FREE AND HYPERBOLIC GROUPS ARE NOT EQUATIONAL

Z. Sela[1,2]

In [Se5] we proved that free and torsion-free hyperbolic groups are stable. In this note we give an example of a definable set in each of these groups that is not in the Boolean algebra of equational sets. Hence, the theories of free and torsion-free hyperbolic groups are not equational in the sense of G. Srour.

In [Se5] we proved that free groups and torsion-free hyperbolic groups are stable. We started by proving that a subset of the collection of definable sets in these groups, that are called minimal rank definable sets, are in the Boolean algebra of equational sets, and that general varieties and Diophantine sets in these groups are equational. Then we used the equationality of Diophantine sets and Duo limit groups and their properties (that are presented in section 3 of [Se5]) to prove that free and torsion-free hyperbolic groups are stable.

In this note, we show that our results for minimal rank definable sets are false in general. We give an example of a definable set in each free or non-elementary torsion-free hyperbolic group that is not in the Boolean algebra of equational sets.

Recall that equational sets and theories were defined by Gabriel Srour (see [Pi-Sr]). A definable set $D(p,q)$ is called *equational* if there exists a constant N_D, so that for every sequence of values $\{q_i\}_{i=1}^m$, for which the sequence of sets that corresponds to the intersections: $\{ \cap_{i=1}^j D(p,q_i) \}_{j=1}^m$ is a strictly decreasing sequence, satisfies: $m \leq N_D$.

We note that the question of the existence of a theory which is stable but not equational was raised by Pillay and Srour [Pi-Sr], and such examples were constructed by Hrushovski and Srour [Hr-Sr] and Baudisch and Pillay [Ba-Pi].

Theorem 1. *The elementary theory of a non-abelian free group is not equational.*

Proof: Let F_k be a non-abelian free group, and let $NE(p,q)$ be the existential set:

$$NE(p,q) \ = \ \exists x_1, x_2 \ \ qp = x_1^{10} x_2^{-9} \ \wedge \ [x_1, x_2] \neq 1.$$

In section 3 of [Se5] we introduced duo limit groups (see definition 3.1 in [Se5]). With the set $NE(p,q)$ we associate the following natural duo limit group:

$$Duo = < d_1 > *_{<d_0>} < d_2 > = < t, u > *_{<u>} < u, s > ; \ q = s^{10} u \, ; \ p = u^{-1} t^{-9}.$$

Note that if we fix a value of the variable u, i.e., we fix a rectangle that is associated with the duo limit group Duo, then for every (fixed) non-trivial value of the variable

[1]Hebrew University, Jerusalem 91904, Israel.

[2]Partially supported by an Israel academy of sciences fellowship.

s, a couple $(q(s), p(t))$ is in the set $NE(p,q)$ for every generic value of the variable t, and correspondingly for every (fixed) non-trivial value of t and a generic value of s.

Lemma 2. *The set $NE(p,q)$ is not equational.*

Proof: If we fix a value u_0 for the variable u, and a non-trivial value s_0 for the variable s, so that $q(s_0) = s_0^{10}u_0$, then for all values of the variable t for which $[t, s_0] \neq 1$, the couple $(q(s_0), p(t)) \in NE(p,q)$ where $p(t) = u_0^{-1}t^{-9}$. A work of Lyndon and Schutzenberger [Ly-Sch], shows that for $m, n, r \geq 2$, the solutions of the equation $x^m y^n z^r = 1$ in a free group, generate a cyclic subgroup. Hence, for all but at most 2 values t_0 of the variable t, for which $[s_0, t_0] = 1$, $(q(s_0), q(t_0)) \notin NE(p,q)$. Hence, if we consider a sequence, $s_1, s_2, \ldots$, where $[s_{i_1}, s_{i_2}] \neq 1$ for $i_1 < i_2$, then the sequence of intersections: $\{\cap_{i=1}^{j} NE(p, q(s_i))\}_{j=1}^{\infty}$ is a strictly decreasing sequence. Therefore, $NE(p,q)$ is not equational.

□

Since a finite union and a finite intersection of equational sets are equational (see corollary 2.27 in [OH]), lemma 2 implies that the set $NE(p,q)$ is not a finite union nor a finite intersection of equational sets.

Lemma 3. *The set $NE(p,q)$ is not co-equational (i.e., it is not the complement of an equational set).*

Proof: The (infinite) union of the sets $NE(p, q_i)$, for all possible values of q_i, is the entire coefficient group F_k. Every finite subunion of such sets is a proper subset of F_k (since the set $NE(p,q)$ is not of generic type or is negligible in the sense of M. Bestvina and M. Feighn [Be-Fe]). Hence, $NE(p,q)$ can not be co-equational.

□

To prove that the theory of a free group is not equational, we need to show that the set $NE(p,q)$ is not a union nor an intersection of an equational and a co-equational sets.

Lemma 4. *The set $NE(p,q)$ is not a union of an equational and a co-equational sets.*

Proof: Suppose that $NE(p,q) = E_1 \cup (E_2)^c$ where E_1, E_2 are equational sets. Let N_1, N_2 be the bounds on the lengths of sequences with strictly decreasing intersections: $\{E_1(p, q_i)\}$ and $\{E_2(p, q_j)\}$. Let $N = \max(N_1, N_2)$. We look at a sequence of values: $\{q_j^i = (s_j^i)^{10} u_i\}_{i,j=1}^{N+1}$, and $\{p_j^i = u_i^{-1}(s_j^i)^{-9}\}_{i,j=1}^{N+1}$, where $\{s_j^i\}$ and $\{u_i\}$ are distinct elements in a given test sequence (i.e., they satisfy the combinatorial properties that are listed in definition 1.20 in [Se2] for a test sequence. Elements in a test sequence should be considered "generic" elements, that as a set satisfy a very small cancellation condition).

Since the elements $\{u_i\}$ and the elements $\{s_j^i\}$ are chosen to be generic (part of a test sequence), a pair, $(p_j^i, q_{j'}^{i'}) \in NE(p,q)$ if and only if $i = i'$ and $j \neq j'$. Hence, for a given index i, $1 \leq i \leq N+1$, and every index j, $1 \leq j \leq N+1$, $E_2(p, q_j^i)$ contains all the elements $p_j^{i'}$, $i \neq i'$, $1 \leq i, i', j < N+1$.

