strictly solid specializations converge into the same quotient resolution, that is constructed according to the general step of the sieve procedure. Note that the constructed quotient resolution is ungraded and terminates in a free product of elliptic factors and possibly a free group. Furthermore, the quotient resolution that is constructed in the general step of the procedure is not a closure of the quotient resolution that was constructed in the previous step, but rather it is a quotient resolution of strictly smaller complexity according to the sieve procedure [Se6].

By proposition 8.3 this iterative procedure terminates after finitely many steps, that we denote ℓ_1 ($\ell_1 > n_k$). When it terminates we are left with an ungraded resolution, that we denote $ERes_{\ell_1}$. With the ungraded resolution, $ERes_{\ell_1}$ we associate a cover resolution, $CERes_{\ell_1}$, that satisfies the properties of theorem 1.21. With $CERes_{\ell_1}$, there is an associated triangle of specializations, (still denoted)

$\{ep_i^m\}_{i=\ell_1+1}^m$ and $\{q_j^m\}_{j=\ell_1+1}^m$, which is a subtriangle of the original triangle of specializations in $G = A * B$, for which the corresponding specialization, $(p_i^m, q_j^m) \in ES(p, q)$, if and only if $i \leq j$. Furthermore, for every pair, (ep_i^m, q_j^m) , for which $\ell_1 + 1 \leq i \leq j \leq m$, the triple (ep_i^m, v^m, q_j^m) extends to a specialization that factors through the cover resolution $CERes_{\ell_1}$.

At this point we continue as we did in the first part of the argument. First, we swap the rows in the subtriangle of specializations. Note that after this swap, for each pair of indices, $\ell_1 + 1 \leq i, j \leq m$, the corresponding specializations, $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $j \leq i$.

Each specialization (v^m, q_j^m) , $\ell_1 + 1 \leq j \leq m$, belongs to some fiber of the graded resolution, $CERes_{\ell_1}$. If there is a subtriangle of our given triangle, (still denoted) $\{ep_i^m\}$ and $\{q_j^m\}$, $\ell_1 + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $\ell_1 + 1 \leq i < j \leq m$, there is no sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{(p_i^m(s), q_j^m(s))\}$, is a restriction to the pair (p, q) of a test sequence in the fiber that contains q_j^m , we reached a terminal point of the procedure (i.e., a terminal point of the second part of the argument for proving stability of $ES(p, q)$). In this case we can extract a contradiction, and conclude stability, from this subtriangle, and the graded resolution $CERes_{\ell_1+1}$.

Otherwise, if there is no subtriangle with this property, there exists a subtriangle of the given triangle, $\{ep_i^m\}$ and $\{q_j^m\}$, $\ell_1 + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $\ell_1 + 1 \leq i < j \leq m$, there exists a sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{(p_i^m(s), q_j^m(s))\}$, is a restriction to the pair (p, q) of a test sequence in the fiber that contains q_j^m .

In this case there exists a further subtriangle of the triangle that is associated with $CERes_{\ell_1}$, (that we still denote) $\{p_i^m\}$ and $\{q_j^m\}$, $\ell_1 + 1 \leq i, j \leq m$, for which:

- (1) for every pair (j, m) , $\ell_1 + 1 < j \leq m$, there exists a sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{\ell_1+1,j}^m(s), p_{\ell_1+1}^m(s), q_j^m(s))\}$, and an associated test sequence of the cover resolution, $CERes_{\ell_1}$:

$$\{(z_{\ell_1+1,j}^m(s), v^m(s), p_{\ell_1+1}^m(s), q_j^m(s), eq_1^m, \dots, eq_{n_k}^m, ep_{n_k+1}^m, \dots, ep_{\ell_1}^m)\}$$

that is all in the fiber that contains (v^m, q_j^m) .

- (2) every combined sequence:

$$\{(z_{\ell_1+1,j_m}^m(s_m), x_{\ell_1+1,j_m}^m(s_m), v^m(s_m), p_{\ell_1+1}^m(s_m), q_{j_m}^m(s_m), eq_1^m, \dots, eq_{n_k}^m, ep_{n_k+1}^m, \dots, ep_{\ell_1}^m, ep_{\ell_1+1}^m)\}$$

where $\ell_1 + 1 < j_m \leq m$, and $m < s_m$, converges into the same ungraded resolution, $FERes_{\ell_1+1}$. In particular, the sequence:

$$\{(z_{\ell_1+1,j_m}^m(s_m), v^m(s_m), p_{\ell_1+1}^m(s_m), q_{j_m}^m(s_m), eq_1^m, \dots, eq_{n_1}^m, ep_{n_k+1}^m, \dots, ep_{\ell_1}^m)\}$$

converges into the ungraded resolution, $ERes_{\ell_1}$.

The resolution, $FERes_{\ell_1+1}$, can be viewed as a formal (ungraded) resolution, and it has the same structure as the resolution, $ERes_{\ell_1}$. With the resolution, $FERes_{\ell_1+1}$, we associate a cover resolution, with a f.p. completion and terminal limit group, according to theorem 1.21, that we denote, $CFERes_{\ell_1+1}$. We can pass to a further subtriangle, so that the sequences of specializations:

$$\{(z_{\ell_1+1,j_m}^m(s_m), x_{\ell_1+1,j_m}^m(s_m), v^m(s_m), p_{\ell_1+1}^m(s_m), q_{j_m}^m(s_m), eq_1^m, \dots, eq_{n_k}^m, ep_{n_k+1}^m, \dots, ep_{\ell_1}^m, ep_{\ell_1+1}^m)\}$$

actually factor through the cover resolution, $CFERes_{\ell_1+1}$.

As for i, j , $\ell_1 + 1 \leq i, j \leq m$, $(p_i^m, q_j^m) \in E(p, q)$ if and only if $j \leq i$, the tuple, $\{(x_{\ell_1+1,j}^m, p_{\ell_1+1}^m, q_j^m)\}$, $\ell_1 + 1 < j \leq m$, that extends to a specialization of $CFERes_{\ell_1+1}$, can not be weakly strictly solid. Hence, on the specializations that factor through $CFERes_{\ell_1+1}$ we may further impose one of finitely many Diophantine conditions that force the specializations, $\{(x_{\ell_1+1,j}^m, p_{\ell_1+1}^m, q_j^m)\}$, $\ell_1 + 1 < j \leq m$, that extends to a specialization of $CFERes_{\ell_1+1}$, to be non weakly strictly solid.

We further pass to a subtriangle, so that every sequence of specializations, $(x_{\ell_1+1,j_m}^m, p_{\ell_1+1}^m, q_{j_m}^m)$, $\ell_1 + 1 < j_m \leq m$, and their extensions to specializations of $CFERes$, together with the specialization of the Diophantine condition that forces it to be non weakly strictly solid, converges into a quotient resolution of $FERes_{\ell_1+1}$, that we denote, $ERes_{\ell_1+1}$. Since generic points in $FERes_{\ell_1+1}$ restrict to weakly strictly solid specializations, $ERes_{\ell_1+1}$, is a proper quotient resolution of $FERes_{\ell_1+1}$, i.e., it is a quotient resolution of strictly smaller complexity than $ERes_{\ell_1}$ (in light of the sieve procedure).

We continue iteratively, precisely as we did in the iterative procedure that was used in the first part of the argument. At each step we first look for a subtriangle, (still denoted) $\{ep_i^m\}$ and $\{q_j^m\}$, $i, j \leq m$, for which for every triple (i, j, m) , $i < j \leq m$, there is no sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{(p_i^m(s), q_j^m(s))\}$, is a restriction to the elements p, q of a test sequence in the fiber that contains (v^m, q_j^m) in the cover of the ungraded resolution that was constructed in the previous step. If there is such a subtriangle, we reached a terminal point of this part of the procedure.

Otherwise, if there is no subtriangle with this property, there exists a subtriangle of the given triangle, $\{ep_i^m\}$ and q_j^m , $i, j \leq m$, for which for every triple (i, j, m) , $i < j \leq m$, there exists a sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{(p_i^m(s), q_j^m(s))\}$, is a restriction to the elements p, q of a test sequence in the fiber that contains (v^m, q_j^m) in the cover of the graded resolution that was constructed in the previous step.

In this case we (possibly) pass to a further subtriangle, and collect test sequences that extend to weakly strictly solid specializations. The combined specializations converge into a graded resolution that is similar to $FERes_{\ell_1+1}$. On the specializations in a further subtriangle, that extend to specializations that factor through a cover resolution we impose one of finitely many possible Diophantine conditions, and the combined specializations converge into a quotient resolution which is a proper quotient resolution (of strictly smaller complexity) of the resolution that was constructed in the previous step.

By the termination of the sieve procedure (theorem 22 in [Se6], see also the proof of theorem 4.4), this iterative procedure terminates after finitely many steps. We denote its terminating step, ℓ_2 , and clearly $\ell_2 > \ell_1$. When it terminates there exists a subtriangle for which for every triple (i, j, m) , $i < j \leq m$, there is no sequence of weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{(p_i^m(s), q_j^m(s))\}$, is a restriction to the elements p, q of a test sequence in the fiber that contains (v^m, q_j^m) in the cover of the graded resolution that was constructed in step ℓ_2 .

