DIOPHANTINE GEOMETRY OVER GROUPS VII: THE ELEMENTARY THEORY OF A HYPERBOLIC GROUP

Z. Sela[1,2]

This paper generalizes our work on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group, to a general torsion-free (Gromov) hyperbolic group. In particular, we show that every definable set over such a group is in the Boolean algebra generated by AE sets, prove that hyperbolicity is a first order invariant of a finitely generated group, and obtain a classification of the elementary equivalence classes of torsion-free hyperbolic groups. Finally, we present an effective procedure to decide if two given torsion-free hyperbolic groups are elementarily equivalent.

In [18]-[24] we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects that are required for the analysis of sentences and elementary sets defined over a free group. The techniques we developed, enabled us to present an iterative procedure for a quantifier elimination for predicates defined over a free group, and answer some of A. Tarski's problems on the elementary theory of a free group. We were also able to classify those finitely generated groups that are elementarily equivalent to a free group.

In this paper we generalize our entire work from free groups to torsion-free hyperbolic groups. In the first section we study limit groups over hyperbolic groups, generalize Guba's theorem on the equivalence of an infinite system of equations to some finite subsystem [9], and associate a canonical Makanin-Razborov diagram with a given system of equations defined over a torsion-free hyperbolic group. As over free groups, this Makanin-Razborov diagram encodes the entire set of solutions to the system.

In the second section we generalize Merzlyakov theorem on the existence of a *formal solution* associated with a positive sentence [11], and the construction of a formal solution to a general AE sentence which is known to be true over some variety defined over a free group, presented in [19], to AE sentences that are known to be true over some variety defined over a torsion-free hyperbolic group.

In the third section, we generalize our study of exceptional solutions of parametric systems of equations defined over a free group [20], to hyperbolic groups. The main result of this section is the existence of a global bound (independent of the parameters specialization) on the number of families of exceptional solutions to parametric systems of equations defined over a torsion-free hyperbolic group.

In the fourth section, we generalize the iterative procedure for validation of an AE sentence defined over a free group, to AE sentences defined over a torsion-free hyperbolic group. As over free groups, this terminating iterative procedure is the basis for our analysis of elementary sets defined over a hyperbolic group.

In the fifth section we generalize the construction of a *core resolution*, presented in [22], to resolutions defined over a torsion-free hyperbolic group. In the sixth section, we generalize the analysis of the Boolean algebra of AE sets defined over a free group ([22],[23]), to study the Boolean algebra of AE sets defined over a torsion-free hyperbolic group, and finally prove that every definable set over a torsion-free hyperbolic group is in the Boolean algebra generated by AE sets.

In the seventh section we use the quantifier elimination obtained in the sixth section, to study some basic properties of the elementary theory of a torsion-free hyperbolic group. We prove that hyperbolicity is a first order invariant of a f.g. group, i.e., we show that a f.g. group that is elementarily equivalent to a torsion-free hyperbolic group must be hyperbolic as well. We continue by classifying all the f.g. groups that are elementarily equivalent to a given torsion-free hyperbolic group. We obtain this classification, by associating an *elementary core* with a given torsion-free hyperbolic group, and prove that two torsion-free hyperbolic groups are elementarily equivalent if and only if their elementary cores are isomorphic. We conclude this section by proving that the universal theory of a torsion-free hyperbolic group is decidable, and use it to construct an effective procedure to decide if two given torsion-free hyperbolic groups are elementarily equivalent.

Some of the work presented in this paper has been generalized since it first appeared. Emina Alibegovic has studied sets of solutions to systems of equations over limit groups [1]. In a sequence of papers Daniel Groves has studied systems of equations over torsion-free relatively hyperbolic groups with abelian parabolic groups [8]. Francois Dahmani has given a simpler and more elegant solution to the universal theory of a torsion-free hyperbolic group [5].

The work that is presented in this paper, was partially supported by an Israel academy of sciences fellowship, and NSF grant no. DMS9729992 through the IAS.

§1. Γ-Limit Groups and their Makanin-Razborov Diagrams

Let $\Gamma =< \gamma_1, \ldots ,\gamma_k >$ be a torsion-free (Gromov) hyperbolic group, and let X be its Cayley graph. Let $G =< g_1, \ldots ,g_m >$ be a f.g. group. Let $\{h_j\} \subset Hom(G,\Gamma)$ be a set of homomorphisms from G to Γ, and suppose that the homomorphisms $\{h_j\}$ belong to distinct conjugacy classes (i.e., for every j_1, j_2, $1 \leq j_1 < j_2$, and every $\gamma \in \Gamma$, $\gamma h_{j_1} \gamma^{-1} \neq h_{j_2}$). For each index j we fix an element $\gamma_j \in \Gamma$ having "minimal displacement" under the action λ_{h_j} and set μ_j to be:

$$\mu_j = \max_{1 \leq u \leq m} d_X(id., \gamma_j h_j(g_u) {\gamma_j}^{-1}) = \min_{\gamma \in \Gamma} \max_{1 \leq u \leq m} d_X(id., \gamma h_j(g_u) \gamma^{-1})$$

Since the homomorphisms in the sequence $\{h_j\} \subset Hom(G,\Gamma)$ are non-conjugate, the sequence of stretching factors $\{\mu_j\}$ does not contain a bounded subsequence. We set $\{(X_j,x_j)\}_{j=1}^{\infty}$ to be the pointed metric spaces obtained by rescaling the metric on the Cayley graph of Γ, $(X, id.)$, by μ_j. (X_j,x_j) is endowed with a left isometric action of our f.g. group G via the homomorphisms $\tau_{\gamma_j} \circ h_j$ where τ_{γ_j} is the inner automorphism of Γ defined by γ_j. This sequence of actions of G on the metric spaces $\{(X_j,x_j)\}_{j=1}^{\infty}$ allows us to obtain an action of G on a real tree by passing to a Gromov-Hausdorff limit.

Proposition 1.1 ([13], 2.3). *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of δ_j-hyperbolic spaces with $\delta_\infty = \underline{\lim}\, \delta_j = 0$. Let H be a countable group isometrically acting on X_j.*

Suppose there exists a base point x_j in X_j such that for every finite subset P of H, the sets of geodesics between the images of x_j under P form a sequence of totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a δ_∞-hyperbolic space X_∞ endowed with a left isometric action of H.

Our spaces $\{(X_j, x_j)\}_{j=1}^{\infty}$ endowed with the left isometric action of G, satisfy the assumptions of the proposition and they are all trees, so they are 0-hyperbolic, hence, X_∞ is a real tree endowed with an isometric action of G. By construction, the action of G on the real tree X_∞ is non-trivial. Let $\{j_n\}_{n=1}^{\infty}$ be the subsequence for which $\{(X_{j_n}, x_{j_n})\}_{n=1}^{\infty}$ converge to the limit real tree X_∞, and let (Y, y_0) denote this (pointed) limit real tree. For convenience, for the rest of this section we denote the homomorphism $\gamma_{j_n} h_{j_n} {\gamma_{j_n}}^{-1} : G \to \Gamma$, by h_n.

As we did in section 1 of [18], with the limit tree we obtained by using the Gromov-Hausdorff topology we associate natural algebraic objects, the *kernel of the action* of G on this (limit) real tree and the quotient of G by this kernel which we call the (*strict*) Γ-*limit group*.

Definition 1.2. *The kernel of the action of the group G on the limit tree Y is defined to be:*

$$K_\infty = \{g \in G \mid \forall y \in Y \;\; g(y) = y\}$$

Having the kernel of the action, we define the (strict) Γ-limit group to be: $L_\infty = G/K_\infty$ and set $\eta : G \to L_\infty$ to be the natural quotient map.

The following simple facts on the kernel of the action and the (strict) limit group are important observations, and their proof is identical to the proof of lemma 1.3 of [18].

Lemma 1.3. *With notation of definition 1.2:*

(i) *L_∞ is a f.g. group.*
(ii) *If Y is isometric to a real line then the (strict) limit group L_∞ is f.g. free abelian.*
(iii) *If $g \in G$ stabilizes a tripod in Y then for all but at most finitely many n's $g \in ker(h_n)$ (recall that a tripod is a finite tree with 3 endpoints). In particular, if $g \in G$ stabilizes a tripod then $g \in K_\infty$.*
(iv) *Let $g \in G$ be an element which does not belong to K_∞. Then for all but at most finitely many n's $g \notin ker(h_n)$.*
(v) *L_∞ is torsion-free.*
(vi) *Let $[y_1, y_2] \subset [y_3, y_4]$ be a pair of non-degenerate segments of Y and assume the stabilizer of $[y_3, y_4]$ in L_∞, $stab([y_3, y_4])$, is non-trivial. Then $stab([y_3, y_4])$ is an abelian subgroup of L_∞ and:*

$$stab([y_1, y_2]) = stab([y_3, y_4])$$

Hence, the action of L_∞ on the real tree Y is stable.

In a torsion-free hyperbolic group every solvable subgroup is infinite cyclic and every maximal cyclic subgroup is malnormal. These properties are naturally inherited by the Γ-limit group L_∞. The proof is identical to the proof of lemma 1.4 of [18].

Lemma 1.4. *With the notation of definition 1.2:*

(i) *Let u_1, u_2, u_3 be non-trivial elements of L_∞, and suppose that $[u_1, u_2] = 1$ and $[u_1, u_3] = 1$. Then $[u_2, u_3] = 1$. It follows that every abelian subgroup in L_∞ is contained in a unique maximal abelian subgroup.*

(ii) *Every maximal abelian subgroup of L_∞ is malnormal.*

(iii) *Every solvable subgroup of the (strict) Γ-limit group L_∞ is abelian.*

Proposition 1.3 shows that the action of L_∞ on the real tree Y is stable. The original analysis of stable actions of groups on real trees applies to f.p. groups ([3]), and the limit group L_∞ is only known to be f.g. at this point, by part (i) of lemma 1.3. Still, given the basic properties of the action of L_∞ on the real tree Y that we already know, we are able to apply a generalization of Rips' work to f.g. groups obtained in [25]. In [25], the real tree Y is divided into distinct components, where on each component a subgroup of L_∞ acts according to one of several canonical types of actions. The theorem from [25] we present is going to be used extensively in the next sections and its statement uses the notions and basic definitions appear in the appendix of [15]. Hence, we refer a reader who is not yet familiar with these notions to that appendix and to [3] and [2].

Theorem 1.5 ([25],3.1). *Let G be a freely indecomposable f.g. group which admits a stable isometric action on a real tree Y. Assume the stabilizer of each tripod in Y is trivial.*

1) *There exist canonical orbits of subtrees of Y: $Y_1, \dots Y_k$ with the following properties:*
   - *(i) gY_i intersects Y_j at most in one point if $i \neq j$.*
   - *(ii) gY_i is either identical with Y_i or it intersects it at most in one point.*
   - *(iii) The action of stab (Y_i) on Y_i is either discrete or it is of axial type or IET type.*

2) *G is the fundamental group of a graph of groups with:*
   - *(i) Vertices corresponding to orbits of branching points with non-trivial stabilizer in Y.*
   - *(ii) Vertices corresponding to the orbits of the canonical subtrees $Y_1, \dots, Y_k$ which are of axial or IET type. The groups associated with these vertices are conjugates of the stabilizers of these components. To a stabilizer of an IET component there exists an associate 2-orbifold. All boundary components and branching points in this associated 2-orbifold stabilize points in Y. For each such stabilizer we add edges that connect the vertex stabilized by it and the vertices stabilized by its boundary components and branching points.*
   - *(iii) Edges corresponding to orbits of edges between branching points with non-trivial stabilizer in the discrete part of Y with edge groups which are conjugates of the stabilizers of these edges.*
   - *(iv) Edges corresponding to orbits of points of intersection between the orbits of $Y_1, \dots, Y_k$.*

Before concluding our preliminary study of (strict) Γ-limit groups and their action on the limit real tree, we present the following basic fact which is necessary in the sequel. It's proof is identical to the proof of lemma 1.6 of [18].

Lemma 1.6. *If L_∞ is a (strict) Γ-limit group acting on a limit tree Y obtained from a converging sequence of homomorphisms from a f.g. group G into a hyperbolic group Γ, and L_∞ is freely-indecomposable, then stabilizers of non-degenerate segments which lie in the complement of the discrete parts of Y are trivial in L_∞. Stabilizers of segments in the discrete components of Y are abelian subgroups of L_∞.*

By theorem 1.5 and lemma 1.6 a non-trivial (strict) Γ-limit group admits a non-trivial abelian decomposition. To further study the algebraic structure of a (strict) Γ-limit group we need to construct its canonical abelian JSJ decomposition, in a similar way to the construction of the abelian JSJ decomposition of a (F_k) limit group ([18],2). Note that since a non-trivial (strict) Γ-limit group admits an abelian decomposition, if a (strict) Γ-limit group is not abelian nor a surface group, its abelian JSJ decomposition is non-trivial. To construct the JSJ decomposition of a Γ-limit group we need to study some basic properties of abelian splittings. We start with the following lemma, which is identical to lemma 2.1 of [18].

Lemma 1.7. *Let L_∞ be a (strict) Γ-limit group, let M be a maximal abelian subgroup in L_∞, and let A be an abelian subgroup of L_∞. Then:*

(i) *If $L_\infty = U *_A V$ and M is not cyclic then M can be conjugated into either U or V.*
(ii) *If $L_\infty = U*_A$ and M is not cyclic then either M can be conjugated into U, or M can be conjugated to M', so that $L_\infty = U *_A M'$.*

By lemma 1.7 if we replace each abelian splitting of L_∞ of the form $L_\infty = U*_A$ in which A is a subgroup of a non-elliptic maximal abelian subgroup M by the amalgamated product $L_\infty = U*_A M'$, we get that every non-cyclic abelian subgroup of L_∞ is elliptic in all the abelian splittings under consideration. This will allow us to use acylindrical accessibility in analyzing all the abelian splittings of L_∞.

Definition 1.8 [25]. *A splitting of a group H is called k-acylindrical if for every element $h \in H$ which is not the identity, the fixed set of h when acting on the Bass-Serre tree corresponding to the splitting has diameter at most k.*

If a (strict) Γ-limit group $L_\infty = V_1 *_{A_1} V_2 *_{A_2} V_3 *_{A_3} V_4$, where A_1, A_2, A_3 are subgroups of a maximal abelian subgroup M that is a subgroup of V_1, then one can modify the corresponding graph of groups to a tripod of groups with V_1 in the center, and V_2, V_3, V_4 at the 3 roots. Since by lemma 1.4 every maximal abelian subgroup of L_∞ is malnormal, the Bass-Serre tree corresponding to this tripod of groups is 2-acylindrical. This sliding operation generalizes to an arbitrary (finite) abelian splitting of a limit group.

Lemma 1.9. *A splitting of L_∞ in which all edge groups are abelian and all non-cyclic abelian groups are elliptic can always be modified (by modifying boundary monomorphisms by conjugations and sliding operations) to be 2-acylindrical.*

Lemma 1.9 shows that if in all abelian splittings of L_∞ under consideration, all non-cyclic abelian subgroups are elliptic, these abelian splittings are 2-acylindrical. By lemma 1.7 if we replace abelian splittings of L_∞ having the form $L_\infty = U*_A$ in which A is a subgroup of a non-elliptic maximal abelian subgroup by the amalgamated product $L_\infty = U *_A M$, we get that every non-cyclic abelian subgroup of L_∞ is elliptic in all the abelian splittings under consideration. Hence, we may assume that all abelian splittings of L_∞ under consideration are 2-acylindrical. As in our study of (F_k) limit groups, this acylindricity finally enables one to construct the canonical abelian JSJ decomposition of a (strict) Γ-limit group (see section 2 of [18]).

Theorem 1.10 (cf. ([18],2.7)). *Suppose that L_∞ is a freely indecomposable (strict) Γ-limit group. There exists a reduced unfolded splitting of L_∞ with abelian edge groups, which we call an abelian JSJ (Jaco-Shalen-Johannson) decomposition of L_∞ with the following properties:*

- (i) *Every canonical maximal QH subgroup (CMQ) of L_∞ is conjugate to a vertex group in the JSJ decomposition. Every QH subgroup of L_∞ can be conjugated into one of the CMQ subgroups of L_∞. Every vertex group in the JSJ decomposition which is not a CMQ subgroup of L_∞ is elliptic in any abelian splitting of L_∞ under consideration.*
- (ii) *A one edge abelian splitting $L_\infty = D *_A E$ or $H_\infty = D*_A$ under consideration which is hyperbolic in another elementary abelian splitting is obtained from the abelian JSJ decomposition of L_∞ by cutting a 2-orbifold corresponding to a CMQ subgroup of L_∞ along a weakly essential s.c.c..*
- (iii) *Let Θ be a one edge splitting along an abelian subgroup $L_\infty = D *_A E$ or $L_\infty = D*_A$ under consideration, which is elliptic with respect to any other one edge abelian splitting of L_∞ under consideration. Then Θ is obtained from the JSJ decomposition of L_∞ by a sequence of collapsings, foldings, and conjugations.*
- (iv) *If JSJ_1 is another JSJ decomposition of L_∞, then JSJ_1 is obtained from the JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [16] for these notions)*

In section 4 of [18] we were able to use the cyclic JSJ decomposition of a (F_k) limit group, in order to show that (F_k) limit groups are f.p. and that a f.g. group is a limit group if and only if it is ω-residually free. When Γ is a general torsion-free hyperbolic group, Γ may contain f.g. subgroups that are not f.p. in which case there are clearly (strict) Γ-limit groups that are not f.p. Still, the ascending chain conditions that enable us to construct the Makanin-Razborov diagram of a (F_k) limit group remain valid, and enable us to construct the (canonical) Makanin-Razborov diagram of a general Γ-limit group.

Definition 1.11. *Let Γ be a torsion-free hyperbolic group, and G a f.g. group. We say that G is a Γ-limit group if G is isomorphic to a subgroup of Γ or if G is a (strict) Γ-limit group.*

We start the construction of the Makanin-Razborov diagram of a Γ-limit group as in the construction of a Makanin-Razborov diagram of a (F_k) limit group. Let Γ be a torsion-free hyperbolic group, and G a f.g. group. On the set of Γ-limit groups we define a relation. Given two Γ-limit groups, R_1, R_2, that are quotients of G, with prescribed maps $\eta_i : G \to R_i$, $i = 1, 2$ we say that $R_1 > R_2$, if there exists an epimorphism with non-trivial kernel: $\tau : R_1 \to R_2$, so that $\eta_2 = \tau \circ \eta_1$.

Theorem 1.12. *Let Γ be a torsion-free hyperbolic group, and G a f.g. group. Every decreasing sequence of Γ-limit groups that are quotients of G:*

$$R_1 > R_2 > R_3 > \dots$$

terminates after finitely many steps.

Proof: Suppose that there exists a f.g. group G and a torsion-free hyperbolic group Γ, for which there exists an infinite decreasing sequence of Γ-limit groups that are quotients of G: $R_1 > R_2 > R_3 > \dots$. W.l.o.g. we may assume that the f.g. group is a free group F_d, for some integer d. We fix such Γ, fix F_d, where d is the minimal positive integer for which there exists an infinite descending chain of Γ-limit quotients, and fix a free basis for F_d, $F_d =< f_1, \dots , f_d >$. We set C to be the Cayley graph of F_d with respect to the given generating set, and look at an infinite decreasing sequence constructed in the following way. We set R_1 to be a Γ-limit group with the following properties:

(1) R_1 is a proper quotient of F_d.
(2) R_1 can be extended to an infinite decreasing sequence of Γ-limit groups: $R_1 > L_2 > L_3 > \dots$.
(3) The map $\eta_1 : F_d \to R_1$ maps to the identity the maximal number of elements in the ball of radius 1 in the Cayley graph C, among all possible maps from F_d to a Γ-limit group L that satisfies properties (1) and (2).

We continue iteratively. At step n, given the finite decreasing sequence $R_1 > R_2 > \dots > R_{n-1}$, we choose the Γ-limit group R_n to satisfy:

(1) R_n is a proper quotient of R_{n-1}.
(2) The finite decreasing sequence of Γ-limit groups: $R_1 > R_2 > \dots > R_n$ can be extended to an infinite decreasing sequence.
(3) The map $\eta_n : F_d \to R_n$ (that is obtained as a composition of the map $F_d \to R_1$ with the sequence of proper epimorphisms: $R_i \to R_{i+1}$, $i = 1, \dots , n-1$) maps to the identity the maximal number of elements in the ball of radius n in the Cayley graph C, among all the possible maps from F_d to a Γ-limit group L_n that satisfies properties (1) and (2).

With the decreasing sequence $R_1 > R_2 > \dots$ we associate a sequence of homomorphisms $\{h_n : F_d \to \Gamma\}$. For each index n, R_n is a quotient of F_d, hence, R_n is generated by d elements that are the image of the fixed generators of F_d under the quotient map η_n.
R_n is a Γ-limit group, hence, either R_n can be embedded into Γ, or R_n is obtained from a converging sequence of homomorphisms $\{u_s : G_n \to \Gamma\}$, where G_n is a f.g. group. In the second case, in which R_n is a (strict) Γ-limit group, since R_n is generated by the image of the elements $f_1, \dots , f_d$ under the quotient map η_n, for large enough s, the images $u_s(G_n)$ are d-generated subgroups of Γ, and furthermore, they are generated by the images of d elements in the f.g. group G_n, that are mapped by the quotient map $\nu_n : G_n \to R_n$ onto the elements $\eta_n(f_1), \dots , \eta_n(f_d)$. Hence, we may assume that in both cases, the Γ-limit groups R_n are obtained as Γ-limit groups from a sequence of homomorphisms $\{v_s : F_d \to \Gamma\}$, and the image of the fixed generating set of the free group F_d, is the set of elements $\eta_n(f_1), \dots , \eta_n(f_d)$.

For each index n, we pick h_n to be a homomorphism $h_n : F_d \to \Gamma$, so that h_n is a homomorphism $v_s : F_d \to \Gamma$ for some large index s, so that h_n satisfies the following two conditions:

(i) every element in the ball of radius n of C, the Cayley graph of F_d, that is mapped by the quotient map $\eta_n : F_d \to R_n$ to the trivial element, is mapped by h_n to the trivial element in Γ. Every such element that is mapped to a non-trivial element by η_n, is mapped by h_n to a non-trivial element in Γ.

(ii) there exists an element $f \in F_d$, that is mapped to the trivial element by $\eta_{n+1} : F_d \to R_{n+1}$, for which $h_n(f) \neq 1$.

To prove theorem 1.12, we will show that the last descending sequence we constructed terminates after finitely many steps.

By construction, the set of homomorphisms $\{h_n : F_d \to \Gamma\}$ does not belong to a finite set of conjugacy classes. Hence, from the sequence $\{h_n\}$ we can extract a subsequence that converges into a (strict) Γ-limit group, that we denote R_∞. By construction, the Γ-limit group R_∞ is the direct limit of the sequence of (proper) epimorphisms:

$$F_d \to R_1 \to R_2 \to \dots$$

Let $\eta_\infty : F_d \to R_\infty$ be the canonical quotient map. Our approach towards proving the termination of descending chains of Γ-limit groups is based on studying the structure of the (strict) Γ-limit group R_∞, and its associated quotient map η_∞. We start this study by listing some basic properties of them.

Lemma 1.13.

(i) *the set of homomorphisms $\{h_n : F_d \to \Gamma\}$ do not belong to finitely many conjugacy classes, hence, R_∞ is a (strict) Γ-limit group.*

(ii) *R_∞ is not finitely presented.*

(iii) *R_∞ can not be presented as the free product of a f.p. group and freely indecomposable Γ-limit groups that do not admit an abelian splitting.*

(iv) *Let $R_\infty = U_1 * \ldots * U_t * F$ be the most refined (Grushko) free decomposition of R_∞, where F is a f.g. free group. Then there exists an index j, $1 \leq j \leq t$, for which:*

(1) U_j is not finitely presented.

(2) If B is a f.g. subgroup of F_d for which $\eta_\infty(B) = U_j$, then the restrictions $h_n|_B$ do not belong to finitely many conjugacy classes. Furthermore, if $b_1, \dots, b_p$ is a generating set for B, then for every index n, $h_n(B)$ is not isomorphic to $\eta_\infty(B)$ by an isomorphism that sends $h_n(b_j)$ to $\eta_\infty(b_j)$ for $j = 1, \dots, p$.

Proof: Part (i) follows from the construction of the sequence $\{h_n : F_d \to \Gamma\}$, since for every index n_0, there exists some element $f \in F_d$, so that $h_{n_0}(f) \neq 1$ and for some index $n_1 > n_0$ and every index $n > n_1$, $h_n(f) = 1$. To prove part (ii), suppose that R_∞ is f.p. i.e.:

$$R_\infty \;= < g_1, \dots, g_d \,|\, r_1, \dots, r_s > .$$

Then for some index n_0, and every index $n > n_0$, $h_n(r_j) = 1$ for $j = 1, \dots, s$. This implies that for some index $n_1 > n_0$, and every index $n > n_1$, each of the groups R_n is a quotient of R_∞, by a quotient map that send the generating set $g_1, \dots, g_d$ of R_∞ to the elements $\eta_n(f_1), \dots, \eta_n(f_d)$, a contradiction.

Suppose that $R_{\infty} = V_1 * \ldots * V_t * M$ where M is f.p. and each of the factors V_j is freely indecomposable and does not admit an abelian splitting. Let $B_1, \ldots, B_t$ and D be f.g. subgroups of F_d, for which $\eta_{\infty}(B_j) = V_j$ for $j = 1, \ldots, t$, and $\eta_{\infty}(D) = M$. W.l.o.g. we may assume that the free group F_d is generated by the collection of the subgroups $B_1, \ldots, B_t, D$.
Since the freely-indecomposable subgroups V_j do not admit an abelian splitting, they are not strict Γ-limit groups. Hence, for every index j, $j = 1, \ldots, t$, there exists an index n_j, so that for every index $n > n_j$, the image $h_n(B_j)$ is isomorphic to the factor V_j, via the isomorphism $\eta_{\infty} \circ h_n^{-1}$.
The factor M is assumed f.p., hence, if $D = < d_1, \ldots, d_s >$, then $M = < d_1, \ldots, d_s \mid r_1, \ldots, r_u >$. There exists an index n_0, for which for every index $n > n_0$, $h_n(r_i) = 1$, for $i = 1, \ldots, u$.

Let $m_0 > n_j$ for $j = 0, \ldots, t$. By construction, there exists an element $f \in F_d$, for which $h_{m_0}(f) \neq 1$ and $\eta_m(f) = 1$ for every $m > m_0$. Since $h_{m_0}(f) \neq 1$, $\eta_{\infty}(f) \neq 1$, which clearly contradicts the fact that $\eta_m(f) = 1$ for every $m > m_0$, Since R_{∞} is a proper quotient of all the Γ-limit groups R_m, and proves part (iii) of the lemma. Part (iv) follows by exactly the same argument.