Our choice of N guarantees that a sequence of sets of the form, $\{E_2(p, q_j^i)\}$, with strictly decreasing intersections, has length bounded by N. Therefore, there must exist some index i_0, $1 \leq i_0 \leq N+1$, for which for every $1 \leq j, j' \leq N+1$: $p_j^{i_0} \in E_2(p, q_{j'}^{i_0})$.

Hence, for this index i_0, $p_j^{i_0} \in E_1(p, q_{j'}^{i_0})$, if and only if $p_j^{i_0} \in NE(p, q_{j'}^{i_0})$, $1 \leq j, j' \leq N+1$. $p_j^{i_0} \in NE(p, q_{j'}^{i_0})$ if and only if $j \neq j'$. Therefore, the sequence of intersections: $\{\cap_{j=1}^{m} E_1(p, q_j^{i_0})\}_{m=1}^{N+1}$ is a strictly decreasing sequence, a contradiction to the assumption that N is the global bound on the length of such a strictly decreasing sequence.

□

Using a somewhat similar argument one can show that $NE(p,q)$ is not the intersection of an equational and a co-equational sets.

Lemma 5. *The set $NE(p,q)$ is not an intersection between an equational and a co-equational sets.*

Proof: Suppose that $NE(p,q) = E_1 \cap (E_2)^c$ where E_1, E_2 are equational sets. Let N_1, N_2 be the bounds on the lengths of strictly decreasing sequences: $\{E_1(p, q_i)\}$ and $\{E_2(p, q_j)\}$, and let $N = \max(N_1, N_2)$. We look at a sequence of values: $\{s_i = (v_i)^{10}(w_i)^{-9}\}_{i=1}^{N+1}$, $\{q_j^i = s_i^{(10^j)}\}_{i,j=1}^{N+1}$, $\{p_j^i = s_i^{(1-10^j)}\}$, where $\{v_i\}$ and $\{w_i\}$ are distinct elements in a given test sequence (”generic” elements).

Since the elements $\{v_i\}$ and $\{w_i\}$ are chosen to be generic (part of a test sequence), a pair, $(p_j^i, q_{j'}^{i'}) \in NE(p,q)$, if and only if $i \neq i'$ or $i = i'$ and $j = j'$. Hence, for a given index i, $1 \leq i \leq N+1$, and every index j, $1 \leq j \leq N+1$, $E_1(p, q_j^i)$ contains the element p_j^i, and all the elements $p_{j'}^{i'}$, $i \neq i'$, $1 \leq i', j' < N+1$.

Our choice of N guarantees that a sequence of sets of the form, $\{E_1(p, q_j^i)\}$, with strictly decreasing intersections, has length bounded by N. Therefore, there must exist some index i_0, $1 \leq i_0 \leq N+1$, for which for every $1 \leq j, j' \leq N+1$: $p_j^{i_0} \in E_1(p, q_{j'}^{i_0})$.

For this index i_0, $p_j^{i_0} \in (E_2)^c(p, q_{j'}^{i_0})$, if and only if $p_j^{i_0} \in NE(p, q_{j'}^{i_0})$, $1 \leq j, j' \leq N+1$. $p_j^{i_0} \in NE(p, q_{j'}^{i_0})$ if and only if $j = j'$. Hence, $p_j^{i_0} \in E_2(p, q_{j'}^{i_0})$ if and only if $j \neq j'$. Therefore, the sequence of intersections: $\{\cap_{j=1}^{m} E_1(p, q_j^{i_0})\}_{m=1}^{N+1}$ is a strictly decreasing sequence, a contradiction to the assumption that N is the global bound on the length of such a strictly decreasing sequence.

□

So far we proved that the set $NE(p,q)$ is not an intersection nor a union of an equational and a co-equational sets. Since a finite union and a finite intersection of equational sets are equational, to prove that the set $NE(p,q)$ is not in the Boolean algebra of equational sets it is enough to prove that $NE(p,q)$ is not a finite union of intersections between an equational and a co-equational sets. Hence, to prove that the set $NE(p,q)$ is not in the Boolean algebra of equational sets, for every positive integer ℓ, we need to find a collection of elements in the definable set $NE(p,q)$, so that this collection of elements can not be contained in a union of ℓ intersections of an equational set and a co-equational set.

Unlike the previous lemmas, to prove that we (indirectly) use the structure of a definable set over a free group that is the outcome of the quantifier elimination procedure [Se3]. This (geometric) structure of definable sets was used in proving the stability of the theory (see theorems 1.9 and 5.1 in [Se5]). Technically, to prove theorem 1 (proposition 6 and theorem 7) we use *envelopes* of definable sets over a free group, as they appeared and proved to exist in section 1 in [Se6]. For presentation purposes we start by proving that the set $NE(p,q)$ is not the union of two intersections of equational and co-equational sets (proposition 6), and then generalize the argument to prove that $NE(p,q)$ is not a finite union of such intersections (theorem 7).

Proposition 6. *The set $NE(p,q)$ is not a union of two intersections between an equational and a co-equational sets.*

Proof: Our approach to prove the proposition combines the argument that was used to prove lemma 5 with the envelope of a definable set and its properties as they are presented and proved to exist in theorem 1.3 in [Se6].

Suppose that:

$$NE(p,q) \,=\, (E_1^1 \cap (E_2^1)^c) \cup (E_1^2 \cap (E_2^2)^c).$$

Let N be the maximum in the finite set of equationality constants of the sets, E_i^1, E_i^2, and $E_i^1 \cap E_i^2$, $i=1,2$.

We modify the argument that was used to prove lemma 5, and the definition of the elements p_j^i and q_j^i in the proof of that lemma. We set:

$$q \,=\, (((t_1)^{10}(s_1)^{-9})^{10m_1}\,((t_2)^{10}(s_2)^{-9})^{-9m_2})^{10f_1}\,u$$

$$p \,=\, u^{-1}\,(((t_3)^{10}(s_3)^{-9})^{10m_3}\,((t_4)^{10}(s_4)^{-9})^{-9m_4})^{-9f_2}.$$

We look at the collection of all the test sequences of values for the variables, $\{t_j,s_j\}_{j=1}^4$ and u, (recall that elements in a test sequence need to satisfy the properties that are listed in definition 1.20 in [Se2], they satisfy a very small cancellation assumption, and should be considered "generic" elements). For all the values in such test sequences, and arbitrary positive integer values of the elements, $\{m_j\}_{j=1}^4$ and f_1,f_2, the corresponding values of the variables (p,q) are in the set $NE(p,q)$.