We continue as we did in the first part of the argument. At this point we once again swap the rows in the subtriangle of specializations. After this swap, for each pair of indices, $\ell_2 + 1 \leq i, j \leq m$, $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $i \leq j$.

We continue according to the first iterative procedure that was used in the second part of the argument. i.e., at each step we collect weakly strictly solid specializations, that converge into a proper quotient resolution (a resolution of strictly smaller complexity) of the graded resolution that was constructed in the previous step. This iterative procedure terminates after finitely many steps. After it terminates we swap the rows in the triangle once again, and continue iteratively along the steps of the second iterative procedure that was used in the second part of the argument. This (second) iterative procedure terminates after finitely many steps, and we continue iteratively. At each step we first swap the rows, and then continue along the first or second iterative procedure until they terminate (after finitely many steps), swap, and continue along the other procedure.

This iterative procedure of swapping and continuing along the two iterative procedures until they terminate, terminates after finitely many steps, by the proof of theorem 4.4. Once it terminates, we are left with an ungraded resolution, $ERes_{\ell_t}$, a cover of this resolution that satisfies the properties of cover resolutions that are listed in theorem 1.21 (and in particular has a f.p. completion and terminal limit group), $CERes_{\ell_t}$, and a subtriangle of specializations, (still denoted) $\{ep_i^m\}$ and $\{q_j^m\}$, $\ell_t + 1 \leq i, j \leq m$ that extend to specializations that factor through the cover resolution, $CERes_{\ell_t}$, for which (the corresponding specialization) $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $\ell_t + 1 \leq i \leq j \leq m$. Furthermore, each specialization (v^m, q_j^m) extends to a specialization that factors through $CERes_{\ell_t}$. A test sequence of specializations in the fiber of $CERes_{\ell_t}$ that contains (v^m, q_j^m) , restricts to specializations, $\{(p_i^m(s), q_j^m(s))\}$, that extend to weakly strictly solid specializations of $Sld(x, p, q)$, $\{(x_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, if and only if $\ell_t + 1 \leq i \leq j \leq m$.

We denote the ungraded resolution that is constructed in step ℓ_t , $ERes$. With $ERes$ we associate a cover (ungraded) resolution, $CERes$, according to theorem 1.21. As (by theorem 1.21) the completion and the terminal limit group of the cover resolution, $CERes$, are finitely presented, we may pass to a further subtriangle of the tuples of specializations that were used to construct the resolution $ERes$, such that all the specializations in this subtriangle factor through $CERes$.

Recall that the subgroup $\langle q \rangle$ is embedded in the graded cover $CERes$, and the elements p can be expressed as words in the elements that are associated with v^m in $CERes$, with the elliptic elements ep . Given the cover resolution, $CERes$, we look at all its test sequences, for which the restriction of the test sequences to the elements p, q , extend to weakly strictly solid specializations of the weakly solid limit group, $Sld(x, p, a)$. By the techniques that were used to construct the formal Makanin-Razborov diagram, with the collection of these test sequences we can associate a formal Makanin-Razborov diagram (see theorem 2.7), where each of the finitely many resolutions in this formal Makanin-Razborov diagram is a closure of $CERes$, and each such resolution has a f.p. completion and terminal limit group.

A specialization of $Sld(x, p, q)$ is not weakly strictly solid if it satisfies one out of finitely many Diophantine conditions. Given each of the resolutions in this (formal) Makanin-Razborov diagram, we look at all the test sequences that factor through it, for which the elements that are associated with the weakly strictly solid specializations of $Sld(x, p, q)$, satisfy one of the finitely many Diophantine conditions that force them to be non weakly strictly solid specializations. With the collection of these sequences we can once again associate a formal Makanin-Razborov diagram, in which every resolution has a f.p. completion and terminal

limit group.

Now, the subtriangle of specializations that factor through the cover graded resolution, $CERes$, extend to specializations that factor through resolutions in the (formal) Makanin-Razborov diagram that is associated with $CERes$. Specialization in the subtriangle that is associated with $CERes$ satisfy: $(p_i^m, q_j^m) \in ES(p, q)$ if and only if $i \leq j$. By the iterative procedure that was applied to construct the terminal ungraded resolution $ERes$, test sequences in $CERes$ that are contained in the fiber of (v^m, q_j^m) , extend to weakly strictly solid specializations of $Sld(x, p, q)$ if and only if $i \leq j$ as well.

The resolutions in the formal Makanin-Razborov diagrams that are associated with $CERes$ enable one to reduce the question of whether a fiber in $CERes$ contains a test sequence that extends to weakly strictly solid specializations of $Sld(x, p, q)$ to a (finite) disjunction of conjunctions of (fixed) AE predicates over the factors of the free product $G = A * B$. Hence, the existence of the (infinite) subtriangle of specializations that is associated with $CERes$, contradicts the stability of the constructed AE predicates over the factors A and B . Therefore, $ES(p, q)$ is stable, and theorem 8.6 follows. □

Having proved that varieties, Diophantine sets, and sets of specializations of the defining parameters for which there exists a rigid or a weakly strictly solid specialization of a rigid or a weakly solid limit group, are stable over a free product in which the factors are stable, we are finally ready to complete the proof of theorem 8.1, i.e., to prove that the theory of a free product of stable groups is stable.

The argument that we use to prove stability of a general definable set over a free product of stable groups, is based on the argument that was used to prove that the set, $ES(p, q)$, is stable over such free products in theorem 8.6. To adapt this approach, we use the uniform geometric description of a definable set over free products, that is stated and proved in theorem 6.1.

Let $Def(p, q)$ be a coefficient-free definable set. Wlog we will assume that $Def(p, q)$ is an $E(AE)^k$ set. Recall that theorem 6.1 associates with $Def(p, q)$ finitely many graded resolutions, $DRes_1, \dots, DRes_g$, and with each such graded resolution, there are finitely many associated auxiliary resolutions. The graded resolutions, $DRes_i$, have the properties of resolutions in the graded Makanin-Razborov diagram (over free products) of a f.p. group (theorem 1.22), and in particular they terminate in a free product of a rigid or a weakly solid limit group (that can be embedded in a f.p. ungraded completion) with finitely many f.p. elliptic factors. The auxiliary resolutions have the same properties as resolutions in a graded formal Makanin-Razborov diagram (theorem 2.7), and their terminal limit groups have the same properties as the terminal limit groups of the graded resolutions, $DRes_i$.

By theorem 6.1, the question of whether a specialization of the defining parameters is in $Def(p, q)$ or in its complement, over an arbitrary non-trivial free product that is not isomorphic to D_∞ , depends entirely on the set of all possible specializations of the terminal limit groups of the graded resolutions, $DRes_i$ (that are associated with $Def(p, q)$), and their possible extensions to specializations of the terminal limit groups of the (finitely many) associated resolutions (see theorem 6.1 for details). This enables one to reduce a sentence over free products to sentences over its factors (theorem 6.3), and even to get a form of quantifier elimination for predicates over free products (theorem 6.4). Note that the graded resolutions that

are constructed in theorem 6.1 and are associated with $Def(p, q)$ are good for all free products, regardless of the number of factors, whereas the reduction of a sentence to its factors, and the quantifier elimination, do depend on the number of factors in the given free product.

By theorem 6.1 (see also the proof of theorem 6.3), a specialization of the defining parameters p, q is in $Def(p, q)$, if and only if specializations of the terminal limit groups of the graded resolutions, $DRes_i$ (that are associated with $Def(p, q)$), and their associated auxiliary resolutions, satisfy certain properties that are listed in theorem 6.1 (and correspondingly in theorem 6.3).

Hence, our approach to prove stability of the set $Def(p, q)$ over free products consists of two parts, as in proving stability of the set $ES(p, q)$ (theorem 8.6). In each part we apply a sequence of interchanging iterative procedures, where the iterative procedures are similar to the ones that were used in the proof of theorem 8.6, and the way we interchange them depends and follows the requirements on the terminal limit group of the graded resolutions, $DRes$, that are associated with $Def(p, q)$, and the terminal limit groups of their associated auxiliary resolutions, as these requirements appear in theorem 6.1 (and in its proof).

The terminal limit group of any of those (finitely many) graded resolutions, is a free product of a rigid or a weakly solid factor (w.r.t. to the parameter subgroup $\langle p, q \rangle$) with (possibly) finitely many elliptic factors. Furthermore, the rigid or weakly solid factor in such a terminal limit group can be embedded in an ungraded f.p. completion. Hence, each terminal limit group of one of the graded resolutions, $DRes_i$, or one of its associated auxiliary resolutions, can be embedded into a f.p. group, which is the free product of the f.p. ungraded completion into which the rigid or weakly solid factor embeds, free product with the (finitely many) f.p. elliptic factors of the terminal limit group.

Let $G = A * B$ be a non-trivial free product, that is not isomorphic to D_∞ , in which both A and B are stable, and suppose that the coefficient-free definable set, $Def(p, q)$, is unstable over G . Since $Def(p, q)$ is unstable over G , for every positive integer m , there exists two sequences of tuples with elements in G , $\{p_i^m\}_{i=1}^m$ and $\{q_j^m\}_{j=1}^m$, such that $(p_i, q_j) \in Def(p, q)$ if and only if $j \leq i$.

