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DIOPHANTINE GEOMETRY OVER GROUPS V_2: QUANTIFIER ELIMINATION II

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Abstract. This paper is the sixth in a sequence on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the two papers on quantifier elimination we use the iterative procedure that validates the correctness of an AE sentence defined over a free group, presented in the fourth paper, to show that the Boolean algebra of AE sets defined over a free group is invariant under projections, hence, show that every elementary set defined over a free group is in the Boolean algebra of AE sets. The procedures we use for quantifier elimination, presented in this paper, enable us to answer affirmatively some of Tarski's questions on the elementary theory of a free group in the last paper of this sequence.

In the first 4 papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects required for the analysis of sentences and elementary sets defined over a free group. In the first paper in this sequence we studied sets of solutions to systems of equations defined over a free group and parametric families of such sets, and associated a canonical Makanin–Razborov diagram that encodes the entire set of solutions to the system. Later on we studied systems of equations with parameters, and with each such system we associated a (canonical) graded Makanin–Razborov diagram, that encodes the Makanin–Razborov diagrams of the systems of equations associated with each specialization of the defining parameters.

In the second paper we generalized Merzlyakov's theorem on the existence of a formal solution associated with a positive sentence. We first constructed a formal solution for a general AE sentence which is known to be true over some variety, and then presented formal limit groups and graded formal limit groups that enable us to collect and analyze the collection of all such formal solutions.

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In the third paper we studied the structure of exceptional solutions of a parametric system of equations. We proved the existence of a global bound (independent of the specialization of the defining parameters) on the number of rigid solutions of a rigid limit group, and a global bound on the number of strictly-solid families of solutions of a solid limit group. Using these bounds we studied the stratification of the "base" of the "bundle" associated with the set of solutions of a parametric system of equations in a free group, and showed that the set of specializations of the defining parameters in each stratum is in the Boolean algebra of AE sets.

In the fourth paper we applied the structural results obtained in the first two papers in the sequence, to analyze AE sentences. Given a true sentence of the form,

$$\forall y \ \exists x \ \Sigma(x,y,a) = 1 \wedge \Psi(x,y,a) \neq 1 \,,$$

we presented an iterative procedure, that produces a sequence of varieties and formal solutions defined over them. Since in order to define the completions of a variety, and the closures of these completions, additional variables are required, the varieties produced along the iterative procedure are determined by larger and larger sets of variables, and so are the formal solutions defined over them. Still, by carefully analyzing these varieties, and properly measuring the complexity of Diophantine sets associated with them, we were able to show that certain complexity of the varieties produced along the procedure strictly decreases, which finally forces the iterative procedure to terminate after finitely many steps.

The outcome of the terminating iterative procedure is a collection of varieties, together with a collection of formal solutions defined over them. The varieties are determined by the original universal variables y, and extra (auxiliary) variables. The collection of varieties gives a partition of the initial domain of the universal variables y, which is a power of the original free group of coefficients, into sets which are in the Boolean algebra of universal sets, so that on each such set the sentence can be validated using a finite family of formal solutions. Hence, the outcome of the iterative procedure can be viewed as a *stratification theorem* that generalizes Merzlyakov's theorem from positive sentences to general AE ones.

In the two papers on quantifier elimination we apply the tools and techniques presented in the previous 4 papers in the sequence, to prove quantifier elimination in the elementary theory of a free group. In order to prove quantifier elimination we show that the Boolean algebra of AE sets is invariant under projections. The projection of a set that is in the

Boolean algebra of AE sets, is naturally an EAE set, hence, to show that the Boolean algebra of AE sets is invariant under projections, we need to show that a general EAE set is in the Boolean algebra of AE sets.

In the first section of the first paper on quantifier elimination [S5], we presented a couple of terminating iterative procedures that together imply that the Boolean algebra of AE sets is invariant under projection in the minimal rank (rank 0) case. Given a set of the form $EAE(p)$, i.e. a set which is a projection of a set that is in the Boolean algebra of AE sets, the first (terminating) iterative procedure described in that section is devoted to uniformization of proofs. The procedure constructs the *tree of stratified sets* associated with the set $EAE(p)$, that encodes all the (finitely many) *proof systems* (i.e. forms of proofs) that are required in order to prove that a given specialization of the defining parameters is indeed in the elementary set $EAE(p)$ we started with (see Definition 1.20 in [S5]). The second terminating iterative procedure that we call the *sieve* method, constructs a (finite) sequence of bundles of *proof statements* (Definition 1.23 in [S5]) that are supposed to "testify" that a given specialization of the defining parameters is in the set $EAE(p)$. This finite sequence of bundles reduces the question of the existence of a possible *witness* with a valid proof statement (Definition 1.19 in [S5]) for any given specialization of the defining parameters, to the structure of the bases of these bundles of proof statements. Since by section 3 of [S3] it is possible to stratify the base of such a bundle, and the existence of a witness for a given specialization of the defining parameters depends only on the stratum (and not on the specific specialization), the set $EAE(p)$ is the union of finitely many strata in the stratifications of the constructed bundles. Since every stratum in the stratification is in the Boolean algebra of AE sets, we were finally able to conclude that the original EAE set is in the Boolean algebra of AE sets.

Our approach towards the analysis of a general EAE set is conceptually similar to the one applied in the minimal rank case, although it is technically much more involved. In the second section of the first paper [S5], we used the techniques and the iterative procedure presented in section 4 of [S4] for validation of a general AE sentence, and presented a (terminating) iterative procedure for the construction of the tree of stratified sets for general EAE predicates, omitting the minimal rank assumption. In the third section in [S5] we have generalized some of the notions presented in the minimal rank case, and showed that in few cases the procedure used to construct the tree of stratified sets can be slightly modified to give a terminating iterative procedure for the analysis of the bundles of proof statements, which implies that in these cases the EAE sets we started the procedures with are in the Boolean algebra of AE sets.

In the fourth section in [S5] we have presented the (*graded*, *multi-graded*) *core resolution*, which is the major additional tool needed in order to further modify the iterative procedure used for the construction of the tree of stratified sets, to give a (terminating) iterative procedure for the analysis of the bundles of proof statements for general EAE sets, which finally implies that every EAE set is in the Boolean algebra of AE sets.

In this second paper on quantifier elimination, that presents the sieve method in the general case, we will keep the notions and the notation that were used in the first paper on quantifier elimination [S5]. Since the iterative procedure we use is rather involved we present it as we did in presenting the iterative procedure for validation of a sentence in section 4 of [S4]. We start by presenting the first step of the procedure, then the second step, the general step, and finally we prove its termination. Then we apply the procedure we constructed to conclude the proof of Theorems 1.4 and 1.3 in [S5] (Theorems 40 and 41 in this paper), that were proved there in the minimal rank case, to finally show that the Boolean algebra of AE sets is invariant under projection, which is the main goal of this paper. For the benefit of the reader, we add an appendix at the end of the paper, that briefly summarizes the sieve method and describes some of the objects it uses.

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The First Step of the Sieve Procedure

Let $P = \langle p \rangle$ be the group of defining parameters. We start the first step of the sieve procedure with each of the PS limit groups $PSHGH_i$ in parallel, hence, we omit its index (see section 1 in [S5] for the notion of a PS limit group). Given a PS limit group $PSHGH$, we associate its canonical taut graded Makanin–Razborov diagram (with respect to the parameter subgroup P) with it, which contains finitely many graded resolutions which we denote $PSHGHRes_j$, and each graded resolution $PSHGHRes_j$ is defined over the rigid or solid limit group $PB_j(b, p, a)$. In the sequel, we will treat each stratum in the singular locus of the graded resolutions $PSHGHRes$ separately, and do it in parallel.

As we did in the minimal rank case and in the few cases presented in section 3 in [S5], we start our analysis of the set of valid PS statements by associating with (the completion of) the graded PS resolution $PSHGHRes$ a (canonical) finite collection of *Non-rigid* and *Non-solid PS limit groups* (Definition 1.25 in [S5]), which we call the *non-rigid PS limit groups* associated with the PS resolution $PSHGHRes_j$, $NRgdPS_1^j, \dots, NRgdPS_q^j$, and the *non-solid PS limit groups* associated with $PSHGHRes_j$, $NSldPS_1^j, \dots, NSldPS_r^j$.

Recall (Definition 1.25 in [S5]) that the graded formal closures associated with the collection of non-rigid and non-solid PS limit groups, determine those "generic" specializations that factor through and are taut with respect to the various PS resolutions $PSHGHRes_j$, but fail to be valid PS statements with respect to the (fixed) proof system, because certain specializations that are part of the proof statement, and are required to be rigid or strictly-solid (with respect to a given set of closures — see Definition 2.12 in [S3]) specializations of the groups $WPHG$ according to the fixed proof system, actually factor through closures of resolutions that are associated with flexible quotients of the corresponding rigid or solid limit groups.

We continue by collecting all the test sequences that factor through the completion of one of the PS resolutions $PSHGHRes_j$, $Comp(PSHGHRes_j)$, and for which for at least one of the tuples: $(h_t^2(n), g_t^1(n), h_1(n), w_n, p_n, a)$ there exists some specialization $g_t^2(n)$ so that the (combined) specialization: $(g_t^2(n), h_t^2(n), g_t^1(n), h_1(n), w_n, p_n, a)$ factors through (at least) one of the limit groups $WP(HG)^2$ (Definition 1.26 in [S5]).

The collection of all these (graded) test sequences factor through a (canonical) collection of *maximal left PS limit groups*, $LeftPS_1^j,\dots,LeftPS_{m_j}^j$. The analysis of graded formal limit groups presented in section 3 of [S2] associates (canonically) with each Left PS limit group $LeftPS_i^j$ a graded formal Makanin–Razborov diagram, and each such graded formal resolution is in fact a one level graded resolution, which is a graded formal closure of the graded resolution $PSHGHRes_j$, $GFCl(PSHGHRes_j)$. Clearly, no specialization (*virtual proof statement*) that factors through the completion of the resolution $PSHGHRes_j$, $Comp(PSHGHRes_j)$, and which is a valid PS statement with respect to our fixed proof system, factors through one of the $LeftPS$ limit groups $LeftPS_1^j, \dots, LeftPS_{m_j}^j$, and their associated resolutions.

To a valid PS statement we have added additional variables, so that their specializations are supposed to be primitive roots of the specializations of pegs of abelian groups that appear in the graded formal closures associated with the groups $WPHGH$, in order to demonstrate that the given sets of closures (specified by the proof system) form a covering closure (for the specializations given by the proof statement). This demonstration remains valid if the orders of the specializations of the variables, that are supposed to be primitive roots, are prime to the indices of the finite index subgroups associated with the (finitely many) closures. The demonstration may fail to be valid if the orders of these specializations are not prime to the order of the finite index subgroups. To check if this failure occurs for a generic specialization of a PS resolution, $PSHGHRes$, we construct *Root* PS limit groups and resolutions, precisely as we did in the minimal rank case (Definition 1.27 in [S5]). We denote the *Root* PS limit groups, $RootPS$, and the *Root* PS resolutions, $RootPSRes$.

No specialization $(r,(h_1^2,g_1^1),\dots,(h^2_{d(ps)},g^1_{d(ps)}),h_0^1,w_0,p_0,a)$ that factors through the resolution $PSHGH$ (a virtual proof statement), and which is a valid PS statement with respect to our fixed proof system, factors through one of the $RootPS$ limit groups $RootPS_1,\dots,RootPS_m$, and their associated Root PS resolutions.

So far, we have constructed bundles for which, if in a given fiber a "generic" PS statement fails to be a valid PS statement, then any PS statement in that fiber fails to be a valid PS statement, i.e. the whole fiber can be avoided. The next bundles, that we construct using "generic" specializations that fail to be valid PS statements, have the same structure as the previous ones, however, in these bundles it may be that even though "generic" PS statements in a given fiber fail to be valid PS statements, the fiber may contain (non-"generic") valid PS statements.

"Generic" specializations that factor through the PS resolutions $PSHGHRes$ can fail to be valid PS statements also if there exist additional rigid or strictly-solid specializations of the limit groups $PSHG(g_1,h_1,w,p,a)$ that are not specified by the given specializations. As in the minimal rank case and the few cases treated in section 3 of [S5], the "generic" specializations for which there exists "surplus" in rigid or strictly-solid specializations are collected in *Extra* PS *(graded) limit groups* and graded resolutions (Definition 1.28 in [S5]). We denote the *Extra* PS *limit groups* associated with the graded PS resolution $PSHGHRes_j$, $ExtraPS_1^j,\dots,ExtraPS^j_{\ell_j}$.

The Extra PS limit groups and their associated graded formal closures collect all the "generic" specializations (i.e. all the test sequences) of the PS

resolutions $PSHGHRes$ for which there exist rigid or strictly-solid families in addition to those specified by the generic specializations. For a general specialization of the PS limit groups $PSHGH$, i.e. a specialization (virtual proof statement) which is not necessarily "generic", it may as well be that the additional rigid or strictly-solid specializations, collected by the Extra PS limit groups and their associated graded formal closures, do become flexible or do coincide with the rigid or strictly-solid families of the various specializations (g_t^1, h_0^1, w_0, p_0), specified by the (virtual) proof statement.

Let $ExtraPSRes$ be one of the Extra PS graded resolutions associated with one of the Extra PS graded limit groups $ExtraPS_i^j$. We will say that a specialization (virtual proof statement) that factors through and is taut with respect to the Extra PS graded resolution $ExtraPSRes$ is *collapsed* if the variables added for each of the additional rigid or strictly-solid specializations (i.e. the ones that were not specified by the (virtual) proof statement) satisfy one of the following:

(1) A specialization of the variables that were added for one of the additional rigid specializations becomes flexible.
(2) A specialization of the variables that were added for one of the additional rigid specializations becomes equal to one of rigid specializations specified by the (virtual) proof statement, i.e. with one of the specializations g_t^1 in the specialization, $(u, v, r, (h_1^2, g_1^1), \dots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0)$.
(3) A specialization of the variables that were added for one of the additional strictly-solid families of specializations becomes flexible.
(4) A specialization of the variables added for one of the additional strictly solid families of specializations belongs to one of strictly-solid families of specializations specified by the (virtual) proof statement, i.e. with a family of one of the strictly-solid families of specializations g_t^1 in the specialization, $(u, v, r, (h_1^2, g_1^1), \dots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_0^1, w_0, p_0)$.

Note that by definition there are only finitely many ways in which a specialization that factors through and is taut with respect to the Extra PS resolution, $ExtraPSRes$, can become a collapsed specialization. We will call each way a specialization can become collapsed a *collapse form*.

Having defined the finitely many possibilities for collapse forms, we collect all the test sequences of specializations that factor through an Extra PS resolution, $ExtraPSRes$, and are collapsed specializations, in finitely many (graded) closures of the Extra PS resolutions, $ExtraPSRes$, that are also (graded) closures of the resolution $PSHGHRes_j$ we started with, that we call *Generic collapse Extra PS* (graded) *resolutions*, $GenericCollapseExtraPSRes$, precisely as we did in the minimal rank case, and in the cases presented in section 3 in [S5] ([S5, 3.4]).

Having collected all the "collapsed" test sequences in a finite collection of generic collapse Extra PS resolutions, we still need to collect all the collapsed specializations that factor and are taut with respect to a given Extra PS resolution $ExtraPSRes$ (but perhaps not through one of the generic collapse ones). To do that, we first need to associate with each Extra PS resolution its associated collection of *auxiliary resolutions* and their *auxiliary limit groups*.

DEFINITION 1 (cf. Definition 2.1 in [S5]). *Let $ExtraPSRes$ be one of the Extra PS resolutions, associated with a PS resolution $PSHGHRes$. With the Extra PS resolution, $ExtraPSRes$, we associate a collection of (multi-graded)* auxiliary resolutions *and (multi-graded)* auxiliary limit groups.

Recall that an Extra PS resolution is a closure of the PS resolution $PSHGHRes$, hence, it has the structure of a (graded) completed resolution. Suppose that the resolution $ExtraPSRes$ contains ℓ levels. Let $Rlim(r,h_2,g_1,h_1,w,p,a)$ be the image of the PS limit group $PSHGH$ in the limit group associated with $ExtraPSRes$, and let $Rlim(z_i,p,a)$ be its image in the (graded) limit group associated with the i-th level in $ExtraPSRes$, where $1 \leq i \leq \ell$.

With $ExtraPSRes$, we associate a taut multi-graded Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level 2, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) graded abelian decomposition associated with the top level of $ExtraPSRes$, i.e. the graded abelian decomposition associated with $Rlim(r,h_2,g_1,h_1,w,p,a)$. Similarly, with each level i in $ExtraPSRes$, $1 \leq i \leq \ell-1$, we associate a multi-graded taut Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level $i+1$, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) graded abelian decomposition associated with the i-th level in $ExtraPSRes$, i.e. the graded abelian decomposition associated with the subgroup $Rlim(z_i,p,a)$.

We call each of the resolutions in these multi-graded diagrams a (multi-graded) auxiliary resolution, *and its terminating solid or rigid limit group a (multi-graded)* auxiliary limit group, *which we denote: $Aux(ExtraPSRes)$. Naturally, with each auxiliary resolution we associate its modular groups, that we call* auxiliary modular groups. *In the sequel, we call the auxiliary* *resolutions associated with the tower containing all the levels up to level 2 (all the levels except the top level),* highest level.

Note that since an auxiliary limit group is associated with the limit groups that appear in all the levels of ExtraPSRes up to level $i+1$, and it is multi-graded with respect to the non-abelian, non-QH vertex groups and edge groups in the abelian decomposition associated with the i-th limit group, $Rlim(z_i, p, a)$, it can be naturally extended to the limit group associated with (the ambient resolution) ExtraPSRes, by sequentially adding to the auxiliary limit group the limit groups that appear in level i and above, according to the (tower-like) structure of the completed resolution, ExtraPSRes, i.e. by sequentially adding abelian and QH vertex groups as they appear in the level i and above in ExtraPSRes. We call the obtained limit group (which is a quotient of the limit group associated with ExtraPSRes), the extended auxiliary limit group. *By construction, the PS limit group, PSHGH, that is associated with the PS resolution, PSHGHRes (that is associated with ExtraPSRes), is mapped into the extended auxiliary limit group, and so are the subgroups associated with the extra rigid and strictly-solid solutions that are associated with ExtraPSRes.*

By construction, the multi-graded auxiliary resolution that terminates in the auxiliary limit group, extends to the graded extended auxiliary resolution *that terminates in the extended auxiliary limit group, and is graded with respect to the subgroup generated by all the limit groups associated with levels 1 down to level i in ExtraPSRes.*

Lemma 2. *Let ExtraPSRes be one of the Extra PS* graded resolutions *associated with a PS resolution, PSHGHRes. According to Definition 1, with the resolution ExtraPSRes we can associate its canonical collection of (multi-graded) auxiliary resolutions.*

With the Extra *PS* resolution, *ExtraPSRes, we associate one of its associated multi-graded auxiliary resolutions, hence, its associated extended auxiliary resolutions, and one of its possible collapse forms. Recall that the PS limit group, PSHGH, that is associated with the PS resolution, PSHGHRes (that is associated with ExtraPSRes), is mapped into the extended auxiliary limit group, and so are the subgroups associated with the extra rigid and strictly-solid solutions that are associated with ExtraPSRes. Furthermore, the auxiliary resolution naturally extends to the extended auxiliary resolution which is graded with respect to the collection of subgroups that appear in all the (top) levels of ExtraPSRes, from level 1 down to level i. Then:*

(i) *If the subgroup corresponding to an extra rigid specialization is not elliptic along the entire (extended) graded auxiliary resolution, then in any specialization that factors through the Extra PS resolution and through its associated graded auxiliary resolution, the corresponding specialization of the subgroup corresponding to that extra rigid specialization is not rigid (i.e. it is flexible).*

(ii) *If the subgroup corresponding to a non-abelian, non-QH vertex group or to an edge group in the abelian decomposition associated with an extra strictly-solid specialization is not elliptic along the entire (extended) graded auxiliary resolution (i.e. contained in a non-abelian, non-QH vertex group or an edge group along the entire extended auxiliary resolution), then in any specialization that factors through the Extra PS resolution and through its associated graded auxiliary resolution, the corresponding specialization of the subgroup corresponding to that extra strictly-solid specialization is not strictly solid.*

(iii) *Suppose that the subgroup associated with an extra rigid specialization is elliptic along the (extended) graded auxiliary resolution. Then for any specialization that factors through our fixed graded auxiliary resolution, and in which the specialization of this extra rigid specialization becomes flexible or coincides with one of the rigid specializations specified by the given collapse form, the corresponding specialization of the graded auxiliary limit group satisfies an additional non-trivial relation (that does not hold in $ExtraPSRes$).*

(iv) *Suppose that the subgroups corresponding to all non-abelian, non-QH vertex groups and edge groups in the abelian decomposition associated with an extra strictly-solid specialization are elliptic along the entire (extended) graded auxiliary resolution. Then it is possible to add some additional variables that are associated with closures of flexible quotients of the corresponding solid limit groups $WPHG$, or additional variables that are associated with the solid limit group $WPHG$ itself, so that every specialization that factors through our fixed (extended) graded auxiliary resolution, and for which the restriction of the specialization to the extra strictly-solid specialization in $ExtraPSRes$ is non-strictly solid or belongs to one of the strictly solid families of specializations that are specified by the the (virtual) proof statement and given collapse form, satisfies an additional Diophantine condition that can be expressed in terms of the variables of the extended graded auxiliary resolution and the variables associated with the corresponding solid limit group $WPHG$*

and its flexible quotient. This additional Diophantine condition is not valid for a "generic" specialization of the Extra PS resolution, $ExtraPSRes$, we started with.

Proof. If the subgroup corresponding to an extra rigid specialization is not elliptic along the entire extended graded auxiliary resolution, the graded limit group associated with the extended auxiliary resolution and the particular extra rigid specialization is necessarily flexible, and we get part (i). Part (ii) follows from Proposition 1.9 in [S3]. Part (iii) follows by definition. Part (iv) follows from the definition of strictly-solid solutions and the equivalence relation that determines their families, which is given in Definition 1.5 in [S3]. □

To start the first step of the sieve procedure, we go over all the multi-graded auxiliary resolutions associated with the Extra PS resolutions, $ExtraPSRes$, and their associated extended auxiliary resolutions (see Definition 1). With each extended auxiliary resolution we associate all its (finitely many) possible collapse forms. With each Extra strictly-solid specialization associated with the Extra PS resolution, $ExtraPSRes$, we naturally associate its solid limit group $WPHG$.

Given a (graded) extended auxiliary resolution and a collapse form, we add variables that are associated with the various solid limit groups $WPHG$ (that are associated with the extra strictly-solid specializations in $ExtraPSRes$) and their flexible quotients, so that the added variables and the extended auxiliary limit group enable us to express the additional Diophantine conditions imposed by the collapse form (see part (iv) in Lemma 2).

The multi-graded abelian decomposition of our given multi-graded auxiliary limit group can get degenerate in finitely many ways (cf. section 11 of [S1] and section 3 of [S3]); each possible degeneracy gives rise to a set of equations on the specializations of the graded auxiliary limit group. Hence, with each possible degeneracy we associate a (canonical) finite collection of maximal proper quotients of the given graded auxiliary limit group, and treat each of these proper quotients in parallel (as we treated the non-degenerate auxiliary limit group).

Given the extended auxiliary limit group or one of its degenerate quotients, and the variables that were added to express the Diophantine conditions imposed by the given collapse form, we look at all the rigid or strictly-solid specializations of the extended graded auxiliary limit group (or its degenerate quotient), and specializations of the additional variables, so that the combined specializations satisfy the Diophantine conditions imposed by the given collapse form.

By our standard method presented in section 5 of [S1], this collection of specializations factor through a canonical (finite) collection of maximal limit groups, which we call *Collapse extra PS limit groups*, and denote:

$$CollapseExtraPS_1^1, \ldots, CollapseExtraPS_d^1 .$$

Suppose that the auxiliary resolution that is associated with $ExtraPSRes$ is highest level (i.e. it is associated with the limit group that appears in the second level of $ExtraPSRes$). Let $(u_0, v_0, r_0, h_2(0), g_2(0), h_1(0), w_0, p_0, a)$ be a specialization of the Extra PS resolution, $ExtraPSRes$, so that the restriction of this specialization to the limit group that is associated with the second level of $ExtraPSRes$, is a rigid or strictly-solid specialization of the auxiliary limit group, and that the specialization can be extended to a (combined) specialization that satisfies the Diophantine conditions imposed by the given collapse form. Let φ_β be an element in the auxiliary modular group that is associated with the (multi-graded) auxiliary resolution. φ_β acts on specializations of the auxiliary limit group. Hence, given the specialization $(u_0, v_0, r_0, h_2(0), g_2(0), h_1(0), w_0, p_0, a)$, φ_β acts on the restriction of this specialization to the subgroup associated with the second level of $ExtraPSRes$. By the structure of $ExtraPSRes$, which is a completed resolution, and the structure of the auxiliary resolution (that is multi-graded with respect to the non-abelian, non-QH vertex groups, and edge groups in the given graded abelian decomposition of the subgroup $Rlim(z_1, p, a)$), and by the structure of the Diophantine conditions imposed by the collapse form as described in Lemma 2, the specialization of the limit group associated with the second level of $ExtraPSRes$ obtained by the action of φ_β, can be naturally extended to a combined specialization that satisfies the Diophantine conditions imposed by the given collapse form, without changing the specialization of the limit group associated with the top level of $ExtraPSRes$, $Rlim(z_1, p, a) = Rlim(r, h_2, g_1, h_1, w, p, a)$. Clearly, the same holds for general auxiliary resolutions, associated with some level i of $ExtraPSRes$, that are not necessarily highest level.

Hence, if a specialization of the Extra PS resolution, $ExtraPSRes$, $(u, v, r, h_2, g_1, h_1, w, p, a)$, extends to a specialization that factors through one of the collapse Extra PS limit groups, $CollapseExtraPS^1$, and restricts to a rigid or strictly-solid specialization of the associated auxiliary resolution, then the same is true for all the specializations in the same strictly-solid family of the extended auxiliary limit group. Hence, in analyzing the

Collapse extra PS limit groups, we consider the non-abelian, non-QH vertex groups and edge groups in the multi-graded abelian JSJ decomposition of the auxiliary limit group, as determined only up to (appropriate) conjugacy, and the abelian and QH vertex groups as "formal", i.e. we are allowed to act on these with their associated modular groups. Adapting this point of view, which is essential along the entire iterative procedure presented in this section (as in the construction of the tree of stratified sets in section 2 of [S5]), replaces the role of restricting to shortest form specializations in the ungraded case (Definition 4.1 in [S4]), and enables us to exclude the variables that belong to lower levels of the Extra PS resolution from taking part in the analysis of the (top part of the) Collapse extra PS limit group, $CollapseExtraPS^1$, i.e. it allows us to get (certain) "separation of variables" (of different levels) in the analysis of Collapse extra PS limit groups (and in analyzing Diophantine sets in general).

By construction, if $p_0 \in T_2(p)$ then there must exist a valid PS statement of the form, $(r,(h_1^2,g_1^1),\dots,(h^2_{\nu(ps)},g^1_{\nu(ps)}),h_0^1,w_0,p_0,a)$ that factors through one the of the PS resolutions $PSHGHRes_j$, that are associated with the Zariski closure of all the valid proof statements of depth 2. By Proposition 3.7 in [S5], the sets $TSPS(p)$ associated with the various PS resolutions $PSHGHRes$, i.e. the sets of specializations p_0 of the defining parameters $P=\langle p\rangle$ for which there exists a test sequence of specializations,

$$\left(v_n,r_n,(h_1^2(n),g_1^1(n)),\dots,(h^2_{d(ps)}(n),g^1_{\nu(ps)}(n)),h_1(n),w_n,p_0,a\right),$$

that factor through the completion of one of the the PS resolution $PSHGHRes_j$, $Comp(PSHGHRes_j)$, and restricts to valid PS statements:

$$\left((h_1^2(n),g_1^1(n)),\dots,(h^2_{d(ps)}(n),g^1_{\nu(ps)}(n)),h_1(n),w_n,p_0,a\right)$$

is in the Boolean algebra of AE sets.

By Theorem 3.6 in [S5], if there exists a valid PS statement that can be extended to a specialization that factors through a PS resolution $PSHGHRes_j$, then either $p_0 \in TSPS(p)$, or the valid PS statement can be extended to a specialization that factors through one of the Collapse extra PS limit groups, that restricts to a rigid or strictly-solid specialization of its associated extended auxiliary limit group.

To analyze the remaining set of valid PS statements we construct an iterative procedure that produces a sequence of multi-graded well-separated resolutions (Definition 2.1 in [S4]), their completions and core resolutions. The iterative procedure is based on the iterative procedure used for the construction of the tree of stratified sets, presented in section 2 of [S5], which is based on the iterative procedure for validation of a sentence, presented in section 4 of [S4]. We start the iterative procedure with each of the PS limit groups $PSHGH$ in parallel, and with each PS limit group we associate its collection of PS resolutions, $PSHGHRes_j$, their associated Extra PS resolutions, $ExtraPSRes$, the associated multi-graded auxiliary resolutions and auxiliary limit groups, and the associated Collapse extra PS limit groups $CollapseExtraPS^1$. As in the iterative procedure for the construction of the tree of stratified sets and the iterative procedure for validation of a sentence, we divide the construction of the graded *developing resolution* and the associated *anvil* constructed in the first step of our sieve procedure into several cases, depending on the structure of the Extra PS resolution we start the first step with, and the structure of the multi-graded resolutions constructed along the first step.

We start the first step of the procedure with all the collapse Extra PS limit groups, $CollapseExtraPS^1$, that are associated with PS resolutions, $PSHGHRes$, and with auxiliary resolutions associated with the tower containing all the levels in an Extra PS resolution, $ExtraPSRes$, except the top level (i.e. the highest level auxiliary resolutions). Since we analyze these collapse Extra PS limit groups in parallel, we will omit their index.

As parts (1) and (2) of the first step of the sieve procedure indicate, we will analyze only multi-graded resolutions of these collapse Extra PS limit groups that are not of maximal complexity, i.e. resolutions for which their core does not contain a single level with abelian decomposition that has the same structure as the abelian decomposition associated with the top level of the associated Collapse extra PS limit group, $CollapseExtraPS^1$. To analyze (specializations that factor through) multi-graded resolutions of maximal complexity we will need to use the Collapse extra PS limit groups associated with auxiliary resolutions that are not of highest level (this is done in parts (3) and (4) of the first step of the sieve procedure).

(1) Let $CollapseExtraPS^1(t,v,r,h_2,g_1,h_1,w,p,a)$ be the Collapse extra PS resolution we started (this branch of the) first step with. Let $Q^1(r,h_2,g_1,h_1,w,p,a)$ be the the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the Collapse extra PS limit group $CollapseExtraPS^1(t,v,r,h_2,g_1,h_1,w,p,a)$.

Note that $Q^1(r,h_2,g_1,h_1,w,p,a)$ is a quotient of the PS limit group, $PSHGH$, we started this branch with. If $Q^1(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of the PS limit group $PSHGH$, we continue this branch of the iterative procedure, by starting the first step of the procedure with the PS limit group $Q^1(r,h_2,g_1,h_1,w,p,a)$.

(2) At this stage we may assume that $Q^1(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the PS limit group $PSHGH$. We set the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v_1^1}$ to be the (image in $CollapseExtraPS^1$ of the) non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the Extra PS resolution $ExtraPSRes$ (alternatively, the image of the factors in the given free decomposition of the auxiliary limit group $Aux(ExtraPSRes)$). The notation, $Base^{i,j}_{k,\ell}$, assumes that i is the number of the current step, k is the level (or part) with which the auxiliary resolution is associated (the level we analyze plus 1), ℓ is the index of the vertex group in the abelian decomposition associated with the current level, and j is the index of the subgroup associated with the limit group that is associated with the current level (the index is bounded by the step number, and is updated whenever the complexity of the abelian decomposition associated with the current level is reduced — see the description of the general step). Note that the parameter subgroup, $AP=\langle p,a\rangle$, is, by definition, a subgroup of $Base^{1,1}_{2,1}$. With the Collapse extra PS limit group, $CollapseExtraPS^1(t,v,r,h_2,g_1,h_1,w,p,a)$, we associate its taut multi-graded diagram with respect to the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v_1^1}$.

As we remarked earlier, in constructing the multi-graded diagram, we regard the QH and abelian vertex groups in the multi-graded abelian decomposition associated with the auxiliary limit group (that is associated with the Collapse extra PS resolution), that are all contained in the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v_1^1}$, as "formal", i.e. the only relations they satisfy are those coming from the abelian decomposition associated with the auxiliary limit group. We denote the completions of the multi-graded resolutions that appear in the taut multi-graded Makanin–Razborov diagram of $CollapseExtraPSRes^1$ by

$$MGQRes_1(t,v,r,h_2,g_1,h_1,w,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v_1^1},a),\dots$$
$$\dots,MGQRes_q(t,v,r,h_2,g_1,h_1,w,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v_1^1},a)\,.$$

We continue with each of the multi-graded resolutions in parallel, hence, we omit the index of the specific resolution we continue with.

Since the Extra PS resolution $ExtraPSRes$, we started the first step with, is well separated (Definition 2.1 in [S4]), with each QH vertex group in one of the abelian decompositions associated with $ExtraPSRes$ there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the Extra PS resolution $ExtraPSRes$.

Each QH vertex group in the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, naturally inherits a sequence of abelian decompositions from a multi-graded resolution $MGQRes_j$. If, for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the Extra PS resolution $ExtraPSRes$, we omit the multi-graded resolution $MGQRes_j$ from the list of completions of resolutions of the Collapse extra PS limit group $CollapseExtraPS^1$. Since the resolutions $MGQRes_j$ are supposed to collect specializations that factor through $ExtraPSRes$, and can be extended to specializations that satisfy the Diophantine conditions imposed by the collapse form (see Lemma 2), the multi-graded resolutions $MGQRes_j$ that are left after omitting the multi-graded resolutions that are not compatible with the taut structure associated with the top level of $ExtraPSRes$, still collect all such (extended collapsed) specializations.

By Theorem 4.13 in [S5], the complexity of the multi-graded core of the resolution $MGQRes$,

$$MGCore\big(\langle r,h_2,g_1,h_1,g_1,w,p,a\rangle, MGQRes\big)\ ,$$

is bounded by the complexity of the abelian decomposition associated with the top level of the Extra PS resolution $ExtraPSRes$ we started the first step with, and if the complexities are equal, then the structure of the core,

$$MGCore\big(\langle r,h_2,g_1,h_1,g_1,w,p,a\rangle, MGQRes\big)\ ,$$

is similar to the structure of the abelian decomposition associated with the top level of the Extra PS resolution $ExtraPSRes$ (see Definition 4.12 in [S5] for the complexity of the core resolution). In this part of the first step of the procedure we will also assume that the complexity of the core,

$$MGCore\big(\langle r,h_2,g_1,h_1,g_1,w,p,a\rangle, MGQRes\big)\ ,$$

is strictly smaller than the complexity of the abelian decomposition associated with the top level of the Extra PS resolution. The case of equality in these complexities is treated in Parts 3 and 4 of the first step. In parallel with Proposition 2.4 in [S5], the image of the limit group $Q^1(r,h_2,g_1,h_1,w,p,a)$ in the terminal rigid or solid limit group of the multi-graded resolution, $MGQRes$, is a proper quotient of $Q^1(h_2,g_1,h_1,w,p,a)$, unless the terminal limit group is rigid or solid with respect to the defining parameters $\langle p\rangle$ (and not only with respect to the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$).

PROPOSITION 3. *Let* $MGQRes(t,v,r,h_2,g_1,h_1,w,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1},a)$ *be one of the multi-graded resolutions constructed above. By construction, the limit group* $Q^1(r,h_2,g_1,h_1,,w,p,a)$ *is mapped into the limit group associated with each of the levels of the multi-graded resolution* $MGQRes$*.*

Let $Q^1_{term}(r,h_2,g_1,h_1,w,p,a)$ *be the image of* $Q^1(r,h_2,g_1,h_1,w,p,a)$ *in the terminal (rigid or solid) limit group of* $MGQRes$.

Then the multi-graded resolution $MGQRes$ *can be replaced by two finite collections of multi-graded resolutions, that are all compatible with the top level of the resolution* $ExtraPSRes$ *associated with the Collapse extra PS limit group,* $CollapseExtraPS^1$*, and are all obtained from* $MGQRes$ *by adding at most a single (terminal) level. Furthermore, all the resolutions in these collections are not of maximal complexity.*

We denote each of the resolutions in these collections, $MGQ'Res$.

(i) *In the first (possibly empty) collection of multi-graded resolutions, the image of the subgroup* $Q^1(r,h_2,g_1,h_1,w,p,a)$ *in the terminal limit group of* $MGQ'Res$, $Q^1_{term}(r,h_2,g_1,h_1,w,p,a)$, *is a proper quotient of* $Q^1(r,h_2,g_1,h_1,w,p,a)$.