By our assumptions, for each $(p_0,q_0) \in NE(p,q)$, $(p_0,q_0) \in E_1^i \cap (E_2^i)^c$ for $i=1$ or $i=2$, hence, in particular, $(p_0,q_0) \in E_1^i$ for $i=1$ or $i=2$. For $i=1,2$, we look at the collection of all the test sequences of elements, $\{t_j,s_j\}_{j=1}^4$ and u, and sequences of positive integers as values for the elements, $\{m_j\}_{j=1}^4$ and f_1,f_2, for which each of the sequences of values of each of the elements, $\{m_j\}_{j=1}^4$ and f_1,f_2, converges to infinity, and the corresponding sequence of values of the pair (p,q) are all in the set E_1^i.

By definition, the group that is generated by the elements, $\{t_j,s_j\}_{j=1}^4$ and u,p,q, together with elements that are (formally) added for the powers: $((t_j)^{10}(s_j)^{-9})^{10m_j}$, $j=1,\dots,4$, (each such formal power commutes with the corresponding element: $(t_j)^{10}(s_j)^{-9}$, $j=1,\dots,4$), is a completion (i.e., an ω-residually free tower - see section 6 in [Se1] and definition 1.12 in [Se2]), with a bottom level which is a free group of rank 9 (that is generated by the elements, t_j,s_j, $j=1,\dots,4$ and u, and two upper levels, where at the first level above the bottom level, there are 4 abelian vertex groups of rank 2 that are connected to a distinguished vertex, and in the top level there are additional two abelian vertex groups of rank 2 that are connected to a distinguished vertex. We denote this completion, *Comp*.

The subgroup *Comp* is a completion, and generic points (test sequences) in its associated variety project to values of the pair, (p,q), that are in $NE(p,q)$, and hence are in $E_1^i(p,q)$ for $i=1$ or $i=2$. Therefore, by the construction of envelopes of definable sets, as they are presented in theorem 1.3 in [Se6], with the sets, $E_1^i(p,q)$, $i=1,2$, there are associated envelopes, that are closures of the completion, $Comp$.

Hence, with the sets E_1^1, E_1^2, there are associated closures of $Comp$, that we denote $Cl_1,\dots,Cl_d$, that form the envelopes of the sets E_1^1, E_1^2, with respect to generic points (test sequences) of the completion, $Comp$. Furthermore, with each closure, Cl_g, there exists a finite (possibly empty) closures of it, $Cl_{g,1},\dots,Cl_{g,f_g}$. By the properties of envelopes (see theorem 1.3 in [Se6]), the set of closures $Cl_1,\dots,Cl_d$ and their associated closures,$Cl_{1,1},\dots,Cl_{g,f_g}$, specifies precisely what test sequences of the completion, $Comp$, restrict to values of the variables (p,q) that are in the sets, E_1^1 and E_1^2 (and what test sequences restrict to values of (p,q) that are in the complements of each of these sets).

We now look at a specific test sequence of values for the variables, t_j, s_j, $j=1,\dots,4$ and u, together with a specific sequence of positive integers as values for the elements, $\{m_j\}_{j=1}^4$ and f_1, f_2, for which: $m_1=m_3$, $m_2=m_4$, $10m_1-9m_2=1$, and $10f_2-9f_1=1$. In the sequel we will denote this specific test sequence, $\{t_j(n), s_j(n)\}$, $j=1,\dots,4$, $u(n)$, and $m_j(n)$, $j=1,\dots,4$ and $f_1(n), f_2(n)$.

With the collection of all the test sequences of the completion, $Comp$, we have associated finitely many envelope closures, $Cl_1,\dots,Cl_d$, and with each closure Cl_g we have further associated finally many (possibly no) closures of it, $Cl_{g,1},\dots,Cl_{g,f(g)}$. With each of these closures there is an associated finite index subgroup of some fixed f.g. free abelian group, so we may look at the intersection of all these finite index subgroups. Hence, with each closure there are finitely many associated cosets of this finite index subgroup. By passing to a subsequence we may assume that all the elements in the specific test sequence that we chose belong to a fixed coset of that finite index subgroup.

Now, for each index n (of the specific test sequence that we chose), we look at the following sequence of values for $1\leq g,h\leq N+1$ (that is similar to the one that we used in proving lemma 5):

$$q_h^g(n) \;=\; (((t_1(n{+}g))^{10}(s_1(n{+}g))^{-9})^{10m_1(n)}\,((t_2(n{+}g))^{10}(s_2(n{+}g))^{-9})^{-9m_2(n)})^{10f_1(n+h)}\,u(n)$$

$$p_h^g(n) \;=\; u(n)^{-1}\,(((t_1(n{+}g))^{10}(s_1(n{+}g))^{-9})^{10m_1(n)}\,((t_2(n{+}g))^{10}(s_2(n{+}g))^{-9})^{-9m_2(n)})^{-9f_2(n+h)}.$$

By the way the values of the sequence is defined, and like in lemma 5, for every index n: $(p_h^g(n), q_{h'}^{g'}(n))\in NE(p,q)$, if and only if $g\neq g'$ or $g=g'$ and $h=h'$ $(1\leq g,g',h,h'\leq N+1)$.

Since we have chosen the positive integers, $m_j(n)$, $j=1,\dots,4$, and $f_1(n), f_2(n)$, to be from a fixed class of the finite index subgroup that is associated with the closures, $Cl_1,\dots,cl_d$, and $Cl_{1,1},\dots,Cl_{d,f(d)}$, for every index n, and every $g\neq g'$, $1\leq h,h',g,g'\leq N+1$, precisely one of the following two possibilities is valid:

1. $(p_h^g(n), q_{h'}^{g'}(n))\in E_1^1\cap E_1^2$.
2. there exists a fixed index i, (wlog we may assume $i=1$), so that: $(p_h^g(n), q_{h'}^{g'}(n))\in E_1^1$ and $(p_h^g(n), q_{h'}^{g'}(n))\notin E_1^2$.

If possibility (1) is valid, then we apply the same argument that was used to prove lemma 5, to the equational set $E_1^1\cap E_1^2$, and the co-equational set, $(E_2^1\cap E_2^2)^c$, and obtain a contradiction. Hence, we may assume that possibility (2) is valid.