Let $Def(p, q)$ be given by the coefficient-free predicate:

$$Def(p, q) = \exists t \forall y_1 \exists x_1 \dots \forall y_k \exists x_k \Sigma(t, y_1, x_1, \dots, y_k, x_k, p, q) = 1 \wedge \\ \wedge \Psi(t, y_1, x_1, y_k, x_k, p, q) \neq 1$$

By theorem 6.1, given the triangle of pairs, $\{(p_i^m, q_j^m) \mid 1 \leq i, j \leq m\}$, we can pass to a subtriangle (still denoted with the same indices), and with each pair, (p_i^m, q_1^m) , associate a specialization of the terminal limit group of one of the resolutions, $DRes$, that are associated with $Def(p, q)$, that restricts to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of that terminal limit group. Furthermore, these specializations of the terminal limit group of $DRes$, are supposed to testify that the various specializations, (p_i^m, q_1^m) are in the set $Def(p, q)$. we can pass to a further subtriangle (still denoted with the same indices), such that these specializations of the terminal limit group of $DRes$, $(t_{i,1}^m, p_i^m, q_1^m)$, $1 \leq i \leq m$, have the property that every sequence of the form: $\{(t_{i_m,1}^m, p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, converges into the same limit group (over free products), $L(t, p, q)$. By passing to a further subtriangle, we may assume that every such sequence, $\{(t_{i_m,1}^m, p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, converges

into the same graded resolution, with respect to the parameter subgroup $\langle q \rangle$: $L = L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$, where L_s is rigid or solid. We denote this graded resolution, $GRes_1$.

We continue as we did in the first part of proving that $ES(p, q)$ is stable (theorem 8.6). At this stage we look for a further subtriangle (still denoted with the same indices), for which:

- (1) for each pair (p_i^m, q_2^m) , there exists a specialization of the terminal limit group of one of the graded resolutions, $DRes$, that restricts to a rigid or a weakly strictly solid specialization of the rigid or solid factor of that terminal limit group, $(t_{i,2}^m, p_i^m, q_2^m)$, $2 \leq i \leq m$. Furthermore, we may assume that these specializations of the terminal limit group of $DRes$ testify that the specializations, (p_i^m, q_2^m) , $2 \leq i \leq m$, are in the set $Def(p, q)$.
- (2) every sequence of these specializations (of the terminal limit group of $DRes$), $\{(t_{i_m,2}^m, p_{i_m}^m, q_2^m)\}_{m=1}^\infty$, $2 \leq i_m \leq m$, converges into the same limit group, $U(t, p, q)$.
- (3) every sequence of pairs of these specializations of the terminal limit group of $DRes$, $\{(t_{i_m,1}^m, t_{i_m,2}^m, p_{i_m}^m, q_1^m, q_2^m)\}_{m=1}^\infty$, $2 \leq i_m \leq m$, converges into the same quotient resolution, that is constructed according to the first step of the sieve procedure [Se6].
- (4) the quotient resolution that is constructed from the convergent sequences of pairs of specializations of the terminal limit group of $DRes$, is a proper quotient resolution of the graded resolutions that was constructed from the convergent sequences, $\{(t_{i_m,1}^m, p_{i_m}^m, q_1^m)\}_{m=1}^\infty$, in the first step of the iterative procedure, i.e., the quotient resolution that is constructed in the second step of the iterative procedure is not a graded closure of the graded resolution that was constructed in the first step, but rather it is a quotient resolution of strictly smaller complexity according to the sieve procedure [Se6].

We continue iteratively as we did in the first part of the proof of theorem 8.6. At each step n we look for a further subtriangle, for which the corresponding sequences of specializations of the terminal limit group of a resolution $DRes$, that is associated with $Def(p, q)$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factors of this terminal limit group, $\{(t_{i_m,n}^m, p_{i_m}^m, q_n^m)\}_{m=1}^\infty$, $n \leq i_m \leq m$, converge into the same limit group, and the combined specializations converge into the same quotient resolution, that is constructed according to the general step of the sieve procedure. Furthermore, the quotient resolution that is constructed in the n -th step of the procedure is not a closure of the quotient resolution that was constructed in step $n-1$, but rather it is a quotient resolution of strictly smaller complexity according to the sieve procedure [Se6].

By proposition 8.3 this iterative procedure terminates after finitely many steps. When it terminates we are left with a triangle of specializations, (still denoted) $\{p_i^m\}_{i=1}^m$ and $\{q_j^m\}_{j=1}^m$, which is a subtriangle of the original triangle of specializations in $G = A * B$, for which $(p_i, q_j) \in Def(p, q)$ if and only if $j \leq i$. Furthermore, suppose that the iterative procedure terminated at step n_1 . Then:

- (1) for each pair (p_i^m, q_j^m) , there exists a specialization of a terminal limit group of a resolution, $DRes$, $(t_{i,j}^m, p_i^m, q_j^m)$, where $1 \leq j \leq n_1$ and $n_1 + 1 \leq i \leq m$, that restricts to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of that terminal limit group.

(2) every sequence of these specializations:

$$\{(t_{i_m,1}^m, \dots, t_{i_m,n_1}^m, p_{i_m}^m, q_1^m, \dots, q_{n_1}^m)\}_{m=n_1+1}^\infty$$

$n_1 + 1 \leq i_m \leq m$, converges into the same quotient resolution, that was constructed according to the general step of the sieve procedure [Se6].

We denote the graded resolution that is constructed in the n_1 -th step of the iterative procedure, $GRes_{n_1}$. By theorem 1.21 we can associate a cover graded resolution with $GRes_{n_1}$, that we denote, $CGRes_{n_1}$, that satisfy all the properties that are listed in theorem 1.21, and after possibly passing to a further subtriangle, we may assume that all the tuples of specializations,

$$\{(t_{i_m,1}^m, \dots, t_{i_m,n_1}^m, p_{i_m}^m, q_1^m, \dots, q_{n_1}^m)\}_{m=n_1+1}^\infty$$

$n_1 + 1 \leq i_m \leq m$, factor through the cover graded resolution, $CGRes_{n_1}$.

Each specialization p_i^m , $n_1 + 1 \leq i \leq m$, belongs to some fiber of the graded resolution, $CGRes_{n_1}$. If there is a subtriangle of our given triangle, (still denoted) $\{p_i^m\}$ and $\{q_j^m\}$, $n_1 + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $n_1 + 1 \leq j \leq i \leq m$, there is a sequence of specializations of the terminal limit group of $DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m , and these elements testify that the tuples, $(p_i^m(s), q_j^m(s))$, are in the definable set, $Def(p, q)$, we reached a terminal point of the first procedure (for proving the stability of $Def(p, q)$), and we continue with the subtriangle, and the graded resolutions, $CGRes_{n_1}$, and the graded resolution, $GRes_{n_1}$, to the next iterative procedure, that studies those pairs (p_i^m, q_j^m) , $n_1 + 1 \leq i < j \leq m$, that do not belong to $Def(p, q)$.

Otherwise, if there is no subtriangle with this property, there exists a subtriangle of the given triangle, $\{p_i^m\}$ and $\{q_j^m\}$, $n_1 + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $n_1 + 1 \leq j < i \leq m$, there exists a sequence of specializations of the terminal limit group of $DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m , and:

- (1) for every triple, (i, j, m) , $n_1 \leq j \leq i \leq m$, the sequence of specializations, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$, and its associated test sequence of the cover graded resolution, $CGRes_{n_1}$, converge into a closure of $CGRes_{n_1}$, and the given specialization of the terminal limit group of $DRes$, $(t_{i,j}^m, p_i^m, q_j^m)$, that testifies that $(p_i^m, q_j^m) \in Def(p, q)$, is a specialization of a cover of that closure of $CGRes_{n_1}$.
- (2) the sequence of specializations of the terminal limit group of $DRes$:

$$\{(t_{i,j}^m(s), p_i^m(s), q_j^m(s))\}$$

do not testify that the tuples, $(p_i^m(s), q_j^m(s))$ are in the definable set, $Def(p, q)$.

In this case there exists a further subtriangle of the triangle that is associated with $CGRes_{n_1}$, (that we still denote) $\{p_i^m\}$ and $\{q_j^m\}$, $n_1 + 1 \leq i, j \leq m$, for which:

- (1) for every triple (i, j, m) , $n_1 + 1 \leq j \leq i \leq m$, there exists a sequence of specializations of the terminal limit group of an auxiliary resolution, $Y_1 DRes$,

that is associated with $DRes$, that restricts to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group of $Y_1 DRes$, $\{(y1_{i,j}^m(s), t_{i,j}^m(s), p_i^m(s), q_j^m)\}$, that extends the sequence of specializations of the terminal limit group of $DRes$, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m)\}$, that is associated with the pair, (p_i^m, q_j^m) . Furthermore, these specializations of the terminal limit group of the auxiliary resolution $Y_1 DRes$, testify that the corresponding specializations of $DRes$, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m)\}$, do not testify that the tuples, (p_i^m, q_j^m) , are in $Def(p, q)$.

(2) every combined sequence:

$$\{(z_{i_m, n_1+1}^m(s_m), y1_{i_m, n_1+1}^m(s_m), t_{i_m, n_1+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_1}^m, q_{n_1+1}^m)\}$$

where $n_1 + 1 \leq i_m \leq m$ and $m < s_m$, converges into the same graded resolution, $FGRes_{n_1+1}$ (it is graded with respect to the parameter subgroup $\langle q_1, \dots, q_{n_1+1} \rangle$). In particular, the sequence:

$$\{(z_{i_m, n_1+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_1}^m)\}$$

converges into the graded resolution, $GRes_{n_1}$.