(ii) *In the second (possibly empty) finite collection of multi-graded resolutions, the terminal limit group of* $MGQ'Res$ *is either a rigid or a solid limit group with respect to the parameter subgroup* $\langle p\rangle$*, i.e. the terminal limit group is rigid or solid with respect to the parameter subgroup* $\langle p\rangle$*, and not only with respect to the multi-grading with respect to the subgroups* $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$*, that was used in the construction of the resolution,* $MGQRes$.

Proof. Identical to the proof of Proposition 2.4 in [S5]. □

By Proposition 3 we can either omit the graded resolution $MGQRes$ from our list of graded resolutions, or we can replace the resolution $MGQRes$ by finitely many resolutions, that for brevity we still denote $MGQRes$, and for each resolution we may assume that either the image of the subgroup $Q^1(r,h_2,g_1,h_1,w,p,a)$ in the terminal graded limit group of $MGQRes$, $Q^1_{term}(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$, or the terminal graded limit group of $MGQRes$ is rigid or solid with respect to the parameter subgroup $P=\langle p\rangle$.

To continue our treatment of multi-graded resolutions that are not of maximal complexity we also need the following immediate lemma, that is similar to Lemma 2.5 in [S5].

LEMMA 4. *Let* $MGQRes(t,v,r,h_2,g_1,h_1,w,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1},a)$ *be one of the multi-graded resolutions in our list, that is not of maximal complexity. Let* $Q^1(r,h_2,g_1,h_1,w,p,a)$, $Q^1(t,v,r,h_2,g_1,h_1,w,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1},a)$, *be the subgroups generated by* $\langle r,h_2,g_1,h_1,w,p,a\rangle$, $\langle t,v,r,h_2,g_1,h_1,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1},a\rangle$, *in correspondence, in the col-*

lapse PS limit group $CollapseExtraPS^1$. *Let*

$$Q_2^1(r,h_2,g_1,h_1,w,p,a), Q_2^1(t,v,r,h_2,g_1,h_1,w,p,a)$$

be the images in the limit group associated with the second level of $MGQRes$, $GQlim_2(t,v,r,h_2,g_1,h_1,w,Base_{2,1}^{1,1},\dots,Base_{2,v_1^1}^{1,1},a)$, *of the subgroups,* $Q^1(r,h_2,g_1,h_1,w,p,a)$ *and* $Q^1(t,v,r,h_2,g_1,h_1,w,p,a)$, *in correspondence.*

Then $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ *is a quotient of* $Q^1(r,h_2,g_1,h_1,w,p,a)$ *and* $Q_2^1(t,v,r,h_2,g_1,h_1,w,p,a)$ *is a proper quotient of* $Q^1(t,v,r,h_2,g_1,h_1,w,p,a)$.

Proof. The lemma is simply a basic property of a multi-graded resolution. □

Suppose that the image of $Q^1(r,h_2,g_1,h_1,w,p,a)$ in the multi-graded limit group associated with the second level of the multi-graded resolution $MGQRes$, $Q_2^1(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$. In this case we use a "fiber product" similar to the one used in the construction of the anvil in the first step of the construction of the tree of stratified sets in [S5]. Note that in applying this fiber product we do not decrease (at times we do increase) the set of virtual proofs that factor through both the modular block, and the Diophantine set, associated with the well-separated multi-graded resolution $MGQRes$.

With the subgroup $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ we associate the graded resolutions that appear in its graded taut Makanin–Razborov diagram with respect to the parameter subgroup $P=\langle p\rangle$,

$$GQRes_1(r,h_2,g_1,h_1,w,p,a),\dots,GQRes_t(r,h_2,g_1,h_1,w,p,a)\,.$$

We continue with each of the graded resolutions $GQRes_j(r,h_2,g_1,h_1,w,p,a)$ in parallel.

If the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with the resolution $GQRes_j(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q_2^1(r,h_2,g_1,h_1,w,p,a)$, we replace the graded resolution $GQRes_j(r,h_2,g_1,h_1,w,p,a)$ by starting part (2) of the the first step with the multi-graded resolution obtained from the top level of the multi-graded resolution $MGQRes$, by replacing its second-level limit group $Q_2^1(t,v,r,h_2,g_1,h_1,w,p,a)$ with the maximal limit groups obtained from all specializations that factor through both $Q_2^1(t,v,r,h_2,g_1,h_1,w,p,a)$ and the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with the graded resolution $GQRes_j(r,h_2,g_1,h_1,w,p,a)$.

If the subgroup generated by $\langle t,v,r,h_2,g_1,h_1,w,p,a\rangle$ in the obtained (one level) resolution, $QRlim'(t,v,r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $CollapseExtraPS^1$, we replace the obtained resolution by starting the first step of our iterative procedure with the limit group $QRlim'(t,v,r,h_2,g_1,h_1,w,p,a)$ instead of the limit group $CollapseExtraPS^1$. Since the resolution $MGQRes$ is not of maximal complexity, by applying Theorem 4.18 in [S5], in analyzing the limit group $QRlim'$ we need to consider only those resolutions in its multi-graded (taut) Makanin–Razborov diagram that are not of maximal possible complexity, in addition to certain closures of maximal complexity resolutions that are themselves not of maximal complexity (see Theorem 4.18 in [S5]).

Hence, for the rest of this part of the procedure we may assume that the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with $GQRes_j(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q_2^1(r,h_2,g_1,h_1,w,p,a)$.

Let $TMGQRes$ be a one-step resolution corresponding to the top level of the multi-graded resolution $MGQRes$. Let $CRes_j(r,h_2,g_1,h_1,w,p,a)$ be the graded resolution obtained from the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the core, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, TMGQRes)$ followed by the graded resolution $GQRes_j(r,h_2,g_1,h_1,w,p,a)$. Note that the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to $CRes_j(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q^1(r,h_2,g_1,h_1,w,p,a)$.

We now treat each of the graded resolutions $GQRes_j$ and their associated resolutions, $CRes_j$, in parallel. Let $PB^2(b_2,p,a)$ be the terminal rigid or solid limit group of the graded resolution $GQRes_j$ (which is also the terminal rigid or solid limit group of its associated resolution $CRes_j$).

The graded resolution $CRes_j(r,h_2,g_1,h_1,w,p,a)$ is composed from the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the core of the multi-graded resolution $TMGQRes$ followed by the resolution $GQRes_j$. In particular, with every QH vertex group in a graded abelian decomposition associated with one of the levels of the graded resolution $CRes_j$, we can naturally associate either a QH vertex group in an abelian decomposition associated with one of the levels of $GQRes_j$, or a finite index subgroup of a QH vertex group in the abelian decomposition associated with (the core resolution of) the multi-graded resolution $TMGQRes$. Therefore, with each QH vertex group of $CRes_j$ we can naturally associate a modular group, which is the modular group of the QH vertex group in $GQRes_j$ or $TMGQRes$ it is associated with. This assignment of modular groups to the QH vertex groups of $CRes_j$ naturally extends to (*reduced*) modular groups associated with each of the levels of the resolution $CRes_j$.

With the graded resolution $CRes_j$, equipped with these reduced modular groups, we associate a finite collection of *framed resolutions*.

Definition 5. *Let $Res(y,p,a)$ be a well-structured, complete graded resolution with reduced modular groups. By construction, with each QH vertex group Q in one of the levels of $Res(y,p,a)$, there is an associated QH subgroup Q', so that Q embeds into Q' as a subgroup of finite index. Let (y_0,p_0,a) be a specialization that factors through the graded resolution $Res(y,p,a)$ with its reduced modular groups, let Q be a QH subgroup in an abelian decomposition associated with one of the levels of $Res(y,p,a)$, and let Q' be the QH subgroup that contains Q as a subgroup of finite index. The specialization (y_0,p_0,a) naturally gives rise to a specialization of the QH vertex Q, that may be extended to some subgroup $\hat{Q}$, $Q < \hat{Q} < Q'$.*

Let $Q_1,\dots,Q_\ell$ be the collection of QH subgroups associated with the various levels of $Res(y,p,a)$, and let $Q'_1,\dots,Q'_\ell$ be the subgroups containing them as subgroups of finite index. Let $\hat{Q}_1,\dots,\hat{Q}_\ell$ be subgroups for which $Q_i < \hat{Q}_i < Q'_i$, for every i, $1 \leq i \leq \ell$. We say that the collection $\hat{Q}_1,\dots,\hat{Q}_\ell$ is a $frame$ for a specialization (y_0,p_0,a), if the specialization (y_0,p_0,a) extends to the subgroups $\hat{Q}_1,\dots,\hat{Q}_\ell$, but not to any subgroup $\tilde{Q}_i < Q'_i$ that properly contains $\hat{Q}_i$, for any i, $1 \leq i \leq \ell$. We denote the extension of the specialization (y_0,p_0,a) to the subgroups $\hat{Q}_1,\dots,\hat{Q}_\ell$, (q_0,y_0,p_0,a), and say that it is a $framed$ $specialization$ (with respect to the frame $\hat{Q}_1,\dots,\hat{Q}_\ell$). We say that a test sequence that factors through $Res(y,p,a)$ (where a test sequence is with respect to the reduced modular groups associated with $Res(y,p,a)$) is framed *with respect to the frame $\hat{Q}_1,\dots,\hat{Q}_\ell$, if each specialization in the test sequence is framed with respect to the given frame.*

Clearly, there are finitely many frames associated with the graded resolution $Res(y,p,a)$. Fixing a frame $\hat{Q}_1,\dots,\hat{Q}_\ell$ associated with the resolution $Res(y,p,a)$, we look at the collection of framed test sequences associated with it. By the techniques presented in [S2], the entire collection of framed test sequences with respect to the given frame $\hat{Q}_1,\dots,\hat{Q}_\ell$, factor through a (canonical) finite collection of graded resolutions obtained from closures of the graded resolution $Res(y,p,a)$ by enlarging each of the subgroups Q_i to be $\hat{Q}_i$, and for each QH vertex group $\hat{Q}_i$ its image in the lower levels of the corresponding closure of the resolution $Res(y,p,a)$. We call the (finite) collection of the graded resolutions obtained from the entire collection of framed test sequences with respect to all possible frames, framed resolutions *associated with the resolution $Res(y,p,a)$. With each framed resolution we naturally associate its $frame$.*

With the graded resolution $CRes_j(r,h_2,g_1,h_1,w,p,a)$ we have associated reduced modular groups. Hence, with $CRes_j$ we can associate a (canonical) finite collection of framed resolutions. We treat the framed resolutions associated with $CRes_j$ in parallel. Let $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$ be such a framed resolution. The top part of $FrmCRes$ corresponds to the (core of the) multi-graded resolution $TMGQRes$, and its bottom part corresponds to the various levels in the resolution $GQRes_j$. If the limit group associated with the top level of $GQRes_j$ in the framed resolution $FrmCRes$ is a proper quotient of the limit group Q_2^1, we replace the framed resolution $FrmCRes$ by starting part (2) of the first step, with the graded resolution obtained from $TMGQRes$ by replacing its terminal limit group $Q_2^1(t,v,r,h_2,g_1,h_1,w,p,a)$ with the finite collection of maximal limit groups obtained from all specializations that factor through both $Q_2^1(t,v,r,h_2,g_1,h_1,w,p,a)$ and the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with the top level of $GQRes_j$ in the framed resolution $FrmCRes$. Hence, for the continuation we may assume that the limit group associated with the top level of $GQRes_j$ in the framed resolution $FrmCRes$ is isomorphic to the limit group $Q_2^1(r,h_2,g_1,h_1,w,p,a)$.

The framed resolution $FrmCRes$ is a well-separated resolution with reduced modular groups. Still, the techniques presented in section 1 of [S2], that generalize Merzlyakov theorem on the existence of formal solutions to AE sentences defined over an arbitrary variety and over well-structured resolutions [S2, 1.18], and use test sequences extensively, generalize from ordinary (well-structured) resolutions to (well-structured) resolutions with reduced modular groups, assuming these resolutions are framed. These are the techniques that enabled us to construct the various bundles associated with the PS resolution, $PSHGHRes$, and start the first step of the sieve procedure. Hence, we are able to repeat the construction of the bundles associated with the PS resolution $PSHGHRes$ we started the sieve procedure with, and associate with the framed resolution $FrmCRes$ a (canonical) finite collection of Non-Rigid and Non-Solid PS resolutions, a collection of Left and Root PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions.

If every test sequence that factors through one of the Extra PS resolutions associated with the framed resolution $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$, factors through a framed resolution with a bigger frame than the one associated with the framed resolution $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$, we exclude this Extra PS resolution from the finite collection of Extra PS resolutions associated with the framed resolution $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$.

At this point we go over the Extra PS resolutions associated with the framed resolutions $FrmCRes$, and with each Extra PS resolution we associate a *developing* resolution, and a finite collection of anvils, in a similar way to what we did in constructing the tree of stratified sets in the second section of [S5].

Let $ExtraPSRes$ be one of the Extra PS resolutions associated with one of the framed resolutions $FrmCRes$. With $ExtraPSRes$ we associate a *developing* resolution, that is set to be the Extra PS resolution $ExtraPSRes$. We denote the developing resolution, $Dvlp(MGQRes)$. We further set the anvils associated with the developing resolution to be the (canonical) finite set of maximal limit quotients of the group obtained as the amalgamated product of the completion of the developing resolution and the completion of the one-step multi-graded resolution, $TMGQRes$, that corresponds to the top level of the multi-graded resolution $MGQRes$, amalgamated along the top part of the developing resolution, which was set to be the resolution induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$ from the top level of $MGQRes$, enlarged by replacing the subgroup associated with the bottom level in the induced resolution with $Q_2^1(r, h_2, g_1, h_1, w, p, a)$. We denote each of the (finitely many) anvils, $Anv(MGQRes)$. Note that the completion of the developing resolution, $Dvlp(MGQRes)$, is canonically mapped into the anvil, $Anv(MGQRes)$.

With each of the Extra PS resolutions (developing resolutions) associated with the framed resolution, $FrmCRes$, we associate finitely many collapse forms (see Lemma 2). As in our treatment of the graded PS resolutions, $PSHGHRes_j$, we started this procedure with, if a valid PS statement can be extended to a specialization that factors through the multi-graded resolution $MGQRes$, then either there exists "generic" valid PS statement, i.e. a test sequence of specializations that factor through one of the framed resolutions, $FrmCRes$, and restrict to valid PS statements, or there exists a valid PS statement that can be extended to a specialization that factors through one of the associated anvils, $Anv(MGQRes)$, and it satisfies the Diophantine conditions specified by one of the collapse forms associated with the corresponding Extra PS resolution.

LEMMA 6. *Suppose that for a specialization p_0 of the defining parameters p, there exists a valid PS statement that can be extended to a specialization that factors through the multi-graded resolution, $MGQRes$:*

$$\big(t, v, r, (h_1^2, g_1^1), \dots, (h_{\nu(ps)}^2, g_{\nu(ps)}^1), h_1, w, p_0, a\big)\,.$$

Then at least one of the following holds:

(1) *There exists a test sequence of specializations that factor through one of the framed resolutions associated with $MGQRes$, $FrmCRes$, and*

for which the specializations of the defining parameters are all p_0, so that the specializations in the test sequence restrict to valid PS statements.

(2) *The valid PS statement can be extended to a specialization that factors through one of the associated anvils, $Anv(MGQRes)$, and satisfies the Diophantine conditions imposed by one of the collapse forms associated with the corresponding Extra PS resolution.*

Proof. Similar to the proof of Theorems 3.6 and 1.33 in [S5]. □

As in our treatment of the PS resolutions $PSHGHRes$, at this point we analyze the set of specializations of the defining parameters $P = \langle p \rangle$ for which there exists a test sequence of specializations that factor through one of the framed resolutions, $FrmCRes$, and restrict to valid PS statements.

PROPOSITION 7. *Let $TSPS(p)$ be the set of specializations p_0 of the defining parameters $P = \langle p \rangle$, for which there exists a test sequence of specializations that factor through one of the framed resolutions, $FrmCRes$, and restrict to valid PS statements. Then $TSPS(p)$ is in the Boolean algebra of AE sets.*

Proof. Identical to the proof of Propositions 3.7 and 1.34 in [S5]. □

By Lemma 6 and Proposition 7, given an Extra PS resolution associated with the framed resolution $FrmCRes$, one of its associated anvils, $Anv(MGQRes)$, and one of its associated collapse forms, we need to continue only with specializations that factor through the anvil, and satisfy the Diophantine conditions imposed by the collapse form, i.e. with each anvil and a collapse form, we need to associate a finite collection of collapse Extra PS limit groups. In order to construct these collapse Extra PS limit groups from the given set of anvils and collapse forms, we first need to associate a canonical collection of auxiliary resolutions and limit groups with the anvil (see Definition 1).

DEFINITION 8 (cf. Definition 2.6 in [S5]). *Recall that the developing resolution, which is always an Extra PS resolution, has the structure of a completed resolution, and the subgroup associated with each level of the developing resolution is naturally mapped into the subgroup associated with the corresponding level in the anvil, $Anv(MGQRes)$.*

With the anvil, $Anv(MGQRes)$, we associate a taut multi-graded Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level 2, with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian *decomposition associated with the limit group that appears in the top level of the anvil, $Anv(MGQRes)$.*

Similarly, with each level i in the anvil, we associate a multi-graded taut Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level $i+1$, The multi-graded diagram is multi-graded with respect to the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in the i-th level in the anvil.

We call each of the resolutions in these multi-graded diagrams a (multi-graded) auxiliary resolution, *and its terminating solid or rigid limit group a (multi-graded)* auxiliary limit group, *which we denote $Aux(MGQRes)$. With each auxiliary resolution we associate its modular groups, that we call* auxiliary modular groups. *In the sequel, we call the auxiliary resolutions associated with the tower containing all the levels up to level 2 (all the levels except the top level),* highest level.

Since an auxiliary limit group is associated with the limit groups that appear in all the levels of the anvil up to level $i+1$, and it is multi-graded with respect to the non-abelian, non-QH vertex groups and edge groups in the abelian decomposition associated with the i-th limit group, it can be naturally extended to the limit group associated with the (ambient) anvil, by sequentially adding to the auxiliary limit group the limit groups that appear in level i and above, according to the (tower-like) structure of the anvil. We call the obtained limit group (which is a quotient of the anvil), the extended auxiliary limit group. *By construction, the anvil, $Anv(MGQRes)$, the developing resolution, as well as the original PS limit group, $PSHGH$, are mapped into the extended auxiliary limit group.*

By construction, the multi-graded auxiliary resolution that terminates in the auxiliary limit group, extends to the graded extended auxiliary resolution *that terminates in the extended auxiliary limit group, and is graded with respect to the subgroup generated by all the limit groups associated with levels 1 down to level i in $ExtraPSRes$.*

QH and abelian vertex groups in the abelian decomposition associated with the limit group, $Aux(ExtraPSRes)$, that is associated with the Extra PS resolution, $ExtraPSRes$, we started the first step with, are considered "formal" along the analysis of a Collapse extra PS limit group, i.e. it is possible to act on them with their modular group and still get a specialization that factors through the corresponding Collapse extra PS limit group. When we construct the auxiliary resolution and modular groups associated

with an anvil, the QH and abelian vertex groups associated with both the previous auxiliary limit group and the newly constructed one are considered "formal" in the same way.

Since the construction of the Collapse extra PS limit groups, from the (extended) auxiliary limit groups and the collapse forms, does not depend on the part of the first step in which the anvil was constructed, we present the (uniform) construction of the Collapse extra PS limit groups at the end of the first step of the procedure, after going through all the various cases that we need to analyze.

So far we have assumed that $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$. Suppose that $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q^1(r,h_2,g_1,h_1,w,p,a)$. In this case we continue to the next level of the multi-graded quotient resolution $MGQRes$. Note that by Corollary 4.16 in [S5], since the multi-graded resolution $MGQRes$ is not of maximal complexity, the (multi-graded) core associated with each of its levels is not of maximal complexity as well. If, for some level j of the multi-graded resolution $MGQRes$, the image of $Q^1(r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with this level, $Q_j^1(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$, then from the highest such level j, we can continue as in case $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$, and associate with the multi-graded resolution $MGQRes$ a finite collection of resolutions composed from the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the top j levels of the multi-graded resolution $MGQRes$, followed by each of the various resolutions in the taut graded Makanin–Razborov diagram of $Q_j^1(r,h_2,g_1,h_1,w,p,a)$. With each such resolution, we (canonically) associate a finite collection of framed resolutions, and with each framed resolution, a canonical collection of Non-Rigid and Non-Solid PS resolutions, a collection of Root and Left PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions. We set each the Extra PS resolutions to be a developing resolution, and with it we associate a (canonical) finite collection of anvils and auxiliary resolutions, precisely as we did in case $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$.

Finally, suppose that for every level j, the image of $Q^1(r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with the j-th level of the multi-graded resolution $MGQRes$, $Q_j^1(r,h_2,g_1,h_1,w,p,a)$, is isomorphic to $Q^1(r,h_2,g_1,h_1,w,p,a)$. In this case, by Proposition 3, the terminal limit group of the multi-graded resolution $MGQRes$, $Q_{term}^1(t,v,r,h_2,g_1,h_1,w,p,a)$, is rigid or solid with respect to the parameter subgroup $P=\langle p\rangle$.

Let $PB^1(b_1,p,a)$ be the terminal rigid or solid limit group of the graded resolution $MGQRes$. We set the graded resolution $CRes(r,h_2,g_1,h_1,w,p,a)$, to be the resolution induced by the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the sequence of core resolutions associated with the various levels of the resolution $MGQRes$, enlarged by setting its terminal limit group to be (the rigid or solid limit group) $PB^1(b_1,p,a)$ (i.e. we amalgamate the terminal limit group of the corresponding induced resolution with the subgroup $PB^1(b,p,a)$).

With the graded resolution $CRes(r,h_2,g_1,h_1,w,p,a)$ we associate a finite (canonical) collection of framed resolutions (see Definition 5). With each of the framed resolutions associated with $CRes(r,h_2,g_1,h_1,w,p,a)$, we associate a (canonical) finite collection of Non-Rigid and Non-Solid PS resolutions, a collection of Root and Left PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions. If every test sequence that factors through an Extra PS resolution associated with the framed resolution $FrmCRes$, factors through a framed resolution with a bigger frame than the one associated with the framed resolution $FrmCRes$, we exclude this Extra PS resolution from the finite collection of PS resolutions associated with the framed resolution $FrmCRes$.

We set each of the Extra PS resolutions associated with one of the framed resolution $FrmCRes$ to be a developing resolution, and with each such developing resolution we associate a (canonical) finite collection of anvils, that we denote $Anv(MGQRes)$, precisely as we did in case $Q_2^1(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$. Note, once again, that by construction the developing resolution is naturally mapped into the anvil. Finally, with each anvil we associate a finite collection of auxiliary resolutions according to the construction presented in Definition 8.

(3) By part (1) we may assume that $Q^1(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the PS limit group $PSHGH$ we started the first step with. part (2) treats the case in which the multi-graded core of the multi-graded resolution $MGQRes$ is not of maximal possible complexity. Hence, the only case left in presenting the first step of our procedure is the case of a multi-graded core, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, MGQRes)$ of maximal possible complexity. In this case, by Theorem 4.12 in [S5], the maximal complexity graded core has the same structure as the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step with.

Conceptually we start the treatment of this case in a similar way to what we did in the first step of the construction of the tree of stratified sets, i.e. we continue to lower levels of the anvil and analyze it in a similar way to what we did with the top level. In parts (1) and (2), we analyzed multi-graded resolutions, $MGQRes$, of the Collapse extra PS limit group, $CollapseExtraPS^1$, that is associated with a collapse form and with an auxiliary resolution of highest level, i.e. an auxiliary resolution associated with the tower containing all the levels in the associated Extra PS resolution up to level 2 (all levels except the top level).

An auxiliary resolution of highest level (Definition 1) is a multi-graded resolution of the subgroup of the Extra PS resolution (associated with the PS resolution $PSHGHRes$ we started this branch of the procedure with), that is associated with all its levels except the top one, with respect to the subgroups which are the non-abelian, non-QH vertex groups in the graded abelian decomposition of $Q^1(r,h_2,g_1,h_1,w,p,a)$. Hence, such a resolution can be extended to a graded resolution of the limit group associated with the entire Extra PS resolution, $ExtraPSRes$, with respect to the subgroup associated with the top level. In the same way, an auxiliary resolution associated with all the levels up to level 3, can be extended to a graded resolution of the limit group associated with the entire Extra PS resolution, with respect to the subgroup associated with the top two levels of it.

Let $(u,v,r,h_2,g_1,h_1,w,p,a)$ be a specialization of the Extra PS resolution, $ExtraPSRes$, and suppose that it can be extended to a specialization that satisfies the Diophantine conditions imposed by the given collapse form (associated with the Collapse extra PS limit group we started this branch with). If in addition, for every Collapse extra PS limit group that is associated with the given collapse form, and with an auxiliary resolutions of highest level, through which such a specialization factors, it factors only through quotient resolutions of maximal complexity of that Collapse extra collapse PS limit group, then by Theorem 4.18 in [S5], the specialization of the Extra PS resolution can be extended to specializations that factor through quotient resolutions of (top part) maximal complexity of Collapse extra PS limit groups associated with the same collapse form and with auxiliary resolutions that are associated with all the levels up to level 3. Therefore, to analyze such specializations, we can replace the maximal complexity quotient resolutions, $MGQRes$, associated with auxiliary resolutions of highest level, with (top part) maximal complexity resolutions of Collapse extra PS limit groups, that are associated with the same collapse form, and with auxiliary resolutions that are associated with all the levels up to level 3. Furthermore, by Theorem 4.19 in [S5], we can further assume that the maximal complexity resolutions associated with auxiliary resolutions that are associated with all the levels up to level 3, are composed from two parts, the top being multi-graded with respect to the formed part of the core resolution, and the bottom being a one step-resolution corresponding to the formed part of the core resolution (see Theorem 4.19 in [S5]).

Hence, to analyze maximal complexity multi-graded resolutions, we first replace the Collapse extra PS limit groups associated with the given collapse form and with auxiliary resolutions of highest level, by those Collapse extra PS limit groups associated with the Extra PS resolution, the given collapse form, and with auxiliary resolutions that are associated with towers containing all the levels up to level 3, i.e. all the levels except the top two. We continue with those Collapse extra PS limit groups in parallel, hence, we will omit their index, and (still) denote the Collapse extra PS limit group we continue with, $CollapseExtraPS^1$.

We start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^1$, with respect to the subgroups, $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$, where the subgroups $Base^{1,1}_{2,j}$, $1 \leq j \leq v^1_1$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$. We still denote these multi-graded resolutions $MGQRes$.

Since in this part we need to analyze specializations that factor through and are taut with respect to maximal complexity multi-graded resolutions of Collapse extra PS limit groups associated with auxiliary resolutions of highest level, as we have already explained, we can continue only with those multi-graded resolutions in the taut Makanin–Razborov diagram of $CollapseExtraPS^1$ that are of maximal complexity, i.e. with a core that has the same structure as the abelian decomposition associated with the the top level of the Extra PS resolution, $ExtraPSRes$.

If part (1) applies to such a multi-graded resolution $MGQRes$, i.e. if the limit group generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in its completion is a proper quotient of the subgroup $Q^1(r,h_2,g_1,h_1,w,p,a)$ we started this branch of the procedure with, we replace this resolution $MGQRes$, by starting the first step of the procedure with the given proper quotient of $Q^1(r,h_2,g_1,h_1,w,p,a)$.

In case the multi-graded resolution $MGQRes$ is of maximal complexity, i.e. the core has the same structure as the abelian decomposition associated with the top level of $ExtraPSRes$, we use the modular groups associated with the abelian decomposition associated with the formed part of

$MGQRes$ to map the subgroup $Q^1(r,h_2,g_1,h_1,w,p,a)$ into its image in the second level of $ExtraPSRes$, that is mapped into $CollapseExtraPS^1$. We now set the subgroups $Base^{1,1}_{3,1},\dots,Base^{1,1}_{3,t^1_1}$ to be the subgroups of $ExtraPSRes$, corresponding to the non-abelian, non-QH vertex groups in the abelian decomposition associated with its second level (alternatively, the factors in the given free decomposition of the auxiliary limit group).

At this point we analyze the terminal limit group of the multi-graded resolution, $MGQRes$, associated with the top level of $ExtraPSRes$, with respect to the subgroups $Base^{1,1}_{3,1},\dots,Base^{1,1}_{3,t^1_1}$, exactly as we analyzed the collapse Extra PS limit group, $CollapseExtraPS^1$, with respect to the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$ in parts (1) and (2), i.e. we associate with the terminal limit group of $MGQRes$ all its multi-graded quotient resolutions with respect to the subgroups, $Base^{1,1}_{3,1},\dots,Base^{1,1}_{3,t^1_1}$ that are its subgroups, and analyze each of the obtained multi-graded quotient resolutions according to parts (1) to (the first part of) (3). If the multi-graded core of such a multi-graded resolution is of maximal possible complexity, and its associated taut structure is identical to the one associated with the second level of $ExtraPSRes$, i.e. if part (3) applies to an obtained quotient multi-graded resolution, we continue in a similar way to our approach in analyzing multi-graded resolutions that their top level is of maximal complexity.

To analyze multi-graded resolutions that are of maximal complexity in their top two levels, we replace the collapse Extra PS limit groups, and analyze collapse Extra PS limit groups associated with the Extra PS resolution, $ExtraPSRes$, and with auxiliary resolutions that are associated with towers containing all the levels up to level 4, i.e. all the levels apart from the top three.

Given such a quotient limit group, we start with its multi-graded taut Makanin–Razborov diagram with respect to the subgroups, $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$. We continue only with resolutions in this diagram that are of maximal complexity and their taut structure is compatible with that of the top level of $ExtraPSRes$. We look at the multi-graded taut Makanin–Razborov diagram of the terminal limit group of such maximal complexity multi-graded resolutions with respect to the subgroups $Base^{1,1}_{3,1},\dots,Base^{1,1}_{3,t^1_1}$ (where the subgroups $Base^{1,1}_{3,j}$ are the non-QH, non-abelian vertex groups in the abelian decomposition associated with the second level of $ExtraPSRes$). Again, we continue only with resolutions that are of maximal complexity, and the taut structure associated with their core is identical to the taut structure associated with the second level of $ExtraPSRes$.

We continue with the terminal limit groups of the obtained multi-graded resolutions (that are assumed to be with maximal complexity cores in their top two parts, and with compatible taut structures). We set the subgroups $Base^{1,1}_{4,1},\dots,Base^{1,1}_{4,r^1_1}$ to be the subgroups of limit groups associated with $ExtraPSRes$, corresponding to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the third level of $ExtraPSRes$ (alternatively, the factors in the given free decomposition of the auxiliary limit group). At this point we analyze the terminal limit group of a maximal complexity resolution (in its top two parts) with respect to the subgroups $Base^{1,1}_{4,1},\dots,Base^{1,1}_{4,r^1_1}$ exactly as we analyzed the Collapse extra PS limit group, $CollapseExtraPS^1$, with respect to the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$, i.e. we associate with the terminal limit group all its multi-graded quotient resolutions with respect to the subgroups,

$$Base^{1,1}_{4,1},\dots,Base^{1,1}_{4,r^1_1}\,,$$

that are its subgroups, and analyze each of the obtained multi-graded quotient resolutions according to parts (1) to (the first part of) (3).

If the core of a multi-graded resolution with respect to the subgroups, $Base^{1,1}_{4,1},\dots,Base^{1,1}_{4,r^1_1}$ is of maximal possible complexity, and its associated taut structure is identical to the one associated with the third level of $ExtraPSRes$, we continue to the next levels of $ExtraPSRes$ precisely in the same way. At each level i, we consider the Collapse extra PS limit groups associated with the given collapse form and with auxiliary resolutions that are associated with the tower containing all the levels up to level $i+1$. Then we analyze the taut Makanin–Razborov diagrams of the limit groups associated with the various levels (from level 1 to level $i-1$), and continue only with those resolutions that are of maximal complexity in all these levels, and the taut structures associated with their core resolutions are identical to those associated with the corresponding levels of the Extra PS resolution, $ExtraPSRes$. Finally we analyze the resolutions in the taut Makanin–Razborov diagram associated with the i-th level according to parts (1), (2), or (the first part of) (3), and continue iteratively.

Let $MGQRes$ be a multi-graded resolution obtained by the above iterative procedure. Suppose that there exists a level for which one of the parts (1)–(2) applies. We first construct a resolution composed from the resolution induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$ from the parts of the resolution $MGQRes$ above the level for which part (1) or (2) applies (i.e. the parts that are of maximal complexity), followed by the graded resolution constructed at that level according to part (1) or (2) (note that the obtained resolution is graded with respect to the parameter subgroup $\langle p\rangle$). With the obtained graded resolution we associate a canonical finite collection of framed resolutions (see Definition 5). With each framed resolution we associate a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we did in part (2). We continue only with Extra PS resolutions that are not "covered" by framed resolutions with a bigger frame. Finally, we set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (still denoted $Anv(MGQRes)$), and with each anvil we associate a finite collection of auxiliary resolutions and limit groups, precisely as we did in part (2) (Definition 8). Note that when a multi-graded resolution (which is a part of the resolution $MGQRes$) is of maximal complexity, an auxiliary resolution is associated only with its terminal level, and not with each of the intermediate levels along the multi-graded resolution.

(4) Suppose that there exists a sequence of multi-graded core resolutions of the multi-graded resolutions constructed by the process described in part (3), that are all of maximal complexity, i.e. each of these multi-graded core resolutions has the same structure (and taut structure) as the graded abelian decomposition associated with the corresponding level of the Extra PS resolution, $ExtraPSRes$, we started the first step of the procedure with. Such sequences of multi-graded resolutions were not treated in parts (1)–(3).

Recall that by Theorem 4.18 of [S5], given a one level (well-separated) resolution, Res, and two Diophantine sets $D_1 \subset D_2$ that are both contained in the Diophantine set associated with the completion of the one level resolution, $Comp(Res)$, the Diophantine sets associated with maximal complexity resolutions associated with (the smaller Diophantine set) D_1 are contained in the union of the Diophantine sets associated with maximal complexity resolutions that are associated with (the bigger Diophantine set) D_2. However, although this "covering" property holds for the Diophantine sets associated with maximal complexity resolutions, it doesn't seem to be necessarily true for the modular blocks associated with these maximal complexity resolutions.

The construction of the core resolution, the induced resolution, and the complexity of a resolution, use the structure of the resolution (or its associated modular block) and its taut structure in an essential way, and not just the structure of the Diophantine set associated with the resolution. For this reason, in order to analyze the specializations that factor through resolutions obtained from a sequence of multi-graded resolutions with maximal complexity cores, we enlarge the set of resolutions (hence, the set of specializations) that we are going to analyze.

We start the analysis with the collection of Collapse extra PS limit groups obtained from the Extra PS resolution, $ExtraPSRes$ (and not with any of its associated auxiliary limit groups), and its (finitely many) associated collapse forms. We still denote each of the obtained Collapse extra PS limit groups, $CollapseExtraPS^1$.

We first analyze the Collapse extra PS limit groups $CollapseExtraPS^1$, using an iterative process which is similar to the one used in part (3). We start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^1$, with respect to the subgroups, $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$, where the subgroups $Base^{1,1}_{2,j}$, $1 \leq j \leq v^1_1$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, and with respect to the formed part of the abelian decomposition associated with the top level of $ExtraPSRes$ (i.e. the collection of abelian and QH vertex groups that appear in the abelian decomposition associated with the top level of $ExtraPSRes$). We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the subgroups, $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the abelian decomposition associated with the top level of $ExtraPSRes$. We (still) denote these (two parts) multi-graded resolutions, for which the second part is one level and has the same structure as the formed part of the abelian decomposition associated with the top level of $ExtraPSRes$, $MGQRes$. We further use the modular groups associated with the formed part of the top level of $ExtraPSRes$, to map the subgroup associated with this formed part in the resolution $MGQRes$, onto its image in the subgroup associated with the second level of $ExtraPSRes$.

We proceed iteratively to the next levels. At each level i, we start with the terminal limit group of the resolution obtained from the top $i-1$ levels, with respect to the subgroups, $Base^{1,1}_{i+1,1},\dots,Base^{1,1}_{i+1,t^1_1}$, where the subgroups, $Base^{1,1}_{i+1,j}$, $1 \leq j \leq t^1_1$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th level of the Extra PS resolution, $ExtraPSRes$, and with respect to the formed part of the abelian decomposition associated with the i-th level of $ExtraPSRes$ (i.e. the collection of abelian and QH vertex groups that appear in the abelian decomposition associated with the i-th level of $ExtraPSRes$). We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the subgroups, $Base^{1,1}_{i+1,1},\dots,Base^{1,1}_{i+1,t^1_1}$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure as the formed part of the abelian decomposition associated with the i-th level of $ExtraPSRes$. We (still) denote the resolutions obtained from the top i levels, $MGQRes$. We further use the modular groups associated with the formed part of the i-th level of $ExtraPSRes$, to map the subgroup associated with this formed part in the resolution $MGQRes$, onto its image in the subgroup associated with the $i+1$ level of $ExtraPSRes$.