If possibility (2) holds, then by the equationality of the set E_1^1 and the argument that was used to prove lemma 5, there exists an index g_0, $1 \leq g_0 \leq N+1$, for which for every h,h', $1 \leq h,h' \leq N+1$, $(p_h^{g_0}(n), q_{h'}^{g_0}(n)) \in E_1^1$. Since $(p_h^{g_0}(n), q_{h'}^{g_0}(n)) \in NE(p,q)$ if and only if $h=h'$, the equationality of the set E_2^1 and the argument that was used in proving lemma 5, imply that there exists an index h_0, $1 \leq h_0 \leq N+1$, for which for every index h', $1 \leq h' \leq N+1$, $(p_{h'}^{g_0}(n), q_{h_0}^{g_0}(n)) \in E_2^1$. In particular, $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in E_2^1$. Since $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in NE(p,q) \cap E_1^1 \cap E_2^1$ and:

$$NE(p,q) \;=\; (E_1^1 \,\cap\, (E_2^1)^c) \,\cup\, (E_1^2 \,\cap\, (E_2^2)^c)$$

it finally follows that $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in E_1^1 \cap E_1^2$.

Now, note that the pairs, $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n))$, can be written as:

$$q_{h_0}^{g_0}(n) \;=\; (((t_1(n+g_0))^{10}(s_1(n+g_0))^{-9})^{10m_1(n)}\,((t_2(n+g_0))^{10}(s_2(n+g_0))^{-9})^{-9m_2(n)})^{10f_1(n+h_0)}\,u(n) \;=$$

$$=\; ((t_1(n+g_0))^{10}(s_1(n+g_0))^{-9})^{10m_1(n)}\,\hat{u}(n)$$

$$p_{h_0}^{g_0}(n) \;=\; u(n)^{-1}\,(((t_1(n+g_0))^{10}(s_1(n+g_0))^{-9})^{10m_1(n)}\,((t_2(n+g_0))^{10}(s_2(n+g_0))^{-9})^{-9m_2(n)})^{-9f_2(n+h_0)} \;=$$

$$=\; \hat{u}(n)^{-1}\,((t_2(n+g_0))^{10}(s_2(n+g_0))^{-9})^{-9m_2(n)}$$

where the sequences, $\{t_j(n)\}$, $\{s_j(n)\}$, $j=1,2$, and $\{\hat{u}(n)\}$, are test sequences, and the sequence $\{m_1(n), m_2(n)\}$ is a sequence of positive integers for which for every index n, $10m_1(n) - 9m_2(n) = 1$.

At this stage we essentially repeat what we did so far with the previous values of the pair p,q (that were only in the set E_1^1 by possibility (2)), for the new sequence of values of this pair that are now known to be in the intersection $E_1^1 \cap E_1^2$.

By definition, the group that is generated by the elements, $\{t_j, s_j\}_{j=1}^2$ and $\hat{u}, p, q$, together with elements that are (formally) added for the powers: $((t_j)^{10}(s_j)^{-9})^{10m_j}$, $j=1,2$, (each such formal power commutes with the corresponding element: $(t_j)^{10}(s_j)^{-9}$, $j=1,2$), is a completion, with a bottom level which is a free group of rank 5 (that is generated by the elements, t_j, s_j, $j=1,2$ and $\hat{u}$, and an upper level, in which there are 2 abelian vertex groups of rank 2 that are connected to a distinguished vertex. We denote this completion, $Comp_1$.

The subgroup $Comp_1$ is a completion, and the given sequence of values $(t_j(n+g_0), s_j(n+g_0), j=1,2, \hat{u}(n), m_1(n), m_2(n))$ is a test sequence of values of $Comp_1$, that projects to values of the pair, (p,q), that are in $NE(p,q)$, and in $E_1^1(p,q) \cap E_1^2$. By the construction of envelopes of definable sets, as they are presented in theorem 1.3 in [Se6], with the intersection, $E_1^1 \cap E_1^2$, and the collection of all the test sequences of $Comp_1$ that restrict to values of the pair p,q that are in $E_1^1 \cap E_1^2$, it is possible to canonically associate finitely many envelopes, that are closures of the completion, $Comp_1$. Since there exists a test sequence of $Comp_1$ that restricts to values of the pair p,q that are in the set $E_1^1 \cap E_1^2$, this collection of envelopes is not empty.

Hence, with the intersection $E_1^1 \cap E_1^2$, there are associated closures of $Comp_1$, that we denote $Cl_1^1, \ldots, Cl_{d_1}^1$, that form the envelopes of the sets $E_1^1 \cap E_1^2$, with respect to generic points (test sequences) of the completion, $Comp_1$. Furthermore, with each closure, Cl_r^1, there exists a finite (possibly empty) closures of it, $Cl_{r,1}^1, \ldots, Cl_{r,f_r}^1$. By the properties of envelopes (see theorem 1.3 in [Se6]), the set of closures $Cl_1^1, \ldots, Cl_{d_1}^1$ and their associated closures,$Cl_{1,1}^1, \ldots, Cl_{r,f_r}^1$, specifies precisely what test sequences of the completion, $Comp_1$, restrict to values of the variables (p,q) that are in the set, $E_1^1 \cap E_1^2$.

We now look at a subsequence of the original test sequence, $t_j(n), s_j(n)$, $j=1,2$, and $\hat{u}(n)$, together with a specific sequence of positive integers as values for the elements, $\{m_1(n), m_2(n)\}$, for which: $10m_1 - 9m_2 = 1$.

With the collection of all the test sequences of the completion, $Comp_1$, we have associated finitely many envelope closures. With each of these closures there is an associated finite index subgroup of some fixed f.g. free abelian group, so we may look at the intersection of all these finite index subgroups. Hence, with each closure there are finitely many associated cosets of this finite index subgroup. By passing to a subsequence we may assume that all the elements in the specific test sequence that we chose belong to a fixed coset of that finite index subgroup.