□

R_{∞} is a Γ-limit group that is a (proper) quotient of all the Γ-limit groups, R_n. For each index n, the limit group R_n was chosen to maximize the number of elements that are mapped to the identity in the ball of radius n of F_d by the quotient map $\eta_n : F_d \to R_n$, among all the proper Γ-limit quotients of R_{n-1} that admit an infinite descending chain of Γ-limit groups. If R_{∞} admits an infinite descending chain of Γ-limit groups:

$$R_{\infty} \to L_1 \to L_2 \to \ldots$$

then the Γ-limit group L_1 admits an infinite descending chain sequence of Γ-limit groups, and since it is a proper quotient of R_{∞}, for large enough index n, the quotient map $\nu_n : F_d \to L_1$ maps to the identity strictly more elements of the ball of radius n in the Cayley graph of F_d, than the map $\eta_n : F_d \to R_n$, a contradiction. Hence, R_{∞} does not admit an infinite descending chain of Γ-limit groups.

To continue the proof of theorem 1.12, i.e., to contradict the existence of the infinite descending chain of Γ-limit groups that we constructed, we need a modification of the shortening procedure that was used in [18] for (F_k) limit groups. The shortening procedure is presented in section 3 of [18], and is used there to prove that a freely indecomposable, non-abelian (F_k) limit group admits a cyclic splitting. Since the description of the shortening procedure is rather long and involved, we prefer not to repeat it, and refer the interested reader to section 3 of [18]. The same construction that appears in [18] applies to Γ-limit groups (where Γ is torsion-free hyperbolic).
Given a f.g. group G, a torsion-free hyperbolic group Γ, and a sequence of homomorphisms $\{u_s : G \to \Gamma\}$, that converges into a Γ-limit group L_{∞}, the shortening procedure constructs another (sub) sequence of homomorphisms from a free group F_d (where the f.g. group G is generated by d elements), $\{v_{s_n} : F_d \to \Gamma\}$, so that the sequence of homomorphisms v_{s_n} converges to a Γ-limit group SQ_{∞}, and there exists a natural epimorphism $L_{\infty} \to SQ_{\infty}$.

Definition 1.14. *We call the Γ-limit group SQ_{∞}, obtained by the shortening procedure, a shortening quotient of the Γ-limit group L_{∞}.*

By construction, a shortening quotient of a Γ-limit group is, in particular, a quotient of that Γ-limit group. In the case of F_k-limit groups, a shortening quotient is always a proper quotient ([18],5.3). If the Γ-limit group we start with, L_∞, is non-cyclic and freely-indecomposable, and the sequence of homomorphisms do not "stabilize in a finite time", a shortening quotient of L_∞ is a proper quotient of it.

Proposition 1.15. *Let G be a f.g. group, let Γ be a torsion-free hyperbolic group, and let $\{u_s : G \to \Gamma\}$ be a sequence of homomorphisms that converges into an action of a non-cyclic, freely-indecomposable (strict) Γ-limit group L on some real tree Y. If for every index s, $u_s(G)$ is not isomorphic to L by the natural map that sends the images of the generators of G in $u_s(G)$ to the images of these generators in L, then every shortening quotient of L, obtained from the sequence $\{u_s\}$, is a proper quotient of L.*

Proof: Suppose that the f.g. group G is generated by d elements. A shortening quotient SQ of L is obtained from a sequence of homomorphisms $\{v_{s_n} : F_d \to \Gamma\}$ that converges into SQ. By construction, the shortening quotient SQ is a quotient of the Γ-limit group L. If the sequence of homomorphisms $\{v_{s_n}\}$ do not belong to finitely many conjugacy classes, then the shortening quotient SQ is a proper quotient of L by the shortening argument that is used in the proof of theorem 5.3 in [18]. Hence, by possibly passing to a subsequence, we may assume that the homomorphisms $\{v_{s_n}\}$ are pairwise conjugate. In this case, if SQ is isomorphic to the Γ-limit group L, then $v_{s_n}(F_d)$ is isomorphic to L for every index n, so for large enough n, the image $u_{s_n}(G)$ is isomorphic to L by an isomorphism that sends the images of the generators of G in $u_{s_n}(G)$ to the images of these generators in L, a contradiction.

□

The shortening procedure, and proposition 1.15, enable us to show that from the sequence of homomorphisms $\{h_n : F_d \to \Gamma\}$ it is possible to extract a subsequence that factors through a *finite resolution* of the Γ-limit group R_∞.

Proposition 1.16. *Let $\{h_n : F_d \to \Gamma\}$ be the sequence of homomorphisms constructed above. Then there exists a finite sequence of Γ-limit groups:*

$$R_\infty \to L_1 \to L_2 \to \ldots \to L_s$$

for which:

(i) *The epimorphisms along the sequence are proper epimorphisms.*

(ii) *Let $L_s = H_1 * \ldots * H_r * F_t$ be the (possibly trivial) Grushko's free decomposition of the terminal Γ-limit group L_s. Then there exists a subsequence, $\{h_{n_t}\}$, of the homomorphisms $\{h_n : F_d \to \Gamma\}$ so that each of the homomorphisms h_{n_t} can be written in the form:*

$$h_{n_t} = \nu_t \circ \eta_{s-1} \circ \varphi^t_{s-1} \circ \eta_{s-2} \circ \ldots \circ \varphi^t_1 \circ \eta_\infty$$

where φ^t_i is a modular automorphism of the Γ-limit group L_i for $1 \leq i \leq s-1$, and ν_t is a homomorphism $\nu_t : L_s \to \Gamma$, that embeds each of the freely indecomposable factors H_j of L_s into Γ.

(iii) *the sequence of homomorphisms $h_{n_t} : F_d \to \Gamma$ converges into a faithful action of the Γ-limit group R_∞ on a real tree Y. Furthermore, the* *entire sequence of homomorphisms h_{n_t} factors through the epimorphism $\eta_\infty : F_d \to R_\infty$, i.e., for every index t, $h_{n_t} = \hat{h}_t \circ \eta_\infty$, where $\hat{h}_t$ is a homomorphism $\hat{h}_t : R_\infty \to \Gamma$.*

Proof: By lemma 1.13 and proposition 1.15 a shortening quotient of R_∞ is a proper quotient of it. Hence, we set L_1 to be a shortening quotient of R_∞. If from the sequence of (shortened) homomorphisms that was used to construct L_1, it's possible to extract a subsequence that satisfy the properties of lemma 1.13, we continue with this subsequence, and use it to get a shortening quotient L_2 of L_1, which by proposition 1.15, is a proper quotient of L_1. Continuing this process iteratively, and recalling that every sequence of proper Γ-quotients terminates after finitely many steps, we finally get the sequence of proper epimorphisms:

$$R_\infty \to L_1 \to L_2 \to \ldots \to L_s.$$

Let $\{\nu_n : F_d \to \Gamma\}$ be the sequence of homomorphisms that was used to construct the terminal shortening quotient L_s. Let $L_s = H_1 * \ldots * H_r * F_t$ be the (possibly trivial) Grushko's decomposition of L_s. Since L_s is a terminal shortening quotient, from the sequence of homomorphisms $\{\nu_n\}$ that was used to construct L_s, it is not possible to extract a subsequence that satisfies the properties of lemma 1.13. Hence, if B_j is a subgroup of F_d that is mapped onto a factor H_j by the associated quotient map $\lambda_s : F_d \to L_s$, then there exists some index n_j, so that for every index $n > n_j$, $h_n(B_j)$ is isomorphic to H_j by the map $\lambda_s \circ h_n|_{B_j}{}^{-1}$.

We prove part (ii) of the proposition by induction on the number of levels in the sequence: $R_\infty \to L_1 \to \ldots \to L_s$. If the sequence has length 1, i.e., if $R_\infty = L_s$, the claim of part (ii) follows from the argument given above, i.e., it follows since $R_\infty = L_s$ does not admit a proper shortening quotient. Suppose the claim of part (ii) holds for the partial sequence: $L_{j+1} \to \ldots \to L_s$. Since every non-trivial abelian subgroup of a torsion-free hyperbolic group is infinite cyclic, it follows that every abelian subgroup of the Γ-limit group L_j is a f.g. free abelian group. Therefore, every vertex group in the abelian JSJ decomposition of L_j is finitely generated. Hence, by our induction hypothesis, the claim of the proposition holds for the restriction of the homomorphism to the pre-image of every non-abelian, non-QH vertex group in the abelian JSJ decomposition of L_j. Since the claim holds for every non-abelian, non-QH vertex group and all the other vertex groups and edge groups in the abelian JSJ decomposition of L_j are f.p. the claim of the proposition is valid for L_j. Therefore, by induction it holds for a subsequence of the original sequence of homomorphisms from R_∞ to Γ. Finally, part (iii) of the proposition is a direct consequence of part (ii).

□

The homomorphisms $\{h_n : F_d \to \Gamma\}$ were chosen so that for every index n, there exists some element $f \in F_d$, for which $\eta_{n+1}(f) = 1$ and $h_n(f) \neq 1$. Since R_∞ is a (proper) quotient of all the Γ-limit groups R_n, for every index n and every element $f \in F_d$, if $\eta_{n+1}(f) = 1$ then $\eta_\infty(f) = 1$. By part (iii) of proposition 1.16, from the sequence $\{h_n\}$ it is possible to extract a subsequence $\{h_{n_t} : F_d \to \Gamma\}$, that factors through the Γ-limit group R_∞, i.e., $h_{n_t} = \hat{h}_t \circ \eta_\infty$, where $\hat{h}_t : R_\infty \to \Gamma$. Hence, for every index t, and every element $f \in F_d$, if $\eta_{n_t+1}(f) = 1$ then $\eta_\infty(f) = 1$, so $h_{n_t}(f) = 1$, a contradiction to the construction of the homomorphisms $\{h_n\}$, which finally concludes the proof of theorem 1.12.

□

Theorem 1.12 clearly implies that a Γ-limit group is Hopf, i.e., that every epimorphism from a Γ-limit group onto itself is an automorphism. Therefore, the relation defined on Γ-limit groups is indeed a partial order.

Let $\Gamma =< \gamma_1, \ldots, \gamma_k >$ be a torsion-free hyperbolic group with a Cayley graph X, and let $G =< g_1, \ldots, g_m >$ be a f.g. group. To analyze the entire set of homomorphisms from the f.g. group G into the hyperbolic group Γ, we start with the following theorem that generalizes proposition 1.16, and associates a resolution with a subsequence of a given sequence of homomorphisms from G to Γ.

Theorem 1.17. *Let G be a f.g. group, let Γ be a torsion-free hyperbolic group, and let $\{h_n \mid h_n : G \to \Gamma\}$ be a sequence of homomorphisms from G into Γ. Then there exists a finite sequence of Γ-limit groups:*

$$G \to L_1 \to L_2 \to \ldots \to L_s$$

for which:

(i) *$\eta_0 : G \to L_1$ is an epimorphism and $\eta_i : L_i \to L_{i+1}$ is a proper epimorphism for $1 \leq i \leq s-1$.*

(ii) *Let $L_s = H_1 * \ldots * H_r * F_t$ be the (possibly trivial) Grushko's free decomposition of the terminal Γ-limit group L_s. Then there exists a subsequence $\{h_{n_t}\}$ of the homomorphisms $\{h_n : G \to \Gamma\}$ so that each of the homomorphisms h_{n_t} can be written in the form:*

$$h_{n_t} \; = \nu_t \circ \eta_{s-1} \circ \varphi^t_{s-1} \circ \eta_{s-2} \circ \ldots \circ \varphi^t_1 \circ \eta_0$$

where φ^t_i is a modular automorphism of the Γ-limit group L_i for $1 \leq i \leq s-1$, and ν_t is a homomorphism $\nu_t : L_s \to \Gamma$, that embeds each of the freely indecomposable factors H_j of L_s into Γ.

(iii) *the sequence of homomorphisms $h_{n_t} : G \to \Gamma$ converges into a faithful action of the Γ-limit group L_1 on a real tree Y_1. Furthermore, the entire sequence of homomorphisms h_{n_t} factor through the epimorphism $\eta_0 : G \to L_1$, i.e., for every index t, $h_{n_t} = \hat{h}_t \circ \eta_0$, where $\hat{h}_t$ is a homomorphism $\hat{h}_t : L_1 \to \Gamma$.*

Proof: A subsequence of the homomorphisms $\{h_n\}$ converges into an action of a Γ-limit group L_1 on some real tree. If there exists a subsequence of that sequence that satisfies the properties listed in lemma 1.13, we use it to get a shortening quotient L_2, which by proposition 1.15 is a proper quotient of L_1. We continue iteratively obtaining proper shortening quotients that are proper quotients, and by theorem 1.12 the process terminates after finitely many steps. This give us the sequence:

$$G \to L_1 \to L_2 \to \ldots L_s.$$

Parts (ii) and (iii) of the theorem follow by the argument that was used to prove proposition 1.16.

□

Theorems 1.12 and 1.17 enable us to obtain the ascending chain conditions that are needed in order to construct the (canonical) Makanin-Razborov diagram associated with a system of equations over a torsion-free hyperbolic group. We start with two immediate implications.

Let Γ be a non-elementary torsion-free hyperbolic group. A f.g. group G is called ω-residually Γ if for any finite set of elements $g_1, \dots, g_n \in G$, there exists a homomorphism $h : G \to \Gamma$ that maps these elements into distinct elements in Γ.

Proposition 1.18. *Let Γ be a non-elementary torsion-free hyperbolic group. A f.g. group G is ω-residually Γ if and only if it is a Γ-limit group.*

Proof: From the definition of an ω-residually Γ group, an ω-residually Γ group is a Γ-limit group. Suppose G is a Γ-limit group. Then G is obtained as a limit of a sequence of homomorphisms $\{h_n : U \to \Gamma\}$. By theorem 1.17, by further passing to a subsequence (still denoted $\{h_n\}$) we may assume that all these homomorphisms factor through the Γ-limit group G. Since the sequence converges into G, for every finite set of elements of G, there exists some n_0, so that for every $n > n_0$, the homomorphism h_n maps the finite set into distinct elements. Hence, G is ω-residually Γ.

□

Proposition 1.19. *Let Γ be a non-elementary torsion-free hyperbolic group. Then there are countably many Γ-limit groups.*

Proof: A f.g. group has only countable f.g. subgroups. Since by theorem 1.16 every Γ-limit group is obtained from finitely many f.g. subgroups of Γ using finitely many iterations of free products and amalgamated free products and HNN extensions along (f.g.) abelian subgroups, and free products and amalgamated products with surface groups and f.g. abelian groups, there are only countably many Γ-limit groups.

□

We continue by proving that there are maximal elements in the set of all Γ-limit groups that are all quotients of a (fixed) f.g. group G, and that there are only finitely many equivalence classes of such maximal elements.

Proposition 1.20. *Let G be a f.g. group and Γ a torsion-free hyperbolic group. Let $R_1, R_2, \dots$ be a sequence of Γ-limit groups that are all quotients of the f.g. group G, and for which:*

$$R_1 < R_2 < \dots$$

Then there exists a Γ-limit group R that is a quotient of G, so that for every index m, $R > R_m$.

Proof: For each Γ-limit group R_m, we choose a homomorphism $h_m : G \to \Gamma$, that factors through the quotient map $\eta_m : G \to R_m$, i.e., $h_m = h'_m \circ \eta_m$, so that h'_m maps the elements in a ball of radius m in R_m into distinct elements in γ.
A subsequence of the homomorphisms $\{h_m\}$ converges into a Γ-limit group R, which is a Γ-limit quotient of G, and by theorem 1.17 we may assume that they all factor through the quotient map $\eta : G \to R$. Hence, by construction, $R > R_m$ for every index m.

□

The combination of propositions 1.19 and 1.20 implies that there are maximal elements in the set of Γ-limit groups obtained from sequences of homomorphisms from a fixed f.g. group G into Γ. Recall that we say that two Γ-limit quotients of G, $\eta_1 : G \to R_1$, $\eta_2 : G \to R_2$, are equivalent, if there exists an isomorphism $\tau : R_1 \to R_2$, so that $\eta_2 = \tau \circ \eta_1$.

Proposition 1.21. *Let G be a f.g. group, and Γ a torsion-free hyperbolic group. Then there are only finitely many equivalence classes of maximal elements in the set of Γ-limit groups that are quotients of G.*

Proof: Let G be a f.g group, and $R_1, R_2, \dots$ an infinite sequence of (non-equivalent) maximal Γ-limit quotients of it. Each R_i is equipped with a given quotient map $\eta_i : G \to R_i$, hence, fixing a generating set for G, we fix a generating set in each of the R_i's. i.e., we have maps $\nu_i : F_d \to R_i$.

For each index i we look at the collection of words of length 1 in F_d that are mapped to the identity by ν_i. There is a subsequence of the R_i's for which this (finite) collection of words is identical. Starting with this subsequence, for each R_i we look at the collection of words of length 2 in F_d that are mapped to the identity by ν_i, and again there is a subsequence for which this (finite) collection is identical. We continue with this process for all lengths ℓ of words in F_d, and look at the diagonal sequence (that we still denote $R_1, R_2, \dots$).

First, we choose homomorphisms $h_i : F_d \to \Gamma$ so that for words w of length at most i, $h_i(w) = 1$ iff $\eta_i(w) = 1$, and h_i factors through the quotient map $F_d \to G$. (we can do that since R_i is a limit quotient of G). After passing to a subsequence the homo. h_i converge into a Γ-limit quotient M of G. Note that in the (canonical) map $F_d \to M$, the elements of length at most i that are mapped to the identity are precisely those that are mapped to the identity by the map $F_d \to R_i$.

Now, $R_1, R_2, \dots$ are maximal limit quotients, so all (but at most 1) are not equivalent to M (by omitting the one that is equivalent to M we may assume that they are all not equivalent to M). We construct a new set of homo. $\tau_i : F_d \to \Gamma$ that factor through the quotient map $F_d \to G$. First, τ_i has the same property as h_i, i.e., the elements of length at most i that are mapped to the identity by τ_i are precisely those that are mapped to the identity by $F_d \to R_i$. Second, since R_i is maximal and is not equivalent to M, there must exist some element $u_i \in F_d$ so that u_i is mapped to the identity in M, but u_i is mapped to a non-trivial element in R_i. We require that $\tau_i(u_i) \neq 1$.

We look at a subsequence of the homo. τ_i that satisfy the conclusion of 1.17 (still denoted τ_i). This subsequence converges into a limit group that has to be M, and (by 1.17) they all factor through M. But that contradicts $\tau_i(u_i) \neq 1$.

□

As a corollary of proposition 1.21 and theorem 1.12, we get a generalization of Guba's theorem [9] for systems of equations over a free group, to an arbitrary torsion-free hyperbolic group.

Theorem 1.22. *Let Γ be a torsion-free hyperbolic group. Then every infinite system of equations in finitely many variables Σ over Γ, is equivalent to a finite subsystem of Σ.*

Proof: Going over the equations in the system Σ, we iteratively construct a directed locally finite tree. We start with the first equation, and with it we associate a one relator group G_1, generated by the variables of the system, with one relator that corresponds to the first equation of the system Σ. By proposition 1.21, with G_1 we associate its finite collection of maximal Γ-limit quotients $R_1, \dots, R_m$. With this collection, we associate a directed finite rooted tree, where in the root we place the group G_1 and in each of the other nodes we place its maximal Γ-limit quotients, each connected to the root by an edge directed from the root to the node.

We continue with the second equation w_2 in the system Σ, and with each of the maximal Γ-limit quotients $R_1, \ldots, R_m$, in parallel. If w_2 represents the trivial word in the group R_i, we leave it unchanged. If w_2 is non-trivial in R_i, we set $\hat{R}_i$ to be the (proper) quotient group $\hat{R}_i = R_i / < w_2 >$. With $\hat{R}_i$ we associate its finite collection of maximal Γ-limit quotients $L_1, \ldots, L_t$, which are all proper quotients of the Γ-limit group R_i. We further extend the locally finite tree, by adding new vertices labeled by $L_1, \ldots, L_t$, and connecting each of them by an edge directed from vertex labeled by R_i to the vertex labeled by L_i.

Continuing iteratively by adding the equations in Σ, we finally construct a locally finite tree. By theorem 1.12 every (directed) path in this locally finite tree is finite, hence, by Konig's lemma, the locally finite tree is finite. Therefore, the construction of the locally finite tree terminates after finitely many steps, so the infinite system Σ is equivalent to some finite subsystem.

□

To study the entire collection of homomorphisms from a given f.g. group into a hyperbolic group, we need to find a way to "encode" this collection. To get such "encoding" we will construct a canonical diagram associated with a Γ-limit group, in a similar way to the construction of the Makanin-Razborov diagram for an (F_k) limit group ([18],5).

Let R be a freely-indecomposable Γ-limit group and let $r_1, \ldots, r_m \in R$ be a generating set of R. To analyze the entire set of homomorphisms from R to Γ, we will need to look at the set of shortening quotients of R. However, we won't need to look at all the shortening quotients of R, but only those shortening quotients obtained from sequences of homomorphisms $\{h_n : R \to \Gamma\}$, for which for every index n, $h_n(R)$ is a proper quotient of R, i.e., those homomorphisms that do not embed R in Γ. Note that by theorem 1.15 every such shortening quotient is a proper quotient of R. For the rest of this section we will restrict our attention to those (proper) shortening quotients.

Following [18], we say that two (proper) shortening quotients S_1, S_2 of the Γ-limit group R are *equivalent*, if there exists an isomorphism $\tau : S_1 \to S_2$, so that the canonical map $\eta_2 : R \to S_2$ can be expressed as $\eta_2 = \tau \circ \eta_1 \circ \varphi$, where $\varphi \in Mod(R)$ and $\eta_1 : R \to S_1$ is the canonical map associated with the shortening quotient S_1. The notion of equivalent shortening quotients is clearly an equivalence relation on the set of couples of shortening quotients and their associated canonical maps: $\{(S_i, \eta_i : R \to S_i)\}$ of the Γ-limit group R.

Let $SQ(R, r_1, \ldots, r_m)$ be the set of (proper) shortening quotients of R. On the set $SQ(R, r_1, \ldots, r_m)$ we define a partial order similar to the one defined on Γ-limit groups. Given two shortening quotients $S_1, S_2 \in SQ(R, r_1, \ldots, r_m)$, we say that $S_1 > S_2$ if S_2 is a proper quotient of S_1 and the canonical map $\eta_2 : R \to S_2$ splits as $\eta_2 = \nu \circ \eta_1$ where $\eta_1 : R \to S_1$ is the canonical map associated with S_1 and $\nu : S_1 \to S_2$ is a proper epimorphism.

Lemma 1.23. *Let R be a freely-indecomposable Γ-limit group. Let $S_1 < S_2 < S_3 < \ldots$ (where $S_j \in SQ(R, r_1, \ldots, r_m)$) be a properly increasing sequence of (proper) shortening quotients of R. Then there exists a shortening quotient $S \in$*

$SQ(R, r_1, \dots, r_m)$ so that $S > S_j$ for every shortening quotient S_j in the increasing sequence.

Proof: Identical to the proof of proposition 1.20.

□

Lemma 1.23 proves the existence of maximal elements with respect to the partial order on the set of (proper) shortening quotients $SQ(R, r_1, \dots, r_m)$. The next lemma shows that there are only finitely many equivalence classes of maximal elements in the set of (proper) shortening quotients.

Lemma 1.24. *Let R be a freely-indecomposable Γ-limit group. The set of (proper) shortening quotients of R, $SQ(R, r_1, \dots, r_m)$, contains only finitely many equivalence classes of maximal elements with respect to its partial order.*

Proof: Identical to the proof of proposition 1.21.

□

The main significance of maximal shortening quotients is the way they "encode" and simplify all the homomorphisms from a freely-indecomposable Γ-limit group into a (torsion-free) hyperbolic group.

Proposition 1.25. *Let R be a freely-indecomposable Γ-limit group. Let $r_1, \dots, r_m \in R$ be a generating set of R, let $M_1, \dots, M_k$ be a collection of representatives of the (finite) set of equivalence classes of maximal (proper) shortening quotients in $SQ(R, r_1, \dots, r_m)$, and for $i = 1, \dots, k$ let $\eta_i : R \to M_i$ be the canonical quotient maps.*
Let $h : R \to \Gamma$ be a homomorphism which is not an embedding. Then there exists some index $1 \leq i \leq k$ (not necessarily unique!) and a modular automorphism $\varphi_h \in Mod(R)$ so that $h \circ \varphi_h$ splits through the maximal shortening quotient M_i. i.e., $h \circ \varphi_h = h_{M_i} \circ \eta_i$ where $h_{M_i} : M_i \to \Gamma$ is a homomorphism.

Proof: Identical to the proof of proposition 5.6 in [18].

□

The shortening procedure and the lemmas and propositions proved so far in this section finally allow us to present the main goal of this section, the (canonical) *Makanin-Razborov diagram* associated with a f.g. group G and a torsion-free hyperbolic group Γ. The Makanin-Razborov diagram "encodes" all possible homomorphisms from G into Γ, and as we will see later, some of its properties can be stated as a criteria for a general f.g. group to be a Γ-limit group.

Let G be a f.g. limit group and Γ a torsion-free hyperbolic group. By proposition 1.21 G has finitely many (equivalence classes of) maximal Γ-limit quotients, $R_1, \dots, R_s$. We continue with each of the maximal Γ-limit quotients in parallel, so we omit their index, and denote the maximal Γ-limit quotient of G we continue with by R. Suppose that $R = H_1 * \dots H_\ell * F_{g_1}$ is the Grushko's factorization of R, where each of the H_i's is a freely-indecomposable non-cyclic subgroup of R and F_{g_1} is a free group. Let $r_1^1, \dots, r_{m_1}^1 \in R$ be a generating set of H_1, $r_1^2, \dots, r_{m_2}^2$ be a generating set of H_2 etc. . By lemma 1.23 the sets of (proper) shortening quotients of the freely-indecomposable generalized Γ-limit groups $H_1, \dots, H_\ell$ contain maximal elements (with respect to the partial order defined above), and by lemma 1.24 there are only finitely many equivalence classes of maximal (proper) shortening quotients of each of the Γ-limit groups $H_1, \dots, H_\ell$. For $i = 1, \dots, \ell$ let $M_1^i, \dots, M_{k_i}^i$ be a collection of representatives of equivalence classes of maximal (proper) shortening quotients in $SQ(H_i, r_1^i, \dots, r_{m_i}^i)$, and let $\eta_j^i : H_i \to M_j^i$ be the canonical projection maps.

We define the *Makanin-Razborov diagram* of G iteratively. We start by mapping G into its finite collection of maximal Γ-limit quotients, and continue with each of the maximal Γ-limit quotients (which we denote R) in parallel. We continue by factoring R into its freely-indecomposable factors $H_1, \dots, H_\ell$ and the free factor F_g. From each of the factors H_i we associate k_i directed edges, starting at H_i and terminating at the maximal (proper) shortening quotient M_j^i. To each such directed edge we associate the canonical quotient η_j^i. Note that we do not proceed from the free factor F_{g_1}.
We proceed iteratively. We factor each of the M_j^i's into freely-indecomposable factors and associate with each such factor representatives for its equivalence classes of maximal (proper) shortening quotients. Since each (proper) shortening quotient of a Γ-limit group is a proper quotient of that Γ-limit group, and each sequence of properly decreasing sequence of Γ-limit groups terminates by theorem 1.12, the construction of the Makanin-Razborov diagram terminates after finitely many steps. Finally, the Makanin-Razborov diagram of a f.g. group encodes all the homomorphisms from it into a given torsion-free hyperbolic group.