The graded resolution, $FGRes_{n_1+1}$, can be viewed as a formal graded resolution, and it has the same structure as the graded resolution, $GRes_{n_1}$. With the graded resolution, $FGRes_{n_1+1}$, we associate a cover graded resolution according to theorem 1.21, that we denote, $CFGRes_{n_1+1}$. We can pass to a further subtriangle, so that the sequences of specializations:

$$\{(z_{i_m, n_1+1}^m(s_m), y1_{i_m, n_1+1}^m(s_m), t_{i_m, n_1+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_1}^m, q_{n_1+1}^m)\}$$

where $n_1 + 1 \leq j_{i_m} \leq m$, and $m < s_m$, actually factor through the graded cover resolution, $CFGRes_{n_1+1}$.

The sequences, $\{(t_{i, n_1+1}^m(s), p_i^m(s), q_{n_1+1}^m)\}$, do not testify that the tuples, $(p_i^m, q_{n_1+1}^m)$, are in $Def(p, q)$, and their extensions to specializations of the terminal limit group of the auxiliary resolution, $Y_1 DRes$, $\{(y1_{i, n_1+1}^m(s), t_{i, n_1+1}^m(s), p_i^m(s), q_{n_1+1}^m)\}$, testify for that. Furthermore, the specializations, $(t_{i, n_1+1}^m, p_i^m, q_{n_1+1}^m)$, $n_1 + 1 \leq i \leq m$, do testify that $(p_i^m, q_{n_1+1}^m) \in Def(p, q)$. Hence, on the specializations that factor through the cover, $CFGRes_{n_1+1}$, we can further impose either one of finitely many Diophantine conditions that forces the specializations of the elements $y1$, that are specializations of the terminal limit group of an auxiliary resolution, $Y_1 DRes$, to restrict to non-rigid or non weakly strictly solid specialization of the rigid or weakly solid specialization factor of that terminal limit group, or to impose the existence of extensions of these specializations of the terminal limit group of $Y_1 DRes$, to an auxiliary resolution, $X_1 Y_1 DRes$ (see theorem 6.1 for these auxiliary resolutions).

We further pass to a subtriangle, so that every sequence of specializations, $(y1_{i_m, n_1+1}^m, t_{i_m, n_1+1}^m, p_{i_m}^m, q_{n_1+1}^m)$, $n_1 + 1 \leq i_m \leq m$, together with their extension to specializations of $CFGRes$, and together with the specialization of the Diophantine condition that forces them to restrict to non-rigid or non weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $Y_1 DRes$, or together with the corresponding specializations of the terminal limit group of an auxiliary resolution, $X_1 Y_1 DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, converges into a quotient resolution of $FGRes_{n_1+1}$, that we denote, $GRes_{n_1+1}$.

If the obtained quotient resolution is a proper quotient resolution of $GRes_{n_1}$, i.e., not a closure of it, we continue iteratively, by looking for a subtriangle with test sequences that extend to specializations of the terminal limit group of one of the auxiliary resolution, Y_1DRes , and then force either one of finitely many Diophantine conditions or a further extension to the terminal limit group of one of the auxiliary resolutions, X_1Y_1DRes .

If the obtained resolution is not a proper quotient resolution of $GRes_{n_1}$, we pass to a further subtriangle, in which test sequences of a cover of the obtained quotient resolution (which is a closure of $GRes_{n_1}$), extend to specializations of the terminal limit group of one the auxiliary resolutions, $Y_2X_1Y_1DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group.

We continue iteratively, according to the iterative procedure that was applied in the first n_1 steps, and according to the proof of theorem 6.1 that constructed the graded resolutions, $DRes$, and their associated auxiliary resolutions, that are all associated with the definable set, $Def(p, q)$. This iterative procedure terminates after finitely many steps according to proposition 8.3, and the proof of theorem 4.4. When the iterative procedure terminates, at step n_k , we are left with a quotient resolution that we denote, $GRes_{n_k}$, that is graded with respect to the parameters, $q_1^m, \dots, q_{n_k}^m$.

With $GRes_{n_k}$ we associate a cover resolution according to theorem 1.21, that we denote $CGRes_{n_k}$. With $GRes_{n_k}$ and its cover, $CGRes_{n_k}$, there is an associated subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, $n_k + 1 \leq i, j \leq m$, so that every specialization, p_i^m extends to a specialization that factors through the cover $CGRes_{n_k}$. By the properties of the iterative procedure that was used to construct the quotient resolution, $GRes_{n_k}$, for any specialization, q_j^m , $n_k + 1 \leq j \leq m$, and any specialization, p_i^m , $j \leq i \leq m$, there exists a sequence of specializations, $\{(p_i^m(s), q_j^m)\}$, where the specializations, $\{p_i^m(s)\}$, are the restriction to the variables p , of a test sequence of specializations in the fiber that contains p_i^m , so that the sequence, $\{(p_i^m(s), q_j^m)\}$, extend to specializations, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m)\}$, that are all specializations of the terminal limit group of a resolution $DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, and each of these specializations testify that the corresponding specialization, $\{(p_i^m(s), q_j^m)\}$, is in the definable set $Def(p, q)$.

At this point we swap the rows in the subtriangle of specializations, as we did in proving the stability of the set $ES(p, q)$ (theorem 8.6). For each i , $n_k + 1 \leq i \leq m$, we replace p_i^m with $p_{m+n_k+1-i}^m$, and for each j , $n_k + 1 \leq j \leq m$, we replace q_j^m with $q_{m+n_k+1-j}^m$. Note that after this swap, for each pair of indices, $n_k + 1 \leq i, j \leq m$, $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $i \leq j$.

Each specialization p_i^m , $n_k + 1 \leq i \leq m$, belongs to some fiber of the graded resolution, $CGRes_{n_k}$. If there is a subtriangle of our given triangle, (still denoted) $\{p_i^m\}$ and $\{q_j^m\}$, $n_k + 1 \leq i, j \leq m$, for which for every triple (i, j, m) , $n_k + 1 \leq j < i \leq m$, there is no sequence of specializations of the terminal limit group of one of the resolutions, $DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, $\{(t_{i,j}^m(s), p_i^m(s), q_j^m)\}$, where the sequence, $\{p_i^m(s)\}$, is a restriction to the elements p of a test sequence in the fiber that contains p_i^m (in the variety that is associated with $CGRes_{n_k}$), and these specializations of the terminal limit group of $DRes$, tes-

tify that $(p_i^m(s), q_j^m) \in Def(p, q)$, we reached a terminal point of the first procedure (for proving the stability of $Def(p, q)$), and we continue with the subtriangle, and the graded resolution, CGR_{n_k} , and the graded resolution, GR_{n_k} , to the second part of the proof (in which we analyze ungraded resolutions and not graded ones, as we did along the second iterative procedure in proving the stability of $ES(p, q)$).

Otherwise, there is a subtriangle with this property, and we pass to that subtriangle. In this case there exists a further subtriangle of the triangle that is associated with CGR_{n_k} , (that we still denote) $\{p_i^m\}$ and $\{q_j^m\}$, $n_k + 1 \leq i, j \leq m$, for which:

- (1) for every pair (i, m) , $n_k + 1 < i \leq m$, there exists a sequence of specializations of the terminal limit group of one of the resolutions, $DRes$, that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of that terminal limit group, $\{(t_{i, n_k+1}^m(s), p_i^m(s), q_{n_k+1}^m)\}$, and an associated test sequence of the cover graded resolution, CGR_{n_k} , $(z_{i, n_k+1}^m(s), p_i^m(s), q_1^m, \dots, q_{n_k}^m)$, that is all in the fiber that contains p_i^m . Furthermore, the specializations, $\{(t_{i, n_k+1}^m(s), p_i^m(s), q_{n_k+1}^m)\}$, testify that the tuples, $(p_i^m(s), q_{n_k+1}^m) \in Def(p, q)$.
- (2) every combined sequence, $\{(z_{i_m, n_k+1}^m(s_m), t_{i_m, n_k+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_k}^m, q_{n_k+1}^m)\}$, where $n_k + 1 < i_m \leq m$, and $m < s_m$, converges into the same graded resolution, $FGRes_{n_k+1}$ (it is graded with respect to the parameter subgroup $\langle q_1, \dots, q_{n_k+1} \rangle$. In particular, the sequence: $\{(z_{i_m, n_k+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_k}^m)\}$, converges into the graded resolution, GR_{n_k} .

With the graded resolution, $FGRes_{n_k+1}$, which is a graded formal closure of GR_{n_k} , we associate a cover graded resolution according to theorem 1.21, that we denote, $CFGRes_{n_k+1}$. We can pass to a further subtriangle, so that the sequences of specializations: $\{(z_{i_m, n_k+1}^m(s_m), t_{i_m, n_k+1}^m(s_m), p_{i_m}^m(s_m), q_1^m, \dots, q_{n_k}^m, q_{n_k+1}^m)\}$, where $n_k + 1 < i_m \leq m$, and $m < s_m$, actually factor through the graded cover resolution, $CFGRes_{n_k+1}$.