Now, for each index n (of the specific test sequence that we chose), we look at the following sequence of values for $1 \leq g,h \leq N+1$ (that is once again similar to the one that we used in proving lemma 5):

$$q_h^g(n) \;=\; ((t_1(n+g))^{10}(s_1(n+g))^{-9})^{10m_1(n+h)}\,\hat{u}(n)$$

$$p_h^g(n) \;=\; \hat{u}(n)^{-1}\,((t_1(n+g))^{10}(s_1(n+g))^{-9})^{-9m_2(n+h)}.$$

By the way the values of the sequence is defined, and like in lemma 5, for every index n: $(p_h^g(n), q_{h'}^{g'}(n)) \in NE(p,q)$, if and only if $g \neq g'$ or $g=g'$ and $h=h'$ $(1 \leq g,g',h,h' \leq N+1)$. Furthermore, since we have chosen the positive integers, $m_1(n), m_2(n)$, to be from a fixed class of the finite index subgroup that is associated with the envelope closures of $E_2^1 \cap E_1^2$ with respect to the completion $Comp_1$, for every index n, and every $g \neq g'$, $1 \leq h,h',g,g' \leq N+1$, $(p_h^g(n), q_{h'}^{g'}(n)) \in E_1^1 \cap E_1^2$.

At this stage we apply the same argument that was used to prove lemma 5, to the equational set $E_1^1 \cap E_1^2$, and the co-equational set, $(E_2^1 \cap E_2^2)^c$, and obtain a contradiction. Hence, $NE(p,q)$ is not the union of two intersections between equational and co-equational sets, and proposition 6 follows.

□

Proposition 6 proves that the set $NE(p,q)$ is not the union of two intersections of equational and co-equational sets. Theorem 7 generalizes the proof to prove that $NE(p,q)$ is not a finite union of such intersections.

Theorem 7. *The set $NE(p,q)$ is not a finite union of intersections between an equational and a co-equational sets.*

Proof: The argument that we use is a straightforward generalization of the argument that was used to prove proposition 6. Suppose that:

$$NE(p,q) \;=\; (E_1^1 \,\cap\, (E_2^1)^c) \,\cup\, \ldots \,\cup\, (E_1^\ell \,\cap\, (E_2^\ell)^c)$$

Let $M_{a_1,\ldots,a_b}(p,q) \;=\; E_1^{a_1} \cap \ldots \cap E_1^{a_b}$, and $T_{a_1,\ldots,a_b}(p,q) \;=\; E_2^{a_1} \cap \ldots \cap E_2^{a_b}$, for $1 \leq b \leq \ell$, and $1 \leq a_1 < \ldots < a_b \leq \ell$. Since the sets, E_1^a and E_2^a, $a=1,\ldots,\ell$, are equational, so are the sets $M_{a_1,\ldots,a_b}$ and $T_{a_1,\ldots,a_b}$. Let N be a bound on the lengths of strictly decreasing sequences: $\{\cap_{j=1}^f M_{a_1,\ldots,a_d}(p,q_j)\}_{f=1}^r$ and $\{\cap_{j=1}^f T_{a_1,\ldots,a_d}(p,q_j)\}_{f=1}^r$, for all $1 \leq d \leq \ell$ and $1 \leq a_1 < \ldots < a_d \leq \ell$.

We modify the argument that was used to prove proposition 6, and the definition of the elements p_h^g and q_h^g in the proof of that proposition. We set:

$$v_1^0=(t_1)^{10}(s_1)^{-9},\ldots,v_{2^\ell}^0=(t_{2^\ell})^{10}(s_{2^\ell})^{-9}$$

and then define the upper level elements iteratively:

$$v_1^1=(v_1^0)^{10m_1^1}(v_2^0)^{-9k_1^1},\ldots,v_{2^\ell-1}^1=(v_{2^\ell-1}^0)^{10m_{2^\ell-1}^1}(v_{2^\ell}^0)^{-9k_{2^\ell-1}^1}$$

$$v_1^r=(v_1^{r-1})^{10m_1^r}(v_2^{r-1})^{-9k_1^r},\ldots,v_{2^\ell-r}^r=(v_{2^{\ell-r+1}-1}^{r-1})^{10m_{2^\ell-r}^1}(v_{2^{\ell-r+1}}^{r-1})^{-9k_{2^\ell-r}^1}$$

for $1\leq r\leq\ell$. Note that finally: $v_1^\ell=(v_1^{ell-1})^{10m_1^\ell}(v_2^{\ell-1})^{-9k_1^\ell}$. We further set $q=(v_1^{\ell-1})^{10m_1^\ell}u$ and $p=u^{-1}(v_2^{\ell-1})^{-9k_1^\ell}$.

We look at the collection of all the test sequences of values for the variables, $\{t_j,s_j\}_{j=1}^{2^\ell}$ and u. Recall that elements in a test sequence need to satisfy the properties that are listed in definition 1.20 in [Se2], they satisfy a very small cancellation assumption, and should be considered "generic" elements. For all the values in such test sequences, and arbitrary positive integer values of the elements, $\{m_j^r\}$ and k_j^r, the corresponding values of the variables (p,q) are in the set $NE(p,q)$.

By our assumptions, for each $(p_0,q_0)\in NE(p,q)$, there exists (at least one) index a, $1\leq a\leq\ell$, for which $(p_0,q_0)\in E_1^a\cap(E_2^a)^c$. Hence, in particular, $(p_0,q_0)\in E_1^a$. For each index a, $1\leq a\leq\ell$, we look at the collection of all the test sequences of elements, $\{t_j,s_j\}_{j=1}^{2^\ell}$ and u, and sequences of positive integers, $\{m_j^r\}$ and $\{k_j^r\}$, for which each of the sequences, $\{m_j^r\}$ and $\{k_j^r\}$, converges to infinity, and the corresponding sequence of values of the pair (p,q) are all in the set E_1^a.

By the iterative definition of the elements v_j^r, the group that is generated by the elements: t_j,s_j, $j=1,\ldots,2^\ell$, u, and v_j^r, $0\leq r\leq\ell-1$, $1\leq j\leq 2^{\ell-r}$, together with elements that represent formal powers of the elements v_j^r (each such formal power commutes with the corresponding element v_j^r), is a completion (i.e., an ω-residually free tower - see section 6 in [Se1] and definition 1.12 in [Se2]). This completion has a bottom level which is a free group of rank $2^{\ell+1}+1$ (that is generated by u and $\{t_j,s_j\}_{j=1}^{2^\ell}$), and ℓ upper levels, where at level r, $1\leq r\leq\ell$, there are $2^{\ell-r+1}$ abelian vertex groups, each of them of rank 2 and connected to the lower level with a cyclic edge group (that is generated by one of the elements v_j^r). We denote this completion, $Comp$.