Theorem 1.26. *Let G be a f.g. group. All the homomorphisms $h : G \to \Gamma$ are given by compositions of modular automorphisms of the Γ-limit groups in the diagram with the canonical maps from the Γ-limit groups into their maximal (proper) shortening quotients and finally with either embeddings of a Γ-limit group in the diagram into Γ, or general homomorphisms ("substitutions") of the terminal free groups that appear in the diagram into Γ.*

The (canonical) Makanin-Razborov diagram associated with a f.g. group, encodes all the homomorphisms from that f.g. group into the torsion-free hyperbolic group Γ. Some specific subdiagrams of it can be viewed as a criteria for a general f.g. group to be a Γ-limit group.

Definition 1.27. *A subdiagram of the Makanin-Razborov diagram in which we choose a unique maximal shortening quotient R of G, and for each free factor H_i of the limit group R we choose a unique edge connecting H_i to one of its maximal (proper) shortening quotients, and proceed iteratively choosing only one maximal (proper) shortening quotient at each stage, is called a Makanin-Razborov resolution.*

To present a criteria for a f.g. group to be a Γ-limit group we need to show the existence of special type of Makanin-Razborov resolutions which we call *strict Makanin-Razborov resolutions.*

Definition 1.28. *Let R be a freely-indecomposable Γ-limit group. We say that a (proper) shortening quotient S of R is a strict shortening quotient if:*

(i) *The subgroups generated by each non-CMQ, non-abelian vertex group together with centralizers of edge groups connected to it in the graph of groups obtained from the abelian JSJ of R by replacing each abelian vertex group* *by the direct summand containing the edge groups connected to it, and the abelian edge groups in the abelian JSJ decomposition of R are mapped monomorphically into the (proper) shortening quotient S by the canonical map $\eta : R \to S$.*

(ii) *each CMQ subgroup of R is mapped to a non-abelian subgroup of S by the canonical map η, and each boundary element of a CMQ subgroup of R is mapped to a non-trivial element in S by η.*

(iii) *Let A be an abelian vertex group in the JSJ decomposition of R, and let $A_1 < A$ be the subgroup generated by all edge groups connected to the vertex stabilized by A. Then A_1 is mapped monomorphically into S by the canonical map η.*

A Makanin-Razborov resolution is called strict if all the maximal shortening quotients appear in it are strict shortening quotients, and the terminal freely indecomposable non-cyclic groups in the resolution can be embedded into the torsion-free hyperbolic group Γ.

Finally, the existence of strict Makanin-Razborov resolutions is a criteria for a f.g. group to be a Γ-limit group. We first show that a Γ-limit group admits a strict Makanin-Razborov resolution.

Proposition 1.29. *Let R be a Γ-limit group. Then the (canonical) Makanin-Razborov diagram of R contains a strict Makanin-Razborov resolution.*

Proof: Identical to the proof of proposition 5.10 in [18].

□

Proposition 1.29 shows that a Γ-limit group admits a strict Makanin-Razborov resolution. To state a criteria for a f.g. group to be a Γ-limit group we need to define an analogue of a strict Makanin-Razborov resolution in the general context of f.g. groups.

Definition 1.30. *Let G be a f.g. group, and let:*

$$G = G_0 \xrightarrow{\nu_0} G_1 \xrightarrow{\nu_1} G_2 \xrightarrow{\nu_2} \dots \xrightarrow{\nu_{m-2}} G_{m-1} \xrightarrow{\nu_{m-1}} G_m$$

be a resolution of G, where each non-free factor in a free decomposition of the terminal group G_m can be embedded in Γ. We say that the given resolution of G is a strict MR resolution if the epimorphisms ν_i have the following properties.

First, we start with a (possible) free factorization of G, $G = G^1 * \dots * G^n * F^s$, where F^s is a free subgroup of G. Each of the factors G^j of $G = G_0$ is mapped by the epimorphism $\nu_0 : G \to G_1$ onto the factor Q^j in a free factorization $G_1 = Q^1 * \dots * Q^n$ of G_1. We further assume that each G^j can either be embedded in Γ and G^j is mapped isomorphically onto the factor Q^j, or it admits a non-trivial abelian splitting Λ_{G^j} with the following properties:

(i) *each abelian edge group in Λ_{G^j} is a maximal abelian subgroup in at least one of the vertex groups it is connected to.*

(ii) *ν_0 maps each of the subgroups generated by a non-QH (quadratically hanging), non-abelian vertex groups and the centralizers of the edges connected to it in the graph of groups obtained from the given abelian splitting by replacing each abelian vertex group with the direct summand containing the* *edge groups connected to it, and each of the abelian edge groups in Λ_{G^j} monomorphically into Q^j.*

iii) *ν_0 maps each QH vertex group in Λ_{G^j} into a non-abelian subgroup of Q^j.*

(iv) *every abelian vertex group in Λ_{G^j} is non-cyclic free abelian, and if A is an abelian vertex group in Λ_{G^j}, and $A_1 < A$ is the summand in A that contains the subgroup generated by all edge groups connected to the vertex stabilized by A in Λ_{G^j} as a subgroup of finite index, then A_1 is mapped monomorphically into Q^j by the map ν_0.*

Finally, we assume that the epimorphisms ν_i associated with the next levels of the resolution of G satisfy similar conditions to the ones listed for ν_0, and the resolution terminates when the target group admits a free decomposition in which each of the non-free factors can be embedded into the hyperbolic group Γ.

Note that a strict Makanin-Razborov resolution of a Γ-limit group R is a strict MR resolution of R, so by proposition 1.29 a Γ-limit group admits a strict MR resolution. Theorem 1.31 shows that this is also a sufficient condition for a f.g. group to be a Γ-limit group.

Theorem 1.31 (cf. ([18],5.12)). *A f.g. group G is a Γ-limit group if and only if it admits a strict MR resolution.*

Proof: The argument we use is a modification of the argument used over free groups ([18],5.12). Let:

$$G = G_0 \xrightarrow{\nu_0} G_1 \xrightarrow{\nu_1} G_2 \xrightarrow{\nu_2} \dots \xrightarrow{\nu_{m-2}} G_{m-1} \xrightarrow{\nu_{m-1}} G_m$$

be a strict MR-resolution of a f.g. group G over the torsion-free hyperbolic group Γ, and let $\{\Lambda_{G_i}\}$ be the cyclic splittings associated with each of the subgroups G_i. We need to show the existence of sequences of modular automorphisms $\{\varphi_i(n) \in Mod(G_i)\}$ for $i = 0, \dots, m-1$, and embeddings of each of the non-free factors of the terminal group G_m into Γ, and maps of the terminal free factor in G_m into Γ, that we denote $\tau(n)$, so that the limit group corresponding to the sequence of homomorphisms:

$$\{h_n : G \to \Gamma \mid h_n = \tau(n)\nu_{m-1}\varphi_{m-1}(n)\nu_{m-2}\varphi_{m-2}(n)\dots\nu_1\varphi_1(n)\nu_0\varphi_0(n)\}$$

is the group G itself.

To construct the homomorphisms $\tau(n)$ of the terminal group G_m, we fix embeddings of each of the non-free factors of G_m, and pick a couple of elements $a, b \in \Gamma$, that generate a free quasi-convex subgroup in Γ. We define $\tau(n)$, by conjugating each of the fixed embeddings of the non-cyclic, non-free factors of G_m by elements $w_i(a,b)$, and map a free basis of the free factor of G_m into elements $w_j(a,b)$, so that the length of the elements w_i, w_j grows fast with n, and the elements w_i, w_j satisfy a (small cancellation) $C'(\frac{1}{n})$ condition.
If none of the cyclic splittings Λ_{G_i} contains QH vertex groups, we may pick the automorphisms $\{\varphi_i(n)\}$ to be an increasing sequence of Dehn twists corresponding to the edges in the cyclic splittings Λ_{G_i}. In the presence of QH vertex groups we need the following technical lemma.

Lemma 1.32. *Let Q be the fundamental group of a (possibly punctured) surface S_Q of Euler characteristic at most -2, or a punctured torus. Let $\mu : Q \to \Gamma$ be a* *homomorphism and suppose that Q is mapped into a non-abelian subgroup of Γ and the image of every boundary component of Q is non-trivial. Then either:*

(i) *there exists a separating s.c.c. $\gamma \subset S_Q$ such that γ is mapped non-trivially into Γ, and the image in Γ of the fundamental groups of each of the connected components obtained by cutting S_Q along γ is non-abelian.*

(ii) *there exists a non-separating s.c.c. $\gamma \subset S_Q$ such that γ is mapped non-trivially into Γ, and the image of the fundamental group of the connected component obtained by cutting S_Q along γ is non-abelian.*

Proof: Identical to the free group case ([18],5.13).

□

By recursively applying lemma 1.32, for each surface S_{Q_j} corresponding to a QH-vertex group in the cyclic splitting $\Lambda_{G_{m-1}}$, we can find a finite set of s.c.c. on S_{Q_j}, so that each connected component of the surface obtained by cutting S_{Q_j} along this family of s.c.c. has Euler characteristic -1, and the fundamental group of each of these connected components is mapped onto a two generated non-abelian subgroup of Γ, i.e., it is either mapped monomorphically into a free subgroup of Γ, or it is mapped onto a freely indecomposable subgroup of Γ.
Since the fundamental group of each of the connected components is mapped either monomorphically into Γ, or into a freely indecomposable subgroup of Γ, given any finite collection of s.c.c. on the various surfaces associated with the QH vertex groups in $\Lambda_{G_{m-1}}$, if we extend each the cyclic splittings $\Lambda_{G_{m-1}}$ by further splitting the QH-vertex groups along the families of s.c.c. chosen according to lemma 1.32, and perform high powers of Dehn twists along the edges of the obtained cyclic decomposition, the given s.c.c. from our (fixed) finite collection will be mapped to non-trivial elements in Γ.
Hence, after performing a high power of Dehn twists, it is possible to find a new collection of disjoint, non-homotopic, non-boundary-parallel s.c.c. on each of the surfaces associated with QH vertex groups in $\Lambda_{G_{m-1}}$, so that each of the connected components obtained by cutting the surfaces along these s.c.c. has Euler characteristic -1, and the fundamental group of each such connected component is mapped onto a free quasi-convex subgroup of rank 2 in Γ. Therefore, the argument used to prove theorem 5.12 in the absence of QH-subgroups generalizes to prove that G_{m-1} is a Γ-limit group. Continuing iteratively to the upper levels of the strict MR resolution, we conclude that $G_0 = G$ is a Γ-limit group.

□

As in section 8 in [18], given a f.g. group $G =< x_1, \dots, x_n, a_1, \dots, a_k >$, and a torsion-free hyperbolic group $\Gamma =< \gamma_1, \dots, \gamma_k >$, we say that a homomorphism $h : G \to \Gamma$ is a *restricted* homomorphism, if $h(a_i) = \gamma_i$, for $1 \leq i \leq k$. So far, given a f.g. group G and a torsion-free hyperbolic group Γ, we have constructed a Makanin-Razborov diagram that encodes all the homomorphisms from G to Γ. The study of the collection of all homomorphisms from G to Γ, can be easily generalized to study the collection of restricted homomorphisms from G to Γ in the same way it was generalized in section 8 of [18]. Similarly, to encode the collection of all the restricted homomorphisms from G to Γ, we associate with G a *restricted* Makanin-Razborov diagram, constructed in a similar way to our construction of the (non-restricted) Makanin-Razborov diagram.

The understanding of the structure of (F_k) limit groups, enable us to show that every residually-free group is the sub-direct product of finitely many limit groups

([18],7.5). Given a torsion-free hyperbolic group Γ, we say that a f.g. group G is *residually*-Γ, if for each non-trivial element $g \in G$, there exists a homomorphism $h: G \to \Gamma$, for which $h(g) \neq 1$.

Theorem 1.33. *A f.g. group G is residually-Γ, if and only if it is a sub-direct product of finitely many Γ-limit groups.*

Proof: Identical to the proof of claim 7.5 in [18].

□

Studying sets of solutions to systems of equations over a torsion-free hyperbolic group Γ, and their associated (restricted) Γ-limit groups, we continue as in [18] to study parametric system of equations. Equivalently, given a f.g. group $G(x,p)$ and a torsion-free hyperbolic group Γ, we need to encode the collections of restricted homomorphisms $\{h : G(x,p,a) \to \Gamma \,|\, h(p) = p_0\}$. In parallel with what we did in sections 9-11 of [18], with a f.g. group $G(x,p,a)$ and the torsion-free hyperbolic group Γ we associate a finite collection of graded Γ-limit groups, $L_1(x,p,a), \ldots, L_s(x,p,a)$, and with each graded Γ-limit group $L_i(x,p,a)$ we associate a *graded* Makanin-Razborov diagram, precisely as we did in the case of an (F_k) limit groups. Given the graded Makanin-Razborov associated with a Γ-limit group, we can further associate with it a singular locus, as we did in section 11 of [18], and as in section 12 in [18], the entire analysis of graded Γ-limit groups can be generalized to the multi-graded case.

§2. Formal Solutions

In the first section we generalized the structure theory developed in [18] for analyzing sets of solutions to systems of equations defined over a free group, in order to study sets of solutions to systems of equations defined over a torsion-free hyperbolic group.

In [19], in order to analyze general sentences and predicates defined over a free group, we used completions, closures, formal solutions and formal limit groups. In this section we generalize these notions and constructions in the context of a general torsion-free hyperbolic group, to finally obtain a generalization of Merzlyakov's theorem for sentences and predicates defined over a hyperbolic group.

We start this section with a special case of our generalization of Merzlyakov's theorem, which is very similar to the original Merzlyakov's theorem (cf. theorem 1.2 in [19]). The proof is essentially identical to the free group case.

Proposition 2.1. *Let $\Gamma =< a_1, \ldots, a_k >$ be a non-elementary torsion-free hyperbolic group, let $w_1(x,y,a) = 1, \ldots, w_s(x,y,a) = 1$ be a system of equations over Γ, and let $v_1(x,y,a), \ldots, v_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence:*

$$\forall y \; \exists x \quad w_1(x,y,a) = 1, \ldots, w_s(x,y,a) = 1 \;\wedge\; v_1(x,y,a) \neq 1, \ldots, v_r(x,y,a) \neq 1$$

is a truth sentence over Γ. Then there exists a formal solution $x = x(y,a)$ so that each of the words $w_j(x(y,a),y,a)$ is the trivial word in the free product $\Gamma * F_y$, where F_y is the free group generated by the universal variables y, and the sentence:

$$\exists y \quad v_1(x(y,a),y,a) \neq 1, \ldots, v_r(x(y,a),y,a) \neq 1$$

is a truth sentence in Γ.
Furthermore, if the words $w_1, \dots, w_s$ *and* $v_1, \dots, v_r$ *are coefficient-free (i.e., they are words only in the variables* x *and* y *and not in the coefficients* a*), then the formal solution* $x = x(y,a)$ *can be taken to be coefficient-free, i.e.,* $x = x(y)$.

Theorem 2.1 generalizes Merzlyakov's theorem in case the universal variables y, belong to the entire affine set Γ^{ℓ}, and are not restricted to some variety. As in analyzing sentences over a free group, to get a generalization of Merzlyakov's theorem to the case in which the universal variables are restricted to some variety, we need to use completions and closures.

Let Γ be a torsion-free hyperbolic group, and $Rlim(y,a)$ a restricted Γ-limit group. As in [19] (proposition 1.10), we may replace the Makanin-Razborov diagram associated with $Rlim(y,a)$ with its (canonical) *strict* Makanin-Razborov diagram, i.e., in a diagram in which every resolution is a strict resolution. Given a restricted Γ-limit group, the notion of a *well-structured resolution* of it ([19], definition 1.11), directly generalizes to the context of torsion-free hyperbolic groups, and so is the *completion* of a well-structured resolution ([19], definition 1.12) and its basic properties (see [19], lemma 1.14).

Definition 2.2 (cf. ([19],1.16-1.17)). *Let* $\Gamma = < a_1, \dots, a_k >$ *be a torsion-free hyperbolic group,* $Rlim(y,a)$ *a (restricted)* Γ*-limit group,* $Res(y,a)$ *a well-structured resolution of* $Rlim(y,a)$*, and* $Comp(Res)(z,y,a)$ *the completion of the resolution* $Res(y,a)$*. Let* $\Gamma * H_1 * \ldots * H_m * F_s$ *be the terminal* Γ*-limit group of the resolution* $Res(y,a)$*, where* Γ *is the coefficient group,* F_s *is a free group of rank* s*, and* $H_1, \dots, H_m$ *are freely indecomposable groups that are isomorphic to subgroups of* Γ.

A closure of the resolution $Res(y,a)$*, denoted* $Cl(Res)(s,z,y,a)$*, is a well-structured resolution defined over* Γ*, that is obtained from the completion,* $Comp(Res)(z,y,a)$*, by the following modifications.*

- (i) *As in the case of* F_k*-limit groups ([19],1.16), replacing each of the (free) abelian vertex groups that appear in the various abelian decompositions associated with the completion,* $Comp(Res)(z,y,a)$*, by (free) abelian supergroups that contain the original ones as subgroups of finite index.*
- (ii) *replacing each of the factors* H_j *by a freely-indecomposable group* V_j *with an associated embedding* $\nu_j : H_j \to V_j$*, and* V_j *is isomorphic to a subgroup of* Γ.

We say that a finite set of closures of the well-structured resolution $Res(y,a)$*,* $Cl_1(Res)(s,z,y,a), \dots, Cl_r(Res)(s,z,y,a)$ *is a covering closure, if every specialization* (y_0,a) *that factors through the resolution* $Res(y,a)$*, can be completed to a specialization* (s_0,z_0,y_0,a) *that factors through at least one of the closures* $Cl_i(Res)(s,z,y,a)$

Given the notions of a closure and a covering closure of a well-structured resolution over the torsion-free hyperbolic group Γ, we are finally able to obtain the existence of formal solutions associated with a general true AE-sentence defined over Γ.

Theorem 2.3 (cf. ([19],1.18)). *Let* $\Gamma = < a_1, \dots, a_k >$ *be a non-elementary torsion-free hyperbolic group, let* $Rlim(y,a)$ *be a restricted* Γ*-limit group, and let*

V_y be its associated variety. Let $Res(y,a)$ be a well-structured resolution of the restricted Γ-limit group $Rlim(y,a)$, and let $Comp(Res)(z,y,a)$ be the completion of the resolution $Res(y,a)$ with a corresponding completed Γ-limit group $Comp(Rlim)(z,y,a)$.

Let $w_1(x,y,a)=1,\dots,w_s(x,y,a)=1$ be a system of equations over Γ, and let $v_1(x,y,a),\dots,v_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence:

$$\forall y \in V_y \ \exists x \quad w_1(x,y,a)=1,\dots,w_s(x,y,a)=1 \wedge$$

$$\wedge \ v_1(x,y,a)\neq 1,\dots,v_r(x,y,a)\neq 1$$

is a truth sentence over Γ.

Then there exists a covering closure: $Cl_1(Res)(s,z,y,a),\dots,Cl_q(Res)(s,z,y,a)$, and for each index i, $1\leq i\leq q$ there exists a formal solution $x_i(s,z,y,a)$, so that each of the words $w_j(x_i(s,z,y,a),y,a)$ is the trivial word in the restricted limit group corresponding to the i-th closure $Cl_i(Rlim)(s,z,y,a)$.
In addition, for each index i there exists a specialization (s_0^i,z_0^i,y_0^i,a) that factors through the i-th closure $Cl_i(Res)(s,z,y,a)$, so that for every index j:

$$v_j(x_i(s_0^i,z_0^i,y_0^i,a),y_0^i,a)\neq 1$$

(in Γ).

Furthermore, if the Γ-limit group $Rlim(y,a)$ can be written as $Rlim(y,a)=Rlim(y)*\Gamma$, $Rlim(y)$ is not abelian, and the words:

$$w_1(x,y,a),\dots,w_s(x,y,a),v_1(x,y,a),\dots,v_r(x,y,a)$$

are coefficient-free, then the formal solutions can be taken to be coefficient-free, i.e., $x=x_i(u,s,z,y)$, defined over a group of the form $Cl_i(Res)(s,z,y,a)*U_1*\ldots*U_r$, where the subgroups U_j are non-cyclic, freely indecomposable, and isomorphic to subgroups of Γ.

Having constructed formal solutions associated with a truth AE sentence and a well-structured resolution defined over a torsion-free hyperbolic group Γ, we associate with a general AE sentence and a well-structured resolution defined over Γ, a canonical collection of formal Γ-limit groups and their associated formal Makanin-Razborov diagrams, precisely as we did in section 2 of [19], so that the collection of formal resolutions in the formal Makanin-Razborov diagrams encodes the entire collection of formal solutions associated with the well-structured resolution and the given AE sentence.
Given an AE predicate and a graded well-structured resolution defined over Γ, we construct finitely many graded formal Γ-limit groups, and their associated graded formal Makanin-Razborov diagrams, precisely as we did in section 3 of [19].

§3. A Bound on the Number of Rigid and Solid Solutions

In the first section we have generalized the structure theory developed in [18] for F_k-limit groups, to Γ-limit groups, for a general torsion-free hyperbolic group Γ. In the second section, we have generalized the results of [19], to prove the existence of formal solutions associated with a true AE sentence defined over a general torsion-free hyperbolic group, and further collected all such formal solutions in (graded) formal Γ-limit groups.

In this section we generalize the results of [20] to general torsion-free hyperbolic groups. We show that given a torsion-free hyperbolic group Γ, and a rigid (solid) Γ-limit group, $Rgd(x,p,a)(Sld(x,p,a))$, there exists a global bound (independent of the value of the defining parameters) on the number of rigid (strictly solid families of) solutions associated with an arbitrary value of the defining parameters.
As in [20] we prove the existence of such a global bound in two steps. Following section 1 of [20], we first study the combinatorial types of rigid and solid solutions, and prove the existence of a global constant R, for which every rigid or shortest strictly solid solution is R-AP covered. Then we use this combinatorial bound, to get a global bound on the number of rigid or strictly solid families of solutions.

Let S be the Cayley graph of a rigid graded Γ-limit group $Rgd(x,p,a)$ with respect to the generating system $Rgd(x,p,a) = < x,p,a >$, and let X be the Cayley graph (tree) of the torsion-free hyperbolic group Γ with respect to some fixed generating set $\Gamma = < a_1,\dots,a_k >$. Clearly, a homomorphism $h : Rgd(x,p,a) \to \Gamma$ corresponds to a natural equivariant map $\tau : S \to X$, where each edge in S is mapped to a (possibly degenerate) geodesic path in X.

Definition 3.1 (cf. ([20],1.1)). *Let B_R be the ball of radius R in the Cayley graph S of $Rgd(x,p,a)$. We say that a homomorphism $h : Rgd(x,p,a) \to \Gamma$ is R-AP-covered, if the union of the R-neighborhoods of the images in X of the edges labeled by either an element of $\{a\}$ or an element of $\{p\}$ in B_R covers the entire image in X of the ball B_1.*

As in [20], to define and control the combinatorial types of rigid solutions we need the following basic theorem.

Theorem 3.2. *Let $Rgd(x,p,a)$ be a rigid Γ-limit group. There exists a constant R_0 so that every rigid homomorphism $h : Rgd(x,p,a) \to \Gamma$ is R_0-AP-covered.*

Proof: Suppose that there is no such constant R_0. Let $\{h_n \mid h_n : Rgd(x,p,a) \to \Gamma\}$ be a sequence of rigid homomorphisms, so that for every integer n, the n-th homomorphism in the sequence, h_n, is not n-AP-covered. By ([13],2.3), from the sequence of rigid homomorphisms $\{h_n\}$ one can extract a subsequence converging in the Gromov-Hausdorff topology into an action of $Rgd(x,p,a)$ on a (pointed) real tree (Y,y_0).
The action of $Rgd(x,p,a)$ on the real tree Y satisfies the conclusions of lemma 1.3, hence, it is in particular a stable action. If K_∞ denotes the kernel of the action, then $L_\infty = Rgd(x,p,a)/K_\infty$ is a Γ-limit group.

Let $Y_{AP} \subset Y$ be the convex hull of the images of the base point y_0 under the action of the subgroup $AP = < a,p >$ of our rigid Γ-limit group $Rgd(x,p,a)$. Clearly, either there exists a non-degenerate segment $J \subset Y$ which is not in the orbit of Y_{AP} in Y, or the orbit of every non-degenerate segment in Y contains a non-degenerate subsegment which is in the subtree Y_{AP}.

If there exists a non-degenerate segment $J \subset Y$ which does not contain a non-degenerate subsegment that is in the orbit of Y_{AP}, then (by the argument that was used in proving theorem 1.2 in [20]) there exists a graph of groups Λ_{L_∞}, with fundamental group L_∞, for which the subgroup $AP < L_\infty$ is contained in a proper connected subgraph of groups $\Lambda' \subset \Lambda_{L_\infty}$.

Now, since AP is contained in a proper connected subgraph of groups Λ' of a graph of groups Λ_{L_∞} with fundamental group L_∞, and since the real tree Y was constructed from the homomorphisms $\{h_n : Rgd(x,p,a) \to \Gamma\}$, for large enough n, the homomorphisms $\{h_n\}$ are flexible homomorphisms, a contradiction to our assumption of the homomorphisms $\{h_n : Rgd(x,p,a) \to \Gamma\}$ being rigid.

By the argument given above, for the rest of the argument we may assume that the orbit of Y_{AP} in Y intersects any non-degenerate segment in the real tree (Y, y_0) in a non-degenerate segment. Furthermore, given any non-degenerate segment J in the real tree Y, the segment is covered in a finite time (depending only on the segment J) by the images of the edges $(y_0, a_i(y_0))$ and $(y_0, p_j(y_0))$, where the elements a_i are the generators of the coefficient group Γ, and the elements p_j are the given generators of the parameter subgroup P.

In case every non-degenerate segment in the real tree Y intersects the orbit of the tree Y_{AP} in a non-degenerate subsegment, we continue inductively by analyzing the actions of each of the point and edge stabilizers on corresponding real trees as in the proof of theorem 1.2 in [20], to finally conclude that there exists a graph of groups Θ with fundamental group L_∞, for which the subgroup AP is contained in the fundamental group of a proper subgraph Θ' of Θ, a contradiction to the assumption on the rigidity of the sequence of homomorphisms $\{h_n\}$. However, to obtain such contradiction, it is necessary to show that an induction process, similar to the one used in the proof of theorem 1.2 in [20], terminates after finitely many steps.

Let Λ be the graph of groups associated with the action of the limit group L_∞ on the real tree Y. If the graph of groups Λ gives rise to a non-trivial free decomposition of L_∞, we continue the induction process with each of the free factors in parallel, noting that a free factor in a non-trivial free decomposition of of L_∞ is a proper quotient of L_∞.

Suppose that all the edge stabilizers in the graph of groups Λ are non-trivial. With the graph of groups Λ we naturally associate a modular group, that we denote $Mod(\Lambda)$, that is generated by Dehn twists along edge groups in Λ, mapping class groups associated the QH vertex groups in Λ, and the subgroups of automorphisms of abelian vertex groups in Λ that fix (elementwise) the edge groups connected to those vertices.