As for i, j , $n_k < i, j \leq m$, $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $i \leq j$, on the tuples, $(t_{i, j}^m, p_i^m, q_j^m)$, for $j < i$, that extend to specializations that factor through $CFGRes_{n_k+1}$, we may impose one of two conditions. Either they satisfy one of finitely many Diophantine conditions that force these specializations to restrict to non-rigid or non weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of $DRes$, or each of these specializations can be extended to a specialization of the terminal limit group of an auxiliary resolution, Y_1DRes , a specialization that restricts to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of that terminal limit group. Furthermore, these specializations of the terminal limit group of Y_1DRes testify that a specialization, $(t_{i, j}^m, p_i^m, q_j^m)$, does not testify that $(p_i^m, q_j^m) \in Def(p, q)$.

We further pass to a subtriangle, so that every sequence of specializations, $(t_{i_m, n_k+1}^m, p_{i_m}^m, q_{n_k+1}^m)$, $n_k + 1 < i_m \leq m$, together with the specialization of the Diophantine condition that forces it to restrict to non rigid or non weakly strictly solid specialization of the terminal limit group of the associated graded resolution, $DRes$, or the specialization of the terminal limit group of an auxiliary resolution, Y_1DRes , converges into a quotient resolution of $FGRes_{n_k+1}$, that we denote, GR_{n_k+1} . GR_{n_k+1} is either a proper quotient resolution of $FGRes_{n_k+1}$ (which is a closure of GR_{n_k}), i.e., a quotient resolution of strictly smaller complexity than GR_{n_k} according to the sieve procedure [Se6], or it is a closure of $FGRes_{n_k+1}$. If GR_{n_k+1} is a proper quotient resolution of GR_{n_k} we reached a terminal point

of this step of the procedure, and we continue to the next step.

If $GRes_{n_k+1}$ is a closure of $GRes_{n_k}$, we associate with it a cover, $CGRes_{n_k+1}$, according to theorem 1.21, with a f.p. completion and terminal limit group. We further pass to a subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, $n_k + 1 \leq i, j \leq m$, for which, $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $i \leq j$, and the associated specializations, $(y_{i, n_k+1}^m, t_{i, n_k+1}^m, p_i^m, q_{n_k+1}^m)$, $n_k + 1 < i \leq m$, extend to specializations that factor through the cover $CGRes_{n_k+1}$.

In the variety that is associated with the cover, $CGRes_{n_k+1}$, generic points (test sequences) in the fibers that contain the specializations that restrict to the tuples, $(y_{i, n_k+1}^m, t_{i, n_k+1}^m, p_i^m, q_{n_k+1}^m)$, $n_k + 1 < i \leq m$, restrict to specializations of the terminal limit group of the auxiliary resolution, Y_1DRes , that extend to specializations of the terminal limit group of at least one of the auxiliary resolutions, X_1Y_1DRes , that restrict to rigid or weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of that auxiliary resolution. Furthermore, these specializations of the terminal limit group of X_1Y_1DRes , testify that the corresponding specializations of Y_1DRes , do not form an obstacle for proving that $(p_i^m(s), q_{n_k+1}^m) \in Def(p, q)$ for $n_k + 1 < i \leq m$, and restrictions of test sequences of $CGRes_{n_k+1}$ to the variables $p, \{p_i^m(s)\}$.

We pass to a further subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, and use the test sequences that extend to specializations of the terminal limit group of an auxiliary resolution, X_1Y_1DRes , to associate a graded closure with $CGRes_{n_k+1}$. With this graded closure we associate a cover with a f.p. completion and terminal limit group according to theorem 1.21.

We pass to a further subtriangle, for which the specializations,

$$(x_{i, n_k+1}^m, y_{i, n_k+1}^m, t_{i, n_k+1}^m, p_i^m, q_{n_k+1}^m)$$

$n_k + 1 < i \leq m$, extend to specializations that factor through the constructed cover resolution. On the tuples, $(x_{i, n_k+1}^m, y_{i, n_k+1}^m, t_{i, n_k+1}^m, p_i^m, q_{n_k+1}^m)$, that extend to specializations that factor through the constructed cover resolution, we may impose one of two conditions, similar to the ones that we imposed on the specializations of the cover resolution, $CGRes_{n_k+1}$. Either they satisfy one of finitely many Diophantine conditions that force these specializations to restrict to non-rigid or non weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of X_1Y_1DRes , or each of these specializations can be extended to a specialization of the terminal limit group of an auxiliary resolution, $Y_2X_1Y_1DRes$, a specialization that restricts to a rigid or a weakly strictly solid specialization of the rigid or weakly solid factor of that terminal limit group. Furthermore, these specializations of the terminal limit group of $Y_2X_1Y_1DRes$ testify that the specializations, $(x_{i, n_k+1}^m, y_{i, n_k+1}^m, t_{i, n_k+1}^m, p_i^m, q_{n_k+1}^m)$, do not testify that $(p_i^m, q_j^m) \in Def(p, q)$ for $n_k + 1 < i \leq m$.

We further pass to a subtriangle, so that every sequence of specializations, $(x_{i, n_k+1}^m, y_{i, n_k+1}^m, t_{i, n_k+1}^m, p_i^m, q_{n_k+1}^m)$, $n_k + 1 < i \leq m$, together with the specialization of the Diophantine condition that forces them to restrict to non rigid or non weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of the auxiliary resolution, X_1Y_1DRes , or the specialization of the terminal limit group of an auxiliary resolution, $Y_2X_1Y_1DRes$, converges into a quotient resolution of $GRes_{n_k+1}$. This quotient resolution is either a proper quotient resolution of $GRes_{n_k+1}$ (which is a closure of $GRes_{n_k}$), i.e., a quotient

resolution of strictly smaller complexity than $GRes_{n_k}$ according to the sieve procedure [Se6], or it is a closure of $GRes_{n_k+1}$. If the constructed quotient resolution is a proper quotient resolution of $GRes_{n_k}$ we reached a terminal point of this step of the procedure, and we continue to the next step.

If the constructed quotient resolution is a closure of $GRes_{n_k}$, we continue iteratively along the steps of the auxiliary resolutions that are associated with the graded resolution, $DRes$, and the given definable set, $Def(p, q)$, in theorem 6.1. At each step we first pass to a subtriangle and associate a closure with the previously constructed quotient resolution, and then impose either a Diophantine condition that forces the collected specializations to restrict to non rigid or non weakly strictly solid specializations of the rigid or weakly solid factor of the terminal limit group of the corresponding auxiliary resolution, or force the corresponding specializations to extend to specializations of the terminal limit group of an associated auxiliary resolution. Then we pass to a further subtriangle and use it to construct a new quotient resolution that is either a closure of $GRes_{n_k}$ or it is a proper quotient resolution, i.e. a quotient resolution of strictly smaller complexity according to the sieve procedure.

By theorem 6.1, and our assumptions that $\{(p_i^m, q_j^m)\} \in Def(p, q)$ if and only if $n_k + 1 < i \leq m$, after finitely many such steps, we must reach a point in which the constructed quotient resolution is indeed a proper quotient resolution.

We continue iteratively, and this iterative procedure terminates after finitely many steps by proposition 8.3 and the argument that was used to prove theorem 4.4. When it terminates we are left with a subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, for which $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $i \leq j$. Furthermore, with the subtriangle there is an associated quotient resolution, $GRes_{b_1}$, that is graded with respect to a tuple, q_1, \dots, q_{b_1} . With $GRes_{b_1}$ there is an associated cover (graded) resolution, $CGRes_{b_1}$, and all the specializations p_i^m extend to specializations that factor through the cover, $CGRes_{b_1}$. Furthermore, for every $b_1 + 1 \leq j < i \leq m$, there is no test sequence in the fiber that contains p_i^m (in the variety that is associated with the closure, $CGRes_{b_1}$), that restrict to specializations, $\{p_i^m(s)\}$, and the tuples $(p_i^m(s), q_j^m) \in Def(p, q)$.

At this point we once again swap the rows in the subtriangle of specializations, so that after the swap, in the subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, $b_1 + 1 \leq i, j \leq m$, $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $j \leq i$. We continue according to the first iterative procedure, until we obtain a quotient resolution, with an associated cover and subtriangle of specializations, so that for each pair in the remaining subtriangle of specializations, for every $j \leq i$, and p_i^m , a test sequence in the fiber that is associated with p_i^m restricts to a sequence of specializations, $\{p_i^m(s)\}$, so that $(p_i^m(s), q_j^m) \in Def(p, q)$.

After this first iterative procedure terminates we swap the rows in the remaining subtriangle triangle once again, and continue iteratively along the steps of the second iterative procedure. This (second) iterative procedure terminates after finitely many steps, and we continue iteratively. At each step we first swap the rows, and then continue along the first or second iterative procedure until they terminate (after finitely many steps), swap, and continue along the other procedure.