The subgroup $Comp$ is a completion, and the projection of generic points (test sequences) in its associated variety project to values of the pair, (p,q), that are in $NE(p,q)$, and hence are in $E_1^a(p,q)$ for some index a, $1\leq a\leq\ell$. Therefore, by the construction of envelopes of definable sets, as they are presented in section 1 of [Se6] (see theorem 1.3 in [Se6]), with some of the groups, $E_1^a(p,q)$, there are associated envelopes, that are closures of the completion, $Comp$.

Hence, with a (non-empty) subcollection of the sets $E_1^1,\ldots,E_1^\ell$, there are associated closures of $Comp$, that we denote $Cl_1,\ldots,Cl_d$ (which are the envelopes of the completion $Comp$ with respect to the collection of sets, $\{E_1^a\}$). Furthermore, with each closure, Cl_g, there exists a finite (possibly empty) closures of it, $Cl_{g,1},\ldots,Cl_{g,f_g}$. By the properties of envelopes (see theorem 1.3 in [Se6]), the set of closures $Cl_1,\ldots,Cl_d$ and their associated closures,$Cl_{1,1},\ldots,Cl_{g,f(g)}$, specifies precisely what test sequences of the completion, $Comp$, restrict to values of the variables (p,q) that are in the sets, $E_1^1,\ldots,E_1^\ell$.

We now look at a specific test sequence of values for the variables, $\{t_j(n), s_j(n)\}_{j=1}^{2^\ell}$ and $u(n)$, together with a specific sequences of positive integers, $\{m_j^r(n)\}$ and $\{k_j^r(n)\}$, for which the integer values, $m_j^r(n)$ and $k_j^r(n)$, do not depend on the indices r or j, $10m_j^r(n) - 9k_j^r(n) = 1$, and the sequences $m_j^r(n)$ and $k_j^r(n)$, converge to infinity.

With the collection of all the test sequences of the completion, $Comp$, we have associated finitely many envelope closures, $Cl_1, \dots, Cl_d$, and with each closure Cl_g we have further associated finally many (possibly no) closures of it, $Cl_{g,1}, \dots, Cl_{g,f(g)}$. With each of these closures there is an associated finite index subgroup of some fixed f.g. free abelian group, so we may look at the intersection of all these finite index subgroups. Hence, with each closure there are finitely many associated cosets of this finite index subgroup. By passing to a subsequence we may assume that all the elements in the specific test sequence that we chose belong to a fixed coset of that finite index subgroup.

For each index n (of the specific test sequence that we chose), we look at the following sequence of values (that is similar to the one that we used in proving proposition 6 and lemma 5):

$$\{q_h^g(n) = (v_1^{\ell-1}(n+g))^{10m_1^\ell(n+h)}\, u(n)\}_{g,h=1}^{N+1}\,;\; \{p_h^g(n) = u(n)^{-1}\, (v_1^{\ell-1}(n+g))^{-9k_1^\ell(n+h)}\}_{g,h=1}^{N+1}.$$

By the way the elements v_j^r are defined, and since we chose the elements $\{t_j(n), s_j(n)\}$, $m_j^r(n)$ and $k_j^r(n)$,from our fixed test sequence, like in proposition 6 and lemma 5: $(p_h^g(n), q_{h'}^{g'}(n)) \in NE(p,q)$, if and only if $g \neq g'$ or $g = g'$ and $h = h'$.

Since we have chosen the positive integers, $m_j^r(n)$ and $k_j^r(n)$, to be from a fixed class of the finite index subgroup that is associated with the closures, $Cl_1, \dots, cl_d$, and $Cl_{1,1}, \dots, Cl_{d,f(d)}$, there exists a positive integer b, $1 \leq b \leq \ell$, and indices $1 \leq a_1 < \ldots < a_b \leq \ell$, so that for every index n, the pair $(q_h^g(n), p_{h'}^{g'}(n)) \in E_1^a$ for $g \neq g'$ and $1 \leq g, g', h, h' \leq N+1$, if and only if a is one of the indices $a_1, \dots, a_b$.

If $b = \ell$, then we apply the same argument that was used to prove lemma 5, to the equational set $E_1^1 \cap \ldots \cap E_1^\ell$, and the co-equational set, $(E_2^1 \cap \ldots \cap E_2^\ell)^c$, and obtain a contradiction. Hence, we may assume that $1 \leq b < \ell$.

Let $M_b = E_1^{a_1} \cap \ldots \cap E_1^{a_b}$, and $T_b = E_2^{a_1} \cap \ldots \cap E_2^{a_b}$. By the equationality of the set T_b, and the argument that was used to prove lemma 5, there exists an index g_0, $1 \leq g_0 \leq N+1$, for which for every h, h', $1 \leq h, h' \leq N+1$, $(p_h^{g_0}(n), q_{h'}^{g_0}(n)) \in T_b$. Since $(p_h^{g_0}(n), q_{h'}^{g_0}(n)) \in NE(p,q)$ if and only if $h = h'$, the equationality of the set T_b and the argument that was used in proving lemma 5, imply that there exists an index h_0, $1 \leq h_0 \leq N+1$, for which for every index h', $1 \leq h' \leq N+1$, $(p_{h'}^{g_0}(n), q_{h_0}^{g_0}(n)) \in T_b$. In particular, $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in T_b$. Since $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in NE(p,q) \cap M_b \cap T_b$ and:

$$NE(p,q) \;=\; (E_1^1 \,\cap\, (E_2^1)^c) \cup \ldots \cup (E_1^\ell \,\cap\, (E_2^\ell)^c)$$

it finally follows that for each index n, there exists an index $a(n)$, $1 \leq a(n) \leq \ell$, and $a(n) \neq a_1, \dots, a_b$ for which: $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in M_b \cap E_1^{a(n)}$. By passing to a further subsequence of indices (still denoted n), we may clearly assume that $a(n)$ is independent of the index n, and that $a = a(n) \neq a_1, \dots, a_b$.