Let $MGQRes$ be a multi-graded resolution obtained by the above iterative procedure. With each part of the resolution $MGQRes$, we associate its core resolution with respect to the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$. By Theorem 4.13 in [S5], either there exists a part in which the complexity of its associated core resolution is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level in $ExtraPSRes$, and the complexities of all the core resolutions associated with the parts above it are identical to the complexities of the abelian decompositions associated with the corresponding levels of $ExtraPSRes$, or the complexities of the core resolutions associated with the various parts are identical to the complexities of the abelian decompositions associated with the various levels of $ExtraPSRes$, and the structures of these core resolutions are similar to the structures of the corresponding abelian decompositions in $ExtraPSRes$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core is strictly smaller than the complexity of the corresponding abelian decomposition in $ExtraPSRes$. We set $GRes$ to be the completion of the graded resolution composed from the sequence of multi-graded resolutions, $MGQRes$, constructed by the above iterative procedure. Note that $GRes$ is graded with respect to the parameter subgroup $P = \langle p \rangle$. We set $SCRes(s, r, h_2, g_1, h_1, w, p, a)$ to be the image of the Extra PS resolution, $ExtraPSRes$, in the resolution $GRes$.

With each multi-graded resolution $MGQRes$ associated with a part of $GRes$, we associate its core resolution with respect to (the corresponding image of) the subgroup $\langle r,h_2,g_1,h_1,w,p,a \rangle$, $MGCore(\langle r,h_2,g_1,h_1,w,p,a \rangle, GRes)$. We further replace each of these core resolutions with the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. We set the graded resolution $PenSCRes(u, r, h_2, g_1, h_1, w, p, a)$ to be the resolution composed from the resolutions induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$ from the penetrated core resolutions associated in the various levels with the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$.

With the resolution $PenSCRes$ we associate a finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We further associate with each framed resolution the graded resolution $SCRes$, which we denote $SCRes_1^{1,1}(s_1, r, h_2, g_1, h_1, w, p, a)$ and call the *first sculpted resolution* (of *width* 1), and $SCRes_1^{1,2}(s_1, r, h_2, g_1, h_1, w, p, a)$, and call the *first sculpted resolution* (of width 2). We set each of the Extra PS resolutions to be both a developing resolution, and a *penetrated sculpted resolution*, which we denote $PenSCRes_1^{1,2}(u_1, r, h_2, g_1, h_1, w, p, a)$, and with it we associate a finite collection of anvils, that are set to be the finite collection of maximal limit groups associated with an amalgamation of the resolution $GRes$ with the Extra PS resolution (the developing resolution). Finally, we associate with the anvil the resolution $GRes$ and its associated sequence of core and penetrated core resolutions, associated with the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$ and constructed in the various parts, which we call a $Carrier$, and denote it $Carrier_2^1$ (the top index refers to the step number and the bottom index refers to the *width*, which is the index of the corresponding sequence of core resolutions).

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions constructed by the iterative procedure presented above, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, are all of maximal complexity, i.e. each of these multi-graded core resolutions has the same structure (and taut structure) as the graded abelian decomposition associated with the corresponding level of the Extra PS resolution, $ExtraPSRes$, we started the first step of the procedure with.

We set $SCRes(s, r, h_2, g_1, h_1, w, p, a)$ to be the graded resolution induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$ from the core resolutions of the multi-graded resolutions constructed along the various levels of the process described above.

If every ungraded resolution that factors through the graded resolution $SCRes(s, r, h_2, g_1, h_1, w, p, a)$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the Extra PS resolutions we started the first step of the sieve procedure with, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution.* If there are ungraded resolutions that factor through $SCRes$, but do not factor through any of the Non-Rigid, Non-Solid, Root or Left PS resolutions or the Generic collapse extra PS resolutions associated with the Extra PS resolutions we started the first step of the procedure with; we do the following.

We start by replacing the core resolutions associated with the multi-graded resolutions constructed in the various levels, and with the subgroup, $\langle r, h_2, g_1, h_1, w, p, a\rangle$, by the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. We set the graded resolution $PenSCRes(u, r, h_2, g_1, h_1, w, p, a)$ to be the resolution composed from the resolutions induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$ from the penetrated core resolutions associated in the various levels with the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$. From the corresponding finite sequence of the constructed (maximal complexity) multi-graded resolutions, $MGQRes$, we naturally obtain a graded resolution, $GRes(v, r, h_2, g_1, h_1, w, p, a)$, which is graded with respect to the defining parameter subgroup $P = \langle p\rangle$. By construction, the resolution $PenSCRes$ is embedded into the completion of the graded resolution $GRes$.

With the graded resolution $GRes$, we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution

$PSHGHRes$ we started the first step of the procedure with. We set each of the Extra PS resolutions to be both a developing resolution and its associated anvil (that we still denote, $Anv(MGQRes)$). We further associate with the anvil (Extra PS resolution) the graded resolution $SCRes$, which we call *first sculpted resolution* (of widths 1 and 2), and denote as $SCRes_1^{1,1}(s,r,h_2,g_1,h_1,w,p,a)$ and $SCRes_1^{1,2}(s,r,h_2,g_1,h_1,w,p,a)$, resp., and the graded resolution $PenSCRes$, which we call *penetrated sculpted resolution*, and denote $PenSCRes_1^{1,2}(u,r,h_2,g_1,h_1,w,p,a)$.

As in the previous parts of the first step, we still need to associate with each anvil a finite collection of (extended) auxiliary resolutions. In case the anvil contains a sculpted resolution, we associate with the anvil two finite collections of auxiliary resolutions. The first collection is associated with the various levels of the sculpted resolution. The second collection is associated with the levels of the developing resolution in case there is no carrier, and with the levels of the carrier in case it exists. Both auxiliary resolutions are constructed in a similar way to the constructions presented in Definitions 8 and 1.

DEFINITION 9 (cf. Definition 8). *With the anvil, $Anv(MGQRes)$, we associate two finite collections of auxiliary resolutions. The first collection is associated with each level of the sculpted resolution, and is constructed according to the construction presented in Definition 8. We call the auxiliary resolutions that are associated with the levels of the sculpted resolution, auxiliary resolutions of* width *1.*

With each level of the anvil (which is the developing resolution in case there is no carrier, and the carrier otherwise) we further associate auxiliary resolutions of width *2. With the anvil we associate a taut multi-graded Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level 2. The defining parameters for the construction of the diagram are taken to be the subgroup generated by the subgroup $P = \langle p \rangle$, the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of (the completion of) the sculpted resolution, and the vertex and edge groups in the formed part of the abelian decomposition associated with the top level of the sculpted resolution. Given the defining parameters, the diagram is multi-graded with respect to the defining parameters, and the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in the top level of the anvil, $Anv(MGQRes)$, i.e. the Collapse extra PS limit group, $CollapseExtraPS^1$.*

Similarly, with each level i in the anvil, we associate a multi-graded taut Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level $i + 1$. The defining parameters for the construction of the diagram are taken to be the subgroup generated by the subgroup $P = \langle p \rangle$, the non-abelian, non-QH vertex groups in the abelian decomposition associated with the corresponding level of (the completion of) the sculpted resolution (i.e. the top level of the completion of the sculpted resolution for which its formed part lies in level i or below the anvil, $Anv(MGQRes)$), and the vertex and edge groups in the formed part of the abelian decomposition associated with that level of the sculpted resolution, if this formed part lies in level $i+1$ of the anvil, $Anv(MGQRes)$, or below. Given the defining parameters, the diagram is multi-graded with respect to the defining parameters, and the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in level i of the anvil, $Anv(MGQRes)$.

We call each of the resolutions in these multi-graded diagrams a (multi-graded) auxiliary resolution *of* width *2, and its terminating solid or rigid limit group a (multi-graded)* auxiliary limit group *(of width 2), which we denote $Aux(MGQRes)$. With each auxiliary resolution we associate its modular groups, that we call* auxiliary modular groups. *In the sequel, we call the auxiliary resolutions associated with the tower containing all the levels up to level 2 (all the levels except the top level),* highest level. *As we did in Definition 8, with each auxiliary resolution of width 2 we naturally associate an* extended auxiliary resolution. *By construction, the anvil, $Anv(MGQRes)$, the developing and the sculpted resolution, as well as the original PS limit group, $PSHGH$, are mapped into the extended auxiliary limit group.*

Before we conclude the first step of the sieve procedure, and prepare the data-structure for starting the second step, we need to check that the iterative procedure that was used in the first step, and the anvils constructed along it together with the terminal resolutions, collect all the Collapse extra PS specializations that factor through the initial Extra PS resolutions, $ExtraPSRes$, we started the first step with, and through the Diophantine conditions imposed by their associated collapse forms.

Theorem 10. *Let $(r, h_2, g_1, h_1, w, p, a)$ be a valid PS statement that factors through one of the PS limit group $PSHGH$, can be extended to a specialization that factors and is taut with respect to one of the Extra PS resolutions, $ExtraPSRes$, we started the first step with, and the extended* *specialization satisfies the Diophantine conditions imposed by one of the collapse forms associated with the Extra PS resolutions. Then either:*

(i) *The specialization (r,h_2,g_1,h_1,w,p,a) can be extended to a test sequence of one of the PS resolutions $PSHGHRes$, we started the first step with, that projects to a collection of valid PS statements; or*

(ii) *(r,h_2,g_1,h_1,w,p,a) can be extended to a specialization that factors through either one of the anvils or one of the terminal resolutions constructed along the first step of the sieve procedure.*

Proof. Suppose that (i) does not hold, i.e. suppose that the valid PS statement (r,h_2,g_1,h_1,w,p,a) cannot be extended to a test sequence of one of the PS resolutions $PSHGHRes$, we started the first step with, that projects to a collection of valid PS statements. In this case the valid PS statement can be extended to a specialization that factors through one of the Collapse extra PS limit groups, $CollapseExtraPS^1$, that were analyzed along the first step of the procedure.

If the extended specialization factors through a sequence of multi-graded resolutions, $MGQRes$, so that an anvil was assigned with this sequence of multi-graded resolutions according to one of the parts (1)–(3) of the first step, then the valid PS statement (r,h_2,g_1,h_1,w,p,a) can be extended to a specialization that factors through an anvil constructed according to the relevant part (1)–(3).

Otherwise, the extended specialization must factor through a sequence of multi-graded resolutions, $MGQRes$, constructed according to part (3) of the first step, so that all the core resolutions of the multi-graded resolutions, $MGQRes$, are of maximal possible complexity. Note that with such a sequence of multi-graded resolutions no anvil was assigned in part (3) of the first step. However, by the construction of graded formal limit groups and resolutions, presented in section (3) of [S2], all the specializations that factor through and are taut with respect to a sequence of multi-graded resolutions, $MGQRes$, that were constructed according to part (3) of the first step and are all of maximal possible complexity, must factor through at least one sequence of multi-graded resolutions constructed according to part (4) of the first part of the sieve procedure. This follows since every test sequence with respect to the formed parts of the core resolutions associated with the various levels of the sculpted resolution, must factor through (at least one of) the multi-graded resolutions constructed in part (4) of the first step of the procedure. Hence, in this case, the valid PS statement (r,h_2,g_1,h_1,w,p,a) can be extended to a specialization that factors through either one of the anvils or one of the terminal resolutions constructed according to part (4) of the first step of the sieve procedure. □

The collection of multi-graded resolutions, $MGQRes$, the developing resolutions and (possibly) sculpted resolutions and carriers, and the anvils, $Anv(MGQRes)$, associated with them, and their collections of (extended) auxiliary resolutions, limit groups, and modular groups, together with the data-structure constructed before starting the first step of the procedure, form the *data-structure* obtained as a result of the first step.

At this stage we continue in a similar way to what we did in the initial part of the first step of the procedure. Given an anvil, $Anv(MGQRes)$, and an (extended) auxiliary resolution, we associate with them all their (finitely many) possible collapse forms. With each extra solid specialization associated with the developing resolution (which is an Extra PS resolution by construction), we naturally associate its solid limit group $WPHG$.

Given an (extended) auxiliary resolution and a collapse form, we add variables that are associated with the various solid limit groups $WPHG$ (that are associated with the extra solid specializations in the developing resolution) and their flexible quotients, so that the added variables and the extended auxiliary limit group enable us to express the additional Diophantine conditions imposed by the collapse form (see part (iv) in Lemma 2).

Given the (extended) auxiliary limit group or one of its (finitely many) degenerate quotients (see section 11 in [S1]), and the variables that were added to express the Diophantine conditions imposed by the given collapse form, we look at all the rigid or strictly-solid specializations of the (extended) auxiliary limit group (or its degenerate quotient), and specializations of the additional variables, so that the combined specializations satisfy the Diophantine conditions imposed by the given collapse form.

By our standard method presented in section 5 of [S1], this collection of specializations factor through a canonical (finite) collection of maximal limit groups, which we call *collapse Extra PS limit groups*, and denote

$$CollapseExtraPS^2_1, \dots, CollapseExtraPS^2_d \,.$$

As we have pointed out in constructing the Collapse extra PS limit groups that serve as input to the first step of the sieve procedure, if a specialization of the anvil, $Anv(MGQRes)$, extends to a specialization that factors through one of the Collapse extra PS limit groups, $CollapseExtraPS^2$, and restricts to a rigid or strictly-solid specialization of the associated auxiliary resolution, then the same is true for all the specializations in the same strictly-solid family of the (extended) auxiliary limit group. Hence, as in the analysis applied in the first step of the procedure, in analyzing the Collapse extra PS limit groups associated with the auxiliary resolutions, we consider the non-abelian, non-QH vertex groups and edge groups in the multi-graded abelian JSJ decomposition of the auxiliary limit group, as determined only up to (appropriate) conjugacy, and the abelian and QH vertex groups as "formal", i.e. we are allowed to act on these with their associated modular groups. Adapting this point of view, which is essential along the entire sieve procedure presented in this section (as in the construction of the tree of stratified sets in section 2 of [S5]), replaces the role of restricting to shortest form specializations in the ungraded case (Definition 4.1 in [S4]), and enables us to exclude the variables that belong to lower levels of the Extra PS resolution from taking part in the analysis of the (top part) of the Collapse extra PS limit group, $CollapseExtraPS^2$, i.e. it allows us to get (certain) "separation of variables" (of different levels) in the analysis of Collapse extra PS limit groups (and in analyzing Diophantine sets in general).

The Second Step of the Sieve Procedure

In the first step we constructed finitely many multi-graded resolutions and their associated multi-graded core resolutions, and with each multi-graded resolution we associated a (canonical) finite collection of Non-rigid, Non-solid, Root and Left PS resolutions, a finite collection of Extra PS resolutions, which are set to be the developing resolutions, a finite collection of Generic collapse extra PS resolutions, and a finite collection of anvils. Possibly, if there exists a sequence of multi-graded core resolutions that are all of maximal possible complexity, we have associated with the anvil a sculpted resolution (part (4) in the first step), and enlarged its algebraic "envelope" (that is set to be the new developing resolution or the carrier). With each anvil, we have associated a finite collection of auxiliary resolutions and limit groups, a finite collection of collapse forms, and finally a finite collection of Collapse extra PS limit groups.

In this part we present the second step of our iterative sieve procedure. We start the second step of the sieve procedure with the Collapse extra PS limit groups that are associated with auxiliary resolutions of highest level (and width 1, in case there exists an associated sculpted resolution), and analyze them in parallel. The analysis of such a Collapse extra PS limit group considers (and depends on) the data-structure associated with it, i.e. the (finite) collection of multi-graded resolutions constructed in the previous step, their core resolutions, and their associated developing (and possibly sculpted and carrier) resolutions, the (extended) graded auxiliary resolution and the anvil.

At each step of the iterative sieve procedure we analyze only Collapse extra PS limit groups that are not associated with *terminal resolutions* (see part (4) of the first step). Since we treat the Collapse extra PS limit groups in parallel, we present the second step of the sieve procedure with one of them, that we denote, $CollapseExtraPS^2$, and its associated anvil, $Anv(MGQRes)$. As in the first step of the sieve procedure, our aim is to obtain a strict decrease in either the Zariski closure or the complexity of the core resolution associated with some level of the data structure we construct.

(1) Let $Q^2(r,h_2,g_1,h_1,w,p,a)$ be the graded limit group generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the Collapse extra PS limit group, $CollapseExtraPS^2$, associated with the anvil, $Anv(MGQRes)$. If $Q^2(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of the PS limit group, $PSHGH$, we started the first step with, we continue this branch of the iterative procedure, by starting the first step of the sieve procedure with the graded limit group $Q^2(r,h_2,g_1,h_1,w,p,a)$ instead of the PS limit group $PSHGH$.

(2) At this stage we may assume that $Q^2(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $PSHGH$. At this part we assume that the core of the multi-graded resolution $MGQRes$, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle,MGQRes)$, is of maximal possible complexity, i.e. it has the same structure as the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step with. We set the subgroups, $Base^{2,1}_{2,1},\dots,Base^{2,1}_{2,v^2_1}$, to be the factors in the given free decomposition associated with the auxiliary limit group, $Aux(MGQRes)$ (its construction is presented in Definition 8). Let

$$MGQ^2Res_1\big(f,r,h_2,g_1,h_1,w,Base^{2,1}_{2,1},\dots,Base^{2,1}_{2,v^2_1},a\big),\dots$$
$$\dots,MGQ^2Res_q\big(f,r,h_2,g_1,h_1,w,Base^{2,1}_{2,1},\dots,Base^{2,1}_{2,v^2_1},a\big)$$

be the completions of the multi-graded resolutions in the taut multi-graded Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^2$, with respect to the subgroups $Base^{2,1}_{2,1},\dots,Base^{2,1}_{2,v^2_1}$. We will treat the multi-graded resolutions MGQ^2Res_j in parallel, hence, we omit their index.

Since the Extra PS resolution $ExtraPSRes(v, r, h_2, g_1, h_1, w, p, a)$, we started the first step with, is well separated, with each QH vertex groups in one of the abelian decompositions associated with $ExtraPSRes$, there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the Extra PS resolution $ExtraPSRes$.

Each QH vertex group in the graded abelian decomposition associated with the top level of the Extra PS resolution $ExtraPSRes$ naturally inherits a sequence of abelian decompositions from a multi-graded resolution MGQ^2Res_j. If for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the Extra PS resolution $ExtraPSRes$, we omit the multi-graded resolution MGQ^2Res from the list of completions of resolutions of the anvil, $Anv(MGQRes)$.

By Theorem 4.13 in [S5], the complexity of the core resolution associated with the (second) multi-graded resolution MGQ^2Res, $MGCore(\langle r, h_2, g_1, h_1, w, p, a\rangle, MGQ^2Res)$, is bounded by the complexity of the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step with, and if these complexities are equal, then the structure of the core of MGQ^2Res is identical to the structure of the multi-graded abelian decomposition associated with the top level of $ExtraPSRes$. In this part of the second step of the sieve procedure we will also assume that the complexity of the core of the (second) multi-graded resolution MGQ^2Res is strictly smaller than the complexity of the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step with.

In this case, we treat the (second) multi-graded resolution MGQ^2Res precisely as we treated the multi-graded resolutions $MGQRes$ in the first step of sieve procedure, i.e. according to parts (1)–(2) of the first step.

(3) In this part we may assume that $Q^2(r, h_2, g_1, h_1, w, p, a)$ is isomorphic to the PS limit group $PSHGH$ we started with. We also assume that the complexity of the core resolution, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, MGQRes)$, is strictly smaller than the complexity of the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step with. Let $Base^{2,2}_{2,1}, \dots, Base^{2,2}_{2,v^2_2}$ be the the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the anvil, $Anv(MGQRes)$ (alternatively, the factors in the free decomposition of the auxiliary limit group). Note that the image of the subgroups $Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1}$ (that were used in constructing the multi-graded resolutions $MGQRes$) can be naturally conjugated into the subgroups, $Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2}$. Let

$$MGQ^2Res_1\big(f,r,h_2,g_1,h_1,w,Base^{2,2}_{2,1},\dots,,Base^{2,2}_{2,v^2_2},a\big),\dots$$
$$\dots,MGQ^2Res_q\big(f,r,h_2,g_1,h_1,w,Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2},a\big)$$

be the completions of the resolutions in the taut multi-graded Makanin–Razborov diagram of the limit group associated with the anvil, $Anv(MGQRes)$, with respect to the subgroups $Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2}$. Since we treat these multi-graded resolutions in parallel, we will omit their indices in the sequel.

Let $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ be the subgroup generated by $\langle t,v,r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the (second) multi-graded resolution MGQ^2Res. By construction, $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ is a quotient of the collapse Extra PS limit group, $CollapseExtraPS^1(t,v,r,h_2,g_1,h_1,w,p,a)$, we started the first step of the sieve procedure with. If $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $CollapseExtraPS^1$, we replace the (second) multi-graded resolution MGQ^2Res, by starting the first step of the procedure with the subgroup $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$.

By Theorem 4.18 in [S5], only multi-graded resolutions

$$MGQRes\big(t,v,r,h_2,g_1,h_1,w,Base^{1,1}_{2,1},\dots,Base^{1,1}_{2,v^1_1},a\big)$$

with maximal complexity multi-graded core, i.e. with multi-graded core that have the same structure as the graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step with, can be reference resolutions to those multi-graded resolutions in the taut multi-graded Makanin–Razborov diagram of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ that have maximal complexity multi-graded core (see Theorem 3.7 in [S4] for reference resolutions). Hence, all the specializations of the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ that belong to the Diophantine set associated with the completion of a multi-graded resolution of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ that have maximal complexity multi-graded core, belong to the Diophantine set associated with at least one of the completions of the multi-graded resolutions $MGQRes$ in the taut multi-graded Makanin–Razborov diagram of $CollapseExtraPS^1$ that have maximal complexity core. Since in this part we have assumed that the multi-graded resolution $MGQRes$ associated with the anvil we started the second step with, does not have a maximal complexity multi-graded core resolution, we may replace the second multi-graded resolution MGQ^2Res, by only those multi-graded resolutions in the taut graded Makanin–Razborov diagram of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$, for which their associated multi-graded core are not of maximal complexity. We analyze these multi-graded resolutions according to parts (1)–(2) of the first step of the sieve method.

(4) With the notation of part (3), at this part we assume that the multi-graded resolution $MGQRes$ is not of maximal possible complexity, and that for our given second multi-graded resolution MGQ^2Res, the corresponding subgroup $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the collapse PS limit group we started the first step with.

Since the multi-graded resolution $MGQRes$ that is associated with the anvil, $Anv(MGQRes)$, is well separated, with each QH vertex groups in one the abelian decompositions associated with $MGQRes$, there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the multi-graded resolution $MGQRes$.

Each QH vertex group in the formed part of the core of the graded abelian decomposition associated with the top level of the multi-graded resolution $MGQRes$ naturally inherits a sequence of abelian decompositions from a (second) multi-graded resolution MGQ^2Res. If for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the multi-graded resolution $MGQRes$, we omit the multi-graded resolution MGQ^2Res from the list of completions of (multi-graded) resolutions of the anvil, $Anv(MGQRes)$.

By Theorem 4.14 in [S5], the complexity of the (multi-graded) core of the multi-graded resolution MGQ^2Res, is bounded by the complexity of the core of the multi-graded resolution $MGQRes$. In this part, we further assume that

$$\begin{aligned}&Cmplx\big(MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, MGQ^2Res)\big)\\ &\qquad\qquad < Cmplx\big(MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, TMGQRes)\big)\,,\end{aligned}$$

where $TMGQRes$ is the one level resolution corresponding to the top level of the multi-graded resolution $MGQRes$. The case of maximal complexity core will be treated in the next parts of the second step of the sieve procedure. To treat a (second) multi-graded resolution which is not of maximal possible complexity we need the following two observations, which are similar to Proposition 3 and Lemma 4.

LEMMA 11. *Let MGQ^2Res be one of the resolutions in our list of multi-graded resolutions that is not of maximal possible complexity. Let MGQ^2lim_{term} be the terminal rigid or solid limit group of the multi-graded resolution MGQ^2Res, and let $Q^2_{term}(t,v,r,h_2,g_1,w,p,a)$ be the image of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ in the terminal limit group MGQ^2lim_{term}.*

Then the multi-graded resolution MGQ^2Res can be replaced by two finite collections of multi-graded resolutions, that are all compatible with the top level of the resolution $TMGQRes$ (the top level of $MGQRes$), and are all obtained from MGQ^2Res by adding at most a single (terminal) level. Furthermore, all the resolutions in these collections are not of maximal complexity.

We denote each of the resolutions in these collections, MGQ^2Res'.

(i) *In the first (possibly empty) collection of multi-graded resolutions, the image of the subgroup $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ in the terminal limit group of MGQ^2Res', $Q^2_{term}(t,v,r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$.*

(ii) *In the second (possibly empty) finite collection of multi-graded resolutions, the terminal limit group of MGQ^2Res' is either a rigid or a solid limit group with respect to the parameter subgroup $\langle p\rangle$, i.e. the terminal limit group is rigid or solid with respect to the parameter subgroup $\langle p\rangle$, and not only with respect to the multi-grading with respect to the subgroups $Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2}$, that was used in the construction of the resolution, MGQ^2Res.*

Proof. Identical to the proof of Lemma 2.7 in [S5]. □

By Lemma 11 we can either omit the graded resolution MGQ^2Res from our list of multi-graded resolutions, or we can replace the resolution MGQ^2Res by finitely many resolutions, that for brevity we still denote MGQ^2Res, and for each resolution we may assume that either the image of the subgroup $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ in the terminal graded limit group of MGQ^2Res, $Q^2_{term}(t,v,r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$, or the terminal graded limit group of MGQ^2Res is rigid or solid with respect to the parameter subgroup $P=\langle p\rangle$.

LEMMA 12. *Let MGQ^2Res be one of the resolutions in our list of (second) multi-graded resolutions. Let $Q^2_2(r,h_2,g_1,h_1,w,p,a)$, $Q^2_2(t,v,r,h_2,g_1,h_1,w,p,a)$, and $Q^2_2(f,r,h_2,g_1,h_1,w,p,a)$, be the images of the subgroups $Q^2(r,h_2,g_1,h_1,w,p,a)$, $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$, and $Q^2(f,r,h_2,g_1,h_1,w,p,a)$*

in the limit group,

$$MGQ^2lim_2\big(f,r,h_2,g_1,h_1,w,Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2},a\big)\,,$$

associated with the second level of the multi-graded resolution MGQ^2Res.

Then $Q^2_2(r,h_2,g_1,h_1,w,p,a)$ *is a quotient of* $Q^2(r,h_2,g_1,h_1,w,p,a)$, $Q^2_2(t,v,r,h_2,g_1,h_1,w,p,a)$ *is a quotient of* $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$, *and* $Q^2_2(f,r,h_2,g_1,h_1,w,p,a)$ *is a proper quotient of* $Q^2(f,r,h_2,g_1,h_1,w,p,a)$.

Proof. The claim is simply one of the basic properties of a multi-graded resolution. □

Suppose that $Q^2_2(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$. In this case we continue as we did in part (2) of the first step of the sieve procedure. Let $TMGQ^2Res(f,r,h_2,g_1,h_1,w,Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2},a)$ be the resolution that corresponds to the top level of the multi-graded resolution MGQ^2Res. By Corollary 4.16 in [S5], since the multi-graded resolution MGQ^2Res is not of maximal complexity,

$$Cmplx\big(MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle,TMGQ^2Res)\big)$$
$$< Cmplx\big(MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle,TMGQRes)\big)\,,$$

where $TMGQRes$ is the one level resolution that corresponds to the top level of the multi-graded resolution $MGQRes$.

With $Q^2_2(r,h_2,g_1,h_1,w,p,a)$ we associate the resolutions that appear in its taut Makanin–Razborov diagram. Given a taut graded (Makanin–Razborov) resolution $GRes(r,h_2,g_1,h_1,w,p,a)$ of $Q^2_2(r,h_2,g_1,h_1,w,p,a)$, we associate with it a taut graded resolution $CRes(r,h_2,g_1,h_1,w,p,a)$, constructed from the multi-graded resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the multi-graded core,

$$MGCore\big(\langle r,h_2,g_1,h_1,w,p,a\rangle,TMGQ^2Res\big)\,,$$

followed by the graded resolution $GRes$. As we did in part (2) of the first step of the sieve procedure, with the graded resolution $CRes$ we associate a (canonical) finite collection of framed resolutions $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and with each Extra PS resolution, a finite collection of generic collapse PS resolutions. We set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils. With each anvil we associate a finite collection of (extended) auxiliary resolutions, according to the construction presented in Definition 8, precisely as we did in the corresponding case in part (2) of the first step of the sieve procedure.

Suppose that $Q_2^2(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q^2(r,h_2,g_1,h_1,w,p,a)$, and $Q_2^2(t,v,r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$. In this case we do the following. With the subgroup $Q_2^2(t,v,r,h_2,g_1,h_1,w,p,a)$ we associate the collection of completions of the multi-graded resolutions that appear in its taut multi-graded Makanin–Razborov diagram with respect to the subgroups $Base_{2,1}^{1,1},\dots,Base_{2,v_1^1}^{1,1}$,

$$MGQRes_1(t,v,r,h_2,g_1,h_1,w,Base_{2,1}^{1,1},\dots,Base_{2,v_1^1}^{1,1},a),\dots$$
$$\dots,MGQRes_e(t,v,r,h_2,g_1,h_1,w,Base_{2,1}^{1,1},\dots,Base_{2,v_1^1}^{1,1},a)\,.$$

We continue with each of the multi-graded resolutions $MGQRes_j$ in parallel.

We set the multi-graded resolution

$$DRes_j\big(t,v,r,h_2,g_1,h_1,w,Base_{2,1}^{1,1},\dots,Base_{2,v_1^1}^{1,1},a\big)$$

to be the multi-graded resolution induced by the subgroup $\langle t,v,r,h_2,g_1,h_1,w,p,a\rangle$ from (the corresponding core of) the multi-graded resolution $TMGQ^2Res$, followed by the multi-graded resolution $MGQRes_j$.

If the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the resolution $DRes_j$, $Q_D(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of the PS limit group $PSHGH(r,h_2,g_1,h_1,w,p,a)$ we started with, we replace the multi-graded resolution $MGQRes_j$ by starting a new branch of the sieve procedure with $Q_D(r,h_2,g_1,h_1,w,p,a)$ instead of the PS limit group $PSHGH$.

If the subgroup generated by $\langle t,v,r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the resolution $DRes_j$, $Q_D(t,v,r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of the Collapse extra PS limit group,

$$CollapseExtraPS^1(t,v,r,h_2,g_1,h_1,w,p,a)\,,$$

we started the first step with, we replace the multi-graded resolution $MGQRes_j$ by starting the first step of the procedure with the limit group $Q_D(t,v,r,h_2,g_1,h_1,w,p,a)$ instead of the Collapse extra PS limit group $CollapseExtraPS^1$.

Since the Extra PS resolution, $ExtraPSRes$, we started the first step with, is well separated, with each QH vertex group in one of the abelian decompositions associated with $ExtraPSRes$ there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the Extra PS resolution $ExtraPSRes$.

Each QH vertex group in the graded abelian decomposition associated with the top level of the Extra PS resolution $ExtraPSRes$ naturally inherits a sequence of abelian decompositions from a multi-graded resolution $DRes_j$. If for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the Extra PS resolution $ExtraPSRes$, we omit the multi-graded resolution $MGQRes_j$ from the list of completions of resolutions of the limit group $Q_2^2(t,v,r,h_2,g_1,h_1,w,p,a)$.

Suppose that the core of a multi-graded resolution $MGQRes_j$, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, MGQRes_j)$, or the core of the associated multi-graded resolution $DRes_j$, are of maximal possible complexity, i.e. the core has the same structure as the multi-graded abelian decomposition associated with the top level of the Extra PS resolution, $ExtraPSRes$, we started the first step of the sieve procedure with. By Theorem 4.15 in [S5], if the core of the multi-graded resolution $MGQRes_j$ is of maximal complexity, so is the multi-graded core of the associated multi-graded resolution $DRes_j$, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, DRes_j)$. Hence, by Theorem 4.18 in [S5], every specialization of the PS limit group, $PSHGH(r,h_2,g_1,h_1,w,p,a)$, that belongs to the Diophantine set associated with the multi-graded resolution $DRes_j$, belongs at least one of the Diophantine sets associated with the multi-graded resolutions with maximal complexity core, $MGQRes$, of the Collapse extra PS limit group $Q^1(t,v,r,h_2,g_1,h_1,w,p,a)$, we started the first step of the sieve procedure with. Since the multi-graded resolution $MGQRes$, associated with the anvil we started the second step of the sieve procedure with, is assumed to have a core which is not of maximal complexity, we may omit those multi-graded resolutions of $Q_2^2(t,v,r,h_2,g_1,h_1,w,p,a)$, $MGQRes_j$, for which the core of the associated resolution $DRes_j$ has a maximal complexity core.

Suppose that the multi-graded resolutions $MGQRes_j$ and $DRes_j$ do not have a maximal complexity core resolution. If the image of the subgroup $Q^2(r,h_2,g_1,h_1,w,p,a)$, in the limit group associated with the multi-graded resolution $MGQRes_j$, is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, we associate with it a finite collection of developing resolutions, anvils, and auxiliary resolutions precisely as we did in case the subgroup $Q_2^2(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, i.e. as in case (2) of the first step of the sieve procedure.

Since we may assume that the image of the subgroup, $Q^2(r,h_2,g_1,h_1,w,p,a)$, in the limit group associated with the multi-graded resolution $MGQRes_j$, is isomorphic to $Q^2(r,h_2,g_1,h_1,w,p,a)$, and the core associated with the multi-graded resolution $MGQRes_j$ is not of maximal possible complexity, we can apply the construction that appears in part (2) of the first step of the sieve procedure, and associate with the multi-graded resolution $MGQRes_j$, a finite collection of graded resolutions:

$$GRes_1(r,h_2,g_1,h_1,w,p,a),\dots,GRes_r(r,h_2,g_1,h_1,w,p,a)\,.$$

With each of these graded resolutions $GRes_i(r,h_2,g_1,h_1,w,p,a)$, we associate a graded resolution $CRes_i(r,h_2,g_1,h_1,w,p,a)$, constructed from the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the (corresponding core of the) multi-graded resolution $TMGQ^2Res$ (which is the top part of the multi-graded resolution MGQ^2Res), followed by the graded resolution $GRes_i(r,h_2,g_1,h_1,w,p,a)$. With each graded resolution $CRes_i(r,h_2,g_1,h_1,w,p,a)$ we can naturally associate a finite collection of framed resolutions as we did in part (2) of the first step of the sieve procedure. We continue with each of these framed resolutions in parallel, and denote each of them $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$.

If the limit group generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ associated with a framed resolution $FrmCRes(q,r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, we replace the framed resolution $FrmCRes$ by starting the first step of the sieve procedure with that limit group (which is a proper quotient of the PS limit group we started the first step with). Otherwise, we follow what we did in part (2) of the first step of the sieve procedure, and associate with the framed resolution $FrmCRes$ a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and with each Extra PS resolution, a finite collection of generic collapse PS resolutions. We set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils and auxiliary resolutions, precisely as we did in part (2) of the first step of the sieve procedure.

Suppose that $Q^2_2(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q^2(r,h_2,g_1,h_1,w,p,a)$, and suppose further that $Q^2_2(t,v,r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$. In this case we continue to the next level of the (second) multi-graded quotient resolution MGQ^2Res. Note that by Corollary 4.16 of [S5], since the multi-graded resolution MGQ^2Res is not of maximal complexity, the (multi-graded) core associated with each of its levels is not of maximal complexity as well. If for some level j of the multi-graded resolution, MGQ^2Res, the image of $Q^2(r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with this level, $Q^2_j(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, or the image of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with this level, $Q^2_j(t,v,r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, then from the highest such level j, we can continue as in case $Q_2^2(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, or $Q_2^2(t,v,r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$, in correspondence, and associate with the (second) multi-graded resolution MGQ^2Res a (canonical) finite collection of framed resolutions, developing resolutions (that are set to be Extra PS resolutions), anvils, and auxiliary resolutions, precisely as we did in case $Q_2^2(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(r,h_2,g_1,h_1,w,p,a)$, or $Q_2^2(t,v,r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$.