We continue by applying the shortening procedure using the modular group, $Mod(\Lambda)$. With the given sequence of homomorphisms $\{h_n\}$, and the modular group $Mod(\Lambda)$, we associate a finite collection of maximal shortening quotients that are all quotients of L_∞. Let L_1 be one of the maximal shortening quotients, so that infinitely many of the homomorphisms $\{h_n\}$, can be written in the form: $h_n = h_n^1 \circ \phi_n$, where $\phi_n \in Mod(\Lambda)$ and h_n^1 is the shortest in its class under the action of the modular group $Mod(\Lambda)$.

The maximal shortening quotient L_1 is a quotient of the limit group L_∞. If L_1 is a proper quotient of L_∞ we continue to the next step of our inductive process with the limit group L_1. Hence, in the sequel, we may assume that L_1 is isomorphic to L_∞.

From the sequence of homomorphisms $\{h_n^1\}$ that factor through L_1 and are the shortest in their modular class, we can extract a subsequence that convergence into a faithful action of $L_1 = L_\infty$ on some real tree Y_1. With this action, there is an associated graph of groups $\Lambda - 1$ with fundamental group L_1. If Λ_1 gives rise to a non-trivial free product of L_1, we continue with each of the factors in this free decomposition, which is a proper quotient of $L_1 = L_\infty$. Hence, we may assume that all the edge groups in Λ_1, and in the graph of groups λ associated with the action of L_∞ on the real tree Y, are non-trivial.

First, we modify the graphs of groups Λ and Λ_1, according to lemma 1.7, to get graphs of groups in which all non-cyclic abelian groups are elliptic (we still denote the obtained graphs of groups Λ and Λ_1).

If all the edge groups in Λ are elliptic in Λ_1 and vice versa, then from the graphs of groups Λ and Λ_1, we obtain a common refinement graph of groups with fundamental group L_1, that we denote Δ_1. If Δ_1 gives rise to a non-trivial free product, we continue with each of the factors in parallel, and each factor is clearly a proper quotient of the Γ-limit group L_1. If Δ_1 does not give rise to a non-trivial free product of L_1, then Δ_1 is a proper refinement of both graphs of groups Λ and Λ_1. With Δ_1 we associate a modular group of automorphisms of L_1, that we denote $Mod(\Delta_1)$. We continue by further shortening the homomorphisms $\{h^1_n\}$, using the modular group $Mod(\Delta_1)$, and obtain an (infinite) subsequence that converge into an action of a Γ-limit group L_2 on a real tree Y_2. If L_2, which is a quotient of L_∞, is isomorphic to it, and the graph of groups Λ_2, associated with the action of L_2 on Y_2, is compatible with Δ_1 (i.e., every edge group in Δ_1 is elliptic in Λ_2 and vice versa), we construct a common refinement of Λ_2 and Δ_1, that we denote Δ_2, and repeat the process. Since the complexity of the obtained refinements is increasing, and by acylindrical accessibility the complexity of an abelian splitting of L_∞ is globally bounded, this process of properly refining the abelian decomposition of L_∞, terminates after finitely many steps.

Let E_1 be an edge in Λ, and E_2 be an edge in Λ_1. If the edge stabilizer of E_1 is elliptic in Λ_1 and the edge stabilizer of E_2 is hyperbolic in Λ (or vice versa), then by theorem 2.1 in [16], then from the combination of the graphs of groups Λ and Λ_1 it is possible to extract a non-trivial free decomposition of $L_1 = L_\infty$. Hence, in case such two edges E_1 and E_2 do exist, we continue to the next step of the inductive process with each of the factors in the non-trivial free decomposition extracted from Λ and Λ_1.

Finally, we may assume that if E_1 is an edge in Λ and E_2 is an edge in Λ_1, then their stabilizers are either elliptic-elliptic or hyperbolic-hyperbolic. We start analyzing this case, by first applying the JSJ machine presented in section 4 of [16] to analyze the hyperbolic-hyperbolic edge stabilizers. The outcome of the JSJ machine, is either a non-trivial free decomposition of $L_1 = L_\infty$, in which case we continue to the next step of the inductive process with each of the free factors in parallel. Otherwise, the JSJ machine produces a quadratic decomposition similar to the one described in theorem 4.21 in [16]. In this last case, we continue by refining the quadratic decomposition, obtained by the JSJ machine, using the elliptic-elliptic splittings in the graphs of groups Λ and Λ_1, to finally obtain an abelian decomposition of L_1, that we denote Δ_1. If Δ_1 gives rise to a non-trivial free decomposition of $L_1 = L - \infty$, we continue to the next step of the inductive process with each of the factors. Otherwise, we associate with Δ_1 a modular group, that is necessarily bigger than the modular group associated with the original graph of groups Λ, and apply the shortening procedure to obtain an (infinite) subsequence that converge into an action of a Γ-limit group L_2 on a real tree Y_2. If L_2, which is a quotient of L_∞, is isomorphic to it, we continue according to the various cases described above. Since each time in which a non-trivial free decomposition is not obtained, the complexity of the refined graph of groups Δ_n is strictly bigger than the complexity of the previous graph of groups, and by acylindrical accessibility the complexity of an abelian splitting of L_∞ is globally bounded, this process of properly refining the abelian decomposition of L_∞, terminates after finitely many steps. Hence, after finitely many steps, we the inductive process replaces the Γ-limit group L_∞ by a proper quotient of it. Since by theorem 1.12 every decreasing sequence of Γ-limit groups terminates after finitely many steps, the inductive process we've constructed terminates after finitely many steps.

□

In a similar way to rigid homomorphisms of rigid Γ-limit groups we analyze solid homomorphisms of solid Γ-limit groups. To state the analogous theorem for solid homomorphisms we will need the notions of *solid family* of specializations of a solid Γ-limit group, and strictly solid family of a solid Γ-limit group. Those are defined in ([20],1.4-1.5) for solid F_k-limit groups. The definition for solid Γ-limit groups is identical, so we don't repeat the definition of these notions.

Definition 3.3. *Let $Sld(x,p,a)$ be a solid Γ-limit group, let S be the Cayley graph of the solid Γ-limit group $Sld(x,p,a)$ with respect to the generating set $Sld(x,p,a) =< x,p,a >$, and let X be the Cayley graph of Γ with respect to the generating set $\Gamma =< a >$.*
With each abelian vertex group A_i in the graded abelian JSJ decomposition of the solid limit group $Sld(x,p,a)$, we associate a fixed retract $r_i : A_i \to A_i^1$, where A_i^1 is the direct summand of of A_i that contains the subgroup generated by the edge groups connected to the vertex stabilized by A_i as a subgroup of finite index. Clearly, the collection of retracts $\{r_i\}$ extends to a retract $r : Sld(x,p,a) \to Sld(x,p,a)$ which restricts to the identity on all the non-abelian vertex groups, and to the retract r_i on the abelian vertex group A_i.

We say that a solid homomorphism $h : Sld(x,p,a) \to \Gamma$ is among the shortest in its solid family, if $h = \hat{h} \circ r$ for some homomorphism $\hat{h} : Sld(x,p,a) \to \Gamma$, and the image of the generating set of $Sld(x,p,a)$ in the free Γ, is the shortest among all homomorphisms $h_1 : Sld(x,p,a) \to \Gamma$ that are in the same solid family of the homomorphism h, and for which there exists some homomorphism $\hat{h}_1$ for which $h_1 = \hat{h}_1 \circ r$.

In the case of a solid limit group we are able to bound the combinatorial types of strictly solid solutions which are among the shortest in their corresponding solid families.

Theorem 3.4. *Let $Sld(x,p,a)$ be a solid Γ-limit group. There exists a constant R_0 so that every strictly solid homomorphism $h : Sld(x,p,a) \to \Gamma$ which is among the shortest in its solid family, is R_0-AP-covered.*

Proof: Given the argument used to prove the existence of a global bound in the rigid case (theorem 3.2), the modification to the solid case is identical to the modification used over a free group, presented in the proof of theorem 1.7 in [20].

□

So far we have shown that for a rigid (solid) Γ-limit group $Rgd(x,p,a)$ ($Sld(x,p,a)$) there exists a constant R_0, so that all the rigid homomorphisms $h : Rgd(x,p,a) \to \Gamma$

(strictly solid homomorphisms $h : Sld(x,p,a) \to \Gamma$ which are the shortest in their solid family) are R_0-AP-covered. As in section 2 of [20], the existence of this bound on the combinatorial types of states of rigid and shortest strictly solid homomorphisms enable us to get a global bound on the number of possible distinct rigid homomorphisms $h : Rgd(x,p,a) \to \Gamma$ (strictly solid families of homomorphisms, $h : Sld(x,p,a) \to \Gamma$) for any specialization of the defining parameters $h(p) = p_0$.

Theorem 3.5 (cf. ([20],2.5)). *Let $Rgd(x,p,a)$ be a rigid Γ-limit group. There exists a global bound $b_R(Rgd(x,p,a))$ for which for any particular value of the defining parameters p_0, there are at most b_R distinct rigid homomorphisms $h : Rgd(x,p,a) \to \Gamma$ satisfying $h(p) = p_0$.*

Proof: Theorem 3.5 generalizes theorem 2.5 in [20], to rigid Γ-limit groups. The proof of theorem 2.5 in [20], uses the existence of a global constant R, for which every rigid homomorphism of a rigid limit group $Rgd(x,p,a)$ is R-AP-covered, and the finite presentability of an F_k-limit group. The existence of a global constant R, for which every rigid homomorphism of a Γ-limit group is R-AP-covered, is proved in theorem 3.2. A Γ-limit group is not f.p. in general, but by the generalization of Guba's theorem presented in theorem 1.22, a Γ-limit group is f.p. in the class of Γ-limit groups. Hence, the proof of theorem 3.5 follows by an identical argument to the one used to prove theorem 2.5 in [20].

□

In a similar way one can globally bound the number of strictly solid families of homomorphisms of a solid Γ-limit group. The proof is identical to the one presented for theorem 2.9 in [20].

Theorem 3.6 (cf. ([20],2.9)). *Let $Sld(x,p,a)$ be a solid Γ-limit group. For any particular value p_0 of the defining parameters p, there are at most b_S distinct strictly solid families of homomorphisms $h : Sld(x,p,a) \to \Gamma$ satisfying $h(p) = p_0$.*

Strictly solid families of specializations of a solid Γ-limit group $Sld(x,p,a)$ were defined as those families of specializations that do not factor through completions of the resolutions in the graded Makanin-Razborov diagram of the Γ-limit group $Sld(x,p,a)$ (definition 3.3). Theorem 3.6 bounds the number of strictly solid families of specializations of a solid Γ-limit group $Sld(x,p,a)$ for any possible value p_0, of the defining parameters p.

As in the free group case, we will need a similar global bound on the number of families of specializations of a solid Γ-limit group $Sld(x,p,a)$, that do not factor through a covering closure (definition 2.2) of the resolutions that appear in the graded Makanin-Razborov diagram of the solid Γ-limit group $Sld(x,p,a)$, for any possible value of the defining parameters p.

Definition 3.7 (cf. ([20],2.12)). *Let $Sld(x,p,a)$ be a solid Γ-limit group, and let:*

$$GRes_1(x,p,a), \ldots, GRes_r(x,p,a)$$

be the resolutions in its graded Makanin-Razborov diagram. Let:

$$Cl^1_1(s,x,p,a), \ldots, Cl^1_{m_1}(s,x,p,a), \ldots, Cl^r_1(s,x,p,a), \ldots, Cl^r_{m_r}(s,x,p,a)$$

be a finite collection of graded closures of the graded resolutions: $GRes_1(x,p,a), \ldots, GRes_r(x,p,a)$.

A specialization (x_0, p_0, a) of the solid Γ-limit group $Sld(x, p, a)$ is called a strictly solid specialization with respect to the given closure if:

(i) *for $p = p_0$ the given closures form a covering closure of the (ungraded) resolutions associated with the specialization p_0 (see definition 2.2).*

(ii) *the specialization (x_0, p_0, a) does not factor through any of the the given closures.*

Note that a strictly solid solution with respect to the given closures is not necessarily strictly solid with respect to the completions of the resolutions $GRes_1(x, p, a), \dots, GRes_r(x, p, a)$ (cf. definition 1.5 in [20]).

An orbit of specializations of the solid Γ-limit group $Sld(x, p, a)$ under the action of the graded modular group $GMod(x, p, a)$ is called strictly solid family with respect to the given closures, if it contains a strictly solid specialization.

As in [20], in parallel with theorem 3.6, given a solid Γ-limit group $Sld(x, p, a)$ and a closure of the resolutions that appear in its graded Makanin-Razborov diagram,, one can globally bound the number of strictly solid families of the solid Γ-limit group $Sld(x, p, a)$ with respect to the given closure.

Theorem 3.8 (cf. ([20],2.13)). *Let $Sld(x, p, a)$ be a solid Γ-limit group, and let:*

$$GRes_1(x, p, a), \dots, GRes_r(x, p, a)$$

be the resolutions in its graded Makanin-Razborov diagram. Let:

$$Cl_1^1(s, x, p, a), \dots, Cl_{m_1}^1(s, x, p, a), \dots, Cl_1^r(s, x, p, a), \dots, Cl_{m_r}^r(s, x, p, a)$$

be a finite collection of graded closures of the graded resolutions: $GRes_1(x, p, a), \dots, GRes_r(x, p, a)$. There exists a constant b_S, so that for any particular value p_0 of the defining parameters p, there are at most b_S distinct families of homomorphisms $h : Sld(x, p, a) \to F_k$ that are strictly solid with respect to the given closure, and satisfy $h(p) = p_0$.

Given the global bounds on the numbers of rigid and solid solutions, and theorem 1.22 from which one can deduce that a given Γ-limit group is f.p. in the class of Γ-limit groups, we can generalize the properties of the stratification of the set of the defining parameters associated with a given graded resolution presented in section 3 of [20]. In particular, the stratification associated with a graded limit group, or a graded resolution, contains finitely many strata, and each stratum is in the Boolean algebra generated by AE sets.

§4. An Iterative Procedure for Validation of a Sentence

In the first 3 sections we have generalized the results and notions presented in [18], [19] and [20], for studying varieties, sentences and predicates defined over a free group, to general torsion-free hyperbolic groups. In this paper we use these notions to generalize the iterative procedure for validation of an AE sentence defined over a free group, presented in [21], to AE sentences defined over a general torsion-free hyperbolic groups.

The definition of a well-separated resolution over a free group, presented in definition 2.2 of [21], generalizes in a direct way to well-separated resolutions over a torsion-free hyperbolic group Γ. We further generalize the notion of a *geometric subresolution* presented in definition 3.1 of [21].

Definition 4.1. *Let $Res(t,y,a)$ be a a well-separated resolution over a torsion-free hyperbolic group Γ, and let $Comp(Res)(u,t,y,a)$ be its completion. Let $GSRes(g,y,a)$ be a resolution with the following properties:*

(i) *the resolution $GSRes(g,y,a)$ is a well-separated resolution.*
(ii) *the resolution $GSRes(g,y,a)$ is a completed resolution, i.e., the completion of $GSRes(g,y,a)$ is the resolution $GSRes(g,y,a)$ itself.*
(iii) *there exists a (geometric) embedding $\nu : GSRes(g,y,a) \to Comp(Res)(u,t,y,a)$ that maps the subgroup $< y,a >$ of the resolution $GSRes(g,y,a)$ onto the subgroup $< y,a >$ of the completed resolution $Comp(Res)(u,t,y,a)$ elementwise. In addition, the embedding ν has the following properties:*
   1) *every QH subgroup in an abelian decomposition associated with one of the various levels of the resolution $GSRes(g,y,a)$ is embedded geometrically into (a finite index subgroup of) a QH subgroup of $Comp(Res)(u,t,y,a)$.*
   2) *every abelian vertex group in an abelian decomposition associated with one of the levels of the resolution $GSRes(g,y,a)$ is embedded into an abelian vertex group in one of the abelian decompositions associated with $Comp(Res)(u,t,y,a)$.*
   3) *every terminal non-cyclic, freely indecomposable factor of the geometric subresolution is embedded by ν into a terminal non-cyclic freely indecomposable factor of the ambient completion, $Comp(Res)(u,t,y,a)$.*
   4) *except for the terminal free groups and terminal freely-indecomposable factors dropped along the various levels of the geometric subresolution, the free and abelian decompositions associated with each level of the resolution $GSRes(g,y,a)$ are the decompositions induced (using Bass-Serre theory) from the embedding ν and the free and abelian decompositions of $Comp(Res)(u,t,y,a)$. Furthermore, the canonical maps between successive levels in the resolution $GSRes(g,y,a)$ are the ones induced from the canonical maps between successive levels in the completion $Comp(Res)(u,t,y,a)$.*

We call the resolution $GSRes(g,y,a)$ together with the embedding $\nu : GSRes(g,y,a) \to Comp(Res)(u,t,y,a)$, a geometric subresolution of the (completion of the) resolution $Res(t,y,a)$. The modular groups associated with a geometric subresolution are set to be the modular groups induced from those of the completed resolution $Comp(Res)(u,t,y,a)$. In particular, the modular groups associated with each of the QH vertex groups in the abelian decompositions of $GSRes(g,y,a)$, are set to be the modular groups of the corresponding QH vertex groups of $Comp(Res)(u,t,y,a)$ into which they are embedded. Note that the completion, $Comp(Res)(u,t,y,a)$, itself is a geometric subresolution of the (completion of the) resolution $Res(t,y,a)$.

In section 3 of [21], while constructing the induced resolution, we have associated a *complexity* with any geometric subresolution of a well-separated resolution defined over a free group, a complexity that was later used in guaranteeing the termination of the iterative procedure for validation of an AE sentence. The notion of a geometric subresolution generalizes directly to well-separated resolutions defined over a torsion-free hyperbolic group. However, the complexity of such resolutions needs to be slightly modified (cf. definition 3.2 in [20]).

Definition 4.2. *Let $Res(t,a)$ be a well-separated completed resolution over a torsion-free hyperbolic group Γ, with (possibly) reduced modular groups associated with each of its various QH subgroups. Let $Q_1,\ldots,Q_m$ be the QH subgroups that appear in the completion $Comp(Res)(t,y,a)$ and let $S_1,\ldots,S_m$ be the (punctured) surfaces associated with the reduced modular group associated with each of the QH vertex group. To each (punctured) surface S_j we may associate an ordered couple $(genus(S_j),|\chi(S_j)|)$. We will assume that the QH subgroups $Q_1,\ldots,Q_m$ are ordered according to the lexicographical (decreasing) order of the ordered couples associated with their corresponding surfaces. Let $rk(Res(t,a))$ be the rank of the free group that is dropped along the resolution $Res(t,a)$ (see definition 2.1 in [21]), let $fact(Res(t,a))$ be the number of freely-indecomposable, non-cyclic terminal (embedded) factors of the resolution $Res(t,a)$, and let $Abrk(Res(t,a))$ be the sum of the ranks of the kernels of the mappings of (free) abelian groups that appear as vertex groups along the resolution $Res(t,a)$ (see definition 1.15 in [21]).*
We set the complexity of the resolution $Res(t,a)$, denoted $Cmplx(Res(t,a))$, to be:

$$Cmplx(Res(t,a)) = (fact(Res(t,a)) + rk(Res(t,a)),\ (genus(S_1),|\chi(S_1)|),\ldots$$

$$\ldots,(genus(S_m),|\chi(S_m)|),\ Abrk(Res(t,a))).$$

On the set of complexities of completed resolutions with (possibly) reduced modular groups we can define a linear order. Let $Res_1(t_1,a)$ and $Res_2(t_2,a)$ be two completed resolutions with (possibly) reduced modular groups. We say that $Cmplx(Res_1(t_1,a)) = Cmplx(Res_2(t_2,a))$ if the tuples defining the two complexities are identical. We say that $Cmplx(Res_1)(t_1,a)) < Cmplx(Res_2(t_2,a))$ if:

(1) *the "kurosh" rank, $fact(Res_1(t_1,a)) + rk(Res_1(t_1,a))$ is smaller than the kurosh rank $fact(Res_2(t_2,a)) + rk(Res_2(t_2,a))$.*
(2) *the above ranks are equal and the tuple:*

$$((genus(S^1_1),|\chi(S^1_1)|),\ldots,(genus(S^1_{m_1}),|\chi(S^1_{m_1}|))$$

is smaller in the lexicographical order than the tuple:

$$((genus(S^2_1),|\chi(S^2_1)|),\ldots,(genus(S^2_{m_2}),|\chi(S^2_{m_2}|)).$$

(3) *the above ranks and tuples are equal and $Abrk(Res_1(t_1,a)) < Abrk(Res_2(t_2,a))$.*

Given a well-separated resolution $Res(t,y,a)$ over a free group, and a subgroup $< y,a >$ of its associated limit group, we have constructed in section 3 of [21], the induced resolution $Ind(Res(t,y,a))(u,y,a)$. The construction of the induced resolution generalizes directly to well-separated resolutions defined over a torsion-free hyperbolic group Γ, hence, we omit its detailed description.

In section 1 of [21] we presented an iterative procedure for the validation of an AE sentence over a free group, assuming all the (F_k) limit groups that are needed to be treated along the procedure are of minimal rank. This procedure generalizes directly to validate AE sentences over a torsion-free hyperbolic group, in the minimal rank case.

The general procedure for validation of a sentence over a free group, presented in section 4 of [21], needs to be slightly modified in order to be adopted for general torsion-free hyperbolic groups. In the previous sections we have already modified all the tools needed in order to be able to modify the general procedure presented in [21], and in the rest of this section we present the modified procedure.

Let:

$$\forall y \ \exists x \ \Sigma(x,y,a) = 1 \ \wedge \ \Psi(x,y,a) \neq 1$$

be a truth sentence over a torsion-free hyperbolic group Γ. Let $F_y =< y_1, \dots, y_\ell >$ be a the free group with a free basis $y_1, \dots, y_\ell$. By proposition 2.1 there exists a formal solution $x = x(y,a)$, and a finite set of restricted Γ-limit groups: $Rlim_1(y,a), \dots, Rlim_m(y,a)$ for which:

(i) the words corresponding to the equations in the system $\Sigma(x(y,a),y,a)$ represent the trivial word in the free group $\Gamma * F_y$.
(ii) for every index i, $Rlim_i(y,a)$ is a proper quotient of the free group $\Gamma * F_y$.
(iii) let $B_1(y), \dots, B_m(y)$ be the basic sets corresponding to the restricted Γ-limit groups $Rlim_1(y,a), \dots, Rlim_m(y,a)$. If $y \notin B_1(y) \cup \ldots \cup B_m(y)$, and $\psi_j(x,y,a)$ is a word corresponding to one of the equations in the system $\Psi(x,y,a)$, then: $\psi_j(x(y,a),y,a) \neq 1$.

Proposition 2.1 gives a formal solution that proves the validity of the given sentence on a co-basic set $(F_k)^\ell \setminus (B_1(y) \cup \ldots \cup B_m(y))$, hence, the rest of the procedure needs to construct formal solutions that prove the validity of the sentence on the remaining basic sets $B_1(y), \dots, B_m(y)$. We will continue with the Γ-limit groups $Rlim_i(y,a)$ in parallel, hence, for brevity we denote the Γ-limit group we continue with, $Rlim(y,a)$. Note that $Rlim(y,a)$ is a proper quotient of the Γ-limit group $\Gamma * F_y$, hence, the Kurosh rank of every resolution in the (taut) Makanin-Razborov diagram of $Rlim(y,a)$ is strictly less than ℓ, the rank of the free group F_y.

With the Γ-limit group $Rlim(y,a)$ we associate its (canonical) taut Makanin-Razborov diagram. Let $Res_1(y,a), \dots, Res_r(y,a)$ be the resolutions in this taut diagram. With each resolution $Res_i(y,a)$ we can associate its completed resolution $Comp(Res_i)(z,y,a)$. We continue with each of the resolutions in parallel, hence, we will omit the index in the sequel.

By theorem 2.3 from the validity of our given sentence, we get the existence of a covering closure $Cl_1(Res)(s,z,y,a), \dots, Cl_q(Res)(s,z,y,a)$ of the resolution $Res(y,a)$, and for each index $1 \leq n \leq q$ there exists a formal solution $x_n(s,z,y,a)$ for which:

(i) the words corresponding to the equations in the system $\Sigma(x_n(s,z,y,a),y,a)$ represent the trivial word in the closure $Cl_n(Res)(s,z,y,a)$.
(ii) there exists some specialization (s_0^n, z_0^n, y_0^n) for which $\Psi(x_n(s_0^n, z_0^n, y_0^n, a), y_0^n, a) \neq 1$.

Given a resolution $Res(y,a)$, and one of the closures from its given covering closure, we analyze all the specializations (s_0, z_0, y_0, a) that are taut and shortest form with respect to the closure $Cl(Res)(s,z,y,a)$ (see definition 4.1 in [21] for the definition of taut shortest form specializations). The entire collection of taut shortest form specializations (s_0, z_0, y_0, a) that factor through the closure $Cl(Res)(s,z,y,a)$, and satisfy the corresponding system of equations, is contained in a finite set of maximal Γ-limit groups $QRlim_1(s,z,y,a), \dots, QRlim_u(s,z,y,a)$. Our analysis of these quotient Γ-limit groups is conducted in parallel, hence, in the sequel we will omit its index.

Let $Zlim_2 =< s_2, z_2, \ldots, s_\ell, z_\ell >$ be the subgroup associated with all the levels of the closure, $Cl(Res)(s,z,y,a)$, except the top level. $Zlim_2$ is naturally mapped into the closure, $Cl(Res)(s,z,y,a)$, hence, into the quotient Γ-limit group $QRlim(s,z,y,a)$. Let $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$ be the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the closure, $Cl(Res)(s,z,y,a)$. With the quotient Γ-limit group $QRlim(s,z,y,a)$ we associate canonically the multi-graded taut Makanin-Razborov diagram with respect to the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$. In constructing this diagram, we take into consideration only those homomorphisms of the quotient Γ-limit group, $QRlim(s,z,y,a)$, that embed the (images of the) terminal non-cyclic, freely-indecomposable factors of the closure, $Cl(Res)(s,z,y,a)$, into the hyperbolic group Γ. Let:

$$MGQRes_1(s,z,y,Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a), \ldots, MGQRes_q(s,z,y,Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$$

be the quotient multi-graded resolutions in this taut Makanin-Razborov diagram.