Now, note that like what we did in proving proposition 6, the pairs, $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n))$, can be written as:

$$q_{h_0}^{g_0}(n) = (v_1^{\ell-1}(n+g_0))^{10m_1^\ell(n+h_0)}\, u(n) \;=\; (v_1^{\ell-2}(n+g_0))^{10m_1^{\ell-1}(n+g_0)}\hat{u}(n)$$

$$p^{g_0}_{h_0}(n)=u(n)^{-1}\,(v^{\ell-1}_1(n+g_0))^{-9k^{\ell}_1(n+h_0)}\;=\;\hat u(n)^{-1}\,(v^{\ell-2}_2(n+g_0))^{-9k^{\ell-1}_1(n+g_0)}.$$

Like the argument that was used in proving proposition 6, at this stage we essentially repeat what we did so far with the previous values of the pair p,q (that were only in the set M_b), for the new sequence of values of this pair that are now known to be in the intersection $M_b\cap E^a_1$.

By the iterative definition of the elements v^r_j, the group that is generated by the elements: t_j,s_j, $j=1,\dots,2^{\ell-1}$, u, and v^r_j, $0\le r\le \ell-2$, $1\le j\le 2^{\ell-1-r}$, together with elements that represent formal powers of the elements v^r_j (each such formal power commutes with the corresponding element v^r_j), is a completion. This completion has a bottom level which is a free group of rank $2^\ell+1$ (that is generated by u and $\{t_j,s_j\}_{j=1}^{2^{\ell-1}}$), and $\ell-1$ upper levels, where at level r, $1\le r\le \ell-1$, there are $2^{\ell-r}$ abelian vertex groups, each of them of rank 2 and connected to the lower level with a cyclic edge group (that is generated by one of the elements v^r_j). We denote this completion, $Comp_1$.

The subgroup $Comp_1$ is a completion, and the given sequence of values ($t_j(n+g_0),s_j(n+g_0)$, $j=1,\dots,2^{\ell-1}$, $\hat u(n)$, $v^r_j(n+g_0),m^r_j(n+g_0),k^r_j(n+g_0)$, $0\le r\le \ell-2$, $1\le j\le 2^{\ell-1-r}$) is a test sequence of values of $Comp_1$, that projects to values of the pair, (p,q), that are in $NE(p,q)$. Furthermore, by our conclusion, these values of the pair (p,q) are in the set $M_b\cap E^a_1$, where $1\le a\le \ell$, and $a\ne a_1,\dots,a_b$.

The values of the pair (p,q) in our given sequence are contained in $M_b\cap E^a_1$. We pass to a further subsequence, so that the values of (p,q) are contained in a maximal intersection $M_b\cap E^{e_1}_1\ldots\cap E^{e_s}_1$, where $1\le e_1<e_2<\ldots<e_s\le \ell$, $1\le s$, and each of elements $e_i\ne a_1,\dots,a_b$. We set $b'=b+s$ and $M_{b'}=M_b\cap E^{e_1}_1\ldots\cap E^{e_s}_1$. Note that $1\le b<b'\le \ell$. Since we assume that the set, $e_1,\dots,e_s$, is a maximal set, no proper subsequence of the given set of values of the pair (p,q), is contained in a set $M_{b'}\cap E^f_1$, where $1\le f\le \ell$, and $f\ne a_1,\dots,a_b,e_1,\dots,e_s$.

By the construction of envelopes of definable sets, as they are presented in theorem 1.3 in [Se6], with the equational set $M_{b'}$, and the collection of all the test sequences of $Comp_1$ that restrict to values of the pair p,q that are in $M_{b'}$, it is possible to canonically associate finitely many envelopes, that are closures of the completion, $Comp_1$. Since there exists a test sequence of $Comp_1$ that restricts to values of the pair p,q that are in the set $M_{b'}$, this collection of envelopes is not empty.

Hence, with the equational set $M_{b'}$, there are associated closures of $Comp_1$, that form the envelopes of the set $M_{b'}$, with respect to generic points (test sequences) of the completion, $Comp_1$. Furthermore, with each such envelope closure there exists a finite (possibly empty) collection of closures of it. By the properties of envelopes (see theorem 1.3 in [Se6]), the set of envelope closures and their associated closures, specifies precisely what test sequences of the completion, $Comp_1$, restrict to values of the variables (p,q) that are in the set, $M_{b'}$.

We now look at the original test sequence of the completion $Comp_1$. With the collection of all the test sequences of the completion, $Comp_1$, we have associated finitely many envelope closures. With each of these closures there is an associated finite index subgroup of some fixed f.g. free abelian group, so we may look at the intersection of all these finite index subgroups. Hence, with each closure there are finitely many associated cosets of this finite index subgroup. By passing to a subsequence of the original test sequence of $Comp_1$, we may assume that all the elements in the specific test sequence that we chose belong to a fixed coset of that finite index subgroup.

Now, for each index n (of the specific test sequence that we chose), we look at the following sequence of values for the pair (p,q) and for indices, $1 \leq g,h \leq N+1$ (that is once again similar to the one that we used in proving lemma 5):

$$q_h^g(n) \;=\; (v_1^{\ell-2}(n+g))^{10m_1^{\ell-1}(n+h)}\hat{u}(n)$$

$$p_h^g(n) \;=\; \hat{u}(n)^{-1}\,(v_1^{\ell-2}(n+g))^{-9k_1^{\ell-1}(n+h)}.$$

By the way the values of the sequence is defined, and like in proposition 6 and lemma 5, for every index n: $(p_h^g(n), q_{h'}^{g'}(n)) \in NE(p,q)$, if and only if $g \neq g'$ or $g = g'$ and $h = h'$ $(1 \leq g, g', h, h' \leq N+1)$. Furthermore, since we have chosen the positive integers, $m_j^r(n), k_j^r(n)$, to be from a fixed class of the finite index subgroup that is associated with the envelope closures of $M_{b'}$ with respect to the completion $Comp_1$, for every index n, and every $g \neq g'$, $1 \leq h, h', g, g' \leq N+1$, $(p_h^g(n), q_{h'}^{g'}(n)) \in M_{b'}$.

$1 \leq b < b' \leq \ell$. If $b' = \ell$, we get a contradiction by the argument that was used to prove lemma 5. Hence, $b' < \ell$. In that case we repeat the argument that we used for the first test sequence of values of (p,q), and use the equationality of $M_{b'}$ to prove the existence of indices, g_0, h_0 for every index n, so that for every index n there exists an index $a(n)$, $1 \leq a(n) \leq \ell$, and $a(n) \neq a_1, \ldots, a_b, e_1, \ldots, e_s$ for which: $(p_{h_0}^{g_0}(n), q_{h_0}^{g_0}(n)) \in M_b \cap E_1^{a(n)}$.