Finally, suppose that for every level j, the image of $Q^2(r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with the j-th level of the multi-graded resolution MGQ^2Res, $Q_j^2(r,h_2,g_1,h_1,w,p,a)$, is isomorphic to $Q^2(r,h_2,g_1,h_1,w,p,a)$, and for every level j, the image of $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with the j-th level of the multi-graded resolution MGQ^2Res, $Q_j^2(t,v,r,h_2,g_1,h_1,w,p,a)$, is isomorphic to $Q^2(t,v,r,h_2,g_1,h_1,w,p,a)$. In this case, by Lemma 11, the terminal limit group of the multi-graded resolution MGQ^2Res, $Q_{term}^2(u,t,v,r,h_2,g_1,h_1,w,p,a)$, is rigid or solid with respect to the parameter subgroup $P=\langle p\rangle$.

We continue as in part (2) of the first step of the sieve procedure. Let $PB^2(b_2,p,a)$ be the terminal rigid or solid limit group of the multi-graded resolution MGQ^2Res. We set the graded resolution $CRes(r,h_2,g_1,h_1,w,p,a)$, to be the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the sequence of core resolutions associated with the various levels of the multi-graded resolution MGQ^2Res, enlarged by setting its terminal limit group to be (the rigid or solid limit group) $PB^2(b_2,p,a)$ (i.e. we amalgamate the terminal limit group of the corresponding induced resolution with the subgroup $PB^2(b_2,p,a)$).

With the graded resolution $CRes$ we associate a finite (canonical) collection of framed resolutions (see Definition 5), and with each framed resolution we associate a (canonical) finite collection of Non-Rigid and Non-Solid PS resolutions, a collection of Root and Left PS resolutions, Extra PS resolutions, and generic collapse Extra PS resolutions, a developing resolution, a finite collection of anvils, and a finite collection of auxiliary resolutions (constructed according to Definition 8).

(5) By part (1) we may assume that $Q^2(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the limit group $Q^1(r,h_2,g_1,h_1,w,p,a)$ associated with the anvil, $Anv(MGQRes)$. parts (1)–(4) treat all the cases in which the core associated with the second multi-graded resolution, MGQ^2Res, is not of maximal complexity. In this part we assume that the core associated with the multi-graded resolution MGQ^2Res is of maximal complexity, i.e. that the core of the multi-graded resolution MGQ^2Res has the same structure as the core associated with the top level of the multi-graded resolution $MGQRes$. We further assume that there is no sculpted resolution associated with the anvil, $Anv(MGQRes)$. We treat this case by slightly modifying the way we treated the case of a maximal complexity core in part (3) of the first step of the sieve procedure.

In parts (1)–(4), we have analyzed multi-graded resolutions, MGQ^2Res, of the Collapse extra PS limit group, $CollapseExtraPS^2$, that is associated with a collapse form and with an auxiliary resolution of highest level, i.e. an auxiliary resolution associated with the tower containing all the parts in the associated anvil, $Anv(MGQRes)$, up to part 2 (all parts except the top part).

To analyze specializations of the PS limit group $PSHGH$, that belong to the Diophantine set associated with such a Collapse extra PS limit group, $CollapseExtraPS^2$, and belong only to Diophantine sets associated with maximal complexity multi-graded resolutions, MGQ^2Res, we first replace the Collapse extra PS limit groups associated with the given collapse form and with auxiliary resolutions of highest level, by those Collapse extra PS limit groups associated with the Extra PS resolution, the given collapse form, and with auxiliary resolutions that are associated with towers containing all the parts up to part 3, i.e. all the parts except the top two. We continue with those Collapse extra PS limit groups in parallel, hence, we will omit their index, and (still) denote the Collapse extra PS limit group we continue with, $CollapseExtraPS^2$.

As we did in part (3) of the first step, we start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^2$, with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top part of the anvil, $Anv(MGQRes)$. We still denote these multi-graded resolutions MGQ^2Res.

Since in this part we need to analyze specializations that factor through and are taut with respect to maximal complexity multi-graded resolutions of Collapse extra PS limit groups, we continue only with those multi-graded resolutions in the taut Makanin–Razborov diagram of $CollapseExtraPS^2$ that are of maximal complexity, i.e. with a core that has the same structure as the core associated with the top part of the resolution, $MGQRes$.

If part (1) applies to such a multi-graded resolution MGQ^2Res, i.e. if the limit group generated by $\langle r, h_2, g_1, h_1, w, p, a\rangle$ in its completion is a proper quotient of the subgroup $Q^1(r, h_2, g_1, h_1, w, p, a)$ we started this branch of the procedure with, we replace this resolution MGQ^2Res, by starting the first step of the procedure with the given proper quotient of $Q^1(r, h_2, g_1, h_1, w, p, a)$.

If the core of the top part of $MGQRes$, the resolution constructed in the first step of the procedure and associated with the anvil, $Anv(MGQRes)$, is not of maximal complexity, and the limit group generated by $\langle t, v, r, h_2, g_1, h_1, w, p, a\rangle$ in the completion of MGQ^2Res, is a proper quotient of the Collapse extra PS limit group we started this branch of the first step with, $CollapseExtraPS^1$, we replace this resolution, MGQ^2Res, by starting the first step of the procedure with the given proper quotient of $CollapseExtraPS^1$, and analyze only the resolutions that appear in its taut multi-graded Makanin–Razborov diagram that are not of maximal complexity (according to part (2) of the first step).

In case the core of the multi-graded resolution MGQ^2Res is of maximal complexity, i.e. the core has the same structure as the core of the top part of $MGQRes$, we map the formed part of the core of MGQ^2Res into the subgroup of MGQ^2Res that correspond to its image in the second part of $MGQRes$. At this point we analyze the terminal limit group of the multi-graded resolution, MGQ^2Res, with respect to the factors in the given free decomposition of the auxiliary limit group, $Aux(MGQRes)$, exactly as we analyzed the Collapse extra PS limit group, $CollapseExtraPS^2$ in parts (1)–(4). If the multi-graded core of such a multi-graded resolution is of maximal possible complexity, and its associated taut structure is identical to the one associated with the second part of $MGQRes$, i.e. if part (5) applies to an obtained quotient multi-graded resolution, we continue in a similar way to our approach in analyzing multi-graded resolutions that their top part is of maximal complexity (cf. part (3) in the first step of the procedure).

At each part i, we consider the Collapse extra PS limit groups associated with the given collapse form and with auxiliary resolutions that are associated with the tower containing all the parts up to part $i+1$. Then we analyze the taut Makanin–Razborov diagrams of the limit groups associated with the various parts (from part 1 to part $i-1$), and continue only with those resolutions that are of maximal complexity in all these parts, and the taut structures associated with their core resolutions are identical to those associated with the corresponding parts of the resolution, $MGQRes$.

Finally we analyze the resolutions in the taut Makanin–Razborov diagram associated with the i-th part according to parts (1)–(4), or (the first part of) (5), and continue iteratively.

Let MGQ^2Res be a multi-graded resolution obtained by the above iterative procedure. Suppose that there exists a level for which one of the parts (1)–(4) applies. We first construct a resolution composed from the resolution induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$ from the parts of the resolution MGQ^2Res above the level for which parts (1)–(4) apply (i.e. the parts that are of maximal complexity), followed by the graded resolution constructed at that level according to parts (1)–(4) (note that the obtained resolution is graded with respect to the parameter subgroup $\langle p \rangle$). With the obtained graded resolution we associate a canonical finite collection of framed resolutions, a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we did in part (4). We continue only with Extra PS resolutions that are not "covered" by framed resolutions with bigger frame. Finally, we set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (still denoted $Anv(MGQ^2Res)$) precisely as we did in part (4), and auxiliary resolutions (according to the construction presented in Definition 8).

Suppose that the sequence of multi-graded core resolutions of the multi-graded resolutions constructed by the process described above, are all of maximal complexity, i.e. each of these multi-graded core resolutions have the same structure as the core associated with the corresponding level of the developing resolution, in the first step of the procedure, which means that none of the parts (1)–(4) applies to any of these multi-graded resolutions. In this case we examine the structure of the corresponding developing resolution. The developing resolution is a framed resolution of a resolution built from a sequence of resolutions induced from corresponding core resolutions. Each of the induced resolutions is a resolution induced by the (image of the) subgroup $\langle r, h_2, g_1, h_1, w, a \rangle$ from the corresponding core resolution, and with each level of the induced resolution there is an associated (framed) multi-graded abelian decomposition (see section 3 of [S4] for the construction of the induced resolution, and Definition 5 for the construction of framed resolutions).

PROPOSITION 13. *Suppose that all the core resolutions associated with the multi-graded resolutions used for the construction of the developing* *resolution are of maximal possible complexity. Let* $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ *be the subgroup generated by the closure of the developing resolution in the anvil* $Anv(MGQRes)$*. From each of the core resolutions associated with the multi-graded resolutions used to construct the developing resolution, there is a resolution induced by the (image of the) subgroup* $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$.

Then either the structure of the resolution composed from the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ *from the core resolutions associated with the various multi-graded resolutions used to construct the developing resolution, is identical to the structure of the developing resolution we started the second step with, or there exists some level* j*, so that the structure of the graded abelian decompositions associated with the resolutions induced by the subgroup* $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ *above level* j *are identical to the structure of graded abelian decompositions associated with the corresponding levels of the developing resolution, and in level* j*, either the number of factors in the (graded) free decomposition associated with the graded abelian decomposition associated with the resolution induced by* $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ *is strictly smaller than the number of factors in the corresponding (graded) free decomposition associated with the corresponding level of the developing resolution, and in case of equality in the number of factors, the complexity of the graded abelian decomposition associated with the resolution induced by* $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ *is strictly smaller than the complexity of the graded abelian decomposition associated with that level in the developing resolution.*

Proof. Identical to the proof of Proposition 4.11 of [S4]. □

If the structure of the resolution composed from the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the various core resolutions associated with the multi-graded resolutions used to construct the developing resolution in the second step of the procedure, is not identical to the structure of the developing resolution associated with the anvil $Anv(MGQRes)$, Proposition 13 implies that there exists some level j for which the structure of the graded abelian decomposition associated with the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ above level j are identical to the structure of the graded abelian decompositions associated with the corresponding levels in the developing resolutions we started the second step with, and in level j, either the number of factors in the graded free decomposition associated with the graded abelian decomposition associated with the resolution induced by $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the number of factors in the corresponding graded free decomposition associated with the corresponding level in the developing resolution we started the second step with, and in case of equality in the number of factors, the complexity of the graded abelian decomposition associated with the resolution induced by $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the complexity of the graded abelian decomposition associated with the corresponding level of that developing resolution. In this case we do the following.

With the graded resolution constructed from the various resolutions induced by the subgroups $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the core resolutions associated with the various multi-graded resolutions constructed along the various levels of the second step of the sieve procedure, we associate a (canonical) finite collection of framed resolutions (Definition 5). With each framed resolution we associate a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions and generic collapse PS resolutions. Finally, we set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils and graded auxiliary resolutions and limit groups, according to the construction presented in Definition 8.

(6) Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions constructed in part (5) are all of maximal possible complexity, and the structure of the resolution composed from the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the various core resolutions associated with the multi-graded resolutions used to construct the developing resolution in the second step of the procedure, has the same structure as that of the developing resolution associated with the anvil $Anv(MGQRes)$. In this case we modify the analysis that was applied in part (4) of the first step.

We start the analysis, with the collection of Collapse extra PS limit groups obtained from the anvil, $Anv(MGQRes)$ (and not with any of its associated auxiliary limit groups), and its (finitely many) associated collapse forms. We still denote each of the obtained Collapse extraPS limit groups, $CollapseExtraPS^2$.

We first analyze the Collapse extra PS limit groups, $CollapseExtraPS^2$, using an iterative process which is similar to the one used in part (5). We start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^2$, with respect to the subgroups $Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v^2_2}$, where the subgroups $Base^{2,2}_{2,j}$, $1\leq j\leq v^2_2$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top part of the anvil, $Anv(MGQRes)$, and with respect to the formed part of the core resolution, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle MGQRes)$, associated with the top level of the anvil, $Anv(MGQRes)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the subgroups $Base^{2,2}_{2,1},\dots,Base^{2,2}_{2,v_2^2}$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the core, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle,MGQRes)$. We (still) denote these (two parts) multi-graded resolutions, for which the second part is one level and has the same structure as the formed part of the core resolution associated with the top part of $Anv(MGQRes)$, MGQ^2Res. We further use the modular groups associated with the formed part of the core resolution associated with the top part of the anvil, $Anv(MGQRes)$, to map the subgroup associated with this formed part in the resolution MGQ^2Res, onto its image in the subgroup associated with the second level of $Anv(MGQRes)$.

We proceed iteratively to the next levels. At each level i, we start with the Makanin–Razborov diagram of the terminal limit group of the resolution obtained from the top $i-1$ levels, with respect to the subgroups, $Base^{2,2}_{i+1,1},\dots,Base^{2,2}_{i+1,t_2^2}$, where the subgroups $Base^{2,2}_{i+1,j}$, $1\leq j\leq t_2^2$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th part of the anvil, $Anv(MGQRes)$, and with respect to the formed part of the core resolution associated with the i-th part of $Anv(MGQRes)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the subgroups, $Base^{2,2}_{i+1,1},\dots,Base^{2,2}_{i+1,t_2^2}$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the core resolution associated with the i-th level of the anvil, $Anv(MGQRes)$. We (still) denote the resolutions obtained from the top i levels, MGQ^2Res. We further use the modular groups associated with the formed part of the core associated with the i-th part of $Anv(MGQRes)$, to map the subgroup associated with this formed part in the resolution MGQ^2Res, onto its image in the subgroup associated with the $i+1$ level of $Anv(MGQRes)$.

The outcome of the above terminating procedure is a (telescopic) sequence of multi-graded resolutions, that we (still) denote MGQ^2Res. Let $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$ be the subgroup generated by the image of the developing resolution, $Dvlp(q, r, h_2, g_1, h_1, w, p, a)$, associated with the anvil, $Anv(MGQRes)$, and the elements associated with the Diophantine conditions imposed by the given collapse form in the Collapse extra PS limit group, $CollapseExtraPS^2$. With each level of a multi-graded resolution constructed in one of the parts, MGQ^2Res, we associate its core resolution with respect to the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$, and the resolution induced from this (one level) core resolution by the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$. The collection of these induced resolutions, associated with the various levels of the multi-graded resolution MGQ^2Res, gives rise to a resolution (that is embedded in the multi-graded resolution MGQ^2Res), that we denote MGQ^2Res_c, of the image of the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$, that is canonically associated with MGQ^2Res. We denote the graded resolution (with respect to the defining parameters $P = \langle p \rangle$) obtained as the compositions of these (induced) resolutions, MGQ^2Res_c, associated with the (telescopic) sequence of multi-graded resolutions MGQ^2Res, $GRes_2(e, c, r, h_2, g_1, h_1, w, p, a)$. By Proposition 13, we may iteratively repeat this construction of induced resolutions with the subgroup associated with the completion of the obtained resolution $GRes_2$, until we obtain a graded resolution, that we still denote $GRes_2$, which is embedded in the sequence of completions of the multi-graded resolutions MGQ^2Res.

With each resolution MGQ^2Res_c associated with a multi-graded resolution MGQ^2Res, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$. By Proposition 13 and Theorem 4.13 in [S5], either there exists a resolution MGQ^2Res_c for which the complexity of its associated core resolution (with respect to the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level of the developing resolution, $Dvlp$, that is associated with the anvil, $Anv(MGQRes)$, and the complexities of all the core resolutions associated with the resolutions MGQ^2Res_c that are associated with the parts above it, are identical to the complexities of the abelian decompositions associated with the corresponding levels of the developing resolution, $Dvlp$, or the complexities of the core resolutions associated with the various resolutions MGQ^2Res_c are all identical to the complexities of the abelian decompositions associated with the corresponding levels of $Dvlp$, and the structures of these core resolutions are similar to the structures of the corresponding abelian decompositions in $Dvlp$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core of MGQ^2Res_c (with respect to the image of the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$) is strictly smaller than the complexity of the abelian decomposition that is associated with the corresponding level of the developing resolution, $Dvlp$. We set $SCRes(s,r,h_2,g_1,h_1,w,p,a)$ to be the image (in the sequence of completions of the resolutions MGQ^2Res) of the developing resolution, $Dvlp(q,r,h_2,g_1,h_1,w,p,a)$, which is associated with the anvil, $Anv(MGQRes)$. We replace the core resolutions associated with the resolutions MGQ^2Res_c, constructed in the various parts, and with the image of the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, by the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. Then we study the structure of the graded resolution composed from resolutions induced by the (images of the) completion of the developing resolution, $Dvlp$, from the corresponding penetrated core resolutions.

We set the graded resolution, $GRes_1(q,r,h_2,g_1,h_1,w,p,a)$, to be the resolution composed from the resolutions induced by the (images of the) subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the penetrated core resolutions associated in the various parts with that completion. With the resolution $GRes_1$ we associate a finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We further associate with each framed resolution the graded resolution $SCRes$, which we denote $SCRes_1^{2,1}(s_1,r,h_2,g_1,h_1,w,p,a)$ and call the *first sculpted resolution* (of *width* 1). and $SCRes_1^{2,2}(s_1,r,h_2,g_1,h_1,w,p,a)$, and call the *first sculpted resolution* (of width 2). With it we associate a finite collection of developing resolutions (that are set to be the Extra PS resolutions), where each of the developing resolutions is also set to be a *penetrated sculpted resolution*, which we denote $PenSCRes_1^{2,2}(u_1,r,h_2,g_1,h_1,w,p,a)$, and a finite collection of anvils, that we denote $Anv(MGQ^2Res)$. Finally, we associate with the anvil the sequence of core resolutions, associated with the sequence of multi-graded resolutions, MGQ^2Res, and the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$, and the resolutions, MGQ^2Res_c, which we call a

Carrier, and denote it $Carrier_2^2$ (the top index refers to the step number and the bottom index refers to the *width*, which is the index of the corresponding sequence of core resolutions), and the sequence of core resolutions associated with the resolutions MGQ^2Res_c, and the images of the completion of $Dvlp$, that are embedded in the carrier. With each anvil we further associate a finite collection of auxiliary resolutions, using a construction that generalizes the one presented in Definition 9.

DEFINITION 14 (cf. Definition 9). *With the anvil, $Anv(MGQ^2Res)$, we associate two finite collections of auxiliary resolutions, in a similar way to the auxiliary resolutions constructed in Definition 9. The first collection is associated with each level of the (first) sculpted resolution according to the construction presented in Definition 8. We call the auxiliary resolutions that are associated with the levels of the sculpted resolution, auxiliary resolutions of* width *1.*

With each level of the anvil, $Anv(MGQ^2Res)$, we further associate auxiliary resolutions of width *2. With the anvil we associate a taut multi-graded Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level 2. The defining parameters for the construction of the diagram are taken to be the subgroup generated by the subgroup $P = \langle p \rangle$, the non-abelian, non-QH vertex groups in the abelian decomposition associated with the bottom level of the top part of the anvil, $Anv(MGQRes)$, which is the anvil with which the (first) sculpted resolution was associated as a developing resolution (note that the top part of $Anv(MGQRes)$ is the part associated with the top part of the (first) sculpted resolution), and the vertex and edge groups in the formed part of the core resolution associated with the top part of the anvil, $Anv(MGQRes)$. Given the defining parameters, the diagram is multi-graded with respect to the defining parameters, and the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in the top level of the anvil, $Anv(MGQ^2Res)$, i.e. the Collapse extra PS limit group, $CollapseExtraPS^2$.*

Similarly, with each level i in the anvil, we associate a multi-graded taut Makanin–Razborov diagram of the limit group associated with the tower that contains all the levels up to level $i+1$. The defining parameters for the construction of the diagram are taken to be the subgroup generated by the subgroup $P = \langle p \rangle$, the non-abelian, non-QH vertex groups in the abelian decomposition associated with the bottom level of the corresponding part *of the anvil, $Anv(MGQRes)$ (i.e. the top part of $Anv(MGQRes)$ for which the formed part of its core resolution lies in level i or below of the anvil, $Anv(MGQ^2Res)$), and the vertex and edge groups in the formed part of the core resolution associated with that part, if this formed part lies in level $i+1$ of the anvil, $Anv(MGQ^2Res)$, or below. Given the defining parameters, the diagram is multi-graded with respect to the defining parameters, and the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in level i of the anvil, $Anv(MGQ^2Res)$.*

We call each of the resolutions in these multi-graded diagrams a (multi-graded) auxiliary resolution *of* width *2, and its terminating solid or rigid limit group a (multi-graded)* auxiliary limit group *(of width 2), which we denote $Aux(MGQRes)$. With each auxiliary resolution we associate its modular groups, that we call* auxiliary modular groups. *In the sequel, we call the auxiliary resolutions associated with the tower containing all the levels up to level 2 (all the levels except the top level),* highest level. *As we did in Definition 8, with each auxiliary resolution of width 2 we naturally associate an* extended auxiliary resolution. *By construction, the anvils, $Anv(MGQ^2Res)$ and $Anv(MGQRes)$, the developing and the sculpted resolution, as well as the original PS limit group, $PSHGH$, are mapped into the extended auxiliary limit group.*

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions MGQ^2Res_c, that are associated with the multi-graded resolutions, MGQ^2Res, constructed by the iterative procedure presented above, and the images of the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, are all of maximal complexity, i.e. each of these multi-graded core resolutions have the same structure (and taut structure) as the abelian decomposition associated with the corresponding level in the developing resolution $Dvlp$, that is associated with the anvil, $Anv(MGQRes)$.

We set $SCRes(s, r, h_2, g_1, h_1, w, p, a)$ to be the graded resolution (graded with respect to the parameter subgroup $P = \langle p\rangle$), composed from the resolutions induced by the subgroup $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$ from the various core resolutions associated with the multi-graded resolutions, MGQ^2Res_c (with respect to the image of the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$).

If every ungraded resolution that factors through the graded resolution $SCRes$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the developing (Extra PS) resolutions we started the second step of the procedure with, or through a framed resolution, $FrmSCRes$, associated with $SCRes$, where the frame associated with the framed resolution $FrmSCRes$ strictly contains the frame associated with the graded resolution $SCRes$, we call the sequence of multi-graded resolutions from which the graded resolution $SCRes$ was constructed, a *terminal resolution*, and do not continue with it to the next step of the procedure. Otherwise, we do the following.

We replace the core resolutions associated with the multi-graded resolutions MGQ^2Res_c and with the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$, by the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. We denote the graded resolution composed from resolutions induced by the (images of the) subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the corresponding penetrated core resolutions, $GRes_1$.

With the graded resolution $GRes_2$ we associate a (canonical) finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We further associate with the graded resolution $GRes_2$ the graded resolution $SCRes$ which we call *first sculpted resolution*, and denote both as $SCRes_1^{2,1}(s,r,h_2,g_1,h_1,w,p,a)$ and $SCRes_1^{2,2}(s,r,h_2,g_1,h_1,w,p,a)$, and the resolution $GRes_1$ which we call *first penetrated sculpted resolution*, and denote $PenSCRes_1^{2,2}(u_1,r,h_2,g_1,h_1,w,p,a)$. Finally, we set each of the Extra PS resolutions to be a developing resolution, which we denote $Dvlp(q,r,h_2,g_1,h_1,w,p,a)$, and with it we associate a finite collection of anvils. With each anvil we associate a finite collection of auxiliary resolutions of width 1 and 2, according to Definition 14.

Before we continue to part (7) of the second step of the sieve procedure, we present a basic property of framed resolutions of graded resolutions composed from resolutions induced from penetrated core resolutions, that is used repeatedly in the sequel.

PROPOSITION 15. *Let Q be a QH vertex group in one of the abelian decompositions associated with the various levels of the first penetrated sculpted resolution $GRes_1$, and suppose that Q is conjugate to a subgroup of finite index in a QH subgroup Q' in an abelian decomposition associated with one of the levels of the associated resolution MGQ^2Res. Suppose that*

Q is not a QH vertex group in the (multi-graded) core resolution associated with MGQ^2Res, $MGCore(\langle r, h_2, g_1, h_1, w, p, a\rangle, MGQ^2Res)$.

Let $FrmGRes$ be a framed resolution associated with the graded resolution $GRes_1$. Then the QH subgroup containing Q in the framed resolution $FrmGRes$ is Q itself, i.e. Q is not properly contained in a QH vertex group in the framed resolution $FrmGRes$.

Proof. By Theorem 4.20 in [S5], in a penetrated core resolution, each of the QH vertex groups that do not belong to the formed part of the core resolution associated with the penetrated core, is mapped isomorphically into the limit group associated with the terminal level of the penetrated core. Furthermore, each of the abelian decompositions associated with the various levels of the penetrated core, gives rise to a corresponding abelian decomposition of the limit group associated with its terminal level. Hence, the QH vertex group Q is mapped onto a QH vertex group Q_t in some abelian decomposition of the limit group associated with the terminal level of the penetrated core. Since the limit group associated with the terminal level of the penetrated core is a free product of the formed part of the core resolution with some additional finitely generated (f.g.) free group, in any framed resolution of the resolution $GRes_1$, non-trivial elements of the subgroup Q_t that do not have non-trivial roots in Q_t, do not have non-trivial roots in the subgroup associated with the framed resolution. If, in a framed resolution $FrmGRes$ of $GRes_1$, the QH vertex group Q in $GRes_1$ is a finite index subgroup in a QH vertex group $\hat{Q}$ in the framed resolution $FrmGRes$, and $\hat{Q}$ properly contains Q, then there are non-trivial elements in Q that do not have non-trivial roots in Q but have non-trivial roots in $\hat{Q}$. Hence, there are non-trivial elements in Q_t that do not have non-trivial roots in Q_t but have non-trivial roots in the limit group associated with the framed resolution $FrmGRes$, a contradiction. Therefore, the QH vertex group Q is not properly contained in a QH vertex group $\hat{Q}$ in a framed resolution of the resolution $GRes_1$. □

(7) Parts (1)–(4) treat all the cases in which the core associated with the second multi-graded resolution MGQ^2Res is not of maximal complexity. Parts (5) and (6) treat the cases in which the core associated with the second multi-graded resolution MGQ^2Res is of maximal complexity, but there is no sculpted resolution associated with the anvil, $Anv(MGQRes)$.

In this part we treat the remaining case. We assume that there exists a sculpted resolution associated with the anvil, $Anv(MGQRes)$, i.e. that the anvil, $Anv(MGQRes)$, was constructed according to part (4) of the first step of the procedure.

According to Definition 9, if a sculpted resolution is associated with an anvil, $Anv(MGQRes)$, then a finite collection of auxiliary resolutions of both width 1 and width 2 is associated with it. We start by analyzing Collapse extra PS limit groups associated with highest level auxiliary resolutions of width 1. Parts (1)–(4) already analyze multi-graded resolutions of such Collapse extra PS limit groups, MGQ^2Res, that are not of maximal complexity. To analyze specializations that factor through maximal complexity resolutions, MGQ^2Res, we continue analyzing Collapse extra PS limit groups associated with width 1 auxiliary resolutions according to part (5).

If there is no sequence of multi-graded resolutions, MGQ^2Res, obtained using the iterative procedure presented in part (5), that do all have core resolutions (with respect to the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$), that are of maximal possible complexity, we have concluded the second step of the sieve procedure. Otherwise, we continue using the procedures that are presented in part (4) of the first step, and part (6) of the second step.

We continue the analysis by replacing the Collapse extra PS limit groups that were associated with auxiliary resolutions of width 1, by those associated with auxiliary resolutions of width 2 (see Definition 9).

We start with Collapse extra PS limit groups, $CollapseExtraPS^2$, associated with highest level auxiliary resolutions of width 2, and we analyze them in parallel. We apply parts (1)–(4) to study limit groups and core resolutions associated with the top level of the carrier or the developing resolution (depending whether a carrier was associated with the anvil, $Anv(MGQRes)$, in part (4) of the first step), i.e. the core associated with the image of $CollapseExtraPS^1$ in $CollapseExtraPS^2$, and with multi-graded resolutions of such Collapse extra PS limit group, $CollapseExtraPS^2$, with respect to the various factors in the associated (width 2) auxiliary limit group (which, according to Definition 9, is assumed to be rigid or solid multi-graded limit group, with respect to the (multi) parameter subgroups which are the non-QH, non-abelian vertex groups and edge groups in the (given) multi-graded abelian decomposition associated with the limit group that appears in the top level of the anvil, $Anv(MGQRes)$, the various vertex and edge groups in the abelian decomposition associated with the formed part of the core resolution that is associated with the top level of the sculpted resolution, and the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of (the completion of) the sculpted resolution).

Suppose that one of the parts (1)–(4) applies to such a multi-graded resolution (that we still denote MGQ^2Res), and to the limit group associated with the carrier or the developing resolution (the image of $CollapseExtraPS^1$). In this case the construction presented in the parts (1)–(4) that applies, terminates in a limit group $MGTerm_1$, which is a rigid or solid multi-graded limit group with respect to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of (the completion of) the sculpted resolution, and the vertex groups in the abelian decomposition associated with the formed part of the core resolution that is associated with the top level of the sculpted resolution.

We continue with the multi-graded resolutions of the terminal rigid or solid limit group of the resolution MGQ^2Res, $MGTerm_1$, with respect to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of (the completion of) the sculpted resolution. Like in part (6), in this part we need to analyze only resolutions associated with $CollapseExtraPS^2$ (that is associated with a width 2 auxiliary resolution), for which the multi-graded resolution of the terminal limit group of MGQ^2Res, $MGTerm_1$, is identical to the formed part of the abelian decomposition associated with the top level of the sculpted resolution (for more details on why we can restrict to these resolutions see Theorem 18 below — it proves that the collection of anvils and terminal resolutions constructed in the second step of the procedure covers the entire set of Collapsed extra PS specializations). We further use the modular groups associated with the formed part of the top level of the sculpted resolution, to map the subgroup associated with this formed part in the resolution MGQ^2Res, onto its image in the subgroup associated with the second level of the sculpted resolution.

We proceed to lower levels in a similar way to what we did in part (4) of the first step, i.e. we proceed iteratively. At each level i of the sculpted resolution associated with the anvil, $Anv(MGQRes)$, we start with the terminal limit group of the resolution obtained from the top $i-1$ levels, and analyze its multi-graded resolutions with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th level of the sculpted resolution, and with respect to the formed part of the abelian decomposition associated with the i-th level of the sculpted resolution (i.e. the collection of abelian and QH vertex groups that appear in the abelian decomposition associated with the i-th level of the sculpted resolution). We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group of such a resolution, with respect to (only) the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th level of the sculpted resolution. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the abelian decomposition associated with the i-th level of the sculpted resolution. We (still) denote the resolutions obtained from the top i levels, MGQ^2Res. We further use the modular groups associated with the formed part of the i-th level of the sculpted resolution, to map the subgroup associated with this formed part in the resolution MGQ^2Res, onto its image in the subgroup associated with the $i+1$ level of the sculpted resolution.

As in the iterative procedure used in part (6), the outcome of the above terminating procedure is a (telescopic) sequence of multi-graded resolutions, that we (still) denote MGQ^2Res. Let $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$ be the subgroup generated by the image of the first sculpted resolution associated with the anvil, $Anv(MGQRes)$, and the elements associated with the Diophantine conditions imposed by the collapse form associated with it in $Anv(MGQRes)$, i.e. the image of $CollapseExtraPS^1$ in $CollapseExtraPS^2$. With each level of a multi-graded resolution constructed in one of the parts, MGQ^2Res, we associate its core resolution with respect to the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$, and the resolution induced from this (one level) core resolution by the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$. The collection of these induced resolutions, associated with the various levels of the multi-graded resolution MGQ^2Res, gives rise to a resolution (that is embedded in the multi-graded resolution MGQ^2Res), that we denote MGQ^2Res_c, of the image of the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$, that is canonically associated with MGQ^2Res. We denote the graded resolution (with respect to the defining parameters $P = \langle p \rangle$) obtained as the compositions of these (induced) resolutions, MGQ^2Res_c, associated with the (telescopic) sequence of multi-graded resolutions MGQ^2Res, $GRes(e, c, r, h_2, g_1, h_1, w, p, a)$. By Proposition 13, we may iteratively repeat this construction of induced resolutions with the subgroup associated with the completion of the obtained resolution $GRes$, until we obtain a graded resolution, that we still denote $GRes$, which is embedded in the sequence of completions of the multi-graded resolutions MGQ^2Res.

With each resolution MGQ^2Res_c associated with a multi-graded resolution MGQ^2Res, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$. By Theorem 4.13 in [S5], either there exists a resolution MGQ^2Res_c for which the complexity of its associated core resolution (with respect to $\langle r, h_2, g_1, h_1, w, p, a \rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level in the sculpted resolution that is associated with $Anv(MGQRes)$, $SCRes_1^{1,2}$, and the complexities of all the core resolutions associated with the resolutions MGQ^2Res_c that are associated with the parts above it, are identical to the complexities of the abelian decompositions associated with the corresponding levels of the sculpted resolution that is associated with $Anv(MGQRes)$, or the complexities of the core resolutions associated with the various resolutions MGQ^2Res_c are all identical to the complexities of the abelian decompositions associated with the corresponding levels of the sculpted resolution, and the structures of these core resolutions are similar to the structures of the corresponding abelian decompositions in the sculpted resolution that is associated with $Anv(MGQRes)$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core of MGQ^2Res_c (with respect to the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding part in the sculpted resolution, $SCRes_1^{1,2}$, that is associated with the anvil, $Anv(MGQRes)$. We set $SCRes(s, r, h_2, g_1, h_1, w, p, a)$ to be the image of the sculpted resolution associated with the anvil, $Anv(MGQRes)$, in the resolution $GRes$.

We further replace the core resolutions associated with the resolutions, MGQ^2Res_c, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$, with the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. We set the graded resolution $PenSCRes(u, r, h_2, g_1, h_1, w, p, a)$ to be the resolution composed from the resolutions induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$ from the penetrated core resolutions associated with the resolutions MGQ^2Res, and with the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$.

With the resolution $PenSCRes$ we associate a finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We further associate with each framed resolution the graded resolution $SCRes$, which we denote $SCRes_1^{2,1}(s_1,r,h_2,g_1,h_1,w,p,a)$ and call the *first sculpted resolution* (of *width* 1), and $SCRes_1^{2,2}(s_1,r,h_2,g_1,h_1,w,p,a)$, and call the *first sculpted resolution* (of width 2). We set each of the Extra PS resolutions to be both a developing resolution, and a *penetrated sculpted resolution*, which we denote $PenSCRes_1^{1,2}(u_1,r,h_2,g_1,h_1,w,p,a)$, and with it we associate a finite collection of anvils, that are constructed in a similar way to the constructions presented in sections 4 and 5 of the second step of the procedure. Finally, we associate with the anvil the sequence of core resolutions with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$, the resolution $GRes$ (and its parts, MGQ^2Res_c), and the sequence of core and and penetrated core resolutions, associated with the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ and constructed in the various parts, which we call a *Carrier*, and denote it $Carrier_2^2$ (the top index refers to the step number and the bottom index refers to the *width*, which is the index of the corresponding sequence of core resolutions).

Suppose that the sequence of core resolutions associated with the sequence of resolutions MGQ^2Res_c and the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, are all of maximal complexity, i.e. each of these multi-graded core resolutions have the same structure (and taut structure) as the graded abelian decomposition associated with the corresponding level of the sculpted resolution that is associated with the anvil, $Anv(MGQRes)$.

We set $SCRes(s,r,h_2,g_1,h_1,w,p,a)$ to be the graded resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the core resolutions of the multi-graded resolutions, MGQ^2Res_c, with respect to the subgroup $\langle r,h_2,g_1,w,p,a\rangle$.

If every ungraded resolution that factors through the graded resolution $SCRes(s,r,h_2,g_1,h_1,w,p,a)$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the sculpted resolution we started the second step of the sieve procedure with, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution*. If there are ungraded resolutions that factor through $SCRes$, but do not factor through any of the Non-Rigid, Non-Solid, Root nor Left PS resolutions nor the generic collapse Extra PS resolutions associated with the sculpted resolution we started the second step of the procedure with, we do the following.

We set the graded resolution $PenSCRes(u,r,h_2,g_1,h_1,w,p,a)$ to be the resolution composed from the resolutions induced by the subgroup

$\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the penetrated core resolutions associated with the various resolutions MGQ^2Res_c, and with the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$.

Recall that the resolution $GRes$ was set to be the composition of the sequence of resolutions MGQ^2Res_c. By construction, $GRes$ is graded with respect to the parameter subgroup $P=\langle p\rangle$, and the resolution $PenSCRes$ is embedded into the completion of the graded resolution $GRes$.

With the graded resolution $GRes$, we associate a (canonical) finite collection of framed resolutions, and with each framed resolution we associate a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (according to the construction used in sections (4) and (5)), that we still denote, $Anv(MGQ^2Res)$. We further associate with the anvil the graded resolution $SCRes$, which we call *first sculpted resolution* (of widths 1 and 2), and denote both as $SCRes_1^{2,1}(s,r,h_2,g_1,h_1,w,p,a)$ and $SCRes_1^{2,2}(s,r,h_2,g_1,h_1,w,p,a)$, and the graded resolution $PenSCRes$, which we call *penetrated sculpted resolution*, and denote $PenSCRes_1^{2,2}(u,r,h_2,g_1,h_1,w,p,a)$.