We continue with a subset of the resolutions that appear in the multi-graded taut Makanin-Razborov diagram of the quotient limit group $QRlim(s,z,y,a)$, a subset through which all the specializations (y_0,a) that are taut with respect to $Res(y,a)$, and for which there exists a taut shortest form specialization (s_0,z_0,y_0,a) that factor through $QRlim(s,z,y,a)$ do factor, and not with all the quotient multi-graded resolutions in this taut multi-graded diagram. If the subgroup of generated by $< y,a >$ of the limit group (generated by) $< s,z,y,a >$ associated with a quotient multi-graded resolution: $MGQRes_i$ is a proper quotient of the Γ-limit group $Rlim(y,a)$ associated with the resolution $Res(y,a)$ we have started this branch of the first step of the procedure, we include the quotient multi-graded resolution $MGQRes_i$ in the subset we continue with.
Otherwise, for each QH vertex group Q in the abelian decomposition associated with the top level of the closure $Cl(Res)(s,z,y,a)$, the boundary elements of Q can be conjugated (in the limit group associated with the closure $Cl(Res)(s,z,y,a)$) into one of subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$. Since the closure $Cl(Res)(s,z,y,a)$ is canonically mapped onto the quotient Γ-limit group $QRlim(s,z,y,a)$, each of the QH subgroups Q in the abelian decomposition associated with the top level of $Cl(Res)(s,z,y,a)$ is naturally mapped into the quotient Γ-limit group $QRlim(s,z,y,a)$. Hence, the QH subgroup Q and its corresponding quotients naturally inherits a sequence of abelian decompositions from the multi-graded abelian decompositions associated with the various levels of a multi-graded resolution $MGQRes_i$ abelian decompositions in which the boundary elements of Q are all elliptic.
The resolution $Res(y,a)$ was assumed to be well-separated, hence, on each of the QH vertex groups Q associated with its various levels, there exists an additional indication for a collection of s.c.c. on its associated surface S that is mapped to a trivial element in the Γ-limit group associated with the next level of the resolution $Res(y,a)$. Given a multi-graded resolution $MGQRes_i$, if for some QH vertex group Q in the abelian decomposition associated with the top level of the closure $Cl(Res)(s,z,y,a)$, the sequence of abelian decompositions Q and its corresponding quotients inherit from the multi-graded resolution $MGQRes_i$, is not compatible with the specific indication of the collection of s.c.c. on the surface S associated with the QH vertex group Q that are mapped to the trivial element in the resolution

$Res(y,a)$, we omit the quotient multi-graded resolution $MGQRes_i$ from our list of quotient multi-graded resolutions. Otherwise, we include the quotient multi-graded resolution in the subset of multi-graded resolutions we continue with.
By construction, every specialization (y_0, a) that factors and is taut with respect to the resolution $Res(y,a)$ and for which there is a specialization (s_0, z_0, y_0, a) that is taut and shortest form with respect to the closure $Cl(Res)(s,z,y,a)$, there exists a quotient multi-graded resolution $MGQRes_i$ that is included in the subset of quotient multi-graded resolutions we continue with, and for which the specialization (s_0, z_0, y_0, a) factors through and is taut with respect to that quotient multi-graded resolution.

Let $Q_1, \ldots, Q_r$ be the QH vertex groups in the abelian decomposition associated with the top level of the resolution $Res(y,a)$ we have started with, i.e., the QH vertex groups in the abelian JSJ decomposition of $Rlim(y,a)$. As in theorems 1.7 and 2.9 in [21], we can assume that each multi-graded quotient resolution $MGQRes_i$ terminates in a multi-graded abelian Γ-limit group with a multi-graded abelian decomposition containing the entire collection of surviving QH vertex groups $Q_{i_1}, \ldots, Q_{i_{r'}}$, and these surviving QH subgroups are either closed surface subgroups or (in case they correspond to punctured surfaces) they are mapped to their images in the subgroup $< z_2, a >$ in either a rigid or solid multi-graded limit group with respect to the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$. Furthermore, the surviving QH vertex groups $Q_{i_1}, \ldots, Q_{i_{r'}}$ are subgroups of the non-abelian, non-QH vertex groups, the ones stabilized by the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$, in all levels above the two terminating ones (see theorem 1.7 in [21]).
A basic property of a multi-graded quotient resolution, is the following property (cf. proposition 4.2 in [21]).

Proposition 4.3. *Suppose that the subgroup generated by $< y, a >$ in the Γ-limit group associated with a multi-graded resolution $MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$, is isomorphic to the Γ-limit group $Rlim(y,a)$ associated with the resolution $Res(y,a)$ we have started the iterative process with. Then the complexity of each of the multi-graded abelian decompositions associated with the various levels of the multi-graded resolution: $MGQRes$ is bounded by the complexity of the abelian decomposition associated with the top level of the completion $Comp(Res)(z,y,a)$. In case of equality, the multi-graded resolution: $MGQRes$ has only one level and its structure is identical to the structure of the abelian decomposition associated with the top level of the completion $Comp(Res)(z,y,a)$.*

Proof: Identical to the proof of proposition 4.2 in [21].

□

The other properties of the multi-graded resolutions, $MGQRes$, that were proved in the free group case can be modified to hold over a general torsion-free hyperbolic group Γ.

Proposition 4.4 (cf. ([21],4.3)). *Let $MGQRes$ be one of the multi-graded quotient resolutions, for which the subgroup generated by $< y, a >$ in the Γ-limit group associated with that multi-graded resolution is isomorphic to the Γ-limit group $Rlim(y,a)$ associated with the resolution $Res(y,a)$. By construction, the original Γ-limit group $Rlim(y,a)$ is mapped into the Γ-limit groups associated with each of the levels of the multi-graded quotient resolution $MGQRes$. Let $Q_{term}(y,a)$*

be the image of $Rlim(y,a)$ in the terminal (rigid or solid) multi-graded Γ-limit group of $MGQRes$. Then the terminal Γ-limit group of the multi-graded resolution, $MGQRes$, can be replaced by a finite collection of finitely many terminal multi-graded Γ-limit groups, in which either $Q_{term}(y,a)$ is a proper quotient of $Rlim(y,a)$, or one can assume that each of the homomorphisms that factors through the given terminal Γ-limit group embeds each of its non-cyclic, freely-indecomposable factors into the coefficient group Γ.

Proof: Identical to the proof of proposition 4.3 in [21].

□

The procedure we present for the validation of an AE sentence over a torsion-free hyperbolic group, is similar to the one used over a free group in section 4 of [21]. The necessary required modifications are similar to the modifications applied in constructing the tree of stratified sets, presented in section 2 of [22] (and are implied by the modification of proposition 4.4 in comparison to proposition 4.3 in [21]).

I: The first step

(1) Let $Rlim(y,a)$ be the Γ-limit group we have started with. Let $Q(y,a)$ be the Γ-limit group generated by $< y,a >$ in the Γ-limit group associated with the multi-graded quotient resolution $MGQRes$. If $Q(y,a)$ is a proper quotient of $Rlim(y,a)$, we continue this branch of the iterative procedure, by replacing the Γ-limit group $Rlim(y,a)$ by $Q(y,a)$, and continue with the finite collection of resolutions that appear in the taut Makanin-Razborov diagram of the Γ-limit group $Q(y,a)$ in the same way we handled the resolutions in the taut Makanin-Razborov diagram of $Rlim(y,a)$.

(2) At this stage we may assume that $Q(y,a)$ is isomorphic to $Rlim(y,a)$. At this part we will also assume that the multi-graded quotient resolution $MGQRes$ is not of maximal complexity, i.e., that the complexities of the abelian decompositions associated with its various levels are strictly smaller than the complexity of the abelian decomposition associated with the top level of the resolution $Res(y,a)$.

    In this case, either the image of $Rlim(y,a)$ in the Γ-limit group associated with the second level of: $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$, $Q_2(y,a)$, is a proper quotient of $Rlim(y,a)$, or the image of $QRlim(s,z,y,a)$ in the limit group associated with the second level of:

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

    $Q_2(s,z,y,a)$, is a proper quotient of $QRlim(s,z,y,a)$, or the homomorphisms that factor through the given multi-graded resolution, $MGQRes$, embed each of the non-cyclic, freely indecomposable factors of the quotient Γ-limit group, $QRlim$, into the coefficient group Γ.

    If the subgroup $Q_2(y,a)$ is a proper quotient of $Rlim(y,a)$, we associate with it its taut Makanin-Razborov diagram, and with each of the resolutions $QRes_j(y,a)$ in the diagram we associate a developing resolution, an auxiliary resolution, and an anvil, precisely as we did in part (2) of the initial step in the free group case.

    If $Q_2(y,a)$ is isomorphic to $Rlim(y,a)$ we continue to the next levels of the multi-graded quotient resolution $MGQRes$ If at some level j of $MGQRes$, the image of $Rlim(y,a)$ in the limit group associated with the j-th level of $MGQRes$, $Q_j(y,a)$, is a proper quotient of $Rlim(y,a)$, we continue in a similar way to what we did in case $Q_2(y,a)$ is a proper quotient of $Rlim(y,a)$.

By part (i) of proposition 4.4, if for every level j, $Q_j(y,a)$ is isomorphic to the Γ-limit group $Rlim(y,a)$ we have started with, we can assume that each of the homomorphisms that factor through the given terminal Γ-limit group of $MGQRes$ embeds each of its non-cyclic, freely-indecomposable factors into the coefficient group Γ. In this case we associate a developing resolution and an anvil with $MGQRes$, precisely as we did in case (2) of the initial step of the procedure for the construction of stratified sets (section 2 in [22]), in case the map of the original F_k-limit group into the terminal limit group of the multi-graded quotient resolution is an isomorphism, and hence, this terminal limit group is rigid or solid with respect to the defining parameters p..

(3) By part (1) we may assume that $Q(y,a)$ is isomorphic to $Rlim(y,a)$. Part (2) treats the case in which the multi-graded quotient resolution $MGQRes$ is not of maximal complexity. The only case left, that of a maximal complexity quotient resolution, $MGQRes$, is treated in precisely the same way it was treated in the initial step of the procedure over a free group (case (3) of the initial step).

II: The general step

The general step of the iterative procedure is similar to the one over free groups, presented in section 4 of [21]. The ultimate goal of the general step of the iterative procedure is to obtain a strict reduction in either the complexity of certain decompositions and resolutions or a strict reduction in the Zariski closures of certain limit groups associated with the anvils constructed in the previous steps of the procedure. The strict reduction in complexity and Zariski closures will finally guarantee the termination of the iterative procedure after finitely many steps, precisely as in the case of free groups.

The description of the general step of the procedure in the case of free groups appears in section 4 of [21]. The only modifications needed in the case of a torsion-free hyperbolic group are those presented in lemma 4.4, and as a consequence, in case (2) of the initial step, and its parallels in the general case. The argument for the termination of the procedure is essentially identical to the free group case. hence, we prefered not to repeat the details of the general step of the procedure, and refer the interested reader to the detailed description in section 4 of [21].

§5. Core Resolutions

In the first 3 sections of this paper, we have generalized the structure theory developed in [18], [19] and [20] from free groups to torsion-free hyperbolic groups. Section 4 generalizes the iterative procedure for validating an AE sentence to hyperbolic groups, that finally allows us to generalize the quantifier elimination procedure presented in [22] and [23]. In this section we prepare the main tool needed for generalizing the q.e. procedure, the construction of the core resolution for resolutions defined over torsion-free hyperbolic groups.

Definition 5.1 (cf. ([22],4.1). *Let $Res(t,v,a)$ be a a well-separated resolution* *over a torsion-free hyperbolic group Γ, and let $Comp(Res)(u,t,v,a)$ be its completion. Let $CRes(r,v,a)$ be a geometric subresolution of the completed resolution $Res(u,t,v,a)$. We say that the resolution $CRes(r,v,a)$ is a core resolution of the subgroup $<v,a>$ in the completion $Comp(Res)(u,t,v,a)$, if the resolution $CRes(r,v,a)$ is a firm subresolution of the subgroup $<r,v,a>$ ([21], definition 3.9). i.e., it has the following properties:*

(i) *the Kurosh rank of the resolution $CRes(r,v,a)$, i.e., the sum of the number of non-cyclic, freely indecomposable factors and the ranks of the free groups dropped along its levels, is equal to the Kurosh rank of the subgroup $<r,v,a>$ with respect to the completed resolution $Comp(Res)(u,t,v,a)$.*

(ii) *there exists a firm test sequence for the subgroup $<r,v,a>$, i.e., a test sequence of the ambient completion, $Comp(Res)(u,t,v,a)$, in which the Kurosh rank of the subgroup $<r,v,a>$ is precisely the Kurosh rank of the core, $CRes(r,v,a)$.*

(iii) *Let $A_1,\ldots,A_m$ be all the non-cyclic pegged abelian groups that appear along the completed resolution $Comp(Res)(t,v,a)$, let $peg_1,\ldots,peg_m$ be the pegs of the abelian groups $A_1,\ldots,A_m$, and let $\{peg_i,q^i_1,\ldots,q^i_{j_i}\}_{i=1}^m$ be an arbitrary basis for the collection of the subgroups $A_1,\ldots,A_m$. Then for any set of integers $\{(s^i_j,n^i_j)\}$, where $n^i_j\geq 2$ and $0\leq s^i_j\leq n^i_j$, there exist a firm test sequence $\{t_n,v_n,a\}$ of the subgroup $<v,a>$, so that for every index n, the specialization of each of the pegs peg_i is an element h_i with no non-trivial roots, and the specialization of each of the basis elements q^i_j is $h_i^{r^i_j}$ where $r^i_j=u^i_j\cdot n^i_j+s^i_j$ for some positive integer u^i_j.*

We denote a core resolution, $Core(<v,a>,Res(t,v,a))(r,v,a)$. In exactly the same way we define a graded core resolution, and a multi-graded core resolution. Since a core resolution is in particular a geometric subresolution, we set the complexity of a core resolution to be its complexity as a geometric subresolution.

The procedure for the construction of a core resolution over hyperbolic groups, is essentially similar to the one presented in section 4 of [22] over a free group. It is composed from 2 iterative procedures. The procedure used for the first part is essentially identical to the iterative procedure used to construct the induced resolution. In the second part, in a similar way to the procedure over a free group, we use an iterative procedure that either reduces the Kurosh rank of the resolution constructed in the first part, or alternatively, shows that the resolution constructed in the first part is a firm subresolution, hence, it is a core resolution.

Since the first part is identical to the construction of the induced resolution, we do not present it in detail. Let $Res(t,v,a)$ be a complete well-separated resolution. We denote the resolution induced by the subgroup $<v,a>$ from $Res(t,v,a)$, $IRes_f(u,v,a)$. To the induced resolution $IRes_f(u,v,a)$ we iteratively add pegs of its abelian groups using the procedure presented in section 4 of [22]. Using the same procedure, we iteratively add roots to QH vertex groups along our constructed resolution that are finite index subgroups in conjugates of QH vertex groups along the ambient resolution $Res(t,v,a)$, so that the outcome of this iterative procedure is a geometric subresolution of the ambient resolution, $Res(t,v,a)$, in which every QH vertex group is conjugate to a QH vertex group along $Res(t,v,a)$.
We denote the resolution obtained by adding pegs and "completing" QH vertex groups, $IRes_p(u,v,a)$. Note that the procedure that iteratively add pegs, and

"complete" QH vertex groups, is conducted by iteratively adding finite order roots to the terminal group, hence, it does not increase the Kurosh rank. Furthermore, if it preserves the Kurosh rank, and the obtained resolution is firm, so is the induced resolution $IRes_f(u,v,a)$.

As in the free group case, to modify the resolution $IRes_p(u,v,a)$ in order to obtain a firm subresolution, we use an iterative procedure, that is aimed towards sequentially reducing the Kurosh rank of the obtained resolutions. The iterative procedure we present either reduces the Kurosh rank after finitely many steps, or guarantees (after finitely many steps) that the resolution $IRes_p(u,v,a)$ is indeed a firm subresolution. The procedure we present, which is a slight modification of the procedure used in the free group case, starts from the bottom level of the resolution $Res(t,v,a)$ and iteratively climbs towards its top level.

(1) The terminal level (which we denote by ℓ) of the resolution $Res(t,v,a)$ is a free product of freely-indecomposable groups that can be embedded in the torsion-free hyperbolic group Γ, and a free group. The terminal level of the resolution $IRes_p(u,v,a)$, admits a (possibly trivial) free decomposition to a (possibly trivial) free group and a (possibly trivial) factor M, so that M embeds into the terminal level of the ambient resolution, $Res(t,v,a)$. The factor M inherits a (possibly trivial) free decomposition from the given free decomposition of the terminal group of $Res(t,v,a)$, $M = M_1 * \ldots * M_s * F$, where F is a (possibly trivial) free group, and the factors M_i are embedded into conjugates of the non-cyclic, freely-indecomposable factors in the free decomposition of the terminal group of the ambient resolution, $Res(t,v,a)$.

    We modify the resolution $IRes_p(u,v,a)$ by replacing each of the terminal factors M_i, with the non-cyclic, freely-indecomposable factor in the terminal group of $Res(t,v,a)$ that contains it. With the subgroup corresponding to this modified resolution, we associate the induced resolution, and iteratively add pegs and complete QH vertex groups. If the terminal group of the resolution has changed, and there exist new factors that embed into non-cyclic, freely-indecomposable factors in the terminal group of $Res(t,v,a)$, we repeat part (1) of the construction.

    Note that this modification does not increase the Kurosh rank, and if it preserves the Kurosh rank, and the obtained resolution is firm, so is the resolution $IRes_p(u,v,a)$. We denote the resolution obtained after this modification $IRes_c(u,v,a)$.

(2) We continue from the level above the terminal one (level $\ell-1$). According to the construction of an induced resolution, the subgroup associated with the resolution $IRes_c(u,v,a)$ inherits a (possibly trivial) free decomposition from each of the levels of the resolution $IRes_c(u,v,a)$, and it is mapped to a subgroup associated with each of the levels. In particular, in accordance with the free decomposition inherited by the subgroup associated with the resolution $IRes_c(u,v,a)$ from the levels that lie above the $\ell-1$ level, the image of the subgroup associated with $IRes_c(u,v,a)$, $G^{\ell-1}$, admits a free decomposition when mapped into the $\ell-1$ level, $G^{\ell-1} = H_1^{\ell-1} * \ldots * H_{s(\ell-1)}^{\ell-1}$. We treat the factors $H_i^{\ell-1}$ in parallel.

    We fix a system of generators of $H_i^{\ell-1}$, $H_i^{\ell-1} =< h_1^{\ell-1}, \ldots, h_{r(\ell-1,i)}^{\ell-1} >$. If no (non-trivial) subgroup of the factor $H_i^{\ell-1}$ fixes a vertex in the Bass-Serre tree associated with the $\ell-1$ level of the resolution $Res(t,v,a)$, we have started with, we leave the factor $H_i^{\ell-1}$ unchanged. Suppose that a (non-trivial) subgroup of $H_i^{\ell-1}$ fixes a vertex in the abelian decomposition associated with the $\ell-1$ level of the resolution $Res(t,v,a)$.

Let $T_{\ell-1}$ be the Bass-Serre tree corresponding to the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and let $\Lambda_{H_i^{\ell-1}}$ be the graph of groups inherited by $H_i^{\ell-1}$ from its action on the Bass-Serre tree $T_{\ell-1}$. Suppose that the abelian decomposition $\Lambda_{H_i^{\ell-1}}$, inherited by $H_i^{\ell-1}$ from its action on the Bass-Serre tree $T_{\ell-1}$, contains a couple of QH vertex groups, Q_1, Q_2, that satisfy the following conditions:

(i) Q_1 is a conjugate of a QH vertex group in the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and Q_2 is conjugate to the same QH vertex group in the abelian decomposition associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$.

(ii) the QH vertex group Q_1 (hence, also Q_2) is not of minimal rank, i.e., there exists a s.c.c. on S_1 (the surface associated with Q_1) that is mapped to the trivial element in the next level of the resolution $Res(t,v,a)$.

In this case we say that the abelian decomposition $\Lambda_{H_i^{\ell-1}}$ contains a *reducing QH couple*.

If the abelian decomposition $\Lambda_{H_i^{\ell-1}}$ contains a reducing QH couple, Q_1 and Q_2, we set the subgroup $\hat{H}_i$ to be the subgroup generated by $H_i^{\ell-1}$, and an element in the Γ-limit group associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$ that conjugates Q_1 to Q_2. We set H_i' to be the Γ-limit group associated with the resolution induced by the subgroup $\hat{H}_i$ from the ambient resolution $Res(t,v,a)$. Since the QH vertex groups Q_1 and Q_2 are not of minimal rank, and since Q_1 and Q_2 are not conjugate in the subgroup $H_i^{\ell-1}$ we have started with, and Q_1 and Q_2 are conjugate in H_i', the Kurosh rank of H_i' is strictly smaller than the Kurosh rank of $H_i^{\ell-1}$. In this case we replace the factor $H_i^{\ell-1}$ by H_i', and repeat the construction the whole construction of the core resolution starting with the obtained group.

Suppose that the abelian decomposition $\Lambda_{H_i^{\ell-1}}$ does not contain a reducing QH couple. Let $\eta_{\ell-1}$ be the map from the Γ-limit group associated with the $\ell-1$ level of $Res(t,v,a)$ to the Γ-limit group associated with the (terminal) ℓ level of $Res(t,v,a)$. Let $t_0 \in T_{\ell-1}$ be its base point, and let $T'_{\ell-1}$ be the finite subtree of $T_{\ell-1}$ spanned by the points $t_0, h_1^{\ell-1}(t_0), \ldots, h_{r(\ell-1,i)}^{\ell-1}(t_0)$. To continue our treatment of the factor $H_i^{\ell-1}$ we need the notions of *floating* and *absorbed* surfaces.

Definition 5.2 (cf. definition 4.7 in [22]). *Let Q be a QH vertex group in the finite tree $T'_{\ell-1}$, and let S be its associated surface. Since the ambient resolution $Res(t,v,a)$ is well-separated, the image of the QH vertex group Q in the next level of the resolution $Res(t,v,a)$ is non-abelian. Recall (definition 4.7 in [22]) that we say that Q is a floating QH vertex group (S is a floating surface) with respect to the geometric subresolution $IRes(u,v,a)$, if Q does not intersect the subgroup $H_i^{\ell-1}$ in a subgroup of finite index, and one of the three conditions hold:*

(i) *the QH vertex group Q is not of minimal rank, i.e., there exists a s.c.c.*

on the surface $\hat{S}$ that is mapped to the trivial element in the limit group associated with the next level of the ambient resolution $Res(t,v,a)$.

(ii) *none of the vertex groups that are adjacent to Q in $T'_{\ell-1}$ intersect non-trivially the subgroup $H_i^{\ell-1}$.*

(iii) *the QH vertex group Q is of minimal rank, and for every vertex group V in $T'_{\ell-1}$ that is adjacent to Q in $T'_{\ell-1}$ and intersects $H_i^{\ell-1}$ non-trivially, the Kurosh rank of the subgroup: $\eta_{\ell-1}(< H_i^{\ell-1} \cap V, Q >)$ is strictly bigger than the Kurosh rank of the subgroup: $\eta_{\ell-1}(H_i^{\ell-1} \cap V)$.*

We say that Q is an absorbed vertex group (S is an absorbed surface) if it is not floating.

In the case in which there is no reducing QH couple, we set $\hat{H}_i$ to be the subgroup generated by the factor $H_i^{\ell-1}$ and one of the following if it exists:

(i) an abelian vertex group in the finite tree $T'_{\ell-1}$ that is intersected non-trivially by the factor $H_i^{\ell-1}$, but is not contained in it.

(ii) an absorbed QH vertex group in the finite tree $T'_{\ell-1}$ (with respect to the subgroup $H_i^{\ell-1}$), that is not contained in $H_i^{\ell-1}$.

(iii) an (abelian) edge group E in the finite tree $T'_{\ell-1}$, that is not contained in $H_i^{\ell-1}$, and is adjacent to a non-abelian, non-QH vertex group V in $T'_{\ell-1}$ that is intersected non-trivially by $H_i^{\ell-1}$, for which the Kurosh rank of the subgroup: $\eta_{\ell-1}(< H_i^{\ell-1} \cap V, E >)$ is bounded from above by the Kurosh rank of the subgroup: $\eta_{\ell-1}(H_i^{\ell-1} \cap V)$.

(iv) an element $v_0 \in V$, $v_0 \notin H_i^{\ell-1}$, where V is a non-abelian, non-QH vertex group in the finite tree $T'_{\ell-1}$ that is intersected non-trivially by the subgroup $H_i^{\ell-1}$, for which there exist two edge groups E_1, E_2 that are adjacent to V in the finite tree $T'_{\ell-1}$ and are not conjugate in the subgroup $H_i^{\ell-1}$, so that v conjugates E_1 to E_2 in the Γ-limit group associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$, and the Kurosh rank of the subgroup: $\eta_{\ell-1}(< H_i^{\ell-1} \cap V, v_0 >)$ is bounded above by the Kurosh rank of the subgroup: $\eta_{\ell-1}(H_i^{\ell-1} \cap V)$.

We further set H'_i to be the subgroup associated with the resolution induced by the subgroup $\hat{H}_i$. Since $\hat{H}_i$ is generated by $H_i^{\ell-1}$, the absorbed QH vertex groups in $T'_{\ell-1}$ with respect to $H_i^{\ell-1}$, abelian vertex groups that are intersected non-trivially by $H_i^{\ell-1}$, and additional elements that do not increase the Kurosh ranks of the corresponding vertex groups, the Kurosh rank of H'_i is bounded by the Kurosh rank of $H_i^{\ell-1}$, $rk(H'_i) \leq rk(H_i^{\ell-1})$. Furthermore, both groups H'_i and $H_i^{\ell-1}$ inherit free decompositions from the abelian decompositions associated with their corresponding actions on the Bass-Serre tree $T_{\ell-1}$, and both the number of factors as well as the rank of the additional free group in the abelian decomposition associated with H'_i (the Kurosh rank) are bounded by the number of factors and the corresponding rank in the abelian decomposition associated with $H_i^{\ell-1}$.
If H'_i is identical to $H_i^{\ell-1}$ we have concluded our treatment of the factor $H_i^{\ell-1}$. Otherwise, we replace the factor $H_i^{\ell-1}$ by H'_i, and repeat the construction of a subgroup $\hat{H}_i$ associated with the newly obtained subgroup $H_i^{\ell-1}$ (without changing the finite tree $T'_{\ell-1}$), and the construction of the

subgroup H_i' associated with the resolution induced by $\hat{H}_i$ from the ambient resolution $Res(t,v,a)$, and repeat our treatment of the subgroup H_i'.
Since in each step we either reduce the Kurosh rank of the obtained subgroup, or we add a new edge group, an abelian vertex group or absorbed QH vertex group from the finite tree $T'_{\ell-1}$ to the subgroup associated with $H_i^{\ell-1}$, or we add an element that conjugates two edge groups in $T'_{\ell-1}$ that were not conjugated previously, we conclude our treatment of the factor $H_i^{\ell-1}$ after finitely many steps. If the Kurosh rank of at least one of the factors $H_i^{\ell-1}$ strictly decreased by the iterative procedure, we replace the resolution $IRes_c(u,v,a)$ with the resolution induced by the subgroup generated by the subgroup associated with $IRes_c(u,v,a)$ and the newly obtained factors $H_1^{\ell-1},\dots,H_{s(\ell-1)}^{\ell-1}$, and denote the obtained resolution $IRes_{\ell-1}(u,v,a)$. In this case, the Kurosh rank of the resolution $IRes_{\ell-1}(u,v,a)$ is strictly smaller than the Kurosh rank of the resolution $IRes_c(u,v,a)$), and we continue by starting the second step of the construction of the core resolution with the resolution $IRes_{\ell-1}(u,v,a)$ (instead of $IRes_f(u,v,a)$). If none of the Kurosh ranks of the various factors $H_i^{\ell-1}$ decreases, we continue by analyzing the next $(\ell-2)$ level of the resolution $IRes_c(u,v,a)$.