Therefore, after passing to a subsequence and repeating the same argument that we used for the first test sequence, it is possible to enlarge the index b' to $b'' = b'+s'$, and obtain a completion $Comp_2$ with a test sequence that restricts to values of the pair (p,q) that are in the set $M_{b''}$. Repeating this argument iteratively, after less than $\ell - 1$ steps we can construct a sequence with similar properties, so that the values of the pair (p,q) is in the set $E_1^1 \cap \ldots \cap E_1^{\ell}$. Hence the same argument that was used to prove lemma 5, that is applied to the equational set $E_1^1 \cap \ldots \cap E_1^{\ell}$ and the co-equational set $E_2^1 \cap \ldots \cap E_2^{\ell}$ gives a contradiction, and finally proves theorem 7.

□

Since the union and the intersection of equational sets are equational, theorem 7 proves that $NE(p,q)$ is not in the Boolean algebra of equational sets, hence that the theory of a free group is not equational. This concludes the proof of theorem 1.

□

The results of [Se4] allow one to modify the argument that was used to prove theorem 1, and generalize theorem 1 to every non-elementary, torsion-free hyperbolic group.

Theorem 8. *The elementary theory of a non-elementary, torsion-free hyperbolic group is not equational.*

Proof: To modify the argument that was used in the free group case, we only need to modify the proof of theorem 7. By the results of [Se4], the same sieve procedure that was used for quantifier elimination over a free group can be used to obtain quantifier elimination over a general torsion-free, non-elementary hyperbolic group. By the results of [Se6], the sieve procedure and the description of definable sets over a torsion-free hyperbolic group enable one to generalize the concept of an envelope of a definable set from a free group to a general torsion-free, non-elementary hyperbolic group.

As there exist quasi-isometric embeddings of a non-abelian free group into any given non-elementary torsion-free hyperbolic group, test sequences generalize to torsion free hyperbolic groups (and indeed, test sequences in hyperbolic groups are used extensively in [Se4]). Therefore, to generalize the argument that was used to prove theorem 7 to non-elementary, torsion-free hyperbolic groups, the only result that still needs to be generalized to hyperbolic groups is the Lyndon-Schutzenberger theorem [Ly-Sch], that shows that for $m, n, p \geq 2$, the solutions of the equation $x^m y^n z^p = 1$ in a free group, generate a cyclic subgroup. For the purposes of generalizing the proof of theorem 7 we need a slightly weaker result that can be proved easily using Gromov-Hausdorff convergence.

Lemma 9. *Let Γ be a non-elementary, torsion-free hyperbolic group. There exist some positive integers $\ell, b > 0$, so that for every $m > b$, the solutions of the system $x^{\ell+1} y^{-\ell} z^m = 1$ in Γ, generate a cyclic subgroup in Γ.*

Proof: Let Γ be a non-elementary, torsion-free hyperbolic group for which such positive integers ℓ and b do not exist. In that case there exist triples $x_n, y_n, z_n \in \Gamma$, for an increasing sequence of integers n, and for which:

(1) $x_n^{n+1} y_n^{-n} z_n^m = 1$ in Γ.
(2) $m > n$, and x_n, y_n, z_n do not belong to a cyclic subgroup in Γ, hence, they do not commute.

The sequence of values x_n, y_n, z_n, define a sequence of homomorphisms from a free group F generated by $< x, y, z, u, v, w >$, to Γ, $h_n : F \to \Gamma$, $h_n(x) = x_n$, $h_n(y) = y_n$, $h_n(z) = z_n$, $h_n(u) = x_n^{n+1}$, $h_n(v) = y^{-n}$, $h_n(w) = z_n^m$. This sequence of actions subconverges into a non-trivial action (not necessarily faithful) of F on some real tree Y. $x_n^{n+1} y_n^{-n} z_n^m = 1$ in Γ, and this equation corresponds to a (thin) triangle that is associated with the equation: $h_n(u)h_n(v)h_n(w) = 1$ in Γ. In the limit tree Y, this sequence of thin triangles converges in the Gromov-Hausdorff topology to a (possibly degenerated) tripod, and this tripod contains a (non-degenerate) segment that is stabilized by at least two elements from the triple x_n, y_n, z_n. By [Pa], this implies that for large enough index n (from the convergent subsequence), two of the elements x_n, y_n, z_n commute, hence, these two elements are contained in the same cyclic subgroup of Γ. In a torsion-free hyperbolic group, this implies that for large enough n, all the 3 elements, x_n, y_n, z_n, belong to the same cyclic subgroup of Γ, a contradiction to (2).

□

Lemma 9 and the results of [Se4] and [Se6] allow one to generalize the proof of theorem 7 to non-elementary, torsion-free hyperbolic groups, hence, to prove that such groups are not equational.

□

References

[Ba-Pi] A. Baudisch and A. Pillay, *A free pseudospace*, Journal of symbolic logic **65** (2000), 443-460.

[Be-Fe] M. Bestvina and M. Feighn, *Definable and negligible subsets of free groups*, preprint.

[Hr-Sr] E. Hrushovski and G. Srour, *On stable non-equational theories*, preprint, 1989.

[Ly-Sch] R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[OH] A. O'hara, *An introduction to equations and equational sets*, preprint.

[Pi-Sr] A. Pillay and G. Srour, *Closed sets and chain conditions in stable theories*, Journal of symbolic logic **49** (1984), 1350-1362.

[Se1] Z. Sela, *Diophantine geometry over groups I: Makanin-Razborov diagrams*, Publication Math. de l'IHES **93** (2001), 31-105.

[Se2] ______, *Diophantine geometry over groups II: Completions, closures and formal solutions*, Israel jour. of Mathematics **134** (2003), 173-254.

[Se3] ______, *Diophantine geometry over groups* V_2*: Quantifier elimination II*, GAFA **16** (2006), 537-706.

[Se4] ______, *Diophantine geometry over groups VII: The elementary theory of a hyperbolic group*, Proceedings of the LMS **99** (2009), 217-273.

[Se5] ______, *Diophantine geometry over groups VIII: Stability*, preprint.

[Se6] ______, *Diophantine geometry over groups IX: Envelopes and imaginaries*, preprint.