So far we have analyzed multi-graded resolutions of Collapse extra PS limit groups, $CollapseExtraPS^2$, that are associated with highest level width 2 auxiliary resolutions, and for which one of the parts (1)–(4) applies to them (where parts (1)–(4) are applied with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$ (the image of $CollapseExtraPS^1$)).

Suppose that none of the parts (1)–(4) applies to such a multi-graded resolution, MGQ^2Res, with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$. In this case we proceed as in part (3) of the first step and part (5) of the second step. For each level i of the anvil, $Anv(MGQRes)$, we analyze Collapse extra PS limit groups, $CollapseExtraPS^2$, that are associated with width 2 auxiliary resolutions that are associated with the limit group that is associated with all the levels of the anvil, $Anv(MGQRes)$, except the top i levels. As in part (5), given such a Collapse extra PS limit group, $CollapseExtraPS^2$, we analyze its multi-graded resolutions so that none of the parts (1)–(4) applies to the multi-graded resolutions associated with the top $i-1$ levels (with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$, the image of $CollapseExtraPS^1$), and (at least) one of the parts (1)–(4)

applies to the multi-graded resolution associated with the i-th part of the anvil.

Given such a multi-graded resolution, we treat it as we treated multi-graded resolutions associated with highest level width 2 auxiliary resolutions. We continue the analysis of the terminal rigid or solid limit group of the obtained multi-graded resolution, and continue only with those multi-graded resolutions for which the abelian decompositions associated with the parts corresponding to the formed parts of the core resolutions associated with the anvil, $Anv(MGQRes)$, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, have the same structure as these formed parts. Finally, with each such multi-graded resolution (that we still denote MGQ^2Res), we associate sculpted resolutions, (possibly) a carrier, and developing resolutions and anvils, precisely as we constructed these in case one of the parts (1)–(4) applies to the multi-graded resolution, MGQ^2Res, associated with the top level of the anvil, $Anv(MGQRes)$ (i.e. precisely as we did in the beginning of part (7)).

Suppose that for a Collapse extra PS limit group, $CollapseExtraPS^2$, associated with the anvil, $Anv(MGQRes)$, itself (and not with a width 2 auxiliary resolution associated with it), there exists a sequence of multi-graded resolutions constructed by the above iterative procedure, for which none of the parts (1)–(4) applies to the various levels (with respect to the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$). In particular, all the core resolutions associated with the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$ are of maximal complexity. We first suppose that in part (4) of the first step of the procedure, a carrier was associated with the anvil, $Anv(MGQRes)$ (i.e. a carrier and not a developing resolution associated with the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$).

In this case we first examine the structure of the corresponding developing resolution. The developing resolution is a framed resolution of a resolution built from a sequence of resolutions induced from corresponding core resolutions. Each of the induced resolutions is a resolution induced by the (image of the) subgroup $\langle r, h_2, g_1, h_1, w, a\rangle$ from the corresponding core resolution, and with each level of the induced resolution there is an associated (framed) multi-graded abelian decomposition (see section 3 of [S4] for the construction of the induced resolution, and Definition 5 for the construction of framed resolutions).

Let $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$ be the subgroup associated with the developing resolution, $Dvlp(q, r, h_2, g_1, h_1, w, p, a)$, that is associated with the anvil, $Anv(MGQRes)$. If the structure of the resolution composed from the resolutions induced by the subgroup $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$ from the various core resolutions associated with the multi-graded resolutions MGQ^2Res_c (with respect to the image of the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$), is not identical to the structure of the developing resolution associated with the anvil $Anv(MGQRes)$, Proposition 13 implies that there exists some level j for which the structure of the graded abelian decomposition associated with the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ above level j are identical to the structure of the graded abelian decompositions associated with the corresponding levels in the developing resolutions we started the second step with, and in level j, either the number of factors in the graded free decomposition associated with the graded abelian decomposition associated with the resolution induced by $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the number of factors in the corresponding graded free decomposition associated with the corresponding level in the developing resolution we started the second step with, and in case of equality in the number of factors, the complexity of the graded abelian decomposition associated with the resolution induced by $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the complexity of the graded abelian decomposition associated with the corresponding level of that developing resolution. In this case we do the following.

With the graded resolution constructed from the various resolutions induced by the subgroups $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the core resolutions associated with the various multi-graded resolutions, MGQ^2Res_c, and with respect to the images of the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$, we associate a (canonical) finite collection of framed resolutions (Definition 5). With each framed resolution we associate a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions and generic collapse PS resolutions. Finally, we set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils. With each anvil, $Anv(MGQ^2Res)$, we associate the sculpted resolution, and the carrier that are associated with the anvil, $Anv(MGQRes)$, with which we started this branch of the second step of the procedure, as well as the resolution $GRes$, and its various parts MGQ^2Res_c.

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions, MGQ^2Res, and the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$, are all of maximal complexity, and either there is no carrier associated with the anvil, $Anv(MGQRes)$, or in case there is an associated carrier, the structure of the resolution composed from the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the various core resolutions associated with the multi-graded resolutions, MGQ^2Res_c, is the same as that of the developing resolution associated with the anvil, $Anv(MGQRes)$. In this case we proceed in a similar way to what we did in part (4) of the first step, and part (6) of the second step.

We start the analysis with the collection of Collapse extra PS limit groups obtained from the anvil, $Anv(MGQRes)$ (and not with any of its associated auxiliary limit groups), and its (finitely many) associated collapse forms. We (still) denote each of these Collapse extra PS limit groups, $CollapseExtraPS^2$.

We first analyze the Collapse extra PS limit groups, $CollapseExtraPS^2$, using an iterative process which is similar to the one used in part (6). We start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^2$, with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top level of the anvil, $Anv(MGQRes)$, and with respect to the formed part of the abelian decomposition associated with the top level of the anvil, $Anv(MGQRes)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top level of the anvil, $Anv(MGQRes)$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the abelian decomposition associated with the top level of the anvil, $Anv(MGQRes)$. We (still) denote these (two parts) multi-graded resolutions, for which the second part is one level and has the same structure as the formed part of the abelian decomposition associated with the top part of the anvil, MGQ^2Res. We further use the modular groups associated with the formed part of the abelian decomposition associated with the top part of the anvil, $Anv(MGQRes)$, to map the subgroup associated with this formed part in the resolution MGQ^2Res, onto its image in the subgroup associated with the second level of $Anv(MGQRes)$.

We proceed iteratively to the next levels. At each level i, we start with the Makanin–Razborov diagram of the terminal limit group of the resolution obtained from the top $i-1$ levels, with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th level of the anvil, $Anv(MGQRes)$, and with respect to the formed part of the abelian decomposition associated with the i-th level of $Anv(MGQRes)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the non-QH, non-abelian vertex groups in the abelian decomposition associated with the i-th level of the anvil, $Anv(MGQRes)$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the abelian decomposition associated with the i-th level of the anvil, $Anv(MGQRes)$. We (still) denote the resolutions obtained from the top i levels, MGQ^2Res. We further use the modular groups associated with the formed part of the abelian decomposition associated with the i-th level of $Anv(MGQRes)$, to map the subgroup associated with this formed part in the resolution MGQ^2Res, onto its image in the subgroup associated with the $i+1$ level of $Anv(MGQRes)$.

Let GQ^2Res be a graded resolution obtained by the above iterative procedure (note that the sequence of multi-graded resolutions constructed by the above procedure form a resolution which is graded with respect to the defining parameters $P = \langle p \rangle$). We divide our treatment of the resolution GQ^2Res, depending on whether there is a carrier, $Carrier_2^1$, associated with the anvil, $Anv(MGQRes)$, or not. We first assume that there is no carrier associated with the anvil, $Anv(MGQRes)$.

With each part of the graded resolution GQ^2Res, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$. By Theorem 4.14 in [S5], either there exists a part in which the complexity of its associated core resolution is strictly smaller than the complexity of the core resolution associated with the corresponding part in $Anv(MGQRes)$, and the complexities of all the core resolutions associated with the parts above it are identical to the complexities of the core resolutions associated with the corresponding parts of $Anv(MGQRes)$, or the complexities of the core resolutions associated with the various parts are identical to the complexities of the core resolutions associated with the various parts of $Anv(MGQRes)$, and the structures of these core resolutions are similar to the structures of the corresponding core resolutions in $Anv(MGQRes)$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core is strictly smaller than the complexity of the corresponding core resolution in $Anv(MGQRes)$. We set $SCRes(s, r, h_2, g_1, h_1, w, p, a)$ to be the image, in the completion of the resolution GQ^2Res, of the sculpted resolution that is associated with the anvil, $Anv(MGQRes)$.

With each multi-graded resolution MGQ^2Res associated with a part of GQ^2Res, we associate its core resolution with respect to (the corresponding image of) the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, GQ^2Res)$. We further replace each of these core resolutions with the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. We set the graded resolution $PenSCRes(u,r,h_2,g_1,h_1,w,p,a)$ to be the resolution composed from the resolutions induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the penetrated core resolutions associated in the various levels with the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$.

With the resolution $PenSCRes$ we associate a finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and generic Collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We further associate with each framed resolution the graded resolution $SCRes$, which we denote $SCRes_1^{2,1}(s_1,r,h_2,g_1,h_1,w,p,a)$, $SCRes_1^{2,2}$, and $SCRes_1^{2,3}$, and call the *first sculpted resolution* of *widths* 1,2,3, in correspondence. We set the image of the developing resolution associated with the anvil, $Anv(MGQRes)$, to be the *second sculpted resolution* of width 2, and denote it, $SCRes_2^{2,2}$. We set each of the Extra PS resolutions to be both a developing resolution, and a *penetrated sculpted resolution*, which we denote $PenSCRes_1^{2,3}(u_1,r,h_2,g_1,h_1,w,p,a)$, and with it we associate a finite collection of anvils, that are set to be the finite collection of maximal limit groups associated with an amalgamation of the resolution GQ^2Res with the Extra PS resolution (the developing resolution). Finally, we associate with the anvil the resolution GQ^2Res and its associated sequence of core and penetrated core resolutions, associated with the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ and constructed in the various parts, which we call a *Carrier*, and denote it $Carrier_3^2$.

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions, MGQ^2Res, constructed by the iterative procedure presented above, and the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, are all of maximal complexity, i.e. each of these multi-graded core resolutions have the same structure (and taut structure) as the graded abelian decomposition associated with the corresponding level of the sculpted resolution associated with the anvil, $Anv(MGQRes)$.

We set $SCRes_1(s_1, r, h_2, g_1, h_1, w, p, a)$ to be the graded resolution induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$ from the core resolutions of the multi-graded resolutions constructed along the various levels of the process described above.

If every ungraded resolution that factors through the graded resolution $SCRes_1(s_1, r, h_2, g_1, h_1, w, p, a)$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the Extra PS resolutions we started the first step of the sieve procedure with, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution.* If there are ungraded resolutions that factor through $SCRes_1$, but do not factor through any of the Non-Rigid, Non-Solid, Root or Left PS resolutions or the Generic collapse extra PS resolutions associated with the Extra PS resolutions we started the first step of the procedure with, we do the following.

With each part of the graded resolution GQ^2Res, we associate its core resolution with respect to (the image of) the subgroup $CollapseExtraPS^1 = \langle c, r, h_2, g_1, h_1, w, p, a \rangle$. By Theorem 4.14 in [S5], either there exists a part in which the complexity of its associated core resolution is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level in $Anv(MGQRes)$, and the complexities of all the core resolutions associated with the parts above it are identical to the complexities of the abelian decompositions associated with the corresponding levels of $Anv(MGQRes)$, or the complexities of the core resolutions associated with the various parts are identical to the complexities of the abelian decompositions associated with the various levels of $Anv(MGQRes)$, and the structures of these core resolutions are similar to the structures of the corresponding abelian decompositions in $Anv(MGQRes)$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core is strictly smaller than the complexity of the corresponding abelian decomposition in $Anv(MGQRes)$. We set $SCRes_2(s_2, r, h_2, g_1, h_1, w, p, a)$ to be the image, in the completion of the resolution GQ^2Res, of the developing resolution that is associated with the anvil, $Anv(MGQRes)$ (which is $Anv(MGQRes)$ itself).

With each multi-graded resolution MGQ^2Res associated with a part of GQ^2Res, we associate its core resolution with respect to (the corresponding image of) the subgroup $CollapseExtraPS^1$. We further replace each of these core resolutions with the corresponding *penetrated core resolutions* (see Definition 4.20 in [S5]). Note that by construction, the original core resolutions are embedded into the corresponding penetrated core resolutions. We set the graded resolution $PenSCRes_2(u_2, r, h_2, g_1, h_1, w, p, a)$ to be the resolution composed from the resolutions induced by the subgroup $CollapseExtraPS^1$ from the penetrated core resolutions associated in the various levels with the subgroup $CollapseExtraPS^1$.

With the resolution $PenSCRes_2$ we associate a finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions. We further associate with each framed resolution the graded resolution $SCRes_1$, which we denote $SCRes_1^{2,1}(s_1,r,h_2,g_1,h_1,w,p,a)$, $SCRes_1^{2,2}$, and $SCRes_1^{2,3}$, and call the *first sculpted resolution* of *widths* 1,2,3, in correspondence. With each sculpted resolution we associate the corresponding penetrated sculpted resolution. We set the image of the developing resolution associated with the anvil, $Anv(MGQRes)$, to be the *second sculpted resolution* of widths 2,3 and denote it, $SCRes_2^{2,2}$ and $SCRes_2^{2,3}$. We set each of the Extra PS resolutions to be both a developing resolution, and a *penetrated sculpted resolution*, which we denote $PenSCRes_2^{2,3}(u_2, r, h_2, g_1, h_1, w, p, a)$, and with it we associate a finite collection of anvils, that are set to be the finite collection of maximal limit groups associated with an amalgamation of the resolution GQ^2Res with the Extra PS resolution (the developing resolution). Finally, we associate with the anvil the resolution GQ^2Res and its associated sequence of core and penetrated core resolutions, associated with the subgroups $\langle r, h_2, g_1, h_1, w, p, a\rangle$ and $CollapseExtraPS^1$, constructed in the various parts, which we call a *Carrier*, and denote it $Carrier_3^2$.

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions, MGQ^2Res, constructed by the iterative procedure presented above, and the subgroup $CollapseExtraPS^1$, are all of maximal complexity, i.e. each of these multi-graded core resolutions has the same structure (and taut structure) as the graded abelian decomposition associated with the corresponding level of the anvil, $Anv(MGQRes)$.

We set $SCRes_2(s_2, r, h_2, g_1, h_1, w, p, a)$ to be the graded resolution induced by the subgroup $CollapseExtraPS^1$ from the core resolutions of the multi-graded resolutions constructed along the various levels of GQ^2Res.

If every ungraded resolution, that factors through the graded resolution $SCRes_2(s_2, r, h_2, g_1, h_1, w, p, a)$, factors through either one of the

Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the developing resolution we started the second step of the sieve procedure with, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution.* If there are ungraded resolutions that factor through $SCRes_2$, but do not factor through any of the Non-Rigid, Non-Solid, Root or Left PS resolutions or the Generic collapse extra PS resolutions associated with the developing resolution we started the first step of the procedure with, we do the following.

With the ambient resolution GQ^2Res we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions. We further associate with it the graded resolution $SCRes_1$, which we denote $SCRes_1^{2,1}(s_1,r,h_2,g_1,h_1,w,p,a)$, $SCRes_1^{2,2}$, and $SCRes_1^{2,3}$, and call the *first sculpted resolution* of *widths* 1,2,3, in correspondence. We set the image of the developing resolution associated with the anvil, $Anv(MGQRes)$, to be the *second sculpted resolution* of widths 2,3 and denote it, $SCRes_2^{2,2}$ and $SCRes_2^{2,3}$. With each sculpted resolution we associate the corresponding penetrated sculpted resolution. We set each of the Extra PS resolutions to be both a developing resolution, and an anvil, that we denote, $Anv(MGQ^2Res)$.

So far we have treated the case in which no carrier is associated with the anvil, $Anv(MGQRes)$. In this case we could have used Theorem 4.14 in [S5] in order to bound the complexities of the core resolutions associated with the various multi-graded resolutions, MGQ^2Res. In case there is a carrier associated with $Anv(MGQRes)$, we need to bound the complexities of the core resolutions associated with the multi-graded resolutions MGQ^2Res, in terms of the complexities of the core resolutions associated with the various multi-graded resolutions $MGQRes$ that are associated with the anvil, $Anv(MGQRes)$.

Theorem 16. *In case there exists a carrier, $Carrier_2^1$, associated with the anvil, $Anv(MGQRes)$, one of the following two possibilities holds:*

(i) *The complexities of the core resolutions associated with the sequence of multi-graded resolutions, MGQ^2Res, and with the (image of the) PS limit group, $\langle r,h_2,g_1,h_1,w,p,a\rangle$, are equal to the complexities of the core resolutions associated with the corresponding multi-graded resolutions, $MGQRes$, associated with the anvil, $Anv(MGQRes)$, we started the second step with. In this case the structures of the corresponding core resolutions are similar.*

(ii) *There exists some level j, for which the core resolutions associated with the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, and with the multi-graded resolutions, MGQ^2Res, in all the levels above level j, have the same structure as the corresponding core resolutions associated with the multi-graded resolutions $MGQRes$, and in level j, the core resolution associated with the corresponding multi-graded resolution MGQ^2Res has strictly smaller complexity than the core resolution associated with the corresponding multi-graded resolution, $MGQRes$.*

Proof. The iterative procedure that was used to construct the multi-graded resolutions, MGQ^2Res, preserves all the formed parts of the abelian decompositions associated with the various levels of the multi-graded resolutions $MGQRes$, associated with the anvil, $Anv(MGQRes)$, we started the second step with.

When we construct the core resolution of a multi-graded resolution, MGQ^2Res, associated with the top level of the sculpted resolution, using the construction presented in section 4 in [S5], we start with the image of the core resolution associated with the corresponding multi-graded resolution $MGQRes$, associated with the anvil, $Anv(MGQRes)$, we started the second step with. Since the formed parts in the abelian decompositions associated with the multi-graded resolution, $MGQRes$, are preserved in the multi-graded resolution, MGQ^2Res, either the core of MGQ^2Res has the same structure as that of the core of $MGQRes$, or the complexity of the core associated with MGQ^2Res is strictly smaller than the complexity of the core associated with $MGQRes$. If the structures of the core resolutions of MGQ^2Res and $MGQRes$ are similar, we continue inductively to multi-graded resolutions MGQ^2Res associated with lower levels, and the theorem follows. □

Suppose that there is a carrier, $Carrier_2^1$, associated with the anvil, $Anv(MGQRes)$. With each part of the graded resolution GQ^2Res, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$. By Theorem 16, either there exists a part in which the complexity of its associated core resolution is strictly smaller than the complexity of the core resolution associated with the corresponding part in $Anv(MGQRes)$, and the complexities of all the core resolutions associated with the parts above it are identical to the complexities of the core resolutions associated with the corresponding parts of $Anv(MGQRes)$, or the complexities of the core resolutions associated with the various parts are identical to the complexities of the core resolutions associated with the various parts of $Anv(MGQRes)$, and the structures of these core resolutions are similar to the structures of the corresponding core resolutions in $Anv(MGQRes)$.

Suppose that the complexities of the core resolutions associated with the various parts of GQ^2Res, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, are identical to the complexities of the core resolutions associated with the various parts of the anvil, $Anv(MGQRes)$. In this case we set $PenScRes$ to be the resolution induced by image of the subgroup associated with the developing resolution, $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$, that is associated with the anvil, $Anv(MGQRes)$, from the penetrated core resolutions associated with the subgroup, $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$, and the various parts of the resolution GQ^2Res. To study the structure of the resolution $PenSCRes$ we need the following proposition, which is similar to Proposition 13.

PROPOSITION 17. *Let $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$ be the subgroup generated by the developing resolution in the anvil, $Anv(MGQRes)$. From each part of the penetrated core resolutions associated with the multi-graded resolutions, MGQ^2Res, constructed along the levels of the second step of the procedure, and with the subgroup $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$, there is a resolution induced by the (image of the) subgroup $\langle q, r, h_2, g_1, h_1, w, p, a\rangle$. By Proposition 13, we may repeat the construction of the induced resolution using the subgroup associated with the composition of the resolutions associated with the various parts, until we obtain an induced resolution that is embedded in the completion of the resolution GQ^2Res. Let $\Lambda_1, \dots, \Lambda_b$ be the graded abelian decompositions associated with the obtained (induced) resolutions and the terminal levels of the penetrated core resolutions associated with the various parts.*

Then, either the structures of the graded abelian decompositions Λ_i are identical to the structures of the corresponding graded abelian decompositions in the developing resolution we started the second step with, or there exists some level j, so that the structures of the abelian decompositions above level j remain unchanged, and either the number of factors in the (graded) free decomposition associated with Λ_j is strictly smaller than the number of factors in the corresponding (graded) free decomposition associated with the corresponding level of the developing resolution, or, in case of equality in the number of factors, the complexity of the graded abelian decomposition Λ_j is strictly smaller than the complexity of the graded abelian decomposition associated with that level in the developing resolution.

Proof. Identical to the proof of Proposition 13 and Proposition 4.11 in [S4]. □

By Propositions 13 and 17, either the structure of the obtained resolution is identical to the structure of the developing resolution associated with the anvil, $Anv(MGQRes)$, or there exists some level j, so that the structure of the graded abelian decompositions associated with the resolutions induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ above level j are identical to the structure of graded abelian decompositions associated with the corresponding levels of the developing resolution; and in level j, either the number of factors in the (graded) free decomposition associated with the graded abelian decomposition associated with the resolution induced by $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the number of factors in the corresponding (graded) free decomposition associated with the corresponding level of the developing resolution, or in case of equality in the number of factors, the complexity of the graded abelian decomposition associated with the resolution induced by $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the complexity of the graded abelian decomposition associated with that level in the developing resolution.

Suppose that either there exists a part of GQ^2Res for which the complexity of the associated core is strictly smaller than the complexity of the corresponding core resolution in $Anv(MGQRes)$, or in case the structure of the core resolutions associated with the various parts are identical to those associated with the various parts of the anvil, $Anv(MGQRes)$, suppose that there is a reduction in the complexity of an abelian decomposition associated with the resolution induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the various penetrated core resolutions that are associated with the various parts of GQ^2Res.

If there exists a part of GQ^2Res for which the complexity of the associated core is strictly smaller than the complexity of the corresponding core in $Anv(MGQRes)$, we set $PenSCRes$ to be the graded resolution composed from resolutions induced by the images of the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the penetrated core resolutions associated with the various parts of GQ^2Res and with the images of the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$.

In case the structure of the core resolutions associated with the various parts of GQ^2Res are identical to those associated with the various parts of the anvil, $Anv(MGQRes)$, and there is a reduction in the complexity of an abelian decomposition associated with the resolution induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the various penetrated core resolutions that are associated with the various parts of GQ^2Res, we set $PenScRes$ to be the resolution induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the penetrated core resolutions associated with this subgroup, $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$, and the various multi-graded resolutions, MGQ^2Res.

With the resolution $PenSCRes$ we associate a finite collection of framed resolutions, and with each framed resolution we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we associated those with the PS resolution $PSHGHRes$ we started the first step of the procedure with. We further associate with each framed resolution the graded resolution $SCRes_1$, which we denote $SCRes_1^{2,1}(s_1,r,h_2,g_1,h_1,w,p,a)$, $SCRes_1^{2,2}$, and $SCRes_1^{2,3}$, and call the *first sculpted resolution* of *widths* 1,2,3, in correspondence. We set each of the Extra PS resolutions to be both a developing resolution, and a *penetrated sculpted resolution*, which we denote $PenSCRes_1^{2,3}(u_1,r,h_2,g_1,h_1,w,p,a)$, and with it we associate a finite collection of anvils, that are set to be the finite collection of maximal limit groups associated with an amalgamation of the resolution GQ^2Res with the Extra PS resolution (the developing resolution). Finally, we associate with the anvil the resolution GQ^2Res and its associated sequence of core and penetrated core resolutions, associated with the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ and constructed in the various parts, which we call a $Carrier$, and denote it $Carrier_3^2$, and the image of the carrier, $Carrier_2^1$, that is associated with the anvil, $Anv(MGQRes)$, that we also denote $Carrier_2^2$.

Suppose that the sequence of core resolutions associated with the various parts of the resolution, GQ^2Res, and the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, are all of maximal complexity, and the abelian decompositions associated with the various levels of the resolution induced by the image of the developing resolution, $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$, from the penetrated core resolutions that are associated with GQ^2Res and the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$, have the same structure as the abelian decompositions associated with the various levels of the developing resolution $Dvlp$, that is associated with the anvil, $Anv(MGQRes)$. In this case we proceed as in the case in which no carrier is associated with the anvil, $Anv(MGQRes)$.

We set $PenSCRes(u,r,h_2,g_1,h_1,w,p,a)$ to be the graded resolution induced by the subgroup $\langle q,r,h_2,g_1,h_1,w,p,a\rangle$ from the penetrated core resolutions of the various parts of GQ^2Res (with respect to the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$).

If every ungraded resolution that factors through the graded resolution $PenSCRes(u,r,h_2,g_1,h_1,w,p,a)$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the anvil, $Anv(MGQRes)$, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution*. Otherwise we do the following.

With the ambient resolution GQ^2Res we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions. We further associate with it the sculpted resolution that is associated with $Anv(MGQRes)$, that which we denote $SCRes_1^{2,1}$, $SCRes_1^{2,2}$, and $SCRes_1^{2,3}$, and call the *first sculpted resolution* of *widths* 1,2,3, in correspondence. We set the image of the developing resolution associated with the anvil, $Anv(MGQRes)$, to be the $first\ penetrated\ sculpted\ resolution$ of width 2, denoted $PenSCRes_2^{1,2}$, and the resolution $PenSCRes$ to be the $first\ penetrated\ sculpted\ resolution$ of width 3, $SCRes_2^{2,3}$. We set each of the Extra PS resolutions to be both a developing resolution, and an anvil, that we denote, $Anv(MGQ^2Res)$.

As in the previous parts of the first and second steps, we still need to associate with each anvil a finite collection of (extended) auxiliary resolutions. Like in part (4) of the first step, with each anvil, $Anv(MGQ^2Res)$, we associate either 2 or 3 collections of auxiliary resolutions (auxiliary resolutions of widths 1,2 and possibly 3), according to Definition 14, and depending on whether there are 2 or 3 *algebraic envelopes* associated with the constructed anvil, $Anv(MGQ^2Res)$.

Like in the first step, before we conclude the second step of the sieve procedure, and prepare the data-structure for starting the next step, we need to check that the iterative procedure that was used in the second step, and the anvils constructed along it together with the terminal resolutions, collect all the Collapse extra PS specializations that factor through the initial developing resolutions, that are associated with the anvils, $Anv(MGQRes)$, we started the second step with, and through the Diophantine conditions imposed by their associated collapse forms.

Theorem 18. *Let $(r, h_2, g_1, h_1, w, p, a)$ be a valid PS statement that factors through one of the PS limit group $PSHGH$, and can be extended to a specialization that factors and is taut with respect to one of the anvils, $Anv(MGQRes)$, that was constructed in the first step of the sieve procedure, and the extended specialization satisfies the Diophantine conditions imposed by one of the collapse forms associated with the developing resolution associated with the anvil, $Anv(MGQRes)$. Then either:*

(i) *The specialization* $(r, h_2, g_1, h_1, w, p, a)$ *can be extended to a test sequence of one of the developing resolutions, we started the second step with, that projects to a collection of valid* PS *statements.*
(ii) $(r, h_2, g_1, h_1, w, p, a)$ *can be extended to a specialization that either factors through and is taut with respect to one of the anvils or one of the terminal resolutions constructed along the second step of the sieve procedure.*

Proof. The argument is essentially identical to the proof of Theorem 10. Suppose that part (i) does not hold. In this case the valid PS statement can be extended to a specialization that factors through one of the Collapse extra PS limit groups, $CollapseExtraPS^2$, that were analyzed along the second step of the procedure.

If the extended specialization factors through a sequence of multi-graded resolutions, MGQ^2Res, so that an anvil was assigned with this sequence of multi-graded resolutions according to one of the parts (1)–(5) of the second step, then the valid PS statement $(r, h_2, g_1, h_1, w, p, a)$ can be extended to a specialization that factors through an anvil constructed according to the relevant parts (1)–(5).

Otherwise, the extended specialization must factor through a sequence of multi-graded resolutions, MGQ^2Res, constructed according to part (5) of the second step, so that all the core resolutions (with respect to the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$) of the multi-graded resolutions, MGQ^2Res, are of maximal possible complexity. Note that with such a sequence of multi-graded resolutions no anvil was assigned in part (5) of the second step. However, by the construction of graded formal limit groups and resolutions, presented in section (3) of [S2], all the specializations that factor through and are taut with respect to a sequence of multi-graded resolutions, MGQ^2Res, that were constructed according to part (5) of the second step and are all of maximal possible complexity, must factor through at least one sequence of multi-graded resolutions constructed according to parts (6) or (7) of the second step of the sieve procedure. Like in the proof of Theorem 10, this follows since every test sequence with respect to the formed parts of the core resolutions associated with the various levels of the sculpted resolution, must factor through (at least one of) the multi-graded resolutions constructed in parts (6) or (7) of the second step of the procedure. Hence, in this case, the valid PS statement $(r, h_2, g_1, h_1, w, p, a)$ can be extended to a specialization that factors through either one of the anvils or one of the terminal resolutions constructed according to parts (6) or (7) of the second step of the sieve procedure. □

The collection of multi-graded resolutions, MGQ^2Res, the graded resolutions obtained from them, GQ^2Res, the developing resolutions and (possibly) sculpted resolutions and carriers, and the anvils, $Anv(MGQ^2Res)$, associated with them, and their collections of (extended) auxiliary resolutions, limit groups, and modular groups, together with the data-structure constructed before starting the second step of the procedure, form the *data-structure* obtained as a result of the first step.

At this stage we continue in a similar way to what we did in the first step of the procedure. Given an anvil, $Anv(MGQ^2Res)$, and an (extended) auxiliary resolution of it, we associate with them all their (finitely many) possible collapse forms. Given an (extended) auxiliary resolution and a collapse form, we add variables that enable one to express the Diophantine conditions imposed by the collapse form, so that with each (extended) auxiliary resolution and one of its associated collapse forms, we finally associate a finite collection of Collapse extra PS limit groups, that we denote $CollapseExtraPS^3$ (see the corresponding construction at the end of the first step for more details on this construction).

The General Step of the Sieve Procedure

In the first two steps of the sieve procedure, we have finally obtained finitely many developing resolutions and anvils, with which we have possibly associated a finite collection of sculpted resolutions, penetrated sculpted resolutions, and carriers, and a finite collection of associated Non-rigid, Non-solid, Root and Left PS resolutions, Extra PS resolutions, Generic collapse extra PS resolutions, multi-graded auxiliary resolutions, and Collapse extra PS limit groups. After presenting the first steps, we finally present the general step of the sieve procedure, and then prove that the procedure terminates after finitely many steps.

We define the general step of the sieve procedure inductively. For brevity we denote the multi-graded resolutions that were obtained in the previous steps of the sieve procedure, $MGQ^mRes(f_m, r, h_2, g_1, h_1, w, p, a)$, where m is the index of the step in which they were constructed. With each such multi-graded resolution there is an associated developing resolution, which we denote $Dvlp^m(q_m, r, h_2, g_1, h_1, w, p, a)$, an anvil that we denote $Anv(MGQ^mRes)$, and multi-graded auxiliary resolutions. With the developing resolution and the anvil, we associate a positive integer, called *width* and denoted $width(m)$, that denotes the number of algebraic envelopes associated with the anvil, i.e. the number of (nested) sequences of core resolutions associated with the anvil, $Anv(MGQ^m Res)$. $width(m)=1$ if and only if no sculpted resolution is associated with the anvil. In case $width(m) > 1$, for each width d, $1 \leq d \leq width(m)$, i.e. with the (nested) collection of the first d algebraic envelopes (sequences of core resolutions associated with the anvil), there are associated $sc(m,d)$ sculpted resolutions, that we denote $SCRes_e^{m,d}(s_e, r, h_2, g_1, h_1, w, p, a)$ for $e = 1, \dots, sc(m,d)$, and corresponding penetrated sculpted resolutions, that we denote $PenSCRes_e^{m,d}(u_e, r, h_2, g_1, h_1, w, p, a)$. At each width d, there is possibly an associated *carrier* that we denote $Carrier_d^m$, that represent the d-th algebraic envelope, i.e. the d-th sequence of core resolutions associated with the anvil. We start the general step of the sieve procedure with the (finite) collection of multi-graded resolutions constructed in the previous step, and their associated widths, developing resolutions, sculpted resolutions and their penetrated sculpted resolutions and carriers, anvils, multi-graded auxiliary resolutions, and their associated collapse forms and Collapse extra PS limit groups, $CollapseExtraPS^{m+1}(f_{m+1}, r, h_2, g_1, h_1, w, p, a)$.

The ultimate goal of the general step of the iterative procedure is to obtain a strict reduction in either the complexity of certain decompositions and resolutions, or a strict reduction in the Zariski closures of certain limit groups associated with the anvils constructed in the previous steps of the procedure. If such reductions do not occur, then we are forced to increase the number of algebraic envelopes associated with the constructed anvil, i.e. to increase the width. Finally, the strict reduction in complexity and Zariski closures, together with a global bound on the width associated with the anvils constructed along the procedure, that we prove in the sequel, will guarantee the termination of the iterative procedure after finitely many steps.

Since we treat the anvils and Collapse extra PS limit groups in parallel, we present the general (n-th) step of the procedure with one of the anvils, $Anv(MGQ^{n-1}Res)(t_{n-1}, r, h_2, g_1, h_1, w, p, a)$ and one of its associated Collapse extra PS limit groups, $CollapseExtraPS^n(f_n, r, h_2, g_1, h_1, w, p, a)$. We construct iteratively the *developing resolutions (penetrated) sculpted resolutions, carriers* and anvils, starting with the Collapse extra PS limit group.

As we did in the first steps, we start the general step of the sieve procedure with the Collapse extra PS limit groups that are associated with auxiliary resolutions of highest level (and width 1, in case there exists a sculpted resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$), and analyze them in parallel. The analysis of such a Collapse extra PS limit group considers (and depends on) the data-structure associated with it, i.e. the (finite) collection of multi-graded resolutions constructed in the previous steps, their core resolutions, and their associated developing, sculpted, and penetrated sculpted resolutions, and carriers, and the auxiliary resolutions constructed in previous steps of the sieve procedure, in a similar way to what we did in the first two steps of the sieve procedure.

(1) Let $Q^n(r,h_2,g_1,h_1,w,p,a)$ be the limit group generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the the Collapse extra PS limit group, $CollapseExtraPS^n(f_n,r,h_2,g_1,h_1,w,p,a)$ (that is associated with the anvil, $Anv(MGQ^{n-1}Res)$). If $Q^n(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of the PS limit group, $PSHGH$, we started the first step of the procedure with, we continue this branch of the iterative procedure, by starting the first step of the procedure with the graded limit group $Q^n(r,h_2,g_1,h_1,w,p,a)$, instead of the graded (PS) limit group $PSHGH$.

(2) At this stage we may assume that $Q^n(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the PS limit group $PSHGH$. Along the sieve process used to construct the anvil, $Anv(MGQ^{n-1}Res)$, we enlarge the subgroups that serve as (multi) parameter subgroups for the multi-graded Makanin–Razborov diagrams of the corresponding Collapse extra PS limit groups, each time the complexity of the core associated with the corresponding multi-graded resolution is being reduced. At step m, $1\le m\le n-1$, we set the parameter subgroups to be $Base_{2,1}^{m,s(m)},\dots,Base_{2,v^m_{s(m)}}^{m,s(m)}$, and the corresponding multi-graded resolution to be

$$MGQ^mRes\big(f_m,r,h_2,g_1,h_1,w,Base_{2,1}^{m,s(m)},\dots,Base_{2,v^m_{s(m)}}^{m,s(m)},a\big)\,.$$

For each index s, $1\le s\le s(n-1)$, we set $f(s)$ to be the minimal index m, $1\le m\le n-1$, for which $s=s(m)$, and $\ell(s)$ to be the maximal index m for which $s=s(m)$. For each couple of indices m_1,m_2, $1\le m_1\le m_2\le n$, let $Q^{m_2}(f_{m_1},r,h_2,g_1,h_1,w,p,a)$ be the subgroup generated by $\langle f_{m_1},r,h_2,g_1,h_1,w,p,a\rangle$ in the Collapse extra PS limit group $CollapseExtraPS^{m_2}(f_{m_2},r,h_2,g_1,h_1,w,p,a)$, which is associated with the the m_2-1 anvil, $Anv(MGQ^{m_2-1}Res)$.