The iterative procedure presented in part (2) terminates after finitely many steps, and concludes our treatment of the various factors $H_i^{\ell-1}$ of the subgroup $G^{\ell-1} = H_1^{\ell-1} * \dots * H_{s(\ell-1)}^{\ell-1}$, which is the image of the subgroup associated with the resolution $IRes_c(u,v,a)$ in the subgroup associated with the $\ell-1$ level of the ambient resolution $Res(t,v,a)$. We continue by iteratively increasing the index m, and analyzing the various factors of the image of the subgroup $G^{\ell-m} = H_1^{\ell-m} * \dots * H_{s(\ell-m)}^{\ell-m}$, which is the image of the subgroup associated with the resolution $IRes_c(u,v,a)$ in the subgroup associated with the $\ell-m$ level of the ambient resolution $Res(t,v,a)$, according to the procedure presented in part (2). Since given part (2), this iterative construction is identical to the one presented in [22] over free groups, we omit its detail, and refer the interested reader to part (3) of the construction of the core resolution over a free group in section 4 of [22].

The iterative procedure used for the second part of the construction of the core resolution terminates after finitely many steps. Using it we obtain a geometric subresolution of the ambient resolution $Res(t,v,a)$, which we denote $IRes_s(u,v,a)$, which is set to be either the resolution $IRes_f(u,v,a)$ obtained by the first part of the construction, in case the procedure used for the second part of the construction of the core resolution has not reduced the Kurosh rank of the resolution it has constructed, or it is the resolution constructed by the procedure used in the second part of the construction of the core resolution, in case this resolution is of strictly smaller Kurosh rank than the resolution $IRes_f(u,v,a)$, constructed by the procedure used in the first part of the construction. The obtained resolution $IRes_s(u,v,a)$ is a geometric subresolution of $Res(t,v,a)$ by construction, in addition it is guaranteed to be a firm subresolution by the following theorem, hence, it may serve as a core resolution, $Core(<v,a>, Res(t,v,a))$.

Theorem 5.3 (cf. ([22],4.8). *The resolution $IRes_s(u,v,a)$, obtained by the procedure for the construction of a core resolution, is a firm subresolution of the resolution $Res(t,v,a)$.*

Proof: The proof is essentially identical to the proof of theorem 4.8 in [22], replacing rank by Kurosh rank.

□

Once the construction of the core resolution is generalized to torsion-free hyperbolic groups, all the basic properties of the core over a free group, presented in propositions 4.13-4.21 in [22], generalize to torsion-free hyperbolic groups as well. The arguments are essentially identical, hence, we refer the interested reader to section 4 in [22] for the basic properties of the core and their proofs.

§6. Quantifier Elimination

As in the case of a free group, to obtain quantifier elimination of elementary predicates over a torsion-free hyperbolic group, our goal is showing that the Boolean algebra of AE sets is invariant under projections. Our approach to proving this invariance is similar to the quantifier elimination procedure in the free group case, presented in [22] and [23].

Given a predicate defined over a torsion-free hyperbolic group Γ, we start by constructing the tree of stratified sets, in the same way it is constructed in section 2 of [22] over a free group. The procedure for constructing the tree of stratified sets is based on the iterative procedure for validating an AE sentence, and the modifications required for generalizing it to torsion-free hyperbolic groups, are precisely the modifications presented in section 4. Hence, we omit its detailed description and refer the interested reader to section 2 of [22].

The outcome of the tree of stratified sets is an encoding of all the (finitely many) possible sequences of forms of families of formal solutions that are needed in order to validate that a certain specialization p_0 of the defining parameters p is indeed in the set $EAE(p)$. As in the free group case, this stratification is the basis for our analysis of the structure of the set $EAE(p)$.

Let $\Gamma =< a_1, \ldots , a_k >$ be a torsion-free hyperbolic group, and let $EAE(p)$ be the set defined by the predicate:

$$EAE(p) \;=\; \exists w \; \forall y \; \exists x \; \Sigma(x, y, w, p, a) = 1 \, \wedge \, \Psi(x, y, w, p, a) \neq 1.$$

Recall (definition 1.19 in [22]) that a specialization w_0 of the variables w, is said to be a *witness* for a specialization p_0 of the defining parameters p, if the following sentence:

$$\forall y \; \exists x \; \Sigma(x, y, w_0, p_0, a) = 1 \, \wedge \, \Psi(x, y, w_0, p_0, a) \neq 1$$

is a truth sentence. Clearly, if there exists a witness for a specialization p_0 then $p_0 \in EAE(p)$, and every $p_0 \in EAE(p)$ has a witness.

By the construction of the tree of stratified sets, a witness w_0 for a specialization p_0, proves that $p_0 \in EAE(p)$ using a certain "proof system" which is built from a finite sequence of (families of) formal solutions, which correspond to boundedly many paths along the tree of stratified sets (proof system are presented in definition 1.20 in [22]).

Given $p_0 \in EAE(p)$, we are not able to say much about a possible witness for p_0 using the information we have collected so far. However, with each "proof system", i.e., with each path of (families of) formal solutions that goes along the tree of stratified sets, one can associate a certain Diophantine set. The bound on the form and number of all possible "proof systems" associated with all possible witnesses suggested by the tree of stratified sets, forces every possible witness for p_0 to belong to one of the finitely many Diophantine sets associated with the (finite) collection of all proof systems. Furthermore, as in the free group case, we will show that if a specialization $p_0 \in EAE(p)$, and it can be shown that $p_0 \in EAE(p)$ using a witness w_0 with a corresponding proof system, then every "generic" specialization of w that belongs to (a closure of) some modular block associated with the Diophantine set which is associated with the specific proof system is a witness for p_0 using the same proof system as w_0. This will reduce the analysis of the set $EAE(p)$ to the analysis of the Diophantine sets and the modular blocks associated with each "proof system", and eventually will enable us to show that the set $EAE(p)$ is in the Boolean algebra of AE sets.

Definition 6.1. *Let $p_0 \in EAE(p)$ be a specialization of the defining parameters p, and let w_0 be a witness for p_0. By the construction of the tree of stratified sets, one can associate a proof system with the couple (w_0, p_0), which corresponds to a (finite) collection of paths in the tree of stratified sets. Note that there may be several proof systems associated with a given couple (w_0, p_0), but the construction of the tree of stratified sets guarantees that the number of proof systems associated with the couple (w_0, p_0) is globally bounded.*
We will say that a given proof system associated with the couple (w_0, p_0) is of depth d, if all the paths associated with the proof system terminate after d steps (levels) of the tree of stratified sets.

As in the free group case, we start by analyzing those specializations of the defining parameters p that have witnesses with proof systems of depth 1, continue by analyzing the specializations of the defining parameters p for which there are witnesses with proof systems of depth at most 2, and then present the analysis of the entire set $EAE(p)$.

Lemma 6.2. *Let $T_1(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p, that have witnesses with proof system of depth 1. Then $T_1(p)$ is an EA set.*

Proof: Identical to lemma 1.21 in [22].

□

Lemma 6.2 proves that the set of specializations of the defining parameters p that have witnesses with proof statement of depth 1 is an EA set. As in the case of a free group, we continue by showing that the set of p's that have witnesses with proof statement of depth 2 is in the Boolean algebra of AE sets, and then generalize it to the existence of a witness with an arbitrary depth proof statement.

Theorem 6.3. *Let $T_2(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p, that have witnesses with proof system of depth 2. Then $T_2(p)$ is in the Boolean algebra of AE sets.*

Before we start with the proof of theorem 6.3, we need the notion of a *valid PS statement*.

Definition 6.4. *Suppose that a specialization $p_0 \in EAE(p)$ has a witness w_0 with a proof system of depth 2 (i.e., $p_0 \in T_2(p)$). The structure of the tree of stratified* *sets guarantees the existence of a rigid or strictly solid families of specializations (h_0^1, w_0, p_0) of one of the rigid or solid limit groups $WPH(h,w,p,a)$ with the following properties:*

(i) *For every Γ-limit group $WPHG(g_1,h_1,w,p,a)$, which is a terminal Γ-limit group of its associated graded resolution, hence, it is a (possibly trivial) free product of a rigid or solid Γ-limit group with several freely indecomposable groups that are embedded into Γ, there are at most (globally) boundedly many rigid or strictly solid families of specializations of the form (g_0^1, h_0^1, w_0, p_0) of $WPHG(g_1,h_1,w,p,a)$, where the strictly solid families are with respect to the given set of closures associated with (some of the other) Γ-limit groups $WPHG(g_1,h_1,w,p,a)$. The elements $(g_0^1, h_0^1, w_0, p_0, a)$ that appear in a proof system, contains representatives for all the boundedly many rigid and solid classes.*

(ii) *The specialization (h_0^1, w_0, p_0) is rigid or solid specialization of the corresponding Γ-limit group WPH, and it does not factor through any of the modular blocks that factor through one of the Γ-limit groups $\lambda WPGL(y,h,w,p,a)$ and correspond to the entire free group F_y.*

(iii) *For each of the (boundedly many) rigid or strictly solid families of specializations (g_0^1, h_0^1, w_0, p_0) there exist a finite collection of rigid or strictly solid families of specializations $(h_0^2, g_0^1, h_0^1, w_0, p_0)$ of the rigid or strictly solid Γ-limit groups $WPHGH(h_2,g_1,h_1,w,p,a)$, so that the (ungraded) resolutions corresponding to the specializations $(h_0^2, g_0^1, h_0^1, w_0, p_0)$ form a covering closure of the (ungraded) resolution corresponding to the specialization (g_0^1, h_0^1, w_0, p_0).*

(iv) *For each of the (boundedly many) rigid or strictly solid families of specializations $(h_0^2, g_0^1, h_0^1, w_0, p_0)$ there exists no specialization g_0^2 of the variables g_2 so that the specialization $(g_0^2, h_0^2, g_0^1, h_0^1, w_0, p_0)$ factors through one of the (rigid or solid) Γ-limit groups $WPHGHG(g_2,h_1,g_1,h_1,w,p,a)$.*

To a specialization of the form:

$$((h_1^2, g_1^1), \dots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0, a)$$

that satisfies conditions (i)-(iv) above, where the integer $\nu(ps)$ depends on the fixed proof system, we add specializations of new variables r, that testify that the associated specialization satisfies property (iii). We call such a combined specialization:

$$(r, (h_1^2, g_1^1), \dots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0, a)$$

a valid PS statement.

The tree of stratified sets guarantees that there are finitely many proof systems of depth 2. Once we fix a proof system of depth 2, we have fixed the rigid or solid Γ-limit group $WPH(h,w,p,a)$, the number of rigid or strictly solid families of specializations of each of the rigid or solid factor in the given free decomposition of the Γ-limit groups $WPHG(g_1,h_1,w,p,a)$, and the number of families of each of the other factors that do not factor through completions of other (deeper) ungraded resolutions of these factors, and the number of rigid or solid specializations of each of the rigid or solid factor in the Γ-limit groups (h_2,g_1,h_1,w,p,a), and the number of families of embeddings of each of the other factors.

We start the analysis of the set $T_2(p)$ by enumerating all the possible proof systems, and for each proof system we collect all possible (configurations of) valid PS statements: $(r,(h_1^2,g_1^1),\dots,(h^2_{\nu(ps)},g^1_{\nu(ps)}),h_0^1,w_0,p_0,a)$ (the integer $\nu(ps)$ depends on the fixed proof system), which by the standard arguments presented in section 1, factor through a (canonical) collection of maximal Γ-limit groups $PSHGH_1,\dots,PSHGH_m$, which we call PS (proof system) Γ-limit groups.

By construction, for each $p_0 \in T_2(p)$ there exists some witness w_0 and a corresponding proof system, so that a specialization of the form:

$$(r,(h_1^2,g_1^1),\dots,(h^2_{\nu(ps)},g^1_{\nu(ps)}),h_0^1,w_0,p_0,a)$$

that is associated with the specialization p_0, the witness w_0 and that proof system, is a valid PS statement (i.e., it satisfies conditions (i)-(iv) of definition 6.4), and factors through a PS Γ-limit group $PSHGH_j$. Naturally, as in the free group case, we will try to understand the set of valid PS statements:

$$(r,(h_1^2,g_1^1),\dots,(h^2_{\nu(ps)},g^1_{\nu(ps)}),h_0^1,w_0,p_0,a)$$

that factor through a given PS limit group $PSHGH_j$. Our main goal will be to show that these valid PS statements are "generic" in some modular blocks associated in the sequel with each of the PS limit groups $PSHGH$.

Let $P=<p>$ be the group of defining parameters. With each of the limit groups $PSHGH_i$ we associate its canonical graded Makanin-Razborov diagram (with respect to the parameter subgroup P), which contains finitely many graded resolutions which we denote $PSHGHRes$ (omitting their index).
As in the free group case, with (the completion of) the graded PS resolution $PSHGHRes$ we associate a (canonical) finite collection of *Non-Rigid* and *Non-Solid PS Γ-limit groups* (definition 1.25 in [22]), which we call the *non-rigid PS Γ-limit groups* associated with the PS resolution $PSHGHRes$, $NRgdPS_1^j,\dots,NRgdPS_q^j$, and the *non-solid PS Γ-limit groups* associated with $PSHGHRes$, $NSldPS_1^j,\dots,NSldPS_r^j$.

We continue by collecting all the test sequences that factor through the completion of one of the PS resolutions $PSHGHRes$, $Comp(PSHGHRes)$, and for which for at least one of the tuples: $(h_t^2(n),g_t^1(n),h_1(n),w_n,p_n,a)$ there exists some specialization $g_m^2(n)$ so that the (combined) specialization: $(g_t^2(n),h_t^2(n),g_t^1(n),h_1(n),w_n,p_n,a)$ factors through (at least) one of the Γ-limit groups $WP(HG)^2$, or there exists a specialization $g_1(n)$ so that the (combined) specialization: $(g_1(n),h_1(n),w_n,p_n,a)$ factors through (at least) one of the Γ-limit groups $WPHG$, which is not associated with one of the paths associated with our fixed proof system (definition 1.31 in [22]).
The collection of all these (graded) test sequences factor through a (canonical) collection of *maximal left PS Γ-limit groups*: $LeftPS_1^j,\dots,LeftPS_{m_j}^j$, and with them we associate the *Left PS* resolutions that are graded formal closures of the resolutions $PSHGHRes$ (see definition 1.26 in [22]). In a similar way, we construct *Root PS* resolutions that collect all the test sequences for which specializations of what is supposed to be primitive roots have roots of order that divides the least common multiples of the indices of the finite index subgroups that are associated with the closures that are associated with the given proof system (see definition 1.27 in [22]).

"Generic" specializations that factor through the PS resolutions $PSHGHRes$ can fail to be valid PS statements also if there exist additional rigid or strictly solid specializations of the Γ-limit groups $PSHG(g_1, h_1, w, p, a)$ that are not specified by the given specializations. The "generic" specializations for which there exists "surplus" in rigid or strictly solid specializations are collected in *extra PS* (graded) Γ-*limit groups* and graded resolutions (definition 1.28 in [22]).

The extra PS Γ-limit groups and their associated graded formal closures collect all the "generic" specializations (i.e., all the test sequences) of the PS limit groups $PSHGH$ for which there exist rigid or strictly solid families in addition to those specified by the generic specializations. For a general specialization of the PS limit groups $PSHGH$, i.e., a specialization which is not necessarily "generic", it may as well be that the additional rigid or strictly solid specializations, collected by the extra PS limit groups and their associated graded formal closures, do become flexible or do coincide with the rigid or strictly solid families of the various specializations (g_t^1, h_0^1, w_0, p_0).

We collect all the test sequences of specializations that factor through an extra PS resolution, $ExtraPSRes$, and are collapsed specializations, in finitely many closures of the resolution $PSHGHRes$ we have started with, which we call $formal$ *collapsed Extra PS* (graded) *resolutions*.
Collecting all "collapsed" test sequences in a finite collection of formal collapsed extra PS resolutions, we still need to collect all the collapsed specializations that factor and are taut with respect to a given extra PS resolution $ExtraPSRes$. We go over all the graded auxiliary resolutions (definition 2.1 in [23]), and all the collapse forms associated with the extra PS resolution, $ExtraPSRes$. Given a graded auxiliary resolution and a collapse form, we add elements that demonstrate that the given extra rigid or solid specialization is collapsed according to the given collapse form.

Given the auxiliary Γ-limit group or one of its degenerate quotients, and the additional elements that are associated with the given collapse form, we look at the associated collection of combined specializations, so that the (combined) specializations satisfy the system of equations associated with the given collapse form associated with the extra PS resolution $ExtraPSRes$.
By our standard method presented in section 1, this collection of specializations factor through a canonical (finite) collection of maximal limit groups, which we call *collapsed extra PS* Γ-*limit groups*.

By construction, if $p_0 \in T_1(p)$ then there must exist a valid PS statement of the form: $(r, (h_1^2, g_1^1), \dots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0, a)$ that factors through one the of the PS resolutions $PSHGHRes_j$ constructed with respect to all proof systems of depth 2. By proposition 3.7 in [22], that generalizes in a straightforward way to torsion-free hyperbolic groups, the sets $TSPS(p)$ associated with the various PS resolutions $PSHGHRes$, i.e., the sets of specializations p_0 of the defining parameters $P =< p >$ for which there exists a test sequence of valid PS statements that factor through any of the PS resolutions $PSHGHRes_j$, is in the Boolean algebra generated by AE sets. As in the free group case, if there exists a valid PS statement that factors through a PS resolution $PSHGHRes_j$, then either there exists a test sequence of valid PS statements that factor through the PS resolution, or there must exist a specialization that factors through one of the graded auxiliary resolutions associated with one of the PS resolutions $PSHGHRes$, and one of its associated collapsed extra PS limit groups.

To analyze the remaining set of valid PS statements we construct an iterative sieve procedure that is similar to the sieve procedure over a free group, presented in [23]. Having generalized the core resolution to resolutions over a torsion-free hyperbolic group in the previous section, the structure of the sieve procedure over a torsion-free hyperbolic group and the proof of its termination, is identical to the sieve procedure over a free group, presented in [23]. Hence, we omit its details and refer the interested reader to [23].

This sieve procedure finally proves theorem 6.3, i.e., it shows that the set of specializations of the defining parameters p, for which there exists a valid PS statement of depth 2, is in the Boolean algebra of AE sets.

The tree of stratified sets has a finite depth, which (by definition) bounds the depth of all possible proof systems associated with the tree of stratified sets. For each integer d, we set $T_d(p)$ to be the set of specializations p_0 of the defining parameters $P =< p >$ for which there exists a valid PS statement for some proof system of depth d. Clearly: $EAE(p) = T_1(p) \cup \ldots \cup T_{d_0}(p)$ where d_0 is the depth of tree of stratified sets. Each of the sets $T_d(p)$ is in the Boolean algebra generated by AE sets, by applying the same sieve procedure that was used to analyze the set $T_2(p)$ (cf. [23]). Since a Boolean algebra is closed under finite unions, it follows that every EAE set is in the Boolean algebra generated by AE sets.

Theorem 6.5. *Let $\Gamma =< a_1, \ldots, a_k >$ be a torsion-free hyperbolic group, and let the EAE set $EAE(p)$ be defined as:*

$$EAE(p) = \exists w \, \forall y \, \exists x \, (\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1) \vee \ldots$$

$$\ldots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1).$$

Then $EAE(p)$ is in the Boolean algebra generated by AE sets.

§7. The elementary theory of a hyperbolic group

In the first six sections of this paper we have modified the techniques presented in [18]-[23] to prove that every elementary set defined over a torsion-free hyperbolic group is in the Boolean algebra of AE sets (theorem 6.5). In [24] we were able to use the quantifier elimination procedure presented in [23] to classify those f.g. groups that are elementarily equivalent to a free group, i.e., to show that a f.g. group is elementarily equivalent to a free group if and only if it is an ω-residually free tower. In this section we use the quantifier elimination procedure for torsion-free hyperbolic groups, to classify those f.g. groups that are elementarily equivalent to a given torsion-free hyperbolic group. We start with the following observation.

Proposition 7.1. *Let Γ_1, Γ_2 be non-elementary torsion-free rigid hyperbolic groups (i.e., Γ_1 and Γ_2 are freely-indecomposable and do not admit any non-trivial cyclic splitting). Then Γ_1 is elementarily equivalent to Γ_2 if and only if Γ_1 is isomorphic to Γ_2.*

Proof: We fix a system of generators of Γ_1, $\Gamma_1 =< g_1, \ldots, g_t >$, and set X to be the Cayley graph of Γ_1 with respect to the given generating system. By section 8 of [27], since Γ_1 is rigid and Γ_2 is hyperbolic, there exists a positive integer n_1, for which if $h : \Gamma_1 \to \Gamma_2$ is a homomorphism that maps the elements in a ball of radius n_1 in the Cayley graph X of Γ_1 into distinct elements in Γ_2, then h is necessarily a monomorphism.

Suppose that Γ_1 is elementarily equivalent to Γ_2. Since the existence of a homomorphism from Γ_1 to Γ_2 that maps the elements that are in a ball of radius n_1 in X into distinct elements in Γ_2, can be formulated by a (coefficient-free) existential sentence that is true over Γ_1, and Γ_1 is assumed elementarily equivalent to Γ_2, there must exist a monomorphism from Γ_1 to Γ_2. Similarly, there must exist a monomorphism from Γ_2 to Γ_1. By the co-Hopf property for hyperbolic groups ([26]), Γ_1 is necessarily isomorphic to Γ_2.

□

Proposition 7.1 implies that, in particular, a uniform lattice in a real rank 1 semi-simple Lie group that is not $SL_2(R)$ is elementarily equivalent to another such lattice if and only if the two lattices are isomorphic. Hence, by Mostow's rigidity, the two lattices are conjugate in the same Lie group. By Margulis normality and super-rigidity theorems, the same hold in higher rank (real) Lie groups.

Theorem 7.2. *Let L_1, L_2 be uniform lattices in real semi-simple Lie groups that are not $SL_2(R)$. Then L_1 is elementarily equivalent to L_2 if and only if L_1 and L_2 are conjugate lattices in the same real Lie group G.*

Proposition 7.1 shows that rigid hyperbolic groups are elementarily equivalent if and only if they are isomorphic. To classify elementary equivalence classes of hyperbolic groups in general, we need to present *elementary* (hyperbolic) *prototypes.*

Definition 7.3. *Let Γ be a non-elementary torsion-free hyperbolic group. We say that Γ is an elementary-prototype if:*

(i) *Γ is not an ω-residually free tower (i.e., Γ is not elementarily equivalent to a free group).*

(ii) *Γ admits a (Grushko) free decomposition: $\Gamma = H_1 * \ldots * H_\ell$, where $H_1, \ldots, H_\ell$ are freely-indecomposable, non-cyclic and are not ω-residually free towers.*

(iii) *Each of the factors H_i does not admit an endomorphism $h : H_i \to H_i * < c >$ with non-trivial kernel, that maps (elementwise) each of the non-QH vertex groups and each of the edge groups in the JSJ decomposition of H_i to their conjugates (in H_i), and each of the QH vertex groups onto a non-abelian subgroup.*

Like rigid hyperbolic groups, elementary prototypes that are elementarily equivalent have to be isomorphic.

Proposition 7.4. *Let Γ_1, Γ_2 be two (hyperbolic) elementary-prototypes. Γ_1 and Γ_2 are elementarily equivalent if and only if they are isomorphic.*

Proof: First, suppose that both elementary-prototypes, Γ_1 and Γ_2, are freely indecomposable. We fix a system of generators of Γ_1, $\Gamma_1 =< g_1, \ldots, g_t >$, and set X to be the Cayley graph of Γ_1 with respect to the given generating system.
Suppose that Γ_1 is elementarily equivalent to Γ_2. From the universal equivalence of Γ_1 and Γ_2, for every index n, there exists a homomorphism $v_n : \Gamma_1 \to \Gamma_2$ that maps the ball of radius n in X, the Cayley graph of Γ_1, monomorphically into Γ_2. Moreover, since Γ_1 is an elementary-prototype, the equivalence of the EA theories of

Γ_1 and Γ_2 implies that for every integer n, there exists a homomorphism h_n, so that for every modular automorphism, $\varphi \in Mod(\Gamma_1)$, the composition $h_n \circ \varphi : \Gamma_1 \to \Gamma_2$, maps the ball of radius n in X monomorphically into Γ_2.

The sequence of homomorphisms $h_n : \Gamma_1 \to \Gamma_2$, contains a subsequence (still denoted $\{h_n\}$) that converges into an action of Γ_1 on a real tree Y. Hence, Γ_1 is a Γ_2-limit group. Therefore, either Γ_1 can be embedded into Γ_2, or a subsequence of the homomorphisms $h_n : \Gamma_1 \to \Gamma_2$ factors through a non-trivial taut Makanin-Razborov resolution of the Γ_2-limit group Γ_1. However, the existence of a subsequence of homomorphisms $h_n : \Gamma_1 \to \Gamma_2$ that factors through a non-trivial resolution of the Γ_2-limit group Γ_1 contradicts our assumption that for every index n, and for every modular automorphism $\varphi \in Mod(\Gamma_1)$, the composition $h_n \circ \varphi$ embeds the ball of radius n in X, the Cayley graph of Γ_1, into Γ_2. Hence, Γ_1 can be embedded in Γ_2. In a similar way, Γ_2 can be embedded in Γ_1. Since we assumed that Γ_1 and Γ_2 are freely-indecomposable, torsion-free hyperbolic groups, they are co-Hopf by [26], which finally implies that Γ_1 is isomorphic to Γ_2 in case Γ_1 and Γ_2 are elementarily equivalent and freely indecomposable.

Suppose that Γ_1 and Γ_2 are (general) elementarily equivalent prototypes. Let $\Gamma_1 = H_1 * \ldots * H_\ell$ and $\Gamma_2 = M_1 * \ldots * M_s$, where the factors H_i and M_j are (non-elementary) freely indecomposable and not ω-residually free towers. By the equivalence of the EA theories of Γ_1 and Γ_2, for every n, there exists a homomorphism h_n, so that for every modular automorphism $\varphi \in Mod(\Gamma_1)$, the composition $h_n \circ \varphi : \Gamma_1 \to \Gamma_2$, maps the ball of radius n in X monomorphically into Γ_2. Moreover, we can further require that for any possible choice of elements: $\gamma_1, \ldots, \gamma_\ell \in \Gamma_2$, and every modular automorphism $\varphi \in Mod(\Gamma_1)$, the homomorphism $\tau_n : \Gamma_1 \to \Gamma_2$ obtained by setting $\tau_n(H_i) = \gamma_i (h_n \circ \varphi(H_i)) {\gamma_i}^{-1}$, maps the ball of radius n in X monomorphically into Γ_2.

From the sequence of homomorphisms: $h_n : \Gamma_1 \to \Gamma_2$, it is possible to extract a subsequence that converges into a faithful action of Γ_1 on a real tree, hence, Γ_1 and its factors $H_1, \ldots, H_\ell$ are all Γ_2-limit groups. Since for every modular automorphism $\varphi \in Mod(\Gamma_1)$, $h_n \circ \varphi$ embeds the ball of radius n in X, the Cayley graph of Γ_1, we can further extract a subsequence of homomorphisms (still denoted $\{h_n\}$) that embeds each of the factors $H_1, \ldots, H_\ell$ into Γ_2. Similarly, there exist similar subsequence of homomorphisms $v_n : \Gamma_2 \to \Gamma_1$ that have similar properties as the sequence $\{h_n\}$, and each of the homomorphisms v_n embeds the factors $M_1, \ldots, M_s$ into Γ_1.