As in the analysis of the set $ES(p, q)$ in theorem 8.6, this iterative procedure of swapping and continuing along the two iterative procedures until they terminate, terminates after finitely many steps, by the proof of theorem 4.4. Once it terminates, we are left with a graded resolution, $GRes_f$, which is graded with

respect to the parameter subgroup, $\langle q_1, \dots, q_f \rangle$, a cover of this graded resolution that satisfies the properties of cover graded resolutions that are listed in theorem 1.21, $CGRes_f$, and a subtriangle of specializations, (still denoted) $\{p_i^m\}$ and $\{q_j^m\}$, $f+1 \leq i, j \leq m$, that extend to specializations that factor through the cover graded resolution, $CGRes_f$, for which $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $f+1 \leq j \leq i \leq m$. Furthermore, each specialization p_i^m extends to a specialization that factors through $CGRes_f$. A test sequence of specializations in the fiber of $CGRes_f$ that contains p_i^m , restricts to specializations, $\{p_i^m(s)\}$, that satisfy $(p_i^m(s), q_j^m) \in Def(p, q)$ if and only if $f+1 \leq j \leq i \leq m$.

Now, we can continue to the second part as we did in proving the stability of varieties, Diophantine sets, and the sets $ES(p, q)$ (theorems 8.2, 8.5 and 8.6). The (cover) graded resolution, $CGRes_f$, terminates in a free product of a rigid or a weakly solid factor with (possibly) finitely many elliptic factors. By theorems 1.14 and 1.15, with the rigid or solid factor, one can associate finitely many (combinatorial) configurations, so that each configuration contains finitely many fractions, and finitely many elliptic elements, and a rigid or almost shortest (weakly) strictly solid specialization is given by fixed words in the fractions and the elliptic elements. The value of these fractions depend only on the specialization of the defining parameters, and not on the (specific) rigid or almost shortest (weakly) strictly solid specialization, whereas the elliptic elements do depend on the specific rigid or weakly strictly solid specialization (and not only on the specialization of the defining parameters).

We can pass to a further subtriangle of specializations, $\{p_i^m\}_{i=f+1}^m$ and $\{q_j^m\}_{j=f+1}^m$, so that $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $j \leq i$, and assume that the specializations of the rigid or weakly solid factor of the terminal limit group of $CGRes_f$ that are associated with the specializations, $\{(p_i^m, q_j^m)\}$, from the subtriangle, are all associated with one (fixed) combinatorial configuration (out of the finitely many combinatorial configurations) that is presented in theorems 1.14 and 1.15, and is associated with the rigid or weakly solid factor of the terminal limit group of $CGRes_f$.

We denote the fractions that appear in the (fixed) combinatorial configuration (that get the same values for all the rigid or almost shortest strictly solid specializations that are associated with the same specializations of the parameters (q_1^m, \dots, q_f^m) , and with the fixed combinatorial configuration), v_1^m, \dots, v_e^m , and for brevity v^m . We denote the elliptic elements that are associated with the values of the parameters (q_1^m, \dots, q_f^m) , eq_1^m, \dots, eq_d^m , and for brevity eq^m , and with the rigid or almost shortest (weakly) strictly solid specializations: ep_1^m, \dots, ep_g^m , and for brevity ep^m (see the statements of theorems 1.14 and 1.15 for these notions).

At this point we start the second part of our argument that is similar to the second part of the argument that was used to prove the stability of the set $ES(p, q)$ in theorem 8.6. This second part involves two interchanging iterative procedures, that are similar to the ones that were used in the first part, and in which all the resolutions are ungraded, i.e., they terminate in elliptic subgroups.

First, with each element, p_i^m , $f < i \leq m$, from the triangle of specializations that is associated with the cover graded resolution, $CGRes_f$, we associate a tuple of elliptic specializations, ep_i^m , that is associated with it and with the fixed combinatorial configuration that is associated with the rigid or weakly solid factor of the terminal limit group of $CGRes_f$, and with the fixed values of the elements, v^m .

We continue with the triangle of specializations, $\{ep_i^m\}$ and $\{q_j^m\}$, $f < i, j \leq$

m . We swap each line of the triangle. After this swap, the corresponding pair, $(p_i^m, q_j^m) \in Def(p, q)$, $f < i, j \leq m$, if and only if $i \leq j$.

We apply the first iterative procedure for the triangle of specializations. At each step we pass to a further subtriangle, and require that specializations of the terminal limit group of one of the graded resolution, $DRes$, or of the terminal limit group of one of its associated auxiliary resolutions, and are associated with (or extend) the pairs, (p_i^m, q_j^m) , $i \leq j$, will converge into the same limit group, and the specializations of the quotient resolution that was constructed in the previous step, that are extended to specializations of these terminal limit group converge into a quotient resolution of the resolution that was constructed in the previous step.

Precisely as we argued in the first part of the procedure, after finitely many steps we obtain a quotient resolution, $GRes_{\ell_1}$, and a subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, $\ell_1 + 1 \leq i, j \leq m$ that are associated with it, with associated elliptic specializations, $\{ep_i^m\}$. With the quotient (ungraded) resolution, we associate a cover (ungraded) resolution, that terminates in a f.p. group, such that all the subtriangle of specializations factor through that cover. Furthermore, for each triple of indices i, j, m , $\ell_1 \leq i \leq j \leq m$, there exists a test sequence in the fiber that contains (v^m, q_j^m) (in the variety that is associated with the given cover of $GRes_{\ell_1+1}$), that restrict to specializations, $\{(v^m(s), ep_i^m, p_i^m(s), q_j^m(s))\}$, so that for every index s , $(p_i^m(s), q_j^m(s)) \in Def(p, q)$.

We continue as we did in the first part of the argument. First, we swap the rows in the subtriangle of specializations. Note that after this swap, for each pair of indices, $\ell_1 + 1 \leq i, j \leq m$, the corresponding specializations, $(p_i^m, q_j^m) \in Def(p, q)$, if and only if $j \leq i$.

We apply the second iterative procedure that was used in the first part of the argument, for the subtriangle of specializations. After finitely many steps we obtain a quotient resolution, $GRes_{\ell_2}$, and a subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, $\ell_2 + 1 \leq i, j \leq m$, that are associated with it, with associated elliptic specializations, $\{ep_i^m\}$. With the quotient (ungraded) resolution, we associate a cover (ungraded) resolution, that terminates in a f.p. group, such that all the subtriangle of specializations factor through that cover. Furthermore, for each triple of indices i, j, m , $\ell_2 + 1 < i < j \leq m$, there exists a test sequence in the fiber that contains q_j^m (in the variety that is associated with the given cover of $GRes_{\ell_2+1}$), that restrict to specializations, $\{(v^m(s), ep_i^m, p_i^m(s), q_j^m(s))\}$, so that for every index s , $(p_i^m(s), q_j^m(s)) \notin Def(p, q)$.

We continue as we did in the first part of the argument. At this point we once again swap the rows in the subtriangle of specializations. Then we apply the first procedure that was used in the first part of the argument, until we construct a quotient (ungraded) resolution, a cover of this quotient resolution, and a subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, with associated elliptic specializations, $\{ep_i^m\}$, that factor through the cover resolution. Furthermore, for each pair of indices i, m , $i < m$, there exists a test sequence in the fiber that contains (v^m, q_j^m) (in the variety that is associated with the given cover of the quotient resolution), that restrict to specializations, $\{(v^m(s), ep_m^i, p_i^m(s), q_j^m(s))\}$, so that for every index s , and every j , $i \leq j \leq m$, $(p_i^m(s), q_j^m(s)) \in Def(p, q)$.

Then we swap the rows of the subtriangle once again, and apply the second iterative procedure that was used in the first part of the argument. We obtain a quotient (ungraded) resolution, a cover of this quotient resolution (with a f.p.

terminal limit group), and a subtriangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, with associated elliptic specializations, $\{ep_i^m\}$, that factor through the cover resolution. Furthermore, for each triple of indices i, j, m , $i < j \leq m$, there exists a test sequence in the fiber that contains $q(v^m, \frac{m}{j})$ (in the variety that is associated with the given cover of the quotient resolution), $\{(v^m(s), ep_i^m, p_i^m(s), q_j^m(s))\}$, so that for every index s , $(p_i^m(s), q_j^m(s)) \notin Def(p, q)$.

As we argued in the proof of theorem 8.6, this iterative procedure of swapping and continuing along the two iterative procedures until they terminate, terminates after finitely many steps, by the proof of theorem 4.4. Once it terminates, we are left with an ungraded (quotient) resolution, $ERes_{\ell_t}$, a cover of this resolution that satisfies the properties of cover resolutions that are listed in theorem 1.21 (and in particular has a f.p. completion and terminal limit group), $CERes_{\ell_t}$, and a subtriangle of specializations, (still denoted) $\{ep_i^m\}$ and $\{q_j^m\}$, $\ell_t + 1 \leq i, j \leq m$ that extend to specializations that factor through the cover resolution, $CERes_{\ell_t}$, for which (the corresponding specialization) $(p_i^m, q_j^m) \in Def(p, q)$ if and only if $\ell_t + 1 \leq i \leq j \leq m$. Furthermore, each specialization q_j^m extends to a specialization that factors through $CERes_{\ell_t}$. A test sequence of specializations in the fiber of $CERes_{\ell_t}$ that contains q_j^m , restricts to specializations, $\{(v^m(s), ep_i^m, p_i^m(s), q_j^m(s))\}$, for which (for every index s), $(p_i^m(s), q_j^m(s)) \in Def(p, q)$, if and only if $\ell_t + 1 \leq i \leq j \leq m$.