In this part of the general step of the sieve procedure we assume that the core associated with the multi-graded resolution,

$$MGQ^{n-1}Res\big(f_{n-1},r,h_2,g_1,h_1,w,Base_{2,1}^{n-1,s(n-1)},\dots,Base_{2,v^{n-1}_{s(n-1)}}^{n-1,s(n-1)},a\big)\,,$$

is of maximal possible complexity, i.e. it has the same structure as the core associated with the top part of

$$MGQ^{n-2}Res\big(f_{n-2},r,h_2,g_1,h_1,w,Base_{2,1}^{n-2,s(n-2)},\dots,Base_{2,v^{n-2}_{s(n-2)}}^{n-2,s(n-2)},a\big)\,.$$

Suppose that for some index s, $1{\le}s{\le}s(n{-}1){-}1$, $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^{\ell(s)}(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$, and let s be the minimal index for which this happens. Then we omit the Collapse extra PS limit group, $CollapseExtraPS^n$, from the list of Collapse extra PS limit groups we started the n-th step with, and replace it by going back to the $\ell(s)$-th step of the sieve procedure, and start it with the limit group $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ instead of the $\ell(s)$-th Collapse extra PS limit group, $CollapseExtraPS^{\ell(s)}$, that was used in the $\ell(s)$-th step of the process that leads to the construction of the anvil, $Anv(MGQ^{n-1}Res)$. Since by the definition of the index $\ell(s)$, the parameter subgroups were enlarged at step $\ell(s)+1$ (hence, the multi-graded resolution $MGQ^{\ell(s)}Res$ is not of maximal complexity), by Theorem 4.18 in [S5] (that implies that the Diophantine sets associated with maximal complexity resolutions cover the Diophantine sets associated with maximal complexity resolutions of a quotient), in analyzing the limit group $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$, we need to take into account only those multi-graded resolutions for which their associated core resolutions are not of maximal complexity, i.e. only those multi-graded resolutions for which their core resolution does not have the same structure as the core associated with the top part of $MGQ^{\ell(s)-1}Res$.

Suppose that for $s=s(n-1)-1$ (hence, for every index s, $1\le s\le s(n-1)-1$), $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $CollapseExtraPS^{\ell(s)}$. We set $s(n)=s(n-1)$. Let

$$MGQ^nRes_1\big(f_n,r,h_2,g_1,h_1,w,Base_{2,1}^{n,s(n)},\dots,Base_{2,v^n_{s(n)}}^{n,s(n)},a\big),\dots$$
$$\dots,MGQ^nRes_q\big(f_n,r,h_2,g_1,h_1,w,Base_{2,1}^{n,s(n)},\dots,Base_{2,v^n_{s(n)}}^{n,s(n)},a\big)$$

be the completions of the multi-graded resolutions in the multi-graded taut Makanin–Razborov diagram of $CompExtraPS^n$ with respect to the parameter subgroups $Base_{2,1}^{n,s(n)},\dots,Base_{2,v^n_{s(n)}}^{n,s(n)}$. We will treat the multi-graded resolutions MGQ^nRes_j in parallel, hence, we omit their index.

Since the multi-graded resolution $MGQ^{\ell(s(n)-1)}Res$ used in step $\ell(s(n)-1)$ along the branch of the process that constructs the anvil, $Anv(MGQ^{n-1}Res)$, is well separated, with each QH vertex group in one of the abelian decompositions associated with $MGQ^{\ell(s(n)-1)}Res$ there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the multi-graded resolution $MGQ^{\ell(s(n)-1)}Res$.

Each QH vertex group in the formed part of the core resolution that is associated with the top level of the multi-graded resolution $MGQ^{\ell(s(n)-1)}Res$ naturally inherits a sequence of abelian decompositions from an n-th multi-graded resolution MGQ^nRes. If for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the multi-graded resolution $MGQ^{\ell(s(n)-1)}Res$, we omit the multi-graded resolution MGQ^nRes from the list of completions of resolutions of the anvil, $Anv(MGQ^{n-1}Res)$.

By Theorem 4.14 in [S5], the complexity of the core associated with the n-th multi-graded resolution MGQ^nRes are bounded by the complexity of the core associated with the multi-graded resolution $MGQ^{n-1}Res$, and if the complexity of the two-core resolutions are equal, then the structure of the core associated with MGQ^nRes is identical to the structure of the core associated with $MGQ^{n-1}Res$. In this part of the n-th step of the sieve procedure we will also assume that the complexity of the core associated with the multi-graded resolution MGQ^nRes is strictly smaller than the complexity of the core associated with $MGQ^{n-1}Res$, hence, we are able to treat the multi-graded resolution MGQ^nRes according to part (4) of step $n-1$ of the sieve procedure.

(3) At this stage we may assume that $Q^n(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the PS limit group $PSHGH$, we started the first step with. In this part we assume that the core associated with the multi-graded resolution $MGQ^{n-1}Res$ is not of maximal possible complexity.

Suppose that for some index s, $1\leq s\leq s(n-1)$, $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^{\ell(s)}(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$, and suppose that s is the minimal index for which this happens. Then we omit the Collapse extra PS limit group, $CollapseExtraPS^n$, from our list, and replace it by going back to the $\ell(s)$-th step of the sieve procedure, and start it with the limit group $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$, which is the subgroup generated by $\langle f_{\ell(s)},r,h_2,g_1,h_1,w,p,a\rangle$ in $CollapseExtraPS^n$, instead of the $\ell(s)$-th limit group $Q^{\ell(s)}(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ used in the $\ell(s)$-th step of the process that leads to the construction of the anvil, $Anv(MGQ^{n-1}Res)$. Since by the definition of the index $\ell(s)$, in case $\ell(s)<n-1$ the (multi) parameter subgroups were enlarged at step $\ell(s)+1$, and in case $\ell(s)=n-1$ the core associated with the multi-graded resolution $MGQ^{n-1}Res$ is assumed to be not of maximal possible complexity; in analyzing the limit group $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ we need to take into account only those

multi-graded resolutions for which their associated core are not of maximal complexity, i.e. those multi-graded resolutions for which their associated core resolutions have strictly smaller complexity than the complexity of the core associated with the top level of the multi-graded resolution $MGQ^{\ell(s-1)}Res$ used in the process of the construction of the anvil $Anv(MGQ^{n-1}Res)$.

(4) In this part we assume that the core associated with the multi-graded resolution $MGQ^{n-1}Res$ is not of maximal complexity, and that $Q^n(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $PSHGH$. We set $s(n)=s(n-1)+1$, and the (multi) parameter subgroups $Base^{n,s(n)}_{2,1},\dots,Base^{n,s(n)}_{2,v^n_{s(n)}}$ to be the factors in the given free decomposition of the associated auxiliary limit group, $Aux(MGQ^{n-1}Res)$. Let

$$MGQ^nRes_1\big(f_n,r,h_2,g_1,h_1,w,Base^{n,s(n)}_{2,1},\dots,Base^{s(n)}_{2,v^n_{s(n)}},a\big),\dots$$
$$\dots,MGQRes_q\big(f_n,r,h_2,g_1,h_1,w,Base^{n,s(n)}_{2,1},\dots,Base^{n,s(n)}_{2,v^n_{s(n)}},a\big)$$

be the completions of the resolutions in the taut multi-graded Makanin–Razborov diagram of $CollapseExtraPS^n$ with respect to the parameter subgroups $Base^{n,s(n)}_{2,1},\dots,Base^{n,s(n)}_{2,v^n_{s(n)}}$. We analyze the n-th multi-graded resolutions MGQ^nRes_j in parallel, hence, we will omit their index.

Since the multi-graded resolution $MGQ^{n-1}Res$, used in step $n-1$ to construct the anvil $Anv(MGQ^{n-1}Res)$, is well separated, with each QH vertex group in one of the abelian decompositions associated with $MGQ^{n-1}Res$ there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the multi-graded resolution $MGQ^{n-1}Res$.

Each QH vertex group in the formed part of the core resolution that is associated with the top level of the multi-graded resolution $MGQ^{n-1}Res$ naturally inherits a sequence of abelian decompositions from an n-th multi-graded resolution MGQ^nRes. If for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the multi-graded resolution $MGQ^{n-1}Res$, we omit the multi-graded resolution, MGQ^nRes, from the list of completions of resolutions of the anvil, $Anv(MGQ^{n-1}Res)$.

In this part we will also assume that the core associated with the n-th multi-graded resolution MGQ^nRes is not of maximal possible complexity, i.e. it does not have the same structure as the core associated with the top level of the $n-1$ multi-graded resolution, $MGQ^{n-1}Res$. The case of maximal complexity will be treated in the next parts of the general step. To treat an n-th multi-graded resolution that is not of maximal complexity, we need the following two observations, that are similar to Lemmas 11 and 12.

LEMMA 19. *Let MGQ^nRes be an n-th multi-graded resolution that is not of maximal complexity. Let $Q^n_{term}(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ be the image of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ (which is the image of $CollapseExtraPS^{n-1}$) in the terminal (rigid or solid) limit group of MGQ^nRes.*

Then the multi-graded resolution MGQ^nRes can be replaced by two finite collections of multi-graded resolutions, that are all compatible with the top level of the resolution $MGQ^{n-1}Res$, and are all obtained from MGQ^nRes by adding at most a single (terminal) level. Furthermore, all the resolutions in these collections are not of maximal complexity.

We denote each of the resolutions in these collections, MGQ^nRes':

(i) *In the first (possibly empty) collection of multi-graded resolutions, the image of the subgroup $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ in the terminal limit group of MGQ^nRes', is a proper quotient of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$.*

(ii) *In the second (possibly empty) finite collection of multi-graded resolutions, the terminal limit group of MGQ^nRes' is either a rigid or a solid limit group with respect to the parameter subgroup $\langle p\rangle$, i.e. the terminal limit group is rigid or solid with respect to the parameter subgroup $\langle p\rangle$, and not only with respect to the multi-grading with respect to the subgroups $Base^{n,s(n)}_{2,1},\dots,Base^{n,s(n)}_{2,v^n_{s(n)}}$, that was used in the construction of the resolution, MGQ^nRes.*

Proof. Identical to the proof of Lemma 2.7 in [S5]. □

LEMMA 20. *Let MGQ^nRes be one of the resolutions in our list of multi-graded resolutions. Let $Q^n_2(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ and $Q^n_2(f_n,r,h_2,g_1,h_1,w,p,a)$, be the images of the subgroups $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ and $Q^n(f_n,r,h_2,g_1,h_1,w,p,a)$ in the limit group, $MGQ^nlim_2(f_n,r,h_2,g_1,h_1,w,p,a)$, associated with the second level of the multi-graded resolution MGQ^nRes.*

Then $Q^n_2(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ is a quotient of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, and $Q^n_2(f_n,r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^n(f_n,r,h_2,g_1,h_1,w,p,a)$.

Proof. The claim is simply one of the basic properties of a multi-graded resolution. □

By Lemma 19 we can either omit the graded resolution MGQ^nRes from our list of multi-graded resolutions, or we can replace the resolution

MGQ^nRes by finitely many resolutions, that for brevity we still denote MGQ^nRes, and for each resolution we may assume that either the image of the subgroup $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, in the terminal graded limit group of MGQ^nRes, is a proper quotient of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, or the terminal limit group of MGQ^nRes is rigid or solid with respect to the parameter subgroup $P=\langle p\rangle$.

Suppose first that the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group $Q_2^n(f_n,r,h_2,g_1,h_1,w,p,a)$, denoted $Q_2^n(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^n(r,h_2,g_1,h_1,w,p,a)$. With the subgroup $Q_2^n(r,h_2,g_1,h_1,w,p,a)$ we associate the completions of the graded resolutions that appear in its graded taut Makanin–Razborov diagram with respect to the subgroup $P=\langle p\rangle$:

$$GQRes_1(r,h_2,g_1,h_1,w,p,a),\dots,GQRes_c(r,h_2,g_1,h_1,w,p,a)\,.$$

We continue with each of the graded resolutions $GQRes_j(r,h_2,g_1,h_1,w,p,a)$ in parallel.

Let $TMGQ^nRes$ be the (one level) resolution corresponding to the top level of the multi-graded resolution MGQ^nRes. Note that by Corollary 4.16 in [S5], the complexity of the (multi-graded) core of $TMGQ^nRes$, is strictly smaller than the complexity of the core of the multi-graded resolution $MGQ^{n-1}Res$, i.e. the multi-graded resolution $TMGQ^nRes$ is not of maximal complexity.

If the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with the resolution $GQRes_j(r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q_2^n(r,h_2,g_1,h_1,w,p,a)$, we replace the graded resolution $GQRes_j$ by starting part (4) of the the n-th step with the multi-graded resolution obtained from $TMGQ^nRes$ by replacing its terminal limit group $Q_2^n(f_n,r,h_2,g_1,h_1,w,p,a)$ with the maximal limit groups obtained from all the specializations that factor through both $Q_2^n(f_n,r,h_2,g_1,h_1,w,p,a)$ and the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with $GQRes_j$. Hence, for the rest of this part we may assume that the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group associated with $GQRes_j$ is isomorphic to $Q_2^n(r,h_2,g_1,h_1,w,p,a)$.

Let $CRes_j(r,h_2,g_1,h_1,w,p,a)$ be the graded resolution obtained from the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the (core of the) multi-graded graded resolution $TMGQ^nRes$, followed by the graded resolution $GQRes_j$. With the resolution $CRes_j$ we associate a finite collection of framed resolutions (see Definition 5), which we denote $FrmCRes$, and continue with each of them in parallel.

If the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to a framed resolution, $FrmCRes$, associated with the graded resolution $CRes_j$, is a proper quotient of $Q^n(r,h_2,g_1,h_1,w,p,a)$, we replace the framed resolution $FrmCRes$ by starting the first step of the procedure with the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the framed resolution $FrmCRes$.

If the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$, in the limit group associated with the second level of $FrmCRes$, which is the level associated with the terminal (second) level of $TMGQ^nRes$, that we denote, $Q_F(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q_2^n(r,h_2,g_1,h_1,w,p,a)$, we replace the framed resolution $FrmCRes$ by starting part (4) of the the n-th step with the multi-graded resolution obtained from $TMGQ^nRes$ by replacing its terminal limit group $Q_2^n(f_n,r,h_2,g_1,h_1,w,p,a)$ with the maximal limit groups obtained from all specializations that factor through both $Q_2^n(f_n,r,h_2,g_1,h_1,w,p,a)$ and $Q_F(r,h_2,g_1,h_1,w,p,a)$. Hence, we may assume that for the rest of this part the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to $FrmCRes$ is isomorphic to $Q^n(r,h_2,g_1,h_1,w,p,a)$, and $Q_F(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q_2^n(r,h_2,g_1,h_1,w,p,a)$.

We now treat each of the framed resolutions $FrmCRes$ in a similar way to our treatment of multi-graded resolutions in part (4) of the second step of the sieve procedure. With the framed resolution $FrmCRes$ we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions. We set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils, that we denote, $Anv(MGQ^nRes)$, and auxiliary resolutions (according to the construction presented in Definition 8), precisely as we did in part (2) of the first step of the sieve procedure.

Suppose that $Q_2^n(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to $Q^n(r,h_2,g_1,h_1,w,p,a)$. In this case we set s, $1\leq s\leq s(n)-1$, to be the minimal index for which $Q_2^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ $=Q^{\ell(s)}(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ (we assume that there exists such an index s in this part). Let

$$MGQRes_1\big(f_{\ell(s)},r,h_2,g_1,h_1,w,Base_{2,1}^{\ell(s),s},\dots,Base_{2,v_s^{\ell(s)}}^{\ell(s),s},a\big),\dots$$
$$\dots,MGQRes_d\big(f_{\ell(s)},r,h_2,g_1,h_1,w,Base_{2,1}^{\ell(s),s},\dots,Base_{2,v_s^{\ell(s)}}^{\ell(s),s},a\big)$$

be the completions of the resolutions in the taut multi-graded Makanin–

Razborov diagram of $Q_2^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ with respect to the (multi) parameter subgroups $Base_{2,1}^{\ell(s),s},\ldots,Base_{2,v_s^{\ell(s)}}^{\ell(s),s}$. We continue with each of the multi-graded resolutions $MGQRes_j$ in parallel.

We set the multi-graded resolution,

$$DRes_j\big(f_{\ell(s)},r,h_2,g_1,h_1,w,Base_{2,1}^{\ell(s),s},\ldots,Base_{2,v_s^{\ell(s)}}^{\ell(s),s},a\big)\,,$$

to be the multi-graded resolution induced by the subgroup $\langle f_{\ell(s)},r,h_2,g_1,h_1,w,p,a\rangle$ from the n-th multi-graded resolution $TMGQ^nRes$, followed by the multi-graded resolution $MGQRes_j$.

If the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the resolution $DRes_j$, $Q_D(r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of the PS limit group $PSHGH(r,h_2,g_1,h_1,w,p,a)$ we started this branch of the sieve procedure with, we replace the multi-graded resolution $MGQRes_j$ by starting a new branch of the sieve procedure with $Q_D(r,h_2,g_1,h_1,w,p,a)$ instead of $PSHGH(r,h_2,g_1,h_1,w,p,a)$.

If the subgroup generated by $\langle f_{\ell(s)},r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the resolution $DRes_j$, $Q_D(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)=Q^{\ell(s)}(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ we do the following.

We set b, $1\leq b\leq s$, to be the minimal index for which the subgroup generated by $\langle f_{\ell(b)},r,h_2,g_1,h_1,w,p,a\rangle$ in the limit group corresponding to the resolution $DRes_j$, $Q_D(f_{\ell(b)},r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^n(f_{\ell(b)},r,h_2,g_1,h_1,w,p,a)=Q^{\ell(b)}(f_{\ell(b)},r,h_2,g_1,h_1,w,p,a)$. Then we replace the multi-graded resolution $MGQRes_j$ (of the limit group $Q_2^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$), by going back to the $\ell(b)$-th step of the sieve procedure, and start it with the limit group $Q_D(f_{\ell(b)},r,h_2,g_1,h_1,w,p,a)$ instead of the $\ell(b)$-th Collapse extra PS limit group, $CollapseExtraPS^{\ell(b)}$, that was used in the $\ell(b)$-th step of the process that lead to the construction of the anvil, $Anv(MGQ^{n-1}Res)$. Since, by the definition of the index $\ell(b)$, the (multi) parameter subgroups were enlarged at step $\ell(b)+1$, by Theorem 4.18 in [S5], in analyzing the limit group $Q_D(f_{\ell(b)},r,h_2,g_1,h_1,w,p,a)$ we need to take into account only those multi-graded resolutions for which their associated core resolutions are not of maximal complexity, i.e. only those multi-graded resolutions for which their core resolution does not have the same structure as the core associated with the top level of the multi-graded resolution $MGQ^{\ell(b-1)}Res$.

Suppose that $Q_D(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)$ is isomorphic to

$$Q^n(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a)=Q^{\ell(s)}(f_{\ell(s)},r,h_2,g_1,h_1,w,p,a).$$

Since the multi-graded resolution $MGQ^{\ell(s-1)}Res$ used in step $\ell(s-1)$ to construct the anvil, $Anv(MGQ^{\ell(s)-1}Res)$, is well separated, with each QH vertex group in one the abelian decompositions associated with $MGQ^{\ell(s-1)}Res$ there is an associated collection of s.c.c. that are mapped to the trivial element in the next level of the multi-graded resolution $MGQ^{\ell(s-1)}Res$.

Each QH vertex group in the formed part of the core resolution that is associated with the top level of the multi-graded resolution $MGQ^{\ell(s-1)}Res$ naturally inherits a sequence of abelian decompositions from the multi-graded resolution $DRes_j$. If for some such QH vertex group Q, this sequence of multi-graded abelian decompositions is not compatible with the collection of s.c.c. on Q that are mapped to the trivial element in the next level of the multi-graded resolution $MGQ^{\ell(s-1)}Res$, we omit the multi-graded resolution $MGQRes_j$ from the list of completions of resolutions of the limit group $Q_2^n(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$.

Suppose that the core of a multi-graded resolution $MGQRes_j$, $MGCore(\langle r, h_2, g_1, h_1, w, p, a\rangle, MGQRes_j)$, or the core of a multi-graded resolution $DRes_j$, is of maximal possible complexity, i.e. the core has the same structure as the core associated with the top level of the multi-graded resolution $MGQ^{\ell(s-1)}Res$. By Theorem 4.15 in [S5], if the core associated with the multi-graded resolution $MGQRes_j$ is of maximal possible complexity, so is the core associated with the multi-graded resolution $DRes_j$. Hence, by Theorem 4.18 in [S5], every specialization of the PS limit group $\langle r, h_2, g_1, h_1, w, p, a\rangle$ that is contained in the Diophantine set associated with $DRes_j$, is contained in at least one of the Diophantine sets associated with a multi-graded resolution with maximal complexity core, $MGQ^{\ell(s)}Res$, of the Collapse extra PS limit group, $CollapseExtraPS^{\ell(s)}$. Since the multi-graded resolution $MGQ^{\ell(s)}Res$ (that was constructed along the path that leads to $Anv(MGQ^{n-1}Res)$) is assumed to have a core which is not of maximal complexity, we may omit multi-graded resolutions with maximal complexity core from the collection of multi-graded resolutions that appear in the taut Makanin–Razborov diagram of $Q_2^n(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$.

Suppose that the multi-graded resolution $MGQRes_j$, and its associated multi-graded resolution, $DRes_j$, do not have a maximal complexity core resolution. We analyze the multi-graded resolution $MGQRes_j$ as we did in step $\ell(s)$ of our sieve procedure. According to the various parts of step $\ell(s)$ of the sieve procedure, we associate with the multi-graded resolution $MGQRes_j$ a finite collection of graded resolutions,

$$GRes_1(r, h_2, g_1, h_1, w, p, a), \dots, GRes_r(r, h_2, g_1, h_1, w, p, a)\,.$$

With each of these graded resolutions $GRes_i(r,h_2,g_1,h_1,w,p,a)$, we associate a graded resolution $CRes_i(r,h_2,g_1,h_1,w,p,a)$, constructed from the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the top level of the n-th multi-graded resolution MGQ^nRes, $TMGQRes$, followed by the graded resolution $GRes_i(r,h_2,g_1,h_1,w,p,a)$. With each graded resolution $CRes_i(r,h_2,g_1,h_1,w,p,a)$ we can naturally associate a finite collection of framed resolutions as we did in part (2) of the first step of the sieve procedure. We continue with each of these framed resolutions in parallel, and denote each of them $FrmCRes$.

If the limit group generated by $\langle r,h_2,g_1,h_1,w,p,a\rangle$ associated with a framed resolution $FrmCRes$ is a proper quotient of $Q^n(r,h_2,g_1,h_1,w,p,a)$, we replace the framed resolution $FrmCRes$ by starting the first step of the sieve procedure with that limit group (which is a proper quotient of the PS limit group, $PSHGH$, we started the first step with). Otherwise, we follow what we did in part (2) of the first step of the sieve procedure, and associate with the framed resolution $FrmCRes$ a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions. We set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (that we denote, $Anv(MGQ^nRes)$), precisely as we did in part (2) of the first step of the sieve procedure. With each anvil we associate a finite collection of auxiliary resolutions, using the construction presented in Definition 8.

Suppose that $Q_2^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ is isomorphic to

$$Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)\,.$$

In this case we continue to the next level of the n-th multi-graded quotient resolution MGQ^nRes. Note that by Corollary 4.16 of [S5], since the multi-graded resolution MGQ^nRes is not of maximal complexity, the (multi-graded) core associated with each of its levels is not of maximal complexity as well. If for some level j of the multi-graded resolution, the image of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ in the limit group associated with this level, $Q_j^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, is a proper quotient of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, then from the highest such level j, we can continue as in case $Q_2^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, and associate with the n-th multi-graded resolution MGQ^nRes a (canonical) finite collection of framed resolutions, and with each framed resolution a canonical finite collection of Non-Rigid and Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and

Generic collapse extra PS resolutions. Each Extra PS resolution is set to be a developing resolution, and with it we associate a finite collection of anvils, and auxiliary resolutions (again, according to Definition 8), precisely as we did in case $Q_2^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$ is a proper quotient of $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$.

Finally, suppose that for every level j, the image of

$$Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$$

in the limit group associated with the j-th level of the multi-graded resolution MGQ^nRes, $Q_j^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$, is isomorphic to $Q^n(f_{n-1},r,h_2,g_1,h_1,w,p,a)$. In this case, by Lemma 19, the terminal limit group of the multi-graded resolution MGQ^nRes, $Q_{term}^n(f_n,r,h_2,g_1,h_1,w,p,a)$, is rigid or solid with respect to the parameter subgroup $P=\langle p\rangle$.

We continue as in part (2) of the first step of the sieve procedure. Let $PB^n(b_n,p,a)$ be the terminal rigid or solid limit group of the multi-graded resolution MGQ^nRes. We set the graded resolution $CRes(r,h_2,g_1,h_1,w,p,a)$, to be the resolution induced by the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ from the sequence of core resolutions associated with the various levels of the resolution MGQ^nRes, enlarged by setting its terminal limit group to be (the rigid or solid limit group) $PB^n(b_n,p,a)$ (i.e. we amalgamate the terminal limit group of the corresponding induced resolution with the subgroup $PB^n(b_n,p,a)$).

With the graded resolution $CRes(r,h_2,g_1,h_1,w,p,a)$ we associate a finite (canonical) collection of framed resolutions (see Definition 5). With each of the framed resolutions associated with $CRes(r,h_2,g_1,h_1,w,p,a)$ we associate a (canonical) finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions. If every test sequence that factors through an Extra PS resolution associated with the framed resolution $FrmCRes$, factors through a framed resolution with a bigger frame than the one associated with the framed resolution $FrmCRes$, we exclude this Extra PS resolution from the finite collection of Extra PS resolutions associated with the framed resolution $FrmCRes$.

We set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (denoted $Anv(MGQ^nRes)$), and a finite collection of auxiliary resolutions (constructed according to Definition 8), precisely as we did in part (2) of the first step.

(5) By part (1) we may assume that $Q^n(r,h_2,g_1,h_1,w,p,a)$ is isomorphic to the PS limit group, $PSHGH$, we started the first step with. Parts (2)–(4) treat all the cases in which the core associated with the multi-graded resolution MGQ^nRes is not of maximal complexity. In this part we assume that the core associated with the multi-graded resolution MGQ^nRes is of maximal complexity, i.e. that the core of the multi-graded resolution MGQ^nRes has the same structure as the core associated with the top part of the multi-graded resolution $MGQ^{n-1}Res$. As in part (5) of the second step of the procedure, in this part we also assume that there is no sculpted resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$, i.e. that one of the parts (1)–(4) applied to at least one of the core resolutions associated with the multi-graded resolutions constructed along the $n-1$ step of the sieve procedure, or that all these core resolutions are of maximal possible complexity, but the developing resolution composed from the various resolutions induced by the subgroup $\langle q_{n-2},r,h_2,g_1,h_1,w,p,a\rangle$, from this sequence of core resolutions, is not identical to the corresponding (developing or sculpted) resolution associated with the anvil, $Anv(MGQ^{n-2}Res)$, with which we started the $n-1$ step of the procedure (see Proposition 13). In this case, as we will see in the sequel, $width(n-1)=1$, and no sculpted resolution is associated with the anvil, $Anv(MGQ^{n-1}Res)$.

We treat this case as we treated part (5) in the second step of the sieve procedure. In parts (1)–(4), we have analyzed multi-graded resolutions, MGQ^nRes, of the Collapse extra PS limit group, $CollapseExtraPS^n$, that is associated with a collapse form and with an auxiliary resolution of highest level, i.e. an auxiliary resolution associated with the tower containing all the parts in the associated anvil, $Anv(MGQ^{n-1}Res)$, up to part 2 (all parts except the top part).

To analyze specializations of the PS limit group $PSHGH$, that belong to the Diophantine set associated with such a Collapse extra PS limit group, $CollapseExtraPS^n$, and belong only to Diophantine sets associated with maximal complexity multi-graded resolutions, MGQ^nRes, we first replace the Collapse extra PS limit groups associated with the given collapse form and with auxiliary resolutions of highest level, by those Collapse extra PS limit groups associated with the Extra PS resolution, the given collapse form, and with auxiliary resolutions that are associated with towers containing all the parts up to part 3, i.e. all the parts except the top two. We continue with those Collapse extra PS limit groups in parallel, hence, we will omit their index, and (still) denote the Collapse extra PS limit group we continue with, $CollapseExtraPS^n$.

As we did in part (5) of the second step, we start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^n$, with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top part of the anvil, $Anv(MGQ^{n-1}Res)$. We still denote these multi-graded resolutions MGQ^nRes.

Since in this part we need to analyze specializations that factor through and are taut with respect to maximal complexity multi-graded resolutions of Collapse extra PS limit groups, we continue only with those multi-graded resolutions in the taut Makanin–Razborov diagram of $CollapseExtraPS^n$ that are of maximal complexity, i.e. with a core that has the same structure as the core associated with the top part of the resolution, $MGQ^{n-1}Res$.

If part (1) or (3) applies to such a multi-graded resolution MGQ^nRes, i.e. if for some index s, $1 \leq s \leq s(n-1)-1$, the limit group generated by $\langle f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a \rangle$ in the completion of MGQ^nRes is a proper quotient of $CollapseExtraPS^{\ell(s)}$, we replace this resolution MGQ^nRes, by starting the $\ell(s)$-th step of the procedure with the given proper quotient of $CollapseExtraPS^{\ell(s)}$.

If the core of the top part of $MGQ^{n-1}Res$, is not of maximal complexity, and the limit group generated by $\langle f_{n-1}, r, h_2, g_1, h_1, w, p, a \rangle$ in the completion of MGQ^nRes, is a proper quotient of the Collapse extra PS limit group $CollapseExtraPS^{n-1}$, we replace this resolution MGQ^nRes, by starting the $n-1$ step of the procedure with the given proper quotient of $CollapseExtraPS^{n-1}$ (and consider only its multi-graded resolutions that are not of maximal complexity).

In case the core of the multi-graded resolution MGQ^nRes is of maximal complexity, i.e. the core has the same structure as the core of the top part of $MGQ^{n-1}Res$, we map the formed part of the core of MGQ^nRes into the subgroup of MGQ^nRes that correspond to its image in the second part of $MGQ^{n-1}Res$. At this point we analyze the terminal limit group of the multi-graded resolution, MGQ^nRes, with respect to the factors in the given free decomposition of the auxiliary limit group, $Aux(MGQ^{n-1}Res)$, exactly as we analyzed the Collapse extra PS limit group, $CollapseExtraPS^n$ in parts (1)–(4). If the multi-graded core of such a multi-graded resolution is of maximal possible complexity, and its associated taut structure is identical to the one associated with the second part of $MGQ^{n-1}Res$, i.e. if part (5) applies to an obtained multi-graded resolution, we continue in a similar way to our approach in analyzing multi-graded resolutions that their top part is of maximal complexity.

At each part i, we consider the Collapse extra PS limit groups associated with the given collapse form and with auxiliary resolutions that are associated with the tower containing all the parts up to part $i+1$. Then we analyze the taut Makanin–Razborov diagrams of the limit groups associated with the various parts (from part 1 to part $i-1$), and continue only with those resolutions that are of maximal complexity in all these parts, and the taut structures associated with the formed part of their core resolutions are identical to those associated with the formed parts of the core resolutions associated with the corresponding parts of the resolution, $MGQ^{n-1}Res$. Finally we analyze the resolutions in the taut Makanin–Razborov diagram associated with the i-th part according to parts (1)–(4), or (the first part of) (5), and continue iteratively.

Let MGQ^nRes be a multi-graded resolution obtained by the above iterative procedure. Suppose that there exists a level for which one of the parts (1)–(4) applies. We first construct a resolution composed from the resolution induced by the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$ from the parts of the resolution MGQ^nRes above the level for which parts (1)–(4) applies (i.e. the parts that are of maximal complexity), followed by the graded resolution constructed at that level according to part (1)–(4)(note that the obtained resolution is graded with respect to the parameter subgroup $\langle p\rangle$). With the obtained graded resolution we associate a canonical finite collection of framed resolutions, a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions, precisely as we did in part (4). We continue only with Extra PS resolutions that are not "covered" by framed resolutions with bigger frame. Finally, we set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (still denoted $Anv(MGQ^nRes)$) precisely as we did in part (4), and auxiliary resolutions (according to the construction presented in Definition 8).

If the structure of the resolution composed from the resolutions induced by the subgroup $\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ (which is the completion of the developing resolution, $Dvlp(MGQ^{n-1}Res) = Dvlp^{n-1}$) from the various core resolutions associated with the multi-graded resolutions used to construct the developing resolution in the n-th step of the procedure, is not identical to the structure of the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$, Proposition 13 implies that there exists some level j for which the structure of the graded abelian decomposition associated with the resolutions induced by the subgroup

$\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ above level j are identical to the structure of the graded abelian decompositions associated with the corresponding levels in the developing resolution, $Dvlp(MGQ^{n-1}Res)$, and in level j, either the number of factors in the graded free decomposition associated with the graded abelian decomposition associated with the resolution induced by $\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ is strictly smaller than the number of factors in the corresponding graded free decomposition associated with the corresponding level in $Dvlp(MGQ^{n-1}Res)$, and in case of equality in the number of factors, the complexity of the graded abelian decomposition associated with the resolution induced by $\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ is strictly smaller than the complexity of the graded abelian decomposition associated with the corresponding level of $Dvlp(MGQ^{n-1}Res)$. In this case we do the following.

With the graded resolution constructed from the various resolutions induced by the subgroups $\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ from the core resolutions associated with the various multi-graded resolutions constructed along the various levels of the n-th step of the sieve procedure, we associate a (canonical) finite collection of framed resolutions (Definition 5). With each framed resolution we associate a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, Extra PS resolutions, and Generic collapse extra PS resolutions. Finally, we set each of the Extra PS resolutions to be a developing resolution, and with it we associate a finite collection of anvils (denoted $Anv(MGQ^nRes)$), and auxiliary resolutions.

(6) Suppose that the sequence of multi-graded core resolutions of the multi-graded resolutions constructed by the process described above, are all of maximal complexity, and the sequence of abelian decompositions induced by the subgroup $\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ is identical to the corresponding abelian decompositions associated with the developing resolution, $Dvlp(MGQ^{n-1}Res)$. In this case we continue as in part (6) of the second step of the procedure (note that in this part we still assume that there is no sculpted resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$, i.e. $width(n-1)=1$).

We start the analysis, with the collection of Collapse extra PS limit groups obtained from the anvil, $Anv(MGQ^{n-1}Res)$ (and not with any of its associated auxiliary limit groups), and its (finitely many) associated collapse forms. We still denote each of the obtained Collapse extra PS limit groups, $CollapseExtraPS^n$.

We first analyze the Collapse extra PS limit groups, $CollapseExtraPS^n$, using the iterative process that is presented in part (6) of the second step of the procedure. We start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^n$, with respect to the subgroups $Base^{n,s(n)}_{2,1},\dots,Base^{n,s(n)}_{2,v^n_{s(n)}}$, where the subgroups $Base^{n,s(n)}_{2,j}$, $1 \leq j \leq v^n_{s(n)}$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the top part of the anvil, $Anv(MGQ^{n-1}Res)$, and with respect to the formed part of the core resolution, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, MGQ^{n-1}Res)$, associated with the top part of the anvil, $Anv(MGQ^{n-1}Res)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the subgroups $Base^{n,s(n)}_{2,1},\dots,Base^{n,s(n)}_{2,v^n_{s(n)}}$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the core, $MGCore(\langle r,h_2,g_1,h_1,w,p,a\rangle, MGQRes)$. We (still) denote these (two parts) multi-graded resolutions, for which the second part is one level and has the same structure as the formed part of the core resolution associated with the top part of $Anv(MGQ^{n-1}Res)$, MGQ^nRes. We further use the modular groups associated with the formed part of the core resolution associated with the top part of the anvil, $Anv(MGQ^{n-1}Res)$, to map the subgroup associated with this formed part in the resolution MGQ^nRes, onto its image in the subgroup associated with the second part of $Anv(MGQ^{n-1}Res)$.

We proceed iteratively to the next levels. At each level i, we start with the Makanin–Razborov diagram of the terminal limit group of the resolution obtained from the top $i-1$ parts, with respect to the subgroups, $Base^{n,s(n)}_{i+1,1},\dots,Base^{n,s(n)}_{i+1,t^n_{s(n)}}$, where the subgroups $Base^{n,s(n)}_{i+1,j}$, $1 \leq j \leq t^n_{s(n)}$, are the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th part of the anvil, $Anv(MGQ^{n-1}Res)$, and with respect to the formed part of the core resolution associated with the i-th level of $Anv(MGQ^{n-1}Res)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the subgroups, $Base^{n,s(n)}_{i+1,1},\dots,Base^{n,s(n)}_{i+1,t^n_{s(n)}}$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the core resolution associated with the i-th part of the anvil, $Anv(MGQRes)$. We (still) denote the resolutions obtained from the top i levels, MGQ^nRes. We further use the modular groups associated with the formed part of the core associated with the i-th level of $Anv(MGQ^{n-1}Res)$, to map the subgroup associated with this formed part in the resolution MGQ^nRes, onto its image in the subgroup associated with the $i+1$ level of $Anv(MGQ^{n-1}Res)$.