The homomorphisms $\{h_n : \Gamma_1 \to \Gamma_2\}$ embed each of the factors H_i into Γ_2, and the homomorphisms $v_n : \Gamma_2 \to \Gamma_1$ embed each of the factors M_j into Γ_1. Since the factors H_i and M_j are freely indecomposable, for each index i, $1 \leq i \leq \ell$, $h_n(H_i)$ is a subgroup of a conjugate of one of the factors M_j, and for each index j, $1 \leq j \leq s$, $v_n(M_j)$ is a subgroup of a conjugate of one of the factors H_i. Therefore, for each index n, with the pair of homomorphisms h_n, v_n we can associate a two sided directed graph Gr_n, where the vertices of the two sides correspond to the factors $H_1, \ldots, H_\ell$ and $M_1, \ldots, M_s$, from a vertex associated with H_i there is an edge connected to it and directed towards the vertex associated with the factor M_j that contains a conjugate of $h_n(H_i)$, and from a vertex associated with M_j there is an edge connected to it and directed towards the vertex associated with the factor H_i that contains a conjugate of $v_n(M_j)$.

Since there are only finitely many possibilities for the combinatorial types of the graphs Gr_n, we can pass to a further subsequence and assume that they are all

identical, which we denote Gr. Since from each vertex in the bipartite directed graph Gr there exists an edge connected to it and directed towards another vertex in Gr, the graph Gr contains a (directed) circle.
Let C be an innermost circle in the graph Gr. Since Gr is a bipartite graph, C contains an equal number of vertices corresponding to the factors H_i and to the factors M_j. Since the factors H_i and M_j are all assumed to be freely-indecomposable, these factors are co-Hopf. Since, in addition, for every index n, the homomorphism h_n embeds the factors H_i in Γ_2, and the homomorphism v_n embeds the factors M_j in Γ_1, the factors H_i and M_j that correspond to vertices along the innermost circle C in the directed graph Gr, are pairwise isomorphic, and the homomorphisms h_n, v_n restricted to the edges in the circle C are isomorphisms.

Since the homomorphisms h_n, v_n restricted to the edges in the circle C are isomorphisms, and since we assumed that for every index n, and any possible collection of elements $\gamma_1, \ldots, \gamma_\ell \in \Gamma_2$, and every modular automorphism $\varphi \in Mod(\Gamma_1)$, the homomorphism $\tau_n : \Gamma_1 \to \Gamma_2$ obtained by conjugating $h_n \circ \varphi(H_i)$ by γ_i for $1 \leq i \leq \ell$, embeds the ball of radius n in X, the Cayley graph of Γ_1, and correspondingly for the homomorphism v_n, no vertex that is not in the innermost circle C in Gr is connected by an edge to a vertex in C. Applying the same argument for the subgraph $Gr \setminus C$, and continuing inductively, we finally obtain that $\ell = s$, and up to a change of order, the factor H_i of Γ_1 is isomorphic to the factor M_i of Γ_2, hence, Γ_1 is isomorphic to Γ_2.

□

After showing that the elementary class of an elementary prototype determines its isomorphism class (among the set of elementary prototypes), we (canonically) associate with any given torsion-free hyperbolic group a retract of it which is an elementary prototype, a retract that we call *elementary core*. As we will see in the sequel, a torsion-free hyperbolic group that is not elementary equivalent to a free group, is elementary equivalent to its elementary core.

Definition 7.5. *Let Γ be a non-elementary torsion-free hyperbolic group. We construct the elementary core of Γ iteratively. Let the Grushko's decomposition of Γ be: $\Gamma = H_1 * \ldots H_m * F_s$, where each of the factors H_i is non-cyclic and freely-indecomposable, and F_s is a (possibly trivial) free group. We omit the free factor F_s, as well as each of the freely-indecomposable factors H_i that are isomorphic to a (closed) surface group that is elementarily equivalent to a free group, i.e., a hyperbolic surface group, where the surface is not the non-orientable surface of genus 2.*
Let $H_1, \ldots, H_t$ be the remaining factors. We continue by constructing the elementary core of each of the factors H_i, $EC(H_i)$, and then set the core of the hyperbolic group Γ, $EC(\Gamma)$, to be:

$$EC(\Gamma) = EC(H_1) * \ldots * EC(H_t).$$

To construct the elementary core of a freely indecomposable, non-cyclic factor H_i, we associate with it its (cyclic) JSJ decomposition. If H_i is an elementary prototype (see definition 7.3), we set $EC(H_i) = H_i$, and conclude the construction of the elementary core of the factor H_i. If H_i is not an elementary prototype, there must exist an endomorphism $\nu : H_i \to H_i * < c >$ with non-trivial kernel, that maps each of the non-QH vertex groups in the JSJ decomposition of H_i into *a conjugate in H_i (elementwise), and each QH vertex group onto a non-abelian subgroup. The existence of such endomorphism ν, implies the existence of a proper retraction, $r : H_i \to H_i$, that maps the factor H_i onto the fundamental group of a (not necessarily connected) proper subgraph of the JSJ decomposition of H_1, and maps (elementwise) each of the non-QH vertex groups into a conjugate (see the argument that was used to prove proposition 6 in [24] for the existence of the proper retraction r, given the proper map ν).*

The image of the retraction r, is the fundamental group of a subgraph of the JSJ decomposition of the torsion-free hyperbolic factor H_i, hence, it is a torsion-free hyperbolic group. Continuing the construction iteratively, we set the elementary core of the factor H_i to be: $EC(H_i) = EC(r(H_i))$.
In each step along the iterative construction of the elementary core of the torsion-free hyperbolic group Γ, we replace a factor by a proper retract of it, which is a proper quotient of the factor, and a subgroup of the original torsion-free hyperbolic group Γ. Hence, the descending chain condition for Γ-limit groups (theorem 1.12) proves that the construction of the elementary core terminates after finitely many steps.

Note that by construction, the ambient torsion-free hyperbolic group Γ has the structure of a tower, in which the bottom level is a free product of the elementary core, $EC(\Gamma)$, with a (possibly trivial) collection of a free group and surface groups, and in each level of the tower a punctured surface group is being added, so that the boundaries of the punctured surface are identified with non-trivial elements in the group associated with the previous level. In addition there are retractions associated with the tower, mapping the groups associated with the various levels to the groups associated with the previous ones (cf. the structure of ω-residually free towers in [24]).

By construction, the elementary core of a (non-elementary) torsion-free hyperbolic group is trivial if and only if the hyperbolic group is an ω-residually free tower, i.e., if it is elementarily equivalent to a free group.
Along the construction of the elementary core various choices can be made, and indeed, in general the elementary core of a torsion-free hyperbolic group is not necessarily a unique subgroup. The next theorem shows that an elementary core is an elementary submodel of a torsion-free hyperbolic group. Hence, every two elementary cores are elementarily equivalent, and since an elementary core is an elementary prototype (definition 7.3), proposition 7.4 implies that the isomorphism type of an elementary core is uniquely defined.

Theorem 7.6. *Let Γ be a non-elementary torsion-free hyperbolic group that is not an ω-residually free tower, i.e., that is not elementarily equivalent to a free group. Then Γ is elementarily equivalent to its elementary core, $EC(\Gamma)$. Furthermore, the embedding of the elementary core, $EC(\Gamma)$, in the ambient group Γ is an elementary embedding.*

Proof: Let $EC = EC(\Gamma)$. By the construction of the elementary core, EC, from the torsion-free hyperbolic group Γ, it follows that Γ is the fundamental group of a tower over the elementary core, EC, free product with a (possibly trivial) finite collection of surface groups and a free group, that we denote Twr. With the structure of the tower Twr we associate a (completed) resolution over the coefficient group EC, that we denote Res_{Γ}. Note that the terminal group of Res_{Γ} is a free product of EC with a (possibly trivial) collection of surface groups and a free group.

Let $EC =< a_1, \dots , a_k >$, and let $\Sigma(y, a) = 1$ be a system of equations with coefficients in the elementary core, EC. With the system $\Sigma(y, a) = 1$, interpreted as a system of equations over EC, which is in particular a torsion-free hyperbolic group, we have associated a taut Makanin-Razborov diagram. With each resolution in this taut Makanin-Razborov diagram we have associated its completion. Let $Comp(Res_1)(z, y, a), \dots , Comp(Res_d)(z, y, a)$ be the set of these completions.

Lemma 7.7. *Let y' be a solution of the system $\Sigma(y, a) = 1$, interpreted as a system of equations over the ambient torsion-free hyperbolic group Γ. Then there exist elements z' in Γ, so that the tuple (z', y') is a specialization of (at least) one of the completions $Comp(Res_1)(z, y, a), \dots , Comp(Res_d)(z, y, a)$, i.e., it is the image of a homomorphism from the Γ-limit group associated with one of these completions into Γ.*

Proof: We look at a test sequence of the resolution, $Res_\Gamma(b)$, associated with the tower Twr, in the core EC. Let $y'_n \in EC$ be the sequence of specializations of the elements $y' \in \Gamma$ along the given test sequence. Since $\Sigma(y', a) = 1$ in Γ, the sequence y'_n satisfies $\Sigma(y'_n, a) = 1$ in EC, for every index n. Hence, for every index n, there exist elements $z'_n \in EC$, so that the tuple (z'_n, y'_n, a) is a specialization of one of the completions $Comp(Res_1)(z, y, a), \dots , Comp(Res_d)(z, y, a)$. If for every n we choose the shortest such z'_n, then the techniques used for the construction of a formal solution ([19],1.17), prove that there exists a subsequence of tuples (z'_n, y'_n, a) that are all specializations of the same completion, $Comp(Res_i)(z, y, a)$, that converge into tuple $(z', y', a) \in \Gamma$, and the tuple (z', y', a) is a specialization of the completion $Comp(Res_i)(z, y, a)$.

□

Clearly, the same argument used to prove lemma 7.7 applies to specializations of graded and multi-graded systems of equations, and the completions of the corresponding taut graded and multi-graded taut Makanin-Razborov diagrams. Let $\Theta(y, a) = 1$ be a system of equations with coefficients in the elementary core, EC, and let:

$$(\forall y)\, (\Theta(y, a) = 1)\ \exists x\ \Sigma(x, y, a) = 1 \wedge \Psi(x, y, a) \neq 1$$

be a sentence. Suppose that the sentence is a true sentence over the elementary core, EC. By theorem 2.1, with each resolution $Res(y, a)$ in the taut Makanin-Razborov diagram associated with the system $\Theta(y, a) = 1$ over the elementary core, EC, it is possible to associate finitely many formal solutions, $x = x_i(s, z, y, a)$, defined over closures of that resolution, $Cl_i(Res)(s, z, y, a)$, that together form a covering closure of the resolution $Res(y, a)$, for which for every index i:

(i) $\Sigma(x(s, z, y, a), y) = 1$ in the EC-limit group corresponding to the closure, $Cl_i(Res)(s, z, y, a)$.

(ii) there exists a specialization (s_i, z_i, y_i, a) (in the core EC), of the closure $Cl_i(Res)(s, z, y, a)$, for which $\Psi(x(s_i, z_i, y_i, a), y_i, a) \neq 1$ (in EC).

By lemma 7.7, if y' is a solution of the system $\Theta(y, a) = 1$ interpreted as a system of equations over the ambient group Γ, then there exists elements z' in Γ, so that the tuple (z', y') is a specialization of (at least) one of the completions $Comp(Res_1)(z, y, a), \dots , Comp(Res_d)(z, y, a)$ of the resolutions in the taut Makanin-Razborov diagram associated with the system of equations $\Theta(y, a) =$

1. Hence, there exists a closure of some of these completions, a formal solution $x_i(s,z,y,a)$, and elements s', for which: $\Theta(x_i(s',z',y',a),y,a)=1$ over that closure. Clearly, the graded and multi-graded categories are completely analogous.

Proposition 7.8. *Let:*

$$\forall y\ \exists x\ (\Sigma_1(x,y,a)=1 \wedge \Psi_1(x,y,a)\neq 1)\ \vee \ldots \vee\ (\Sigma_r(x,y,a)=1 \wedge \Psi_r(x,y,a)\neq 1)$$

be an AE sentence with coefficients in the elementary core, EC. Then the AE sentence is a true sentence over EC if and only if it is a true sentence over the ambient hyperbolic group Γ, i.e., the embedding of the elementary core of a torsion-free hyperbolic group into the ambient hyperbolic group is an AE embedding.

Proof: Suppose that

$$\forall y\ \exists x\ (\Sigma_1(x,y,a)=1 \wedge \Psi_1(x,y,a)\neq 1)\ \vee \ldots \vee\ (\Sigma_r(x,y,a)=1 \wedge \Psi_r(x,y,a)\neq 1)$$

is a sentence with coefficients in the elementary core, EC. If the sentence is a true sentence over EC, then it is possible to apply the iterative procedure presented in section 4 (based on [21]), and associate iteratively with the sentence a (finite) sequence of anvils, developing resolutions and formal solutions defined over them, that prove the validity of the sentence over the elementary core EC. By the arguments given above, the proof given by the sequence of anvils, developing resolutions, and formal solutions defined over them, is valid over the non-elementary torsion-free hyperbolic group Γ.

Suppose that the given sentence is false over the elementary core, EC. By applying the iterative procedure for validation of a sentence over EC, there exists a resolution $Res(z,y,a)$ over EC, so that there exists a test sequence of specializations of this resolution (over EC), for which for the corresponding specializations of the variables y, there is no specialization of the variables x (in EC) for which:

$$(\Sigma_1(x,y,a)=1 \wedge \Psi_1(x,y,a)\neq 1)\ \vee \ldots \vee\ (\Sigma_r(x,y,a)=1 \wedge \Psi_r(x,y,a)\neq 1).$$

We want to show that the given sentence is false over the ambient hyperbolic group Γ. The resolution $Res(z,y,a)$ is a resolution over the elementary core, EC, hence, it is also a resolution over the ambient hyperbolic group Γ. By our assumptions, the resolution $Res(z,y,a)$ has a test sequence of specializations in EC, so that for the corresponding sequence of specializations of the variables y, there are no specializations of the variables x in EC, for which:

$$(\Sigma_1(x,y)=1 \wedge \Psi_1(x,y)\neq 1)\ \vee \ldots \vee\ (\Sigma_r(x,y)=1 \wedge \Psi_r(x,y)\neq 1).$$

EC is a non-elementary quasi-convex subgroup of Γ, so the test sequence of specializations of the resolution $Res(z,y,a)$ in EC is also a test sequence of specializations of the resolution $Res(z,y,a)$ in the ambient group Γ. Suppose that our given sentence is a true sentence over Γ. Then for each specialization of the variables y from our given test sequence, there exists a specialization of the variables x in Γ for which:

$$(\Sigma_1(x,y,a)=1 \wedge \Psi_1(x,y,a)\neq 1)\ \vee \ldots \vee\ (\Sigma_r(x,y,a)=1 \wedge \Psi_r(x,y,a)\neq 1)$$

where the equalities and inequalities are in Γ.
Using our machinery for constructing a formal solution (theorem 2.1), and after possibly passing to a subsequence of the test sequence in Γ, we obtain a formal solution $x = x(s, z, y, a)$ defined over some closure $Cl(s, z, y, a)$ of the resolution $Res(z, y, a)$ over Γ. Since Γ retracts onto its elementary core, the existence of a formal solution over the ambient group Γ, implies the existence of a formal solution $x' = x'(s, z, y, a)$ defined over a closure $Cl'(s, z, y, a)$ of the resolution $Res(z, y, a)$ over the elementary core, EC, where the same test subsequence that factored through the closure, $Cl(s, z, y, a)$, factors through the closure $Cl'(s, z, y, a)$. This is a contradiction, since we assumed that for every specialization of the variables y in the test sequence, there does not exist a specialization of the variables x in the elementary core, EC, for which the set of equalities and inequalities defining the sentence hold.

□

Proposition 7.8 proves that the embedding of the elementary core into the ambient torsion-free hyperbolic group is an AE embedding (assuming the ambient group is not elementarily equivalent to a free group). To prove that the embedding is an elementary embedding, it remains to prove that the quantifier elimination procedure conducted over EC, for predicates defined over the elementary core, EC, is valid over the the ambient hyperbolic group Γ. The quantifier elimination procedure presented in [22] and [23] (and section 6) is composed from two parts, the procedure for the construction of the tree of stratified sets, and the sieve procedure. We start by showing that the procedure for the construction of the tree of stratified sets remains valid over the ambient group.

The procedure for the construction of the tree of stratified sets analyzes the remaining set of y's using multi-graded resolutions, and their associated developing resolutions, and then we associate with each developing resolution, its entire collection of formal solutions, which are encoded in the graded formal diagram. Since the predicates we consider are defined over the elementary core, EC, and the construction of the tree of stratified sets is conducted over EC, all the resolutions are defined over EC, and all the limit groups are EC-limit groups.

Lemma 7.7 proves that the set of the remaining y's over the group Γ at each step, can be completed to specializations that factor through the completions of the multi-graded resolutions at each step of the procedure for the construction of the tree of stratified sets (over EC). We have also argued that if p_0 is in the definable set of the given predicate, interpreted as a predicate over Γ, then with each ungraded resolution associated with the (graded) developing resolution and p_0, there exists a formal solution (over Γ) that satisfies the properties of theorem 2.1.
The ungraded resolution defined over Γ, can be approximated by a sequence of ungraded resolutions over the core, EC. From the existence of a formal solution defined over the given ungraded resolution over Γ, it follows that all the approximating resolutions over EC admit formal solutions that approximate the formal solution over Γ. Hence, the given formal solution defined over Γ, factors through the completion of at least one of the graded formal resolutions associated with the given developing resolution over the elementary core EC.

The iterative procedure for validation of an AE sentence, shows that if an AE sentence over EC is a true sentence, then it has a proof using a finite collection of formal solutions. By the proof of lemma 7.7, the same is true for sentences defined over Γ.

Let $L(p,w)$ be an AE predicate defined over the elementary core, EC. Let $Q_{EC}(p,w)$ be the set defined by $L(p,w)$ over the EC, and $Q_\Gamma(p,w)$ be the set defined by $L(p,w)$ over Γ. The tree of stratified sets constructed over the free group EC, shows that if $(p_0,w_0) \in Q_{EC}(p,w)$, then the sentence corresponding to the specialization (p_0,w_0) can be proved using a sequence of formal solutions according to one of the proof systems given by the tree of stratified sets over EC. The argument given above, shows that if $(p_0,w_0) \in Q_\Gamma(p,w)$, then the sentence corresponding to the specialization (p_0,w_0) can be proved using a sequence of formal solutions according to one of the proof systems given by the same tree of stratified sets (over EC).

Furthermore, if $Rgd(x,p,a)$ ($Sld(x,p,a)$) is a rigid (solid) EC-limit group, then the maximal number of rigid (families of strictly solid) specializations of $Rgd(x,p,a)$ ($Sld(x,p,a)$) for a specialization of the defining parameter p is identical over the elementary core EC, and the ambient group Γ. Hence, the collection of proof systems associated with the tree of stratified sets is identical over EC and over Γ, i.e., for the two sets $Q_{EC}(w,p)$ and $Q_\Gamma(w,p)$.

The collection of proof systems associated with the tree of stratified sets over EC and over Γ are identical. Any valid PS statement over Γ can be approximated by a sequence of valid PS statements over EC. Furthermore, any valid PS statement over Γ does not factor through any of the Non-Rigid, Non-Solid, Root or Left PS resolutions constructed along the Sieve procedure over EC, and if it factors through an extra PS resolution, it has to factor through either one of the collapse extra PS EC-limit groups associated with it, or one of the Generic collapse extra PS resolutions associated with it (constructed over EC). Hence, any valid PS statement over Γ, belongs to at least one of the $TSPS(p)$ sets constructed along the sieve procedure (over the core EC. See proposition 1.34 in [22] for the definition of the sets $TSPS(p)$). Therefore, the reduction of a predicate defined over the elementary core, EC, to a predicate in the Boolean algebra of AE sets, can be done uniformly for the towers defined over the elementary core EC, i.e., for all the torsion-free hyperbolic groups for which EC is their elementary core. Since the AE theories of these towers are equivalent by proposition 7.8, the elementary core EC is an elementary submodel of the ambient torsion-free hyperbolic group Γ.

□

Corollary 7.9. *Let $\Gamma_1, \Gamma 2$ be two non-elementary torsion-free hyperbolic groups. Then Γ_1 and Γ_2 are elementarily equivalent if and only if their elementary cores, $EC(\Gamma_1)$ and $EC(\Gamma_2)$, are isomorphic.*

Proof: By the construction of the elementary core (definition 7.5), the elementary core of a torsion-free hyperbolic group is trivial if and only if the torsion-free hyperbolic group is a hyperbolic ω-residually free tower, i.e., if and only if it is elementarily equivalent to a free group. Hence, to prove the corollary we may assume that both hyperbolic groups, Γ_1 and Γ_2, are not elementarily equivalent to a free group.

By theorem 7.6, Γ_1 and Γ_2 are elementarily equivalent to their elementary cores, $EC(\Gamma_1)$ and $EC(\Gamma_2)$, in correspondence. An elementary core of a non-elementary, torsion-free hyperbolic group that is not elementarily equivalent to a free group is an elementary prototype, hence, by proposition 7.4, $EC(\Gamma_1)$ is elementarily equivalent to $EC(\Gamma_2)$ if and only if $EC(\Gamma_1)$ is isomorphic to $EC(\Gamma_2)$, and the corollary follows.

□

Remark: The iterative procedure that was used to construct the the elementary core of a torsion-free hyperbolic group (see definition 7.5), can be applied in the case of a general f.g. group. It is not difficult to see that the construction terminates after finitely many steps in the case of a general f.g. group as well. However, for a f.g. group we do not know if the elementary core is unique up to isomorphism, and whether there is a connection between the structure of the elementary core and the first order theory of the group.

Corollary 7.9 determines the elementary classes of (torsion-free) hyperbolic groups. The next theorem shows that hyperbolicity is preserved under elementary equivalence.

Theorem 7.10. *Let Γ be a torsion-free hyperbolic group, and let G be a f.g. group. If G is elementarily equivalent to Γ, then G is a torsion-free hyperbolic group. Hence, either:*

(i) *Both G and Γ are either trivial or (infinite) cyclic.*

(ii) *Both G and Γ are ω-residually free towers, i.e., they are both elementarily equivalent to a non-abelian free group.*

(iii) *Both G and Γ are non-elementary and not ω-residually free towers, and the elementary core of Γ, $EC(\Gamma)$, is isomorphic to the elementary core of G, $EC(G)$.*

Proof: Let Γ be a torsion-free hyperbolic group and G a f.g. group, and suppose that Γ is elementarily equivalent to G. A f.g. group that is elementarily equivalent to a trivial or an infinite cyclic group is trivial or infinite cyclic in correspondence. By [24] a f.g. group that is elementarily equivalent to a free group is an ω-residually free tower, which is, in particular, a torsion-free hyperbolic group. Hence, to prove the theorem we may assume that the torsion-free hyperbolic group Γ is non-elementary, and not an ω-residually free tower, i.e., not elementarily equivalent to a free group. Under these assumptions, we can associate with Γ its elementary core, $EC(\Gamma)$, and by theorem 7.6 Γ is elementarily equivalent to $EC(\Gamma)$, so $EC(\Gamma)$ is elementarily equivalent to G.

Since the elementary core, $EC(\Gamma)$, is a torsion-free hyperbolic group, and G is elementarily equivalent to $EC(\Gamma)$, G is, in particular, universally equivalent to $EC(\Gamma)$, which implies that there exists a sequence of homomorphisms $u_n : G \to EC(\Gamma)$, so that for each index n, u_n embeds the elements in the ball of radius n in G into $EC(\Gamma)$, hence, G is an $EC(\Gamma)$-limit group. Since G is an $EC(\Gamma)$-limit group, and G is elementarily equivalent to a torsion-free hyperbolic group, G contains no non-cyclic abelian subgroups. Hence, by applying the construction presented in definition 7.5, one can associate with G a core, which we denote $C(G)$, together with a sequence of retractions of G:

$$G \to G_1 \to G_2 \to \ldots \to G_t = C(G)$$

that are used in the construction of the core, $C(G)$, and in which all the retractions G_i are $EC(\Gamma)$-limit groups.

At this point we modify the argument that was used to prove that elementarily equivalent elementary prototypes are isomorphic (proposition 7.4), in order to prove that $EC(\Gamma)$ and the core of G, $C(G)$, are isomorphic. This implies that the core,

$C(G)$, is a torsion-free hyperbolic group, which further implies, by the construction of the core and the combination theorem of M. Bestvina and M. Feighn [4], that G is a torsion-free hyperbolic group.

Let $EC(\Gamma) = H_1 * \ldots * H_\ell$ and $C(G) = M_1 * \ldots * M_s$, where the factors H_i and M_j are (non-elementary) freely indecomposable and not ω-residually free towers. By the equivalence of the EA theories of $EC(\Gamma)$ and G, for every n, there exists a homomorphism $h_n : C(G) \to EC(\Gamma)$, so that for every modular automorphism $\varphi \in Mod(C(G))$, the composition $h_n \circ \varphi : C(G) \to EC(\Gamma)$, maps the ball of radius n in the Cayley graph of $C(G)$ monomorphically into $EC(\Gamma)$. Moreover, we can further require that for any possible choice of elements: $\gamma_1, \ldots, \gamma_s \in EC(\Gamma)$, and every modular automorphism $\varphi \in Mod(C(G))$, the homomorphism $\tau_n : C(G) \to EC(\Gamma)$ obtained by setting $\tau_n(M_i) = \gamma_i(h_n \circ \varphi(M_i)){\gamma_i}^{-1}$, maps the ball of radius n in X monomorphically into $EC(\Gamma)$.
From the sequence of homomorphisms: $h_n : C(G) \to EC(\Gamma)$, it is possible to extract a subsequence that converges into a faithful action of $C(G)$ on a real tree. Since for every modular automorphism $\varphi \in Mod(C(G))$, $h_n \circ \varphi$ embeds the ball of radius n in the Cayley graph of $C(G)$, we can further extract a subsequence of homomorphisms (still denoted $\{h_n\}$) that embeds each of the factors $M_1, \ldots, M_s$ into $EC(\Gamma)$.

We continue by constructing a torsion-free hyperbolic group L, into which the f.g. group G is naturally embedded. By the above argument, each of the factors M_i, $1 \leq i \leq s$, of the core, $C(G)$, is embedded in the elementary core $EC(\Gamma)$. Hence, for each index i, $1 \leq i \leq s$, we fix an embedding: $\nu_i : M_i \to EC(\Gamma)$. By the construction of the core of the $EC(\Gamma)$-limit group G, $C(G)$ (see definition 7.5), the $EC(\Gamma)$-limit group G is obtained from its core, $C(G)$, which is a retract of G, by starting with a free product of $C(G)$ with a (possibly trivial) free product of a free group and closed hyperbolic surface groups, and then iteratively adding hyperbolic punctured-surface groups that are amalgamated to the group associated with the previous steps along their boundary subgroups.