For brevity, we denote the ungraded resolution that is constructed in step ℓ_t , $ERes$, and its closure, $CERes$. Recall that the subgroup $\langle q \rangle$ is embedded in the graded cover $CERes$, and the elements p can be expressed as words in the elements that are associated with v^m in $CERes$, with the elliptic elements ep_i^m . Given the cover resolution, $CERes$, we look at all its test sequences, for which the restriction of the test sequences to the elements p, q , extend to specializations of the various terminal limit groups of the resolutions, $DRes$, and of their associated auxiliary resolutions, that are all associated with $Def(p, q)$ according to theorem 6.1. All these test sequences, together with their extended specializations (to the terminal limit groups of the graded resolutions $DRes$ and their auxiliary resolutions), can be collected in finitely many graded resolutions that have similar properties to the resolutions in the formal Makanin-Razborov diagram in theorems 2.6 and 2.7. In particular, all the resolutions that are associated with these test sequences and with the closure, $CERes$, do all have f.p. completion and terminal limit groups. We denote these ungraded resolutions that are associated with the closure, $CERes$, $ACERes$.

The specializations in the subtriangle that is associated with the closure, $CERes$, have the property that a generic point (test sequence) in the fiber that contains a pair (p_i^m, q_j^m) , restricts to specializations, $(p_i^m(s), q_j^m(s)) \in Def(p, q)$, if and only if $i \leq j \leq m$. The properties of the graded resolutions, $DRes$, and their associated auxiliary resolutions, as listed in the statement of theorem 6.1, enable one to reduce a sentence over free products to a finite disjunction of conjunctions of sentences over the factors of the free product in theorem 6.3. In exactly the same way (as in the argument that was used to prove theorem 6.3), the constructed resolutions, $ACERes$, enable one to reduce the question of whether or not a tuple, (p_i^m, q_j^m) , is in $Def(p, q)$, to a finite disjunction of conjunctions of predicates in the free variables (parameters), ep_i^m and eq_j^m , over the factors A and B of the ambient free product $G = A * B$.

Hence, the existence of the (infinite) subtriangle of specializations that is as-

sociated with $CERes$, contradicts the stability of the constructed predicates over the factors A and B . Therefore, $Def(p, q)$ is stable, and theorem 8.1 follows for coefficient-free definable sets (predicates).

The stability of all coefficient-free predicates over free products of stable groups implies the stability of all predicates over such free products. Indeed, we can replace each predicate by a coefficient-free predicate, by replacing coefficients with free variables (parameters). The stability of the obtained coefficient-free predicate clearly implies the stability of the original predicate. This proves the stability of free products of two stable groups and concludes the proof of theorem 8.1. \square

Theorem 8.1 proves the stability of free products of finitely many stable groups. In case we look at a sequence of groups for which every sentence is uniformly stable for the sequence, we obtain stability of the free product of the entire sequence.

Theorem 8.7. *Let G_1, G_2, \dots be a sequence of groups. Suppose that every sentence (over groups) Φ is uniformly stable over the sequence $\{G_i\}$. Then the infinite free product, $G_1 * G_2 * \dots$, is stable.*

Proof: Let $G = G_1 * G_2 * \dots$, and let $Def(p, q)$ be a coefficient-free definable set over G . The graded resolutions, $DRes$, and their associated auxiliary resolutions, that were associated with $Def(p, q)$ in theorem 6.1, are universal (although not canonical), hence, the conclusion of theorem 6.1 is valid for all countable free products and not just for finite free products.

The iterative procedures that were used to analyze the set, $Def(p, q)$, over a free product of the form $A * B$, in the argument that was used to prove theorem 8.1, work without a change over a countable free product. Hence, if $Def(p, q)$ is not stable over G , we can associate with it a quotient ungraded resolution, $ERes$, a cover of this ungraded resolution, $CERes$, and a triangle of specializations, $\{p_i^m\}$ and $\{q_j^m\}$, $i, j \leq m$, that factor through that cover, $CERes$, precisely as we did with non-stable predicate over a finite free product.

With the cover $CERes$, we associate finitely many ungraded resolutions, $ACERes$, precisely as we did in the proof of theorem 8.1. Now, the instability of $Def(p, q)$, that is translated into the subtriangle of specializations that factor through the cover, $CERes$, implies the existence of a predicate over the various factors in the free product $G = G_1 * G_2 * \dots$, that is not uniformly stable. This contradicts our assumption on the sequence of groups, $\{G_i\}$.

As we noted in the proof of theorem 8.1, every predicate can be transformed into a coefficient-free predicate by replacing the constants by free variables (parameters). Hence, if all coefficient-free predicates are stable over the countable free product G , all the predicates are stable. \square

§9. Noetherianity

In this section we prove another basic property that a free product inherits from its factors, the Noetherianity of varieties. This problem was brought to our attention by Ilya Kazachkov [Ca-Ka]. Its proof doesn't really require what we already proved in this paper, but it does rely on the techniques that were used in constructing the Makanin-Razborov diagram for varieties over a free product [Ja-

Se]. Abderezak Ould-Houcine has informed us that he has an alternative proof of the problem.

Theorem 9.1. *Let A, B be equationally Noetherian groups. Then $A * B$ is equationally Noetherian.*

Proof: As an infinite dihedral group is linear, the theorem follows for D_∞ by Guba's theorem [Gu]. Hence, we may assume that A and B are equationally Noetherian, and $A * B$ is a non-trivial free product that is not isomorphic to D_∞ .

Suppose that there exists a coefficient-free infinite system, $\Sigma(x) = 1$, where x is a finite tuple of variables, that is not equivalent to a finite subsystem. We denote by $\Sigma_n(x) = 1$ the finite subsystem of $\Sigma(x) = 1$ that contains the first n equations (in Σ).

Since Σ is not equivalent to a finite subsystem, there must exist a sequence of integers: $n_1 < n_2 < \dots$, for which there exist specializations of the tuple x in $A * B$: x_1, x_2, \dots , such that for every index k , $\Sigma_{n_k}(x_k) = 1$ and $\Sigma_{n_{k+1}}(x_k) \neq 1$, in $A * B$.

By theorem 18 in [Ja-Se], from the sequence of specializations, $\{x_k\}$, it is possible to extract a subsequence that converges into a resolution *Res* (over free products): $L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_s$, where the L_i 's are limit groups over free products, the initial limit group, L_0 , is naturally generated by the elements x , that correspond to the variables in the system of equations, $\Sigma(x) = 1$, and L_s is a free product of finitely many elliptic factors and a (possibly trivial) free group. Note that the elliptic factors in L_s need not be finitely presented, hence, we can not deduce from theorem 18 in [Ja-Se] that a subsequence of the specializations $\{x_k\}$ factor through the resolution *Res*.

Also, since for each index k , $\Sigma_{n_k}(x_k) = 1$, and the resolution, *Res*, is obtained from a convergent subsequence of the specializations, $\{x_k\}$, all the equations in the infinite system, $\Sigma(x)$, hold as relations of the initial limit group L_0 of *Res*.

L_s is the terminal limit group of the resolution, *Res*. L_s need not be finitely presented. Let L_s be the free product, $A_1 * \dots * A_\ell * F_m$, where A_1, \dots, A_ℓ are elliptic, and F_m is a (possibly trivial) free group. Each of the elliptic factors, A_i , $i = 1, \dots, \ell$, is finitely generated and not necessarily finitely presented. Since both factors A and B of the free product $G = A * B$, are equationally Noetherian, for each factor A_i , there exists a f.p. group, FPA_i , and an epimorphism, $\tau_i : FPA_i \rightarrow A_i$, so that every homomorphism $h : FPA_i \rightarrow A$ or $f : FPA_i \rightarrow B$, factors as $h = \hat{h} \circ \tau_i$ or $f = \hat{f} \circ \tau_i$, where \hat{h} and \hat{f} are homomorphisms from A_i to A and B in correspondence.

With the (well-structured) resolution, *Res*, we associate its completion, $Comp(Res)$. We set the completion, $Comp(FPRes)$, to be the completion, that is obtained from $Comp(Res)$ by replacing the terminal limit group of *Res*, L_s , by the f.p. group, $FPL_s = FPA_1 * \dots * FPA_\ell * F_m$. L_0 is naturally embedded in $Comp(Res)$, and is generated by the elements x that correspond to the variables in the system, $\Sigma(x) = 1$. Similarly, there is a subgroup generated by the elements x in the completion, $Comp(FPRes)$.

Since a subsequence of the specializations, $\{x_k\}$ converges into the resolution, *Res*, and $Comp(FPRes)$ (and FPL_s) are finitely presented, there exists a subsequence of this subsequence (of $\{x_k\}$) that factors through $Comp(FPRes)$, i.e., each of these specializations can be extended to a specialization that factors through $Comp(FPRes)$. Since $Comp(FPRes)$ differs from $Comp(Res)$ only in the terminal limit group (FPL_s versus L_s), and since every homomorphism of the terminal

limit group of $Comp(FPRes)$, FPL_s , factors through the epimorphism onto the limit group L_s , the subsequence of the sequence of specializations, $\{x_k\}$, that factor through $Comp(FPRes)$, factor through the original completion, $Comp(Res)$ (and through the original resolution, Res). However, all the equations of the system $\Sigma(x) = 1$, hold as relations for the limit group L_0 , that is generated by the elements x that correspond to the variables of Σ . This contradicts the assumption that for each k , $\Sigma_{n_k}(x_k) = 1$, but $\Sigma_{n_{k+1}}(x_k) \neq 1$.