The outcome of the above terminating procedure is a (telescopic) sequence of multi-graded resolutions, that we (still) denote MGQ^nRes. Let $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$ be the subgroup generated by the image of the developing resolution, $Dvlp^{n-1}(q_{n-1}, r, h_2, g_1, h_1, w, p, a)$, associated with the anvil, $Anv(MGQ^{n-1}Res)$, and the elements associated with the Diophantine conditions imposed by the given collapse form in the Collapse extra PS limit group, $CollapseExtraPS^n$. With each level of a multi-graded resolution constructed in one of the parts, MGQ^nRes, we associate its core resolution with respect to the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$, and the resolution induced from this (one level) core resolution by the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$. The collection of these induced resolutions, associated with the various levels of the multi-graded resolution MGQ^nRes, gives rise to a resolution (that is embedded in the multi-graded resolution MGQ^nRes), that we denote MGQ^nRes_c, of the image of the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$, that is canonically associated with MGQ^nRes. We denote the graded resolution (with respect to the defining parameters $P = \langle p\rangle$) obtained as the compositions of these (induced) resolutions, MGQ^nRes_c, associated with the (telescopic) sequence of multi-graded resolutions MGQ^nRes, $GRes(e, c, r, h_2, g_1, h_1, w, p, a)$. By Proposition 13, we may iteratively repeat this construction of induced resolutions with the subgroup associated with the completion of the obtained resolution $GRes$, until we obtain a graded resolution, that we still denote $GRes$, which is embedded in the sequence of completions of the multi-graded resolutions MGQ^nRes.

With each resolution MGQ^nRes_c associated with a multi-graded resolution MGQ^nRes, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$. By Theorem 4.13 in [S5], either there exists a resolution MGQ^nRes_c for which the complexity of its associated core resolution (with respect to the image of $\langle r, h_2, g_1, h_1, w, p, a\rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level of the developing resolution, $Dvlp^{n-1}$, associated with the anvil, $Anv(MGQ^{n-1}Res)$, and the complexities of all the core resolutions associated with the resolutions MGQ^nRes_c that are associated with the parts above it, are identical to the complexities of the abelian decompositions associated with the corresponding levels of $Dvlp^{n-1}$, or the complexities of the core resolutions associated with the various resolutions MGQ^nRes_c are all identical to the complexities of the abelian decompositions associated with the corresponding levels of $Dvlp^{n-1}$, and the structures of these core resolutions are similar to the structures of the corresponding abelian decompositions in $Anv(MGQ^{n-1}Res)$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core of MGQ^nRes_c (with respect to the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level in the developing resolution, $Dvlp^{n-1}$. In this case we associate with the resolution MGQ^nRes a finite collection of developing resolutions and anvils, and with each anvil we associate sculpted resolutions (of widths 1 and 2), a carrier, that we denote $Carrier_2^n$, and auxiliary resolutions, precisely as we did in this case in part (6) of the second step of the procedure. In this case, $width(n) = 2$, and $sc(n,1) = sc(n,2) = 1$.

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions MGQ^nRes_c, that are associated with the multi-graded resolutions, MGQ^nRes, constructed by the iterative procedure presented above, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, are all of maximal complexity, i.e. each of these multi-graded core resolutions has the same structure (and taut structure) as the abelian decomposition associated with the corresponding level in $Dvlp^{n-1}$.

We set $SCRes(s, r, h_2, g_1, h_1, w, p, a)$ to be the graded resolution (graded with respect to the parameter subgroup $P = \langle p\rangle$), composed from the resolutions induced by the subgroup $\langle q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ (that is associated with the developing resolution, $Dvlp^{n-1}$) from the various core resolutions associated with the multi-graded resolutions, MGQ^nRes_c (with respect to the images of the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$).

If every ungraded resolution that factors through the graded resolution $SCRes$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic Collapse extra PS resolutions associated with the developing (extra PS) resolutions we started the n-th step of the procedure with, or through a framed resolution, $FrmSCRes$, associated with $SCRes$, where the frame associated with the framed resolution $FrmSCRes$ strictly contains the frame associated with the graded resolution $SCRes$, we call the sequence of multi-graded resolutions from which the graded resolution $SCRes$ was constructed, a *terminal resolution*, and do not continue with it to the next step of the procedure. Otherwise, we do the following.

With the sequence of multi-graded resolutions, MGQ^nRes, and their associated resolution, $GRes$, we associate developing resolutions and anvils, and with each anvil we further associate sculpted resolutions (of widths 1 and 2), and a penetrated sculpted resolution (of width 2), and auxiliary resolutions, precisely as we did in this case in part (6) of the second step of the procedure. In this case, $width(n) = 2$, and $sc(n,1) = sc(n,2) = 1$.

(7) Parts (1)–(6) treat all the cases in which there is no sculpted resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$, i.e. the case in which $width(n-1) = 1$. In this part we treat the general case. We assume that the anvil $Anv(MGQ^{n-1}Res)$ has width, $width(n-1) \geq 2$, and with it there are associated a developing resolution, $Dvlp^{n-1}(q_{n-1}, r, h_2, g_1, h_1, w, p, a)$, sculpted resolutions, that we denote $SCRes_e^{n-1,d}(s_e, r, h_2, g_1, h_1, w, p, a)$, and corresponding penetrated sculpted resolutions,

$$PenSCRes_e^{n-1,d}(u_e, r, h_2, g_1, h_1, w, p, a)\,,$$

where $1 \leq d \leq width(n-1)$, and for each d, $1 \leq e \leq sc(n-1,d)$, where $sc(n-1,d)$ is the number of sculpted resolutions in width d and step $n-1$. With the anvil, $Anv(MGQ^{n-1}Res)$, and a subset of the width indices d, $1 \leq d \leq width(n-1)$, there are also associated carriers, $Carrier_d^{n-1}$. Our analysis of this general case is based on part (7) of the second step of the procedure.

Recall, that according to Definition 14, if $width(n-1) > 1$, then a finite collection of auxiliary resolutions of widths 1 to $width(n-1)$ is associated with the anvil, $Anv(MGQ^{n-1}Res)$. We start by analyzing Collapse extra PS limit groups associated with highest level auxiliary resolutions of width 1. Parts (1)–(4) already analyze multi-graded resolutions of such Collapse extra PS limit groups, MGQ^nRes, that are not of maximal complexity. To analyze specializations that factor through maximal complexity resolutions, MGQ^nRes, we continue analyzing Collapse extra PS limit groups associated with width 1 auxiliary resolutions according to part (5).

Suppose that there exists a Collapse extra PS limit group associated with the anvil, $Anv(MGQ^{n-1}Res)$, itself (and not with a width 1 auxiliary limit group of it), and a sequence of multi-graded resolutions of it, MGQ^nRes, that are constructed according to the iterative process presented in part (5), for which none of the parts (1)–(4) applies to any of the levels that are analyzed according to the iterative process (hence, in particular, all the core resolutions associated with the multi-graded resolutions MGQ^nRes and the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$ are of maximal possible complexity). Furthermore, suppose that the resolution induced by the subgroup that is associated with the first sculpted resolution, $SCRes_1^{n-1,1}$, from the various core resolutions that are associated with the sequence of multi-graded resolutions, MGQ^nRes, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$, has the same structure as the first sculpted resolution $SCRes_1^{n-1,1}$ (i.e. no reduction in the complexity of the abelian decompositions associated with the induced resolution according to Proposition 13 occurs). These are precisely the assumptions on the sequence of multi-graded resolutions, MGQ^nRes, which the process presented in part (5) is not built to handle. Hence, we proceed as in part (7) of the second step.

We continue the analysis by replacing the Collapse extra PS limit groups that were associated with auxiliary resolutions of width 1, by those associated with auxiliary resolutions of width 2 (see Definition 14). We start with Collapse extra PS limit groups, $CollapseExtraPS^n$, associated with highest level auxiliary resolutions of width 2, and we analyze them in parallel. We apply parts (1)–(4) to study limit groups and core resolutions associated with the top level of the second algebraic envelope that is associated with the anvil, $Anv(MGQ^{n-1}Res)$, i.e. the core associated with the image of the limit group associated with the second algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$, in $CollapseExtraPS^n$, and with multi-graded resolutions of such Collapse extra PS limit group, $CollapseExtraPS^n$, with respect to the various factors in the associated (width 2) auxiliary limit group (cf. part (7) in the second step).

Suppose that one of the parts (1)–(4) applies to such a multi-graded resolution (that we still denote MGQ^nRes), and to the limit group associated with the second algebraic envelope associated with the anvil, $Anv(MGQ^{n-1}Res)$. In this case the construction, presented in the parts (1)–(4) that apply, terminates in a limit group $MGTerm_1$, which is a rigid or solid multi-graded limit group with respect to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of (the completion of) the first sculpted resolution of width 2 (i.e. the first sculpted resolution that is associated with the second algebraic envelope), $SCRes_1^{n-1,2}$, and the abelian and QH vertex groups in the formed parts of the abelian decompositions associated with the core resolutions that are associated with the various levels of the first sculpted resolution of width 2, $SCRes_1^{n-1,2}$.

We continue with the multi-graded resolutions of the terminal rigid or solid limit group of the resolution MGQ^nRes, $MGTerm_1$, with respect to the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of (the completion of) the sculpted resolution, $SCRes_1^{n-1,2}$, and the abelian and QH vertex groups in the formed parts of the abelian decompositions associated with the core resolutions that are associated with all the levels of the sculpted resolution, $SCRes_1^{n-1,2}$, except the top level. Like in part (6), and parts (6)–(7) in the second step, in this part we need to analyze only resolutions associated with $CollapseExtraPS^n$ (that is associated with a width 2 auxiliary resolution), for which the multi-graded resolution of the terminal limit group of MGQ^nRes, $MGTerm_1$, is identical to the formed part of the abelian decomposition associated with the core resolution that is associated with the top level of the sculpted resolution, $SCRes_1^{n-1,2}$ (for more details on why we can restrict to these resolutions see Theorem 21 below — it proves that the collection of anvils and terminal resolutions constructed in the general step of the procedure covers the entire set of Collapsed extra PS specializations). We further use the modular groups associated with the formed part of the core resolution associated with the top level of the sculpted resolution, $SCRes_1^{n-1,2}$, to map the subgroup associated with this formed part in the resolution MGQ^nRes, onto its image in the subgroup associated with the second level of the sculpted resolution.

We proceed to lower levels in a similar way to what we did in part (4) of the first step, i.e. we proceed iteratively. At each level i of the sculpted resolution, $SCRes_1^{n-1,2}$, we start with the terminal limit group of the resolution obtained from the top $i-1$ levels, and analyze its multi-graded resolutions with respect to the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th level of the sculpted resolution, and with respect to the formed parts of the abelian decompositions associated with the core resolutions that are associated with all the levels of the sculpted resolution, $SCRes_1^{n-1,2}$, up to level i. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group of such a resolution, with respect to (only) the non-QH, non-abelian vertex groups in the graded abelian decomposition associated with the i-th level of the sculpted resolution, $SCRes_2^{n-1}$, and the formed parts of the abelian decompositions associated with the core resolutions that are associated with all the levels of the sculpted resolution, $SCRes_2^{n-1}$, up to level $i+1$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the abelian decomposition associated with the core resolution that is associated with the i-th level of the sculpted resolution, $SCRes_1^{n-1,2}$. We (still) denote the resolutions obtained from the top i levels, MGQ^nRes. We further use the modular groups associated with the formed part of the core resolution that is associated with the i-th level of the sculpted resolution, to map the subgroup associated with this formed part in the resolution MGQ^nRes, onto its image in the subgroup associated with the $i+1$ level of the sculpted resolution.

Given a (telescopic) sequence of multi-graded resolutions, MGQ^nRes, constructed by the above procedure, we continue as in part (7) of the second step, and part (6) of the general step. Let $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$ be the subgroup associated with the second algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$, i.e. the subgroup generated by the image of the (completion of the) sculpted resolution, $SCRes_1^{n-1,2}$, associated with the anvil, $Anv(MGQ^{n-1}Res)$, and the elements associated with the Diophantine conditions imposed by the given collapse form associated with this sculpted resolution in $Anv(MGQ^{n-1}Res)$. With each level of a multi-graded resolution constructed in one of the parts, MGQ^nRes, we associate its core resolution with respect to the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$, and the resolution induced from this (one level) core resolution by the (image of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$. The collection of these induced resolutions, associated with the various levels of the multi-graded resolution MGQ^nRes, gives rise to a resolution (that is embedded in the multi-graded resolution MGQ^nRes), that we denote MGQ^nRes_c, of the image of the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a\rangle$, that is canonically associated with MGQ^nRes. We denote the graded resolution (with respect to the defining parameters $P = \langle p\rangle$) obtained as the compositions of these (induced) resolutions, MGQ^nRes_c, associated with the (telescopic) sequence of multi-graded resolutions MGQ^nRes, $GRes(e, c, r, h_2, g_1, h_1, w, p, a)$. By Proposition 13, we may iteratively repeat this construction of induced resolutions with the subgroup associated with the completion of the obtained resolution $GRes$, until we obtain a graded resolution, that we still denote $GRes$, which is embedded in the sequence of completions of the multi-graded resolutions MGQ^nRes.

With each resolution MGQ^nRes_c associated with a multi-graded resolution MGQ^nRes, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$. By Theorem 4.13 in [S5], either there exists a resolution MGQ^nRes_c for which the complexity of its associated core resolution (with respect to the image of $\langle r, h_2, g_1, h_1, w, p, a\rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level of the sculpted resolution, $SCRes_1^{n-1,2}$, and the complexities of all the core resolutions associated with the resolutions MGQ^nRes_c that are associated with the parts above it, are identical to the complexities of the abelian decompositions associated with the corresponding levels of $SCRes_1^{n-1,2}$, or the complexities of the core resolutions associated with the various resolutions MGQ^nRes_c are all identical to the complexities of the abelian decompositions associated with the corresponding levels of $SCRes_1^{n-1,2}$, and the structures of these core resolutions are similar to the structures of the corresponding abelian decompositions associated with $SCRes_1^{n-1,2}$ (see Definition 4.12 in [S5] for the complexity of a core resolution).

Suppose that there exists a part for which the complexity of the associated core of MGQ^nRes_c (with respect to the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$) is strictly smaller than the complexity of the abelian decomposition associated with the corresponding level in $SCRes_1^{n-1,2}$. In this case we associate with the resolution MGQ^nRes a finite collection of developing resolutions and anvils, and with each anvil we associate sculpted resolutions (of widths 1 and 2), a penetrated sculpted resolution (of width 2), a carrier, that we denote $Carrier_2^n$, and auxiliary resolutions, precisely as we did in this case in part (6) of the second step of the procedure. In this case, $width(n)=2$, and $sc(n,1)=sc(n,2)=1$.

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions MGQ^nRes_c, that are associated with the multi-graded resolutions, MGQ^nRes, constructed by the iterative procedure presented above, and with respect to the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, are all of maximal complexity, i.e. each of these multi-graded core resolutions have the same structure (and taut structure) as the abelian decomposition associated with the corresponding level of the sculpted resolution $SCRes_1^{n-1,2}$, that is associated with the anvil, $Anv(MGQ^{n-1}Res)$.

We set $SCRes(s,r,h_2,g_1,h_1,w,p,a)$ to be the graded resolution induced by the image of the completion of the sculpted resolution, $SCRes_1^{n-1,2}$, from the core resolutions of the multi-graded resolutions, MGQ^nRes_c, constructed along the various levels of the process described above, with respect to the subgroup $\langle r,h_2,g_1,w,p,a\rangle$. Note that since we assume that the core resolutions associated with the resolutions, MGQ^nRes_c, and the subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$, have the same structure as the abelian decompositions associated with the various levels of the sculpted resolution, $SCRes_1^{n-1,2}$, the resolution $SCRes$ has the same structure as the sculpted resolution, $SCRes_1^{n-1,2}$.

If every ungraded resolution that factors through the graded resolution $SCRes$, factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or generic Collapse extra PS resolutions associated with the sculpted resolution $SCRes_1^{n-1,2}$, or through a framed resolution, $FrmSCRes$, associated with $SCRes_1^{n-1,2}$, where the frame associated with the framed resolution $FrmSCRes$ strictly contains the frame associated with the graded resolution $SCRes_1^{n-1,2}$, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution*. Otherwise we do the following.

Recall that we denote the graded resolution composed from the sequence of resolutions, MGQ^nRes_c, that are associated with the multi-graded resolutions, MGQ^nRes, $GRes$. Note that $GRes$ is a resolution of the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$ that is associated with the second algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$. With the resolution $GRes$ we associate developing resolutions and anvils, and with each anvil we further associate sculpted resolutions (of widths 1 and 2), and a penetrated sculpted resolution (of width 2), and auxiliary resolutions, precisely as we did in this case in part (6) of the second step of the procedure. In this case, $width(n)=2$, and $sc(n,1)=sc(n,2)=1$.

So far we have analyzed multi-graded resolutions of Collapse extra PS limit groups, $CollapseExtraPS^n$, for which parts (1)–(4) are applied with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$ (the subgroup associated with the second algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$).

Suppose that none of the parts (1)–(4) applies to such a multi-graded resolution, MGQ^nRes, with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$. In this case we proceed as in part (3) of the first step and part (5) of the second step. For each level i of the anvil, $Anv(MGQ^{n-1}Res)$, we analyze Collapse extra PS limit groups, $CollapseExtraPS^n$, that are associated with width 2 auxiliary resolutions that are associated with the limit group that is associated with all the levels of the anvil, $Anv(MGQRes)$, except the top i levels. As in part (5) of the second step, given such a Collapse extra PS limit group, $CollapseExtraPS^n$, we analyze its multi-graded resolutions so that none of the parts (1)–(4) applies to the multi-graded resolutions associated with the top $i-1$ levels (with respect to the subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$), and (at least) one of the parts (1)–(4) applies to the multi-graded resolution associated with the i-th part of the anvil.

Given such a multi-graded resolution, we treat it as we treated multi-graded resolutions associated with highest level width 2 auxiliary resolutions.

We continue the analysis of the terminal rigid or solid limit group of the obtained multi-graded resolution, and continue only with those multi-graded resolutions for which the abelian decompositions associated with the parts corresponding to the formed parts of the core resolutions associated with the anvil, $Anv(MGQ^{n-1}Res)$, and the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$, have the same structure as these formed parts. Finally, with each such multi-graded resolution (that we still denote MGQ^nRes), we associate sculpted resolutions, (possibly) a carrier, and developing resolutions and anvils, precisely as we constructed these in case one of the parts (1)–(4) applies to the multi-graded resolution, MGQ^nRes, associated with the top level of the anvil, $Anv(MGQ^{n-1}Res)$ (i.e. precisely as we did in the beginning of part (7)).

Suppose that for a Collapse extra PS limit group, $CollapseExtraPS^n$, associated with the anvil, $Anv(MGQ^{n-1}Res)$, itself (and not with a width 2 auxiliary resolution associated with it), there exists a sequence of multi-graded resolutions constructed by the above iterative procedure, for which none of the parts (1)–(4) applies to the various levels (with respect to the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$). In particular, all the core resolutions associated with the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$ are of maximal complexity.

In this case we examine the structure of the resolution associated with the second algebraic envelope that is associated with the anvil, $Anv(MGQ^{n-1}Res)$. Recall that (according to the construction presented in part (6)) with the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$, that is associated with the second algebraic envelope in $Anv(MGQ^{n-1}Res)$, and with the sequence of multi-graded resolutions, $MGQ^{n-1}Res$, that were constructed in step $n-1$, there are associated core resolutions, and a resolution that is composed from resolutions induced by the subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$ from the sequence of core resolutions. Let $\langle e, c, r, h_2, g_1, h_1, w, p, a \rangle$ be the completion of this composed (induced) resolution.

If the structure of the resolution composed from the resolutions induced by the (image of the) subgroup $\langle e, c, r, h_2, g_1, h_1, w, p, a \rangle$ from the various core resolutions associated with the resolutions MGQ^nRes and the (images of the) subgroup $\langle c, r, h_2, g_1, h_1, w, p, a \rangle$, is not identical to the structure of the resolution that is associated with the second algebraic envelope that is associated with the anvil, $Anv(MGQ^{n-1}Res)$, Proposition 13 implies that there exists some level j for which the structure of the graded abelian decompositions associated with the resolution induced by the subgroup $\langle e, c, r, h_2, g_1, h_1, w, p, a \rangle$ above level j are identical to the structure of the graded abelian decompositions associated with the corresponding levels in the resolution that is associated with the second algebraic envelope in $Anv(MGQ^{n-1}Res)$, and in level j, either the number of factors in the graded free decomposition associated with the graded abelian decomposition associated with the resolution induced by $\langle e,c,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the number of factors in the corresponding graded free decomposition associated with the corresponding level in the resolution that is associated with the second algebraic envelope, and in case of equality in the number of factors, the complexity of the graded abelian decomposition associated with the resolution induced by $\langle e,c,r,h_2,g_1,h_1,w,p,a\rangle$ is strictly smaller than the complexity of the graded abelian decomposition associated with level j of that resolution.

In this case we treat the obtained induced resolution, as in the case one of the cases (1)–(4) applies in one of the parts of the iterative procedure that was used (in part (7)) to construct the sequence of multi-graded resolutions MGQ^nRes. In this case $width(n)=2$, and $sc(n,2)$ is either 1 or 2.

Suppose that the structure of the resolution induced by the subgroup $\langle e,c,r,h_2,g_1,h_1,w,p,a\rangle$ from the various core resolutions associated with the resolutions MGQ^nRes and the (images of the) subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$, is identical to the structure of the resolution that is associated with the second algebraic envelope that is associated with the anvil, $Anv(MGQ^{n-1}Res)$.

Suppose that there is no carrier, $Carrier_2^{n-1}$, associated with the second algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$. In this case we set $GRes_2$ to be the resolution induced by the subgroup $\langle e,c,r,h_2,g_1,h_1,w,p,a\rangle$ from the various core resolutions associated with the resolutions MGQ^nRes and the (images of the) subgroup $\langle c,r,h_2,g_1,h_1,w,p,a\rangle$. We further set $GRes_1$ to be the resolution induced by the image of the first sculpted resolution (of width 2), $SCRes_1^{n-1,2}$, from the core resolutions associated with the (images of the) subgroup $\langle r,h_2,g_1,h_1,w,p,a\rangle$ and the resolution $GRes_2$. Note that by our assumptions $GRes_1$ has the same structure as the first sculpted resolution, $SCRes_1^{n-1,2}$, and $GRes_2$ has the same structure as the second sculpted resolution, $SCRes_2^{n-1,2}$.

If every ungraded resolution that factors through the graded resolution $GRes_1$ factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the sculpted resolution $SCRes_1^{n-1,2}$ ($SCRes_2^{n-1,2}$), or through a framed resolution associated with $SCRes_1^{n-1,2}$ ($SCRes_2^{n-1,2}$) that strictly contains the frame associated with $SCRes_1^{n-1,2}$, ($SCRes_2^{n-1,2}$), we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution*. Otherwise we continue by studying Collapse extra PS limit groups associated with the next algebraic envelopes that are associated with the anvil, $Anv(MGQ^{n-1}Res)$.

Suppose that there is a carrier, $Carrier_2^{n-1}$, associated with the second algebraic envelope associated with the anvil, $Anv(MGQ^{n-1}Res)$. In this case we study the structure of the resolution induced by the subgroup associated with the penetrated core resolution, $PenSCRes_1^{n-1,2}$, from the core resolutions associated with that subgroup, and with the resolution induced by the subgroup $\langle e,c,r,h_2,g_1,h_1,w,p,a\rangle$ from the various core resolutions that are associated with the sequence of multi-graded resolutions, MGQ^nRes. We denote this induced resolution, $PenSCRes$. If the structure of this induced resolution, $PenSCRes$, is not identical to the structure of the penetrated sculpted resolution, $PenSCRes_1^{n-1,2}$, we proceed as we did in the case that one of the parts (1)–(4) applies to the resolutions MGQ^nRes.

Suppose that the structure of the induced resolution, $PenSCRes$, is identical to the structure of the penetrated sculpted resolution, $PenSCRes_1^{n-1,2}$. If every ungraded resolution that factors through the graded resolution $PenSCRes$ factors through either one of the Non-Rigid, Non-Solid, Root, Left PS or Generic collapse extra PS resolutions associated with the sculpted resolution $PenSCRes_1^{n-1,2}$, or through a framed resolution associated with $PenSCRes_1^{n-1,2}$ that strictly contains the frame associated with $PenSCRes_1^{n-1,2}$, we do not continue to the next step of the sieve procedure with the given sequence of multi-graded resolutions we have constructed, and call them a *terminal resolution*. Otherwise we continue by studying Collapse extra PS limit groups associated with the next algebraic envelopes that are associated with the anvil, $Anv(MGQ^{n-1}Res)$.

So far we have analyzed the sculpted, and (possibly) penetrated sculpted and developing resolutions associated with the first two algebraic envelopes that are associated with the anvil, $Anv(MGQ^{n-1}Res)$. For the rest of this section, we assume that there exists a Collapse extra PS limit group, $CollapseExtraPS^n$, that is associated with the anvil, $Anv(MGQ^{n-1}Res)$, itself (and not with a width 2 auxiliary resolution associated with it), with a resolution constructed by the iterative procedure presented in part (7) of the general step, for which all the core resolutions, and the resolutions induced by the subgroups that are associated with the sculpted, penetrated sculpted and (possibly) developing resolutions of width at most 2, have the same structure as the corresponding resolutions that are associated with the anvil, $Anv(MGQ^{n-1}Res)$.

We continue by induction on the index of the algebraic envelopes that are associated with the anvil, $Anv(MGQ^{n-1}Res)$. Recall that we assume that the anvil $Anv(MGQ^{n-1}Res)$ has width, $width(n-1) \geq 2$, and with it there are associated a developing resolution, $Dvlp^{n-1}(q_{n-1}, r, h_2, g_1, h_1, w, p, a)$, sculpted resolutions, which we denote $SCRes_e^{n-1,d}(s_e, r, h_2, g_1, h_1, w, p, a)$, and corresponding penetrated sculpted resolutions, $PenSCRes_e^{n-1,d}(r_e, r, h_2, g_1, h_1, w, p, a)$, where $1 \leq d \leq width(n-1)$, and for each d, $1 \leq e \leq sc(n-1, d)$, where $sc(n-1, d)$ is the number of sculpted resolutions in width d and step $n-1$. With the anvil, $Anv(MGQ^{n-1}Res)$, and a subset of the width indices d, $1 \leq d \leq width(n-1)$, there are also associated carriers, $Carrier_d^{n-1}$.

We continue by induction on the index of the associated algebraic envelope, and for each index d, $2 < d \leq width(n-1)$, we construct resolutions of Collapse extra PS limit groups, $CollapseExtraPS^n$, that are associated with width d auxiliary resolutions according to the iterative procedure presented in part (7). We further analyze each of the constructed resolutions, with respect to the subgroups associated with the core resolutions, penetrated core resolutions, and (possibly) developing resolution that are associated with the first d algebraic envelopes that are associated with the anvil, $Anv(MGQ^{n-1}Res)$. If for such a resolution that is associated with the anvil, $Anv(MGQ^{n-1}Res)$, itself (and not with a width d auxiliary resolution) all the core resolutions, and the resolutions induced by the subgroups that are associated with the sculpted, penetrated sculpted and (possibly) developing resolution of width at most d, have the same structure as the corresponding resolutions that are associated with the anvil, $Anv(MGQ^{n-1}Res)$, we continue by analyzing Collapse extra PS limit groups associated with auxiliary resolutions of width $d+1$.

Suppose that there exists a Collapse extra PS limit group, $CollapseExtraPS^n$, that is associated with the anvil, $Anv(MGQ^{n-1}Res)$, itself (and not with a width $width(n-1)$ auxiliary resolution of it), with a resolution constructed by the iterative procedure presented in part (7) of the general step, for which all the core resolutions, and the resolutions induced by the subgroups that are associated with the sculpted, penetrated sculpted and developing resolution of width at most $width(n-1)$, have the same structure as the corresponding resolutions that are associated with the anvil, $Anv(MGQ^{n-1}Res)$.

We essentially generalize what we did in part (7) of the second step in this case. We start the analysis with the collection of Collapse extra PS limit groups, $CollapseExtraPS^n$, obtained from the anvil, $Anv(MGQ^{n-1}Res)$ (and not with any of its associated auxiliary limit groups), and its (finitely many) associated collapse forms.

We first analyze the Collapse extra PS limit groups, $CollapseExtraPS^n$, using an iterative process which is similar to the one used in part (6) and in the beginning of part (7). We start with the multi-graded taut Makanin–Razborov diagram of the Collapse extra PS limit group, $CollapseExtraPS^n$, with respect to the non-QH, non-abelian vertex groups in the multi-graded abelian decomposition associated with the top part of the anvil, $Anv(MGQRes)$, and with respect to the formed part of the core resolution that is associated with the subgroup associated with the algebraic envelope of width, $width(n-1)$, and with the top part of the anvil, $Anv(MGQ^{n-1}Res)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the non-QH, non-abelian vertex groups in the multi-graded abelian decomposition associated with the top part of the anvil, $Anv(MGQRes)$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the core resolution that is associated with the subgroup associated with the algebraic envelope of width, $width(n-1)$, and with the top part of the anvil, $Anv(MGQ^{n-1}Res)$. We (still) denote these (two parts) multi-graded resolutions, MGQ^nRes. We further use the modular groups associated with the formed part of the core resolution associated with the algebraic envelope of width, $width(n-1)$, and with the top part of the anvil, $Anv(MGQ^{n-1}Res)$, to map the subgroup associated with this formed part in the resolution MGQ^nRes, onto its image in the subgroup associated with the second part of $Anv(MGQ^{n-1}Res)$.

We proceed iteratively to the next levels. At each part i, we start with the Makanin–Razborov diagram of the terminal limit group of the resolution obtained from the top $i-1$ parts, with respect to the non-QH, non-abelian vertex groups in the multi-graded abelian decomposition associated with the i-th part of the anvil, $Anv(MGQRes)$, and with respect to the formed part of the core resolution associated with the algebraic envelope of width, $width(n-1)$, and with the i-th part of the anvil, $Anv(MGQ^{n-1}Res)$. We proceed with the multi-graded taut Makanin–Razborov diagram of the terminal (rigid or solid) limit group with respect to the non-QH, non-abelian vertex groups in the abelian decomposition associated with the i-th part of the anvil, $Anv(MGQ^{n-1}Res)$. We continue only with those multi-graded resolutions in the second taut Makanin–Razborov diagram that are of maximal possible complexity, i.e. those resolutions that are one level and have the same structure, and the same taut structure, as the formed part of the core resolution associated with the algebraic envelope of width, $width(n-1)$, and with the i-th part of the anvil, $Anv(MGQ^{n-1}Res)$. We (still) denote the resolutions obtained from the top i levels, MGQ^nRes. We further use the modular groups associated with the formed part of the corresponding core resolution (that is associated with the i-th part of $Anv(MGQ^{n-1}Res)$), to map the subgroup associated with this formed part in the resolution MGQ^nRes, onto its image in the subgroup associated with the $i+1$ part of $Anv(MGQ^{n-1}Res)$.

Given a (telescopic) sequence of multi-graded resolutions, MGQ^nRes, constructed by the above procedure, we continue as in analyzing the second algebraic envelope. Let $\langle c, q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$ be the subgroup generated by the image of the (completion of the) developing resolution, $Dvlp^{n-1}$, associated with the anvil, $Anv(MGQ^{n-1}Res)$, and the elements associated with the Diophantine conditions imposed by the given collapse form associated with this developing resolution in the Collapse extra PS limit group, $CollapseExtraPS^n$. With each level of a multi-graded resolution constructed in one of the parts, MGQ^nRes, we associate its core resolution with respect to the (image of the) subgroup $\langle c, q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$, and the resolution induced from this (one level) core resolution by the (image of the) subgroup $\langle c, q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$. The collection of these induced resolutions, associated with the various levels of the multi-graded resolution MGQ^nRes, gives rise to a resolution (that is embedded in the multi-graded resolution MGQ^nRes), that we denote MGQ^nRes_c, of the image of the subgroup $\langle c, q_{n-1}, r, h_2, g_1, h_1, w, p, a\rangle$, that is canonically associated with MGQ^nRes. We denote the graded resolution (with respect to the defining parameters $P=\langle p\rangle$) obtained as the compositions of these (induced) resolutions, MGQ^nRes_c, associated with the (telescopic) sequence of multi-graded resolutions MGQ^nRes, $GRes(e, c, q_{n-1}, r, h_2, g_1, h_1, w, p, a)$. By Proposition 13, we may iteratively repeat this construction of induced resolutions with the subgroup associated with the completion of the obtained resolution $GRes$, until we obtain a graded resolution, that we still denote $GRes$, which is embedded in the sequence of completions of the multi-graded resolutions MGQ^nRes.

With each resolution MGQ^nRes_c associated with a multi-graded resolution MGQ^nRes, we associate its core resolution with respect to (the image of) the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$. By Theorem 16 either there exists a resolution $MGQ^n Res_c$ for which the complexity of its associated core resolution (with respect to the image of $\langle r, h_2, g_1, h_1, w, p, a\rangle$) is strictly smaller than the complexity of the core resolution associated with the corresponding part (and the image of the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$) in the resolution associated with the $width(n-1)$ algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$, and the complexities of all the core resolutions associated with the resolutions $MGQ^n Res_c$ that are associated with the parts above it, are identical to the complexities of core resolutions associated with the corresponding parts in the resolution associated with the $width(n-1)$ envelope in $Anv(MGQ^{n-1}Res)$, or the structures of the core resolutions associated with the various resolutions $MGQ^n Res_c$ are all identical to the structures of the corresponding core resolutions associated with the $width(n-1)$ algebraic envelope in $Anv(MGQ^{n-1}Res)$.

Suppose that there exists a part for which the complexity of the associated core of $MGQ^n Res_c$ (with respect to the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$) is strictly smaller than the complexity of the core resolution associated with the corresponding part in the resolution associated with the $width(n-1)$ algebraic envelope associated with the anvil, $Anv(MGQ^{n-1}Res)$. In this case we associate with the sequence of resolutions $MGQ^n Res$ a finite collection of developing resolutions and anvils, precisely as we associate these in part (6), and with each anvil we associate the sculpted resolutions, penetrated sculpted resolutions, and carriers that are associated with the anvil, $Anv(MGQ^{n-1}Res)$ (they are all of widths 1 to $width(n-1)$). We further set $width(n) = width(n-1)+1$, and with each anvil we associate a carrier, that we denote $Carrier^n_{width(n)}$. In this case, $sc(n,d) = sc(n-1,d)$ for $1 \leq d \leq width(n-1)-1$, $sc(n, width(n-1)) = sc(n-1, width(n-1))$ if there is a carrier, $Carrier^{n-1}_{width(n-1)}$, associated with $Anv(MGQ^{n-1}Res)$, and $sc(n, width(n-1)) = sc(n-1, width(n-1)) + 1$ otherwise, and $sc(n, width(n)) = 1$.

Suppose that the sequence of core resolutions associated with the sequence of multi-graded resolutions $MGQ^n Res_c$, and the images of the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, have the same structure as the corresponding core resolutions associated with the resolution that is associated with the $width(n-1)$ algebraic envelope in the anvil, $Anv(MGQ^{n-1}Res)$.

In this case we first examine the structure of the resolution induced from the sequence of resolutions, $MGQ^n Res_c$, by the image of the penetrated core resolution, $PenSCRes_1^{n-1,width(n-1)}$. If the structure of the resolution composed from the resolutions induced by the (image of the) completion of $PenSCRes_1^{n-1,width(n-1)}$ from the various core resolutions associated with the resolutions MGQ^nRes_c, and with the images of this completion, is not identical to the structure of the penetrated sculpted resolution, $PenSCRes_1^{n-1,width(n-1)}$, Proposition 13 implies that there exists some level j for which the structure of the graded abelian decomposition associated with the resolutions induced by the image of the completion of $PenSCRes_1^{n-1,width(n-1)}$ above level j are identical to the structure of the graded abelian decompositions associated with $PenSCRes_1^{n-1,width(n-1)}$, and in level j, either the number of factors drops, or the complexity of the associated abelian decomposition decreases.