We construct the hyperbolic group L, by imitating the construction of G from $C(G)$. We start with s copies of the elementary core, $EC(\Gamma)$, where with the i-th copy of $EC(\Gamma)$ we associate the (fixed) image of the factor M_i by the i-th embedding: $\nu_i : M_i \to EC(\Gamma)$. We continue with a free product of the s copies of $EC(\Gamma)$ free product with a (possibly trivial) free product of a free group and closed hyperbolic surface groups, as in the free product associated with the bottom level of the construction of G from its core, $C(G)$. We further continue the construction of the group L by iteratively adding hyperbolic punctured-surface groups that are amalgamated to the group associated with the previous steps along their boundary components, precisely as in the construction of G from its core, $C(G)$. By construction, L is torsion-free and G is a subgroup of L, and by the combination theorem of M. Bestvina and M. Feighn [4], L is a hyperbolic group.

Recall that $EC(\Gamma) = H_1 * \ldots * H_\ell$ and $C(G) = M_1 * \ldots * M_s$, where the factors H_i and M_j are (non-elementary) freely indecomposable and not ω-residually free towers. By the equivalence of the EA theories of $EC(\Gamma)$ and G, for every n, there exists a homomorphism $v_n : EC(\Gamma) \to G$, so that for every modular automorphism $\varphi \in Mod(EC(\Gamma))$, the composition $v_n \circ \varphi : EC(\Gamma) \to G$, maps the ball of radius n in the Cayley graph of $EC(\Gamma)$ monomorphically into G. Moreover, we can further require that for any possible choice of elements: $g_1, \ldots, g_\ell \in G$, and every modular automorphism $\varphi \in Mod(EC(\Gamma))$, the homomorphism $\tau_n : EC(\Gamma) \to G$ obtained by setting $\tau_n(H_i) = g_i(v_n \circ \varphi(H_i)){g_i}^{-1}$, maps the ball of radius n in X monomorphically into G.

Since G is embedded in the hyperbolic group L, we can further compose the sequence of homomorphisms $\{v_n\}$, with the embedding $\nu : G \to L$, and obtain a sequence of homomorphisms $\hat{v}_n : EC(\Gamma) \to L$. From the sequence of homomorphisms: $\hat{v}_n : EC(\Gamma) \to L$, it is possible to extract a subsequence that converges into a faithful action of $EC(\Gamma)$ on a real tree. Since for every modular automorphism $\varphi \in Mod(EC(\Gamma))$, $v_n \circ \varphi$ embeds the ball of radius n in the Cayley graph of $EC(\Gamma)$, we can further extract a subsequence of homomorphisms (still denoted $\{v_n\}$) that embeds each of the factors $H_1, \ldots, H_\ell$ into G. By the construction of the core of the $EC(\Gamma)$-limit group G, $C(G)$, since $EC(\Gamma)$ is an elementary prototype and for every index n, the homomorphism $v_n : EC(\Gamma) \to G$ embeds each of the factors $H_1, \ldots, H_\ell$ of $EC(\Gamma)$ into G, for every index n, the composition of v_n with the projection from the $EC(\Gamma)$-limit group G to its core $C(g)$, $\eta : G \to C(G)$, $v'_n = \eta \circ v_n : EC(\Gamma) \to C(G)$, embeds each of the factors $H_1, \ldots, H_\ell$ into the core $C(G)$.

At this point we can continue in a similar way to the argument that was used to prove proposition 7.4. The homomorphisms $h_n : C(G) \to EC(\Gamma)$ embed each of the factors M_j into $EC(\Gamma)$, and the homomorphisms $v'_n : EC(\Gamma) \to C(G)$ embed each of the factors H_i into $C(G)$. Since the factors H_i and M_j are freely indecomposable, for each index i, $1 \leq i \leq \ell$, $v'_n(H_i)$ is a subgroup of a conjugate of one of the factors M_j, and for each index j, $1 \leq j \leq s$, $h_n(M_j)$ is a subgroup of a conjugate of one of the factors H_i. Therefore, for each index n, with the pair of homomorphisms h_n, v'_n we can associate a bipartite directed graph Gr_n, where the vertices of the two sides correspond to the factors $H_1, \ldots, H_\ell$ and $M_1, \ldots, M_s$, from a vertex associated with H_i there is an edge connected to it and directed towards the vertex associated with the factor M_j that contains a conjugate of $v'_n(H_i)$, and from a vertex associated with M_j there is an edge connected to it and directed towards the vertex associated with the factor H_i that contains a conjugate of $h_n(M_j)$.

Since there are only finitely many possibilities for the combinatorial types of the graphs Gr_n, we can pass to a further subsequence and assume that they are all identical, which we denote Gr. As in the proof of proposition 7.4, the directed graph Gr contains a (directed) circle, and an innermost circle, which we denote C, contains an equal number of vertices corresponding to the factors H_i and to the factors M_j. Since the factors H_i are all assumed to be freely-indecomposable (and they are torsion-free hyperbolic), these factors are co-Hopf. Since, in addition, for every index n, the homomorphism v'_n embeds the factors H_i into $C(G)$, and the homomorphism h_n embeds the factors M_j into $EC(\Gamma)$, the factors H_i and M_j that correspond to vertices along the innermost circle C in the directed graph Gr, are pairwise isomorphic, and the homomorphisms h_n, v'_n restricted to the edges in the circle C are isomorphisms.

Since the homomorphisms h_n, v'_n restricted to the edges in the circle C are isomorphisms, and since we assumed that for every index n and any possible collection of elements $g_1, \ldots, g_\ell \in G$, and a modular automorphism $\varphi \in Mod(EC(\Gamma))$, the homomorphism $\tau_n : EC(\Gamma) \to C(G)$ obtained by conjugating $v'_n \circ \varphi(H_i)$ by γ_i for $1 \leq i \leq \ell$, embeds the ball of radius n in the Cayley graph of $EC(\Gamma)$, and correspondingly for the homomorphism h_n, no vertex that is not in the innermost circle C in Gr is connected by an edge to a vertex in C. Applying the same argument for the subgraph $Gr \setminus C$, and continuing inductively, we finally obtain that $\ell = s$, and up to a change of order, the factor H_i of $EC(\Gamma)$ is isomorphic to the factor M_i of $C(G)$. Hence, $EC(\Gamma)$ is isomorphic to $C(G)$, and in particular, the core $C(G)$ is a torsion-free hyperbolic group. By the construction of the core $C(G)$ from the $EC(\Gamma)$-limit group G, the combination theorem of Bestvina-Feighn [4] implies that G is a torsion-free hyperbolic group. Therefore, the rest of the theorem follows from corollary 7.9.

□

Corollary 7.9 asserts that the elementary class of a torsion-free hyperbolic group that is not elementary equivalent to a free group (among all f.g. groups) is determined by the isomorphism class of its elementary core. Hence, in order to be able to decide if two torsion-free hyperbolic groups are elementary equivalent one needs to decide if they are elementarily equivalent to a free group, and if they are not, to compute their elementary core, and to decide if the two elementary cores are isomorphic.

Theorem 7.11. *Let $\Gamma_1, \Gamma 2$ be two torsion-free hyperbolic groups. Then it is decidable if Γ_1 is elementarily equivalent to Γ_2.*

Proof: According to corollary 7.9, in order to decide if two torsion-free hyperbolic groups, Γ_1 and Γ_2, are elementarily equivalent, we need to check the following:

- (i) Decide if either Γ_1 or Γ_2 are trivial or cyclic. To be elementarily equivalent, if one of them is trivial or cyclic, the other must be as well.
- (ii) Assuming both Γ_1 and Γ_2 are non-elementary, we need to construct effectively their elementary cores, $EC(\Gamma_1)$ and $EC(\Gamma_2)$.
- (iii) Once the elementary cores, $EC(\Gamma_1)$ and $EC(\Gamma_2)$, are constructed, we need to check effectively if they are isomorphic.

Given a hyperbolic group there is an algorithm to find its hyperbolicity constant δ (see [12]). Given δ, the word problem is decidable, hence, it is possible to decide if any of the two given torsion-free hyperbolic groups, Γ_1 and Γ_2, are trivial or infinite cyclic. Hence, for the rest of the argument, we may assume that both Γ_1 and Γ_2 are non-elementary.

By the construction of the elementary core (definition 7.5), the core itself is a torsion-free hyperbolic group. Hence, once the cores of Γ_1 and Γ_2 are constructed, the solution to the isomorphism problem for torsion-free hyperbolic groups ([28],[6]) decides effectively if the cores, $EC(\Gamma_1)$ and $EC(\Gamma_2)$, are isomorphic. Therefore, to prove theorem 7.11, we are only required to give a procedure to construct the elementary core of a torsion-free hyperbolic group effectively.

The construction of the elementary core of a torsion-free hyperbolic group Γ is conducted iteratively. Given the group Γ, we first need to find effectively its (possibly trivial) Grushko's decomposition: $\Gamma = H_1 * \ldots * H_m * F_s$, where the factors H_i are non-elementary and freely indecomposable. We need to decide effectively which of the freely-indecomposable factors is a (non-exceptional) surface groups, and for those that are not surface groups, we need to compute effectively the (essential) JSJ decomposition of each of the factors H_i, and once the JSJ is given, we need to check if there is proper retraction of a given factor into the fundamental group of a proper subgraph of the JSJ decomposition, so that the retraction maps each non-QH vertex group and each edge group in the JSJ to a conjugate (elementwise), and so that the retraction factors through a proper map of the factor to the image of the retract free product with an infinite cyclic group, so that under this map each

QH vertex group is mapped into a non-abelian subgroup (see the construction of the elementary core in definition 7.5). Once such a proper retraction exists, for at least one of the factors, we continue iteratively. By theorem 1.12, the construction terminates after finitely many steps (see definition 7.5).

The Grushko factorization of a torsion-free hyperbolic group can be found effectively by either the procedure suggested by Gerasimov in [7], or the procedure presented in [28] for the same purpose, as part of the solution of the isomorphism problem. The JSJ decomposition of a freely-indecomposable, torsion-free hyperbolic group is constructed effectively in [28] and [6] (see also section 9 in [27]), and using its construction it is also possible to decide which of the freely indecomposable factors in the Grushko's decomposition are (non-exceptional) surface groups.

Once the JSJ decomposition of the various factors is constructed, we still need to decide effectively which of them admits a proper map into a free product of the fundamental group of a proper subgraph of the JSJ decomposition and an infinite cyclic group, so that each non-QH vertex group and each edge group is mapped to a conjugate (elementwise) in the fundamental group of the proper subgraph of the JSJ decomposition, and each surface group is mapped into a non-abelian subgroup.

Given a factor in the Grushko decomposition of Γ, to check if there exists such a proper map, we go over all the proper (not necessarily connected) subgraphs of the JSJ that contain all the non-QH vertex groups and all the edges between them, in parallel. Given such a subgraph, we need to check effectively if there is a map of each of the remaining QH vertex groups into the fundamental group of the subgraph free product with an infinite cyclic group, that maps the boundary of each of the QH vertex groups into (a conjugate of) the edge group it is connected to, and so that the image of each of the QH vertex groups is non-abelian.

The fundamental group of a subgraph of the JSJ decomposition is a torsion-free hyperbolic group, and so is its free product with an infinite cyclic group. The existence of a homomorphism of a QH vertex group into the fundamental group of the given subgraph free product with an infinite cyclic group, that maps the boundary elements of the QH vertex into (conjugates of) the edge groups it is connected to, naturally translates into a finite system of equations (with coefficients), over the fundamental group of the subgraph free product with an infinite cyclic group. To impose the image of each QH vertex group to be non-abelian, we further need to replace the finite system of equations, by a finite disjunction of finite systems of equalities and inequalities. Hence, it is sufficient to show that it is decidable if a finite system of equalities and inequalities has a solution, i.e., to show that the universal theory of a torsion-free hyperbolic group is decidable (note that the Diophantine theory of a torsion-free hyperbolic group is decidable by [17]). We would like to note that a more elegant solution to the universal theory of a torsion-free hyperbolic group than the one described below was given by F. Dahmani in [5].

Theorem 7.12 (cf. [5]). *The universal theory of a torsion-free hyperbolic group is decidable.*

Proof: Let $\Gamma =< a_1, \dots, a_k >$ be a non-elementary torsion-free hyperbolic group, and let $F_k =< f_1, \dots, f_k >$ be a free group of rank k. With the hyperbolic group Γ, we naturally associate the quotient map $\tau : F_k \to \Gamma$, that maps f_i to a_i, for $i = 1, \dots, k$.
Let $\Sigma(y, a) = 1$ be a finite system of equations over Γ, and let $\Psi(y, a) \neq 1$ be a

finite system of inequalities over Γ. To show that the universal theory over Γ is decidable, we need an effective procedure to decide if the conjunction of Σ and Ψ has a solution.

By [17], with the system $\Sigma(y,a)$ it is possible to associate effectively finitely many systems of equations over the free group F_k:

$$\Sigma_1(x_1,f)=1,\ldots,\Sigma_\ell(x_\ell,f)=1$$

so that for every index i, $1\leq i\leq \ell$, there exists a tuple of elements $\hat{y}_i \in <x_i,f>$, and for every solution x_i^0 of the system of equations Σ_i, i.e., for every x_0^i that satisfies $\Sigma_i(x_i^0,a)=1$, $\tau(y_i^0)$ is a solution of the system Σ, i.e., $\Sigma(\tau(y_i^0),a)=1$. Furthermore, every solution of the system $\Sigma(y,a)=1$ over Γ, is obtained in that way from one of the systems $\Sigma_i(x_i,a)=1$ over the free group F_k.

By the Bulitko lemma (cf. lemma 1.4 in [Ma]), if a system of equations over a free group has a solution, then it has a solution with periodicity bounded by a constant that can be computed from the combinatorial complexity of the system of equations (see lemma 1.4 in [Ma] for this constant). The Bulitko lemma naturally generalizes to a conjunction of a system of equalities and inequalities over a free group, and using the canonical representatives presented in [17], it generalizes to the conjunction of systems of equalities and inequalities over a torsion-free hyperbolic group.

Therefore, if the conjunction of the systems $\Sigma(y,a)=1$ and $\Psi(y,a)\neq 1$ has a solution in the hyperbolic group Γ, then there exists an index i, $1\leq i\leq\ell$, and a solution x_i^0 of the system $\Sigma_i(x_i,f)=1$ for which:

(i) $\Sigma(\tau(y_i^0),a)=1$ and $\Psi(\tau(y_i^0),a)\neq 1$ (in Γ).

(ii) the periodicity of x_0^i is bounded by a constant that can be effectively computed, and depends only on the given presentation of Γ, $\Gamma=<a_1,\ldots,a_k>$, and the combinatorial structures of the systems Σ and Ψ.

Given the generalization of the Bulitko lemma, to decide effectively if the conjunction of the systems $\Sigma(y,a)=1$ and $\Psi(x,a)\neq 1$ has a solution over the torsion-free hyperbolic group Γ, we use [17] to construct effectively the systems of equations:

$$\Sigma_1(x_1,f)=1,\ldots,\Sigma_\ell(x_\ell,f)=1.$$

By the work of Makanin [10] and A. Razborov [14], given the systems of equations, Σ_i, over the free group F_k, and the effective bound on the periodicity given by the Bulitko lemma, it is possible to extract finitely many *fundamental sequences*, so that all the solutions of the systems Σ_i with periodicity bounded by the prescribed bound, factor through.

These fundamental sequences, constructed by the Makanin-Razborov machinery, correspond to resolutions of the (F_k) limit groups associated with the various systems Σ_i. To decide if the conjunction of the systems $\Sigma(y,a)=1$ and $\Psi(y,a)\neq 1$ has a solution, we continue by replacing each resolution constructed by the Makanin-Razborov machinery from the various systems Σ_i, with a strict resolution over the hyperbolic group Γ. Once the resolutions over Γ are strict, it is rather straightforward to decide effectively if there is a homomorphism that factors through one of them and satisfy the system of inequalities Ψ (the equalities in the system Σ hold by construction for any homomorphism that factors through one of the strict resolutions we construct).

With the systems of equations $\Sigma_1, \dots, \Sigma_\ell$, the Makanin-Razborov machinery associates finitely many resolutions (over the free group F_k). Given one of these resolutions, $Res(x,f)$, that is defined over the free group F_k, we transform it into a strict resolution over the torsion-free hyperbolic group Γ iteratively, from bottom to top.
The terminal group of the resolution $Res(x,f)$ is a free group of the form $F_k * F_s$, where F_s may be trivial, and F_k is the coefficient group. With the coefficient group F_k, there is an associated quotient map onto Γ. We start the modification of $Res(x,f)$ by replacing the terminal group $F_k * F_s$ with a group of the form $\Gamma * F_s$, with a naturally associated quotient map, and where Γ is considered to be the coefficient group of the newly obtained resolution.

At this stage we start modifying the next level of the resolution $Res(x,f)$, i.e., the group L_{t-1}, the abelian decomposition Λ_{t-1}, and the quotient map $\eta_{t-1} : L_{t-1} \to F_k * F_s$, associated with the level above the terminal level. L_{t-1} is mapped by the quotient map η_{t-1} onto the terminal group $F_k * F_s$, so this map extends naturally to an epimorphism $\nu_{t_1} : L_{t-1} \to \Gamma * F_s$. We continue by checking all the possible degeneracies of the abelian decomposition Λ_{t-1} given the map ν_{t-1}.

(1) an edge group in Λ_{t-1} is mapped by ν_{t-1} to the trivial subgroup.
(2) a non-abelian vertex group in Λ_{t-1} is mapped to an abelian subgroup or the trivial subgroup by ν_{t-1}.
(3) some of the boundary elements of a QH vertex group in Λ_{t-1} are mapped to the trivial subgroup by ν_{t-1}.
(4) the image of a QH vertex group in Λ_{t-1} is mapped to an abelian (cyclic) subgroup or to the trivial subgroup by ν_{t-1}.

Clearly, there are finitely many possibilities for degenerations in the abelian decomposition Λ_{t-1}, and they can all be checked using the solution to the word problem in the hyperbolic group $\Gamma * F_s$.

If there are no degenerations in the graph of groups Λ_{t-1}, we associate with it a completion over the torsion-free hyperbolic group Γ, constructed by starting with the group $\Gamma * F_s$ associated with the terminal level, and then follow the construction of the completion presented in definition 1.12 of [19] according to the abelian decomposition Λ_{t-1}.
Suppose that Λ_{t-1} has degenerations, and contains no QH vertex groups. In this case we modify Λ_{t-1} by collapsing all the edge groups that are mapped to the trivial subgroup in $\Gamma * F_s$ by ν_{t-1}, all the abelian vertex groups that are mapped to the trivial subgroup, and all the non-QH, non-abelian vertex groups that are mapped to an abelian subgroup in Λ_{t-1}. The obtained graph of groups, denoted Δ_{t-1}, admits no degenerations, hence, we can associate a completion over Γ with it, as in the non-degenerate case.

Suppose that Λ_{t-1} contains QH vertex groups. We start by checking which QH vertex groups are mapped into trivial or abelian subgroups in the terminal group $\Gamma * F_s$ by the map ν_{t-1}. We modify Λ_{t-1} by collapsing each of the QH vertex groups that are mapped into the trivial subgroup, as well as all the edges connected to such a QH vertex group. We replace each of the QH vertex groups that are mapped into non-trivial abelian subgroups, with the maximal free abelian quotient of the QH vertex group, in which the boundary elements are fixed. We denote the obtained abelian decomposition Λ'_{t-1}. Note that every QH vertex group in Λ'_{t-1} is mapped onto a non-abelian subgroup by ν_{t-1}.

We continue by modifying the abelian decomposition Λ'_{t-1} as we did in case there were no QH subgroups. We collapse all edge groups that connect between two non-QH vertex groups and are mapped to the trivial group by ν_{t-1}. If V is a non-abelian, non-QH vertex group in Λ'_{t-1} that is mapped to an abelian subgroup by ν_{t-1}, and V is connected to a non-QH vertex group, we collapse the edge that connects these two vertices. We further iteratively erase all edge groups that are connected to QH vertex groups in the obtained graph of groups, and are mapped to the trivial group by ν_{t-1}. If such an edge is contained in a circle in the obtained graph of groups, we introduce a new free factor in our abelian decomposition, generated by the Bass-Serre generator associated with that circle. Let U be a vertex group that remains isolated after we erase the edge groups that are connected to QH vertex groups and mapped to the trivial subgroup by ν_{t-1}. If U is mapped to the trivial subgroup by ν_{t-1}, we erase it.

We continue by modifying QH vertex groups that are mapped to non-abelian subgroups by ν_{t-1}. Let Q be such a vertex group and let S_Q be its associated surface. If none of the edge groups connected to Q is mapped to the trivial subgroup by ν_{t-1}, we leave Q unchanged. Suppose that there exist edge groups connected to Q that are mapped into the trivial subgroup. We modify the QH subgroup Q, by the fundamental group of the surface obtained from S_Q by filling with disks all the boundary components that are mapped into the trivial subgroup by ν_{t-1}. If the obtained surface is a closed non-orientable surface of genus 2, a 3 punctured sphere, or a 2 punctured projective plane, we declare the obtained group as a non-QH, non-abelian vertex group. If the obtained surface is a 1-punctured Klein bottle, we replace it by one non-abelian, non-QH vertex group and a loop based on it, obtained by cutting the 1-punctured Klein bottle along its non-boundary parallel s.c.c.

For each connected component in the obtained graph of groups that does not contain the coefficient group Γ, we add a new generator, so that the fundamental group of that connected component is mapped into $\Gamma * F_s$ conjugated by the new free generator. With the obtained abelian decomposition, we associate a completion over the hyperbolic group Γ, constructed by starting with the terminal group $\Gamma * F_s$, and continuing using the given abelian decomposition according to definition 1.12 in [19].

The obtained completion over Γ, associated with the 2 bottom levels of given resolution over F_k, is a Γ-limit group by theorem 1.31, and it is f.p. by construction. The word problem is solvable for f.p. Γ-limit groups, In the similar way to its solution for f.p. residually finite groups, i.e., by using two parallel processes, the first looks for a presentation of the given word in the normal closure of the given relations, and the second looks for a homomorphism into Γ that maps the given word into a non-trivial element (clearly, one of the two processes must terminate). Hence, the word problem for the obtained completion over Γ is decidable.
Since in order to construct the completion associated with the bottom 2 levels, only a solution to the word problem in the terminal hyperbolic group is required, the construction used for the bottom 2 levels generalizes to the higher levels of the resolution, and finally associates with it a completion over Γ.

Once completions over Γ are associated with the resolutions associated with the systems of equations $\Sigma_1, \dots, \Sigma_\ell$, to decide if there exists a solution of the system Σ over Γ, for which the inequalities Ψ hold, we only need to check if the words that appear in the system Ψ are all non-trivial in at least one of the completions over Γ we constructed. Since the word problem is decidable over these completions, this can be decided effectively. Hence, it is possible to decide effectively if the conjunction of the systems Σ and Ψ have a solution in the torsion-free hyperbolic group Γ.

□

Theorem 7.12 enables us to decide if there exists a proper map from a freely-indecomposable factor in the Grushko decomposition of a torsion-free hyperbolic group into the fundamental group of a proper subgrpaph of its JSJ decomposition free product with an infinite cyclic group, that maps each of the non-QH vertex groups and all the edge groups in the JSJ decomposition into their conjugates (elementwise), and so that the image of each QH is non-abelian.

If no such retract exists for any of the factors, the construction of the elementary core is completed. If such a retract exists for at least one of the factors, we continue iteratively. The construction terminates after finitely many iterations, by the descending chain condition for Γ-limit groups (theorem 1.12) and the co-Hopf property for non-elementary, freely-indecomposable, torsion-free hyperbolic groups. Hence, the effective construction of the elementary core of a torsion-free hyperbolic group is completed, and so is the procedure for deciding if two such groups are elementarily equivalent (theorem 7.11).

□

References

[1] E. Alibegovic, *Makanin-Razborov diagrams for limit groups*, preprint.

[2] M. Bestvina, *R-trees in topology, geometry, and group theory*, Handbook of geometric topology, 55-91, North Holland, Amsterdam, 2002.

[3] M. Bestvina and M. Feighn, *Stable actions of groups on real trees*, Inventiones Math. **121** (1995), 287-321.

[4] ______, *A combination theorem for negatively curved groups*, Jour. of Differential Geometry **35** (1992), 85-101.

[5] F. Dahmani, *On equations in relatively hyperbolic groups*, preprint.

[6] F. Dahmani and D. Groves, *The isomorphism problem for toral relatively hyperbolic groups*, preprint.

[7] V. Gerasimov, *Detecting connectedness of the boundary of a hyperbolic group*, preprint.

[8] D. Groves, *Limit groups for relatively hyperbolic groups II. Makanin-Razborov diagrams*, Geometric Topology **9** (2005), 2319-2358.

[9] V. S. Guba, *Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems*, Math. Zametki **40** (1986), 321-324.

[10] G. S. Makanin, *Equations in a free group*, Math. USSR Izvestiya **21** (1983), 449-469.

[11] Yu. I. Merzlyakov, *Positive formulae on free groups*, Algebra i Logika **5** (1966), 257-266.

[12] P. Papasoglu, *An algorithm detecting hyperbolicity, Geometric and computational perspectives on infinite groups, Dimacs Ser. Discrete math. theoretical computer science no.25, 193-200.*

[13] F. Paulin, *Outer automorphisms of hyperbolic groups and small actions on R-trees, Arboreal Group Theory (ed. R. C. Alperin), 331-343.*

[14] A. A. Razborov, *On systems of equations in a free group*, Math. USSR Izvestiya **25** (1985), 115-162.

[15] E. Rips and Z. Sela, *Structure and rigidity in hyperbolic groups I*, GAFA **4** (1994), 337-371.

[16] ______, *Cyclic splittings of finitely presented groups and the canonical JSJ decomposition*, Annals of Mathematics **146** (1997), 53-104.

[17] ______, *Canonical representatives and equations in hyperbolic groups*, Inventiones Math. **120** (1995), 489-512.

[18] Z. Sela, *Diophantine geometry over groups I: Makanin-Razborov diagrams*, Publication Math. de l'IHES **93** (2001), 31-105.

[19] ______, *Diophantine geometry over groups II: Completions, closures and formal solutions*, Israel jour. of Mathematics **134** (2003), 173-254.

[20] ______, *Diophantine geometry over groups III: Rigid and solid solutions*, Israel jour. of Mathematics **147** (2005), 1-73.

[21] ______, *Diophantine geometry over groups IV: An iterative procedure for validation of a sentence*, Israel jour. of Mathematics **143** (2004), 1-130.

[22] ______, *Diophantine geometry over groups V_1: Quantifier elimination I*, Israel jour. of Mathematics **150** (2005), 1-197.

[23] ______, *Diophantine geometry over groups V_2: Quantifier elimination II*, GAFA **16** (2006), 537-706.

[24] ______, *Diophantine geometry over groups VI: The elementary theory of a free group*, GAFA **16** (2006), 707-730.

[25] ______, *Acylindrical accessibility for groups*, Inventiones Mathematicae **129** (1997), 527-565.

[26] ______, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie Groups II*, GAFA **7** (1997), 561-593.

[27] ______, *The isomorphism problem for hyperbolic groups I*, Annals of Math. **141** (1995), 217-283.

[28] ______, *The isomorphism problem for hyperbolic groups II*, in preparation.