This proves that every coefficient free system of equations over $A*B$ is equivalent to a finite subsystem. For systems of equations with coefficients, we need to repeat the same argument, working over the specific free product $G = A * B$. In this case, we first assume that both A and B are countable (and then generalize the argument for any A and B). We start with an infinite system of equations with coefficients, $\Sigma(x, a_j, b_j) = 1$, where x is a finite tuple of variables, and a_j and b_j are (finite) tuples of elements in A and B , in correspondence, for each equation j , and the system Σ is not equivalent to a finite subsystem.

We choose the specializations $\{x_k\}$, and indices $n_1 < n_2 < \dots$, in the same way as in the coefficient-free case. i.e., $\Sigma_{n_k}(x_k, a_j, b_j) = 1$, and $\Sigma_{n_{k+1}}(x_k, a_j, b_j) \neq 1$, for each index k . By modifying the proof of theorem 18 in [Ja-Se] for systems of equations with coefficients (over the same coefficient group), there exists a subsequence of the specializations, $\{x_k\}$, that converges into a resolution with coefficients, $L_0 \rightarrow \dots \rightarrow L_s$, where L_0 is a countable limit group, that is generated by the coefficient group $A * B$ and elements x that correspond to the variables in the system Σ , and L_s , the terminal limit group, is a free product: $C * D * A_1 * \dots * A_\ell * F_m$, where $A < C$, $B < D$, the A_i 's are elliptic factors, and F_m is a possibly trivial free group. Furthermore, C is f.g. relative to A , D is f.g. relative to B , and the A_i 's are all finitely generated. Since the initial limit group L_0 is generated by A , B , and elements x that correspond to the variables in Σ , and since L_0 is obtained as a limit from a subsequence of the specializations, $\{x_k\}$, all the equations in Σ hold as relations for the limit group, L_0 (note that L_0 is a countable, not necessarily f.g. limit group, that is f.g. relative to the elliptic subgroups A and B . Still all the basic results on limit groups over free products that were obtained in [Ja-Se], including theorem 18, hold for such limit groups. The modification of the basic results of [Ja-Se] from f.g. groups to countable groups that are f.g. relative to finitely many elliptic (coefficient) groups, requires (in addition to [Ja-Se]), the techniques of [Gu] and [Se11], that allow one to analyze super-stable actions of such groups on real trees, by presenting them as inductive limits of f.g. groups, where these f.g. groups are defined iteratively by enlarging the stabilizer of a single vertex in each step).

Since both factors A and B are equationally Noetherian, we can replace the completion of the constructed resolution (with coefficients), with a completion that is f.p. relative to A and B . This means that we can replace the terminal limit group, L_s , by a free product, $FPC * FPD * FPA_1 * \dots * FPA_\ell * F_m$, where:

- (1) FPC is f.p. relative to A , and FPD is f.p. relative to B . There exist natural epimorphisms, $FPC \rightarrow C$, that restricts to the identity homomorphism on A , and $FPD \rightarrow D$, that restricts to the identity homomorphism on B , such that every homomorphism, $h : FPC \rightarrow A$ and $f : FPD \rightarrow B$, that restricts to the identity homomorphism on A and B in correspondence, factors through the corresponding natural epimorphism: $FPC \rightarrow C$ or $FPD \rightarrow D$.

- (2) A_1, \dots, A_ℓ are all finitely presented. For every index i , $i = 1, \dots, \ell$, there exists a natural epimorphism, $FPA_i \rightarrow A_i$, such that every homomorphism, $h : FPA_i \rightarrow A$ and $f : FPA_i \rightarrow B$, factors through the natural epimorphism: $FPA_i \rightarrow A_i$.

Since the modified completion is f.p. relative to A and B , there exists a subsequence of the specializations, $\{x_k\}$, that factor through this modified completion. Since the modified completion differs from the original completion only in the terminal limit group, and since every homomorphism of the terminal limit group of the modified completion into $A * B$, that restricts to the identity on A and B , factors through the epimorphism onto the limit group L_s , the terminal limit group of the original completion, the subsequence of the sequence of specializations, $\{x_k\}$, that factor through the modified completion, factor through the original completion. As in the coefficient-free case, this contradicts the fact that all the equations in the system $\Sigma(x, a_j, b_j) = 1$, hold as relations for the limit group L_0 , the initial limit group of the original resolution, that is generated by the elements x that correspond to the variables of Σ , together with A and B , as we have assumed that for each index k , $\Sigma_{n_k}(x_k, a_j, b_j) = 1$, but $\Sigma_{n_{k+1}}(x_k, a_j, b_j) \neq 1$.

This proves that every system of equations with coefficients over $A * B$, where both A and B are countable, is equivalent to a finite subsystem. For system of equations over free products $A * B$, for general equationally Noetherian A and B , we need to slightly modify our argument, and essentially reduce it to the countable case.

Once again, we start with an infinite system of equations with coefficients, $\Sigma(x, a_j, b_j) = 1$, where x is a finite tuple of variables, and a_j and b_j are (finite) tuples of elements in A and B , in correspondence, for each equation i , and the system Σ is not equivalent to a finite subsystem (A and B are general groups).

We choose the specializations $\{x_k\}$, and indices $n_1 < n_2 < \dots$, in the same way as in the countable case. i.e., $\Sigma_{n_k}(x_k, a_j, b_j) = 1$, and $\Sigma_{n_{k+1}}(x_k, a_j, b_j) \neq 1$, for each index k .

Each specialization x_k is an element in the free product $A * B$. We set A_c to be a countable subgroup of A that is generated by the elements, $\{a_j\}$, that appear in the various equations of the system, $\Sigma(x, a_j, b_j) = 1$, together with all the elements of A that appear in the words in $A * B$ that are associated with the countable collection of elements $\{x_k\}$. Similarly, we set B_c to be a countable subgroup of B that is generated by $\{b_j\}$, and the elements of B that appear in the words in $A * B$ that are associated with the elements $\{x_k\}$.

Now, both the coefficients in the equations of $\Sigma(x, a_j, b_j) = 1$, and the specializations, $\{x_k\}$, are all contained in the countable subgroup, $A_c * B_c$. Hence, we repeat the construction of the resolution in the countable case, and extract a subsequence of the specializations $\{x_k\}$, that converges into a resolution: $L_0^c \rightarrow \dots \rightarrow L_s^c$ over the countable free product, $A_c * B_c$. The initial limit group, L_0^c , is generated by A_c , B_c , and finitely many elements that are associated with the variables x . Since L_0^c is obtained as a limit of a subsequence of the specializations, $\{x_k\}$, all the equations in the system, $\Sigma(x, a_j, b_j)$, hold as relations in the limit group, L_s^c .

L_s^c , the terminal limit group, is f.g. over the groups A_c and B_c , and admit a free product: $C_c * D_c * A_1 * \dots * A_\ell * F_m$, where $A_c < C_c$, $B_c < D_c$, the A_i 's are elliptic factors, and F_m is a possibly trivial free group. We replace the subgroups C_c and D_c , by the groups, $C = C_c *_{A_c} A$, and $D = D_c *_{B_c} B$. As C_c and D_c are f.g. over

A_c and B_c in correspondence, C and D are f.g. over A and B , in correspondence. We set L_s to be $L_s = C * D * A_1 * \dots * A_\ell * F_m$. L_s is f.g. over A and B .

With the resolution $Res_c: L_0^c \rightarrow \dots \rightarrow L_s^c$, we can associate its completion, $Comp(Res_c)$. With $Comp(Res_c)$ we associate another completion, that we denote $Comp(Res)$, that is obtained from the completion, $Comp(Res_c)$, by replacing the terminal limit group of the resolution Res_c , L_s^c , with the group, L_s . Note that $Comp(Res_c) < Comp(Res)$, and since all the equations in the system, Σ , hold as relations in the limit group L_0^c , and hence in $Comp(Res_c)$, they hold as relations in the completion, $Comp(Res)$, as well.

Since both factors A and B are equationally Noetherian, we can replace the completion, $Comp(Res)$, with a completion that is f.p. relative to A and B . This means that we can replace the terminal limit group, L_s , by a free product, $FPC * FPD * FPA_1 * \dots * FPA_\ell * F_m$, that satisfy properties (1) and (2) that were used in modifying the completion in case A and B are countable. In particular, the modified free product is f.p. relative to A and B , and any homomorphism of the modified free product into $A * B$, that restricts to the identity on A and B , factors through the completion, $Comp(Res)$.

Since the modified completion is f.p. relative to A and B , there exists a subsequence of the specializations, $\{x_k\}$, that factor through this modified completion. Hence, this subsequence factor (i.e., can be extended to specializations that factor) through the completion, $Comp(Res)$. As in the coefficient-free and countable cases, this contradicts the fact that all the equations in the system $\Sigma(x, a_j, b_j) = 1$, hold as relations in $Comp(Res)$, whereas we assumed that for each index k , $\Sigma_{n_k}(x_k, a_j, b_j) = 1$, but $\Sigma_{n_{k+1}}(x_k, a_j, b_j) \neq 1$. This finally proves that $A * B$ is equationally Noetherian. □

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