In this case we associate with the (telescopic) sequence of resolutions MGQ^nRes a finite collection of developing resolutions and anvils, precisely as we did in this case in part (7) of the second step of the procedure (i.e. the developing resolutions are Extra PS resolutions associated with framed resolutions of the resolution induced by the image of the completion of $PenSCRes_1^{n-1,width(n-1)}$), and with each anvil we associate the sculpted resolutions, penetrated sculpted resolutions, and carriers that are associated with the anvil, $Anv(MGQ^{n-1}Res)$ (they are all of widths 1 to $width(n-1)$). We set the resolution induced by the image of the completion of $PenSCRes_1^{n-1,width(n-1)}$ from the core resolutions associated with the sequence of resolutions MGQ^nRes_c, to be the first penetrated sculpted resolution of width $width(n-1)$, that we denote, $PenSCRes_1^{n,width(n-1)}$. We further set $width(n) = width(n-1)$, and with each anvil we associate a carrier, that we denote $Carrier^n_{width(n)}$ (and which is the $width(n) = width(n-1)$ algebraic envelope). In this case, $sc(n,d) = sc(n-1,d)$ for $1 \leq d \leq width(n-1)-1$, and $sc(n, width(n)) = 1$.

Suppose that the structure of the resolution composed from the resolutions induced by the (image of the) completion of $PenSCRes_1^{n-1,width(n-1)}$ from the various core resolutions associated with the resolutions MGQ^nRes_c, and with the images of this completion, is identical to the structure of the penetrated sculpted resolution, $PenSCRes_1^{n-1,width(n-1)}$. We set $PenScRes$ to be the resolution induced by image of the completion of $PenSCRes_1^{n-1,width(n-1)}$ from the penetrated core resolutions associated with the image of this completion, and the various resolutions MGQ^nRes_c.

By Proposition 17, either the structure of the obtained resolution, $PenSCRes$, is identical to the structure of $PenSCRes_1^{n-1,width(n-1)}$, that is associated with the anvil, $Anv(MGQ^{n-1}Res)$, or there exists some level j, so that the structure of the graded abelian decompositions associated with $PenSCRes$ above level j are identical to the structure of the graded abelian decompositions associated with $PenSCRes_1^{n-1,width(n-1)}$, and in level j, either the number of factors drops, or the complexity of the associated abelian decomposition decreases.

In this case we associate with the (telescopic) sequence of resolutions $MGQ^n Res$, and with the resolution $PenSCRes$, a finite collection of developing resolutions and anvils, precisely as we did in this case in part (7) of the second step of the procedure (i.e. the developing resolutions are Extra PS resolutions associated with framed resolutions of $PenSCRes$), and with each anvil we associate the sculpted resolutions, penetrated sculpted resolutions, and carriers that are associated with the anvil, $Anv(MGQ^{n-1}Res)$ (they are all of widths 1 to $width(n-1)$). We set $width(n) = width(n-1)+1$, and the resolution $PenSCRes$ to be the first penetrated sculpted resolution of width $width(n)$, that we denote, $PenSCRes_1^{n,width(n)}$. With each anvil we associate a carrier, that we denote $Carrier^n_{width(n)}$. In this case, $sc(n,d) = sc(n-1,d)$ for $1 \leq d \leq width(n-1)-1$, $sc(n, width(n-1)) = sc(n-1, width(n-1))$ if there is a carrier, $Carrier^{n-1}_{width(n-1)}$, associated with $Anv(MGQ^{n-1}Res)$, and $sc(n, width(n-1)) = sc(n-1, width(n-1)) + 1$ otherwise, and $sc(n, width(n)) = 1$.

Suppose that the structure of the obtained resolution, $PenSCRes$, is identical to the structure of the first penetrated resolution, $PenSCRes_1^{n-1,width(n-1)}$. We first verify that the obtained resolution is not a terminal resolution. In this case we inductively repeat the examination of the images of the first sculpted resolution and the first penetrated sculpted resolution (of width $width(n-1)$), by examining the images of the e-th sculpted and penetrated sculpted resolutions of $width(n-1)$, $SCRes_e^{n-1,width(n-1)}$ and $PenSCRes_e^{n-1,width(n-1)}$, for $2 \leq e \leq sc(n-1, width(n-1))$. Finally, if all the obtained resolutions associated with these sculpted and penetrated sculpted resolutions have the same structures as those associated with the corresponding sculpted and penetrated sculpted resolutions, we examine the image of the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$, if this developing resolution is associated with the $width(n-1)$ algebraic envelope.

Suppose that all the resolutions obtained from the images of the sculpted, penetrated sculpted and (possibly) developing resolutions (of $width(n-1)$ in $Anv(MGQ^{n-1}Res)$), have the same structure as the corresponding resolutions that are associated with these sculpted, penetrated sculpted, and developing resolutions. We further assume that all the obtained resolutions are not terminal resolutions (in case such a resolution is a terminal resolution we do not continue with the sequence of multi-graded resolutions MGQ^nRes to the next step of our iterative procedure).

In this case we proceed as in part (7) of the second step, and associate with the resolution $GRes$, that is composed from the sequence of multi-graded resolutions, MGQ^nRes_c, a (canonical) finite collection of framed resolutions. With each framed resolution we canonically associate a finite collection of Non-Rigid, Non-Solid, Root and Left PS resolutions, and a finite collection of Extra PS resolutions, and Generic collapse extra PS resolutions. We set each of the Extra PS resolutions to be a developing resolution, and with a developing resolution we associate a finite collection of anvils. With each anvil we associate the sculpted resolutions, penetrated sculpted resolutions, and carriers that are associated with the anvil, $Anv(MGQ^{n-1}Res)$ (they are all of widths 1 to $width(n-1)$). We further set $width(n) = width(n-1) + 1$. In this case, $sc(n,d) = sc(n-1,d)$ for $1 \leq d \leq width(n-1) - 1$, $sc(n, width(n-1)) = sc(n-1, width(n-1))$ if there is a carrier, $Carrier^{n-1}_{width(n-1)}$, associated with $Anv(MGQ^{n-1}Res)$, and $sc(n, width(n-1)) = sc(n-1, width(n-1)) + 1$ otherwise, and $sc(n, width(n)) = sc(n, width(n-1))$.

As in the previous parts of the general step, we still need to associate with each anvil a finite collection of (extended) auxiliary resolutions. Like in part (4) of the first step, and part (7) of the second step, with each anvil, $Anv(MGQ^nRes)$, we associate $width(n)$ collections of auxiliary resolutions (auxiliary resolutions of widths 1 to $width(n)$), according to Definition 14.

Like in the first and second steps, before we conclude the general step of the sieve procedure, and prepare the data-structure for starting the next step, we need to check that the iterative procedure that was used in the general step, and the anvils constructed along it together with the terminal resolutions, collect all the Collapse extra PS specializations that factor through the initial developing resolutions, that are associated with the anvils, $Anv(MGQ^{n-1}Res)$, we started the n-th step with, and through the Diophantine conditions imposed by their associated collapse forms.

Theorem 21. *Let $(r, h_2, g_1, h_1, w, p, a)$ be a valid PS statement that factors through one of the PS limit group $PSHGH$, and can be extended to a specialization that factors and is taut with respect to one of the anvils, $Anv(MGQ^{n-1}Res)$, that was constructed in the $n-1$ step of the sieve* *procedure, and the extended specialization satisfies the Diophantine conditions imposed by one of the collapse forms associated with the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)$. Then either:*

(i) *The specialization $(r, h_2, g_1, h_1, w, p, a)$ can be extended to a test sequence of one of the developing resolutions, we started the n-th step with, that projects to a collection of valid PS statements; or*

(ii) *$(r, h_2, g_1, h_1, w, p, a)$ can be extended to a specialization that either factors through and is taut with respect to one of the anvils or one of the terminal resolutions constructed along the n-th step of the sieve procedure.*

Proof. Identical to the proof of Theorem 18. □

Termination of the Sieve Procedure

Defining the first, second and general steps of the sieve procedure, we are still required to prove its termination. To prove termination of our iterative procedure in the minimal rank case (section 1), we used the strict decrease in the complexity of the resolutions associated with successive steps of the procedure, a strict decrease that forces termination. In the procedures used to construct the tree of stratified sets and to validate a general AE sentence, we proved termination by combining the decrease in either the Zariski closures or the complexities of the resolutions and decompositions associated with successive steps of these procedures. In presenting the general step of the sieve procedure, we have considered the possibility that both the Zariski closures and the complexities of the various core resolutions and developing resolutions associated with an anvil do not decrease. In this case, if the constructed developing resolution is not a terminal resolution, we have associated additional sculpted resolutions, penetrated sculpted resolutions and carriers with the corresponding anvil. As we will see in the sequel, in addition to the arguments used to prove the termination of the procedure for the construction of the tree of stratified sets, in order to prove termination of the sieve procedure, we will also need to obtain a global bound on the number of sculpted resolutions (of given width) associated with an anvil.

Theorem 22. *The sieve procedure terminates after finitely many steps.*

Proof. Suppose that the sieve procedure does not terminate after finitely many steps, hence, the procedure must contain an infinite path. At step

n along the infinite path of the sieve procedure, there are associated anvil and developing resolution, a width, $width(n)$, $width(n)$ collections of core resolutions, and possibly sculpted resolutions, penetrated sculpted resolutions, and carriers. To get a contradiction to the existence of an infinite path, we start by applying the arguments used to prove the termination of the iterative procedure for validation of a sentence [Se4, 4.12], to show that the subgroup and the sequence of core resolutions associated with a given algebraic envelope along an infinite path, can be changed only finitely many times, i.e. the subgroup associated with such an algebraic envelope, and parts (1)–(3) of the general step of the sieve procedure, can be applied to the subgroup associated with a fixed algebraic envelope and its associated sequence of core resolutions, only finitely many times. We continue, by deducing from this "stabilization" of the subgroup and the sequence of core resolutions associated with a given algebraic envelope, that the number of sculpted resolutions of the same width along the infinite path, i.e. the set of non-negative integers $\{sc(n,d) \mid 1 \leq d \leq width(n)\}_{n=1}^{\infty}$, is not bounded. To obtain a contradiction, we apply the approach used in [S3] to obtain a global bound on the number of rigid and strictly-solid families of solutions, to get a bound on the number of sculpted resolutions of the same width along the infinite path, that clearly contradicts the unboundedness of the set $\{sc(n,d) \mid 1 \leq d \leq width(n)\}_{n=1}^{\infty}$.

PROPOSITION 23. *Let $d_0 \geq 1$ be an integer index, and let M_{d_0} be either the subgroup $\langle r, h_2, g_1, h_1, w, p, a\rangle$, or a limit group $\langle c, q_{n_0-1}, r, h_2, g_1, h_1, w, p, a\rangle$ that is associated in step n_0 with the d_0-th algebraic envelope that is associated with an anvil, $Anv(MGQ^{n_0}Res)$, constructed at step n_0 along our given infinite path of the sieve procedure (i.e. the subgroup $\langle c, q_{n_0-1}, r, h_2, g_1, h_1, w, p, a\rangle$ is generated by the resolution associated with the $d_0 - 1$ algebraic envelope in $Anv(MGQ^{n_0-1}Res)$, and elements associated with the collapse form that is associated with the Collapse extra PS limit group, $CollapseExtraPS^{n_0}$, along our given infinite path).*

Suppose that the subgroup M_{d_0} is associated with the d_0-th algebraic envelope that is associated with the anvils, $Anv(MGQ^{n}Res)$, for all $n > n_0$ (i.e. it is not being replaced after step n_0, except perhaps by a proper quotient). Then the structure of the core resolutions associated with M_{d_0} (the width d_0 sequence of core resolutions), can be changed in only finitely many steps along the given infinite path of the sieve procedure, i.e. parts (1)–(4) can be applied only in finitely many steps along the infinite path to the core resolutions associated with the subgroup M_{d_0}.

Proof. Since for all $n > n_0$, the width-d_0 sequence of core resolutions, is the sequence of core resolutions associated with the subgroup M_{d_0}, either $d_0 = 1$, in which case $M_{d_0} = \langle r, h_2, g_1, h_1, w, p, a \rangle$, or the sequences of the formed parts of the depth d core resolutions are unchanged for all $n > n_0$ and $d < d_0$. Hence, the claim of the proposition for $d_0 > 1$ is similar to the case $d_0 = 1$. So, to prove the proposition it is enough to consider the case $d_0 = 1$, in which case $M_1 = \langle r, h_2, g_1, h_1, w, p, a \rangle$. The argument we apply in this case, is a modification of the argument used to prove the termination of the iterative procedure for validation of a sentence [S4, 4.12].

Each time part (1) of the general step of the procedure is applied to the subgroup M_1 along the given infinite path of the sieve procedure, the limit group $Q^n(r,h_2,g_1,h_1,w,p,a)$ (the subgroup generated by $\langle r,h_2,g_1,h_1,w,p,a \rangle$ in $ExtraCollapsePS^n$) is replaced by its proper quotient. Hence, by the descending chain condition for limit groups, for the rest of the argument we may assume that part (1) of the general step is not applied to the the subgroup M_1 along our given path of the procedure. If part (2) or (3) of the general step is applied to the core resolution associated with the top part and with the subgroup M_1 along the infinite path, then the limit group $Q^{\ell(s)}(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$, is replaced by its proper quotient, $Q^n(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$, for some index s, $1 \leq s \leq s(n-1)$. Hence, by the descending chain condition for limit groups, for any fixed s, parts (2) and (3) can be applied to the limit group $Q^n(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$ (associated with the top part and the subgroup M_1) only finitely many times along the infinite path.

Each time part (4) of the general step of the sieve procedure is applied to a multi-graded resolution associated with the top part and the subgroup M_1 along the infinite path, the index of the based subgroups, $s(n)$, is increased by 1, and the complexity of the associated (multi-graded) core resolution strictly decreases. Since a strictly decreasing sequence of complexities of (multi-graded) core resolutions terminates after finitely many steps, parts (1)–(4), can be applied only finitely many times to the multi-graded resolution associated with the top part and the subgroup M_1 in the anvils, $Anv(MGQ^nRes)$, along our given infinite path of the sieve procedure.

Hence, there exists some step n_1 along the given path of the sieve procedure, so that for every step $n > n_1$ (for which one of the parts (1)–(4) is applied to the core resolutions associated with the subgroup M_1), and for the (top part) core resolution associated with the subgroup M_1:

(i) $s(n) = s(n_1)$.

(ii) For every $1 \leq s \leq s(n_1)-1$, $Q^{\ell(s)}(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$ is isomorphic to $Q^n(f_{\ell(s)}, r, h_2, g_1, h_1, h_1, w, p, a)$.

(iii) The multi-graded core resolution associated with the top level of the developing resolution and the subgroup M_1 constructed in step n of the sieve procedure, $MGCore(M_1, MGQ^n Res)$, is of maximal complexity.

Since for every $n > n_1$ (for which one of the parts (1)–(4) is applied to the core resolutions associated with the subgroup M_1), the multi-graded core $MGCore(M_1, MGQ^n Res)$ associated with the top part of the developing resolution and the subgroup M_1 is of maximal complexity, if we restrict our attention to the core resolutions associated with the subgroup M_1, then for every such $n > n_1$, the procedure is effectively applied to the limit group associated with the terminal level of the maximal complexity multi-graded resolutions, $MGQ^n Res$ (associated with the top part). By construction, in the limit group associated with the terminal level of the maximal complexity resolutions $MGQ^n Res$, the image of the limit group $Q^n(f_{\ell(s(n_1)-1)}, r, h_2, g_1, h_1, w, p, a)$, $Q^n_{term}(f_{\ell(s(n_1)-1)}, r, h_2, g_1, h_1, w, p, a)$, is a proper quotient of $Q^{\ell(s)}(f_{\ell(s(n_1)-1)}, r, h_2, g_1, h_1, w, p, a)$. If at some step $n > n_1$, the image of $Q^n(r,h_2,g_1,h_1,w,p,a)$ in the terminal level of $MGQ^n Res$, $Q^n_{term}(r, h_2, g_1, h_1, w, p, a)$, is a proper quotient of $Q^1(r, h_2, g_1, h_1, w, p, a)$, then for some n_2, for $n > n_2$ (for which one of the parts (1)–(4) is applied to the core resolutions associated with the subgroup M_1), the infinite path of the iterative procedure is effectively applied to a proper quotient of the limit group $Q^1(r, h_2, g_1, h_1, w, p, a)$ we started with. Otherwise, let $s_1 \leq s(n_1)-1$ be the minimal index s, for which $Q^n_{term}(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$, is a proper quotient of $Q^{\ell(s)}(f_{\ell(s)}, y, a)$ for some step $n > n_1$ (for which one of the parts (1)–(4) is applied to the core resolutions associated with M_1). Repeating the arguments used for analyzing the multi-graded resolutions associated with the top level of the first sculpted (or developing) resolution along our infinite path, parts (1)–(4) can be applied only finitely many times to the multi-graded resolutions associated with the second level of the first sculpted (or developing) resolution associated with the subgroup M_1, along our given infinite path of the sieve procedure; hence, there must exists some index n_2, so that for every $n > n_2$ (for which parts (1)–(4) are applied to the core resolutions associated with M_1) the multi-graded resolutions associated with the top two levels of the developing resolution, that is associated with the subgroup M_1, along the given infinite path of the sieve procedure, are of maximal complexity.

Suppose that parts (1)–(4) are applied to the core resolutions associated with the subgroup M_1 at infinitely many steps along our given infinite path of the sieve procedure. By the descending chain condition for limit groups [S1, 5.1], if we repeat this argument inductively, we obtain either

(i) a level ℓ, and an index n_ℓ, for which the multi-graded resolutions associated with the top ℓ levels of the associated developing resolution associated with the subgroup M_1 are of maximal complexity at all steps $n > n_\ell$ (for which parts (1)–(4) are applied to the core resolutions associated with M_1), and the image of $Q^n(r, h_2, g_1, h_1, w, p, a)$ in the limit group associated with the $\ell + 1$ level of the developing resolution associated with M_1, is a proper quotient of $Q^1(r, h_2, g_1, h_1, w, p, a)$ for all such $n > n_\ell$; or

(ii) an infinite sequence of limit groups $Q(f_{\ell(s)}, r, h_2, g_1, h_1, w, p, a)$, $s = 1, 2, \dots$, with a corresponding sequence of (multi-graded) core resolutions with respect to the subgroup M_1, so that the sequence of complexities of these core resolutions is strictly decreasing.

Since a sequence of strictly decreasing complexities of core resolutions terminates after finitely many steps, case (ii) cannot exist. Hence, case (i) holds, so there exists some index n_q for which for every step $n > n_q$ along the infinite path of the sieve procedure (for which parts (1)–(4) are applied to the core resolutions associated with M_1), the procedure is effectively applied to a proper quotient of the limit group $Q^1(r, h_2, g_1, h_1, w, p, a)$ we started the procedure with. Therefore, the descending chain condition for limit groups, contradicts the existence of an infinite path, for which the core resolutions associated with the subgroup M_1 are being changed at infinitely many steps using parts (1)–(4), and the proposition follows. □

By inductively applying Proposition 23, and the descending chain conditions for the complexities of core and induced resolutions, for each width d, the subgroups and the sequences of core resolutions of widths at most d, associated with the various anvils along our given infinite path of the sieve procedure, stabilize after finitely many steps. After the subgroups and the core resolutions stabilize, each time the resolution associated with the algebraic envelope of width d (which is a resolution composed from resolutions induced from the sequence of (stable) core resolutions), the complexity of at least one of the abelian decompositions along the associated resolution decreases. Hence, by the descending chain condition for complexities of abelian decompositions, for each width d, the resolutions associated with all the algebraic envelopes of widths at most d, stabilize after finitely many steps along a given infinite path of the sieve procedure.

Similarly, by the descending chain condition for the complexities of core resolutions, the core resolutions associated with a subgroup associated with an algebraic envelope of width $e < d$, and the resolution associated with the algebraic envelope of width d, stabilize after finitely many steps. By Proposition 17 and the descending chain condition for the complexities of resolutions, the penetrated sculpted resolutions of width at most d stabilize after finitely many steps along an infinite path of the sieve procedure.

DEFINITION 24. *Given an infinite path of the sieve procedure, and a positive integer d, the subgroups and the core resolutions associated with all the algebraic envelopes of width at most d, as well as the resolutions associated with these envelopes and the sculpted and penetrated sculpted resolutions of width at most d do not change after finitely many steps along the given infinite path. We call these subgroups, core resolutions, sculpted and penetrated sculpted resolutions,* stable.

By the structure of the general step of the sieve procedure, the first sculpted and penetrated sculpted resolutions of arbitrary width are built from the PS limit group $\langle r, h_2, g_1, h_1, w, p, a \rangle$. By Theorem 16, for each step d for which there exists a change in the stable core resolutions associated with the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$, and the stable resolution associated with the algebraic envelope of width d, compared with the corresponding stable core resolutions of width $d-1$, a decrease in the complexity of the associated core resolutions occurs. Similarly, by Proposition 17, for each step d for which the stable core resolutions associated with the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$ remain unchanged, but there is a change in the structure of the stable first penetrated core resolution (of width d), compared with the structure of the corresponding one of width $d-1$, a decrease of the complexity of at least one of the abelian decompositions associated with the first penetrated sculpted resolution occurs. By the descending chain condition for the complexities of core resolutions, and abelian decompositions, there exists some width d_1, for which the structure of the stable core resolutions associated with the subgroup $\langle r, h_2, g_1, h_1, w, p, a \rangle$, and the structure of the stable first penetrated sculpted resolution remain unchanged for all algebraic envelopes of width $d \geq d_1$.

Hence, a stable second sculpted resolution is associated with the algebraic envelope of width $d_1 + 1$. By the arguments used for the stable first penetrated sculpted resolution, there exists some width $d_2 > d_1$, for which the structure of the stable core resolutions associated with the first two sculpted resolutions, and the structure of the first two stable penetrated sculpted resolutions, remain unchanged for all widths, $d \geq d_2$. Continuing inductively, for each positive integer m, there exists some width d_m, so that the structure of the stable core resolutions associated with the first m sculpted resolutions, and the structure of the first m stable penetrated sculpted resolutions, remain unchanged for all widths, $d \geq d_m$.

Definition 25. *Given an infinite path of the sieve procedure, and a positive integer m, there exists a width d_m, for which the structure of the first m stable sculpted and penetrated sculpted resolutions of width d_m do not change for all $d \geq d_m$. We call this collection of (first m) stable sculpted and penetrated sculpted resolutions, eventual.*

The stabilization of the subgroups and core and penetrated sculpted resolutions of width at most d along a given infinite path of the sieve procedure, together with the structure of the general step of the sieve procedure, guarantee that for every given infinite path along it, there is no global bound on the number of (eventual) sculpted resolutions of the same width associated with the anvils constructed along the infinite path.

Proposition 26. *Given an infinite path of the sieve procedure, for each positive integer m, there exists a step n_m and width d_m, so that the sculpted and penetrated sculpted resolutions of width d_m at step n_m, are all eventual, and the number of n_m sculpted resolutions of width d_m, $sc(n_m, d_m)$, satisfies $sc(n_m, d_m) = m$ (hence, $sc(n, d_m) = m$ for all $n \geq n_m$).*

Proof. Given an infinite path of the sieve procedure, for each integer m, there exists a smallest width, d_m, so that all the first m stable sculpted and penetrated sculpted resolutions of width d_m are eventual. For each width d, there exists some step n_d along the given infinite path, for which for all $n \geq n_d$, the subgroups, core resolutions, and the sculpted and penetrated sculpted resolutions, associated with all the algebraic envelopes of width at most d are stable. Therefore, for every positive integer m, the subgroups, core resolutions, sculpted and penetrated sculpted resolutions, and carriers, associated with all the algebraic envelopes of widths at most d_m, are the stable ones for $n \geq n_{d_m}$, the first m sculpted and penetrated sculpted resolutions are eventual, and $sc(n_{d_m}, d_m) = sc(n, d_m) = m$ for all $n \geq n_{d_m}$. □

Proposition 26 shows that if the sieve procedure does not terminate after finitely many steps, there is no bound on the number of eventual sculpted (and penetrated sculpted) resolutions of the same width associated with the anvils along an infinite path of the sieve procedure. Therefore, to conclude the proof of Theorem 22, i.e. to prove the termination of the sieve procedure after finitely many steps, we need to show the existence of a global bound on the number of eventual sculpted resolutions of the same width associated with the anvils along an infinite path of the procedure.

Theorem 27. *Given an infinite path of the sieve procedure, there exists a global bound (independent of the width) on the number of eventual sculpted resolutions of the same width, i.e. on the numbers $sc(n,d)$ associated with the anvils along the infinite path, where n is large enough, so that all the subgroups, core resolutions, sculpted and penetrated sculpted resolutions associated with the first d algebraic envelopes are stable, and for which the $sc(n,d)$ sculpted and penetrated sculpted resolutions of width d are eventual.*

Proof. Our approach towards obtaining a bound on the number of eventual sculpted resolutions with the same width along an infinite path of the sieve procedure, is based on the argument used to obtain a bound on the number of rigid and strictly-solid families of solutions (with respect to a given covering closure) of rigid and solid limit groups, presented in the first two sections of [S3] (Theorems 2.5,2.9 and 2.13 in [S3]).

Suppose that there is no bound (independent of the width) on the number of eventual sculpted resolutions of the same width associated with an anvil along the given infinite path of the sieve procedure. Then for each positive integer m, there exists some index n_m and width d_m, so that the m-th stable sculpted resolution of width d_m constructed in the n_m step along the given path, $SCRes_m^{n_m,d_m}(s_m,r,h_2,g_1,g_1,h_1,w,p,a)$, contains the $m-1$ eventual penetrated sculpted resolutions,

$$PenSCRes_1^{n_m,d_m}(r_1,r,h_2,g_1,h_1,w,p,a),\dots,$$
$$PenSCRes_{m-1}^{n_m,d_m}(r_{m-1},r,h_2,g_1,h_1,w,p,a)\,,$$

that are naturally (geometrically) embedded in the m-th stable sculpted resolution $SCRes_m^{n_m,d_m}(s_m,r,h_2,g_1,h_1,w,p,a)$. Each penetrated sculpted resolution added along the sieve procedure is an Extra PS resolution. Hence, in constructing the i-th (eventual) penetrated sculpted resolution, $PenSCRes_i^{n_m,d_m}(s_i,r,h_2,g_1,h_1,w,p,a)$, several (boundedly many) extra rigid and almost shortest strictly solid solutions are being added, while the extra rigid solutions associated with the previously constructed eventual penetrated sculpted resolution become flexible, and extra strictly solid solutions associated with previously constructed penetrated sculpted resolutions become non-strictly solid, or belong to the families of strictly-solid solutions that are declared in the proof statement (see Definition 2.8 in [S3] for almost shortest strictly-solid specializations). Note that by construction, the variables that represent an extra rigid or strictly-solid solution that is being added to the i-th eventual penetrated sculpted resolution of width d_m, $PenSCRes_i^{n_m,d_m}$, (x_i, h_1, w, p, a), differ from the variables that represent any other extra rigid or strictly-solid solution that is being added to any of the previous eventual penetrated sculpted resolutions of the same width that are embedded into it.

Suppose that for some index $i < m$, to the i-th eventual penetrated sculpted resolution of width d_m, $PenSCRes_i^{n_m,d_m}$ constructed along the infinite path, an additional extra rigid solution is being added. Since the eventual penetrated sculpted resolution $PenSCRes_i^{n_m,d_m}$ is embedded in the next eventual penetrated sculpted resolution of width d_m, $PenSCRes_{i+1}^{n_m,d_m}$, the extra rigid solution that was added to the i-th eventual penetrated sculpted resolution, $PenSCRes_i^{n_m,d_m}$, does not become flexible when embedded into the $i+1$ eventual penetrated sculpted resolution, $PenSCRes_{i+1}^{n_m,d_m}$, which clearly contradicts the properties of the $i+1$ eventual penetrated sculpted resolution. Hence, the only extra solutions that are being added to the various eventual penetrated sculpted resolutions that are embedded in $PenSCRes_m^{n_m,d_m}$, are almost shortest strictly-solid solutions (with respect to the given covering closures associated with the solid limit groups $WPHG$, see Definition 2.12 in [S3]).

Since our given path of the sieve procedure is infinite, none of the resolutions constructed in the various steps along the infinite path are terminal resolutions. Hence, at each step of the infinite path, there exists a test sequence that factors through the developing resolution constructed at that step, so that for the corresponding test sequences that factor through each of the penetrated sculpted resolutions associated with the given developing resolution, the specialization of at least one of the variables associated with an extra strictly-solid solution associated with that penetrated sculpted resolution is indeed strictly solid.

The extra strictly-solid solutions that are being added to the various penetrated sculpted resolutions along the infinite path of the sieve procedure are strictly-solid solutions of a finite collection of solid limit groups, $WPHG(g_1, h_1, w, p, a)$, associated with the tree of stratified sets. Hence, for each positive integer m, there exists an index n_m, width d_m, and a sequence of integers, $1 \leq i_1 < \cdots < i_m$, and an associated sequence of (variables corresponding to) extra strictly-solid specializations $(x_{i_1}, h_1, w, p, a), \dots, (x_{i_m}, h_1, w, p, a)$ that are being added to the stable penetrated sculpted resolutions of width d_m,

$$PenSCRes_{i_1}^{n_m,d_m} \dots, PenSCRes_{i_m}^{n_m,d_m}$$

in correspondence, so that these extra strictly-solid specializations are extra strictly-solid specializations of the same solid limit group $WPHG$, and there exists a test sequence that factors through the n_m-th eventual penetrated sculpted resolution of width d_m, $SCRes_{i_m}^{n_m,d_m}$, and for each j, $1 \leq j \leq m$, the specializations of the variables (x_{i_j}, h_1, w, p, a) along the corresponding test sequence of the eventual penetrated sculpted resolution $PenSCRes_{i_j}^{n_m,d_m}$, are all strictly-solid specializations of the (same) solid limit group $WPHG$.

Given such a test sequence that factors through the eventual penetrated sculpted resolution of width d_m, $PenSCRes_{i_m}^{n_m,d_m}$, we naturally associate with it test sequences of the eventual penetrated sculpted resolutions, $PenSCRes_{i_1}^{n_m,d_m}, \dots, PenSCRes_{i_m}^{n_m,d_m}$ (by restricting the modular automorphisms associated with the given (ambient) test sequence to each of the penetrated sculpted resolutions). Given the corresponding test sequence of the eventual penetrated sculpted resolution, $PenSCRes_{i_1}^{n_m,d_m}$, we further associate with the eventual penetrated sculpted resolution, $PenSCRes_{i_1}^{n_m,d_m}$, and the subgroup associated with the corresponding extra solid solution, $\langle x_{i_1}, h_1, w, p, a\rangle$, a sequence of strictly-solid solutions, $\{(q_{i_1}, h_1, w, p, a)_\ell\}$, that are almost shortest in the strictly-solid families of the solutions $\{(x_{i_1}, h_1, w, p, a)_\ell\}$ (where the metric according to which we measure the length of a specialization, is the metric induced by the eventual penetrated sculpted resolution from its embedding into the limit group associated with the eventual penetrated sculpted resolution $PenSCRes_{i_m}^{n_m,d_m}$, and the action of the limit group associated with this penetrated sculpted resolution on a real tree obtained from the test sequence of specializations).

Since penetrated sculpted resolutions are well separated, by passing to a subsequence of the given test sequence, we may assume that the sequence $\{(q_{i_1}, h_1, w, p, a)_\ell\}$ converge into a subgroup $\langle q_{i_1}, h_1, w, p, a\rangle$ of a closure of the eventual penetrated sculpted resolution, $PenSCRes_{i_1}^{n_m,d_m}$, and so are the variables that impose the Diophantine condition that demonstrates that specializations of the subgroups, $\langle x_{i_1}, h_1, w, p, a\rangle$ and $\langle q_{i_1}, h_1, w, p, a\rangle$, belong to same (strictly-solid) family of the solid limit group $WPHG$. Given that subsequence, we associate a similar subgroup, $\langle q_{i_2}, h_1, w, p, a\rangle$, in a closure of the eventual penetrated sculpted resolution $PenSCRes_{i_2}^{n_m,d_m}$, and so on, until we associate a subgroup, $\langle q_{i_m}, h_1, w, p, a\rangle$, with the penetrated sculpted resolution $PenSCRes_{i_m}^{n_m,d_m}$.

Definition 28. *Given the test sequences of specializations associated with the various eventual sculpted resolutions, we say that a sequence* *of integers* $\{m_u\}_{u=1}^{\infty}$, *together with tuples of integers* $1 \leq i_{u,j_1} < i_{u,j_2} < \cdots < i_{u,j_u} < m_u$, *and test subsequences that factor through the various (eventual) sculpted resolutions,* $SCRes_{m_u}^{n_{m_u},d_{m_u}}$, *are a* $fenced$ *subsequence, if there exists a (fixed) proper quotient,* $WPHG'(g_1,h_1,w,p,a)$, *of the solid limit group* $WPHG$, *so that for every subgroup,* $\langle q_{u,i_j},h_1,w,p,a\rangle$, *of an eventual penetrated sculpted resolution,* $PenSCRes_{u,i_j}^{n_{m_u},d_{m_u}}$, *that is associated with an extra solid specialization, there exists a subgroup* $\langle \hat{q}_{u,i_j},h_1,w,p,a\rangle$ *of a closure of the same eventual penetrated sculpted resolution,* $PenSCRes_{u,i_j}^{n_{m_u},d_{m_u}}$, *so that the specializations of the subgroups,* $\langle q_{u,i_j},h_1,w,p,a\rangle$ *and* $\langle \hat{q}_{u,i_j},h_1,w,p,a\rangle$, *represent elements that belong to the same strictly-solid families of the solid limit group* $WPHG$, *(i.e. there exist elements in the corresponding closure of* $PenSCRes_{u,i_j}^{n_{m_u},d_{m_u}}$ *that demonstrate the Diophantine condition that connects between (specializations of) the two subgroups), and all the subgroups* $\langle \hat{q}_{u,i_j},h_1,w,p,a\rangle$ *factor through the (fixed) proper quotient* $WPHG'$ *of* $WPHG$.

If given the test sequences of specializations we started with, the sequence of eventual sculpted and penetrated sculpted resolutions contains a fenced subsequence, we continue with the fenced subsequence, replace the solid limit group $WPHG$ by the proper quotient corresponding to the subsequence which we denote $WPHG'$, and for each couple of indices (u,i_j), we replace the subgroup $\langle \hat{q}_{u,i_j},h_1,w,p,a\rangle$, by a subgroup $\langle \tilde{q}_{u,i_j},h_1,w,p,a\rangle$, obtained as a limit from a test sequence of specializations, $\{(\tilde{q}_{u,i_j},h_1,w,p,a)_\ell\}_{\ell=1}^{\infty}$, that are the almost shortest specializations in the strictly-solid family of the corresponding sequence of specializations, $\{(\hat{q}_{u,i_j},h_1,w,p,a)_\ell\}_{\ell=1}^{\infty}$, with respect to the graded abelian decomposition of the proper quotient $WPHG'$ of $WPHG$ (where the parameter subgroup is $\langle w,p\rangle$). To the subgroups $\langle q_{u,i_j},h_1,w,p,a\rangle$, $\langle \hat{q}_{u,i_j},h_1,w,p,a\rangle$, and $\langle \tilde{q}_{u,i_j},h_1,w,p,a\rangle$, of (the corresponding closure of) $PenSCRes_{u,i_j}^{n_{m_u},d_{m_u}}$, we add variables that demonstrate the Diophantine conditions their specializations have to satisfy (i.e. that force their specializations to belong to the same strictly-solid family of the original solid limit group $WPHG$).

If the obtained sequence of eventual sculpted and penetrated sculpted resolutions and their given test sequences, contain a further fenced subsequence, we continue with it and replace the obtained limit group $WPHG'$ by its corresponding proper quotient. Since limit groups satisfy the descending chain condition, after finitely many replacements, we may assume that our sequence of eventual sculpted and penetrated sculpted resolutions and their given test sequences contain no fenced subsequence. We call the obtained collection that has no fenced subsequences, and its corresponding limit group (still denoted $WPHG$), a *stable collection* and a *stable* (*solid*) *limit group* in correspondence. We continue denoting the subgroups corresponding to extra strictly-solid solutions in the various penetrated sculpted resolutions, $PenSCRes_{u,i_j}^{n_{m_u},d_{m_u}}$, that factor through a stable limit group, $\langle q_{u,i_j}, h_1, w, p, a\rangle$.

Let η be the natural quotient map from the original solid limit group onto the stable limit group $WPHG$. Let Θ_0 be the graded abelian decomposition of the original solid limit group with respect to the parameter subgroup $APW = \langle w, p, a\rangle$, and let Θ be the graded abelian decomposition of the stable limit group $WPHG$. Suppose that Θ contains an abelian vertex group V, that is not the distinguished vertex group (i.e. V does not contain the subgroup APW), and the subgroup generated by the edge groups connected to the vertex stabilized by V in Θ is not of finite index in V. By Proposition 1.9 in [S3], since the stable limit group $WPHG$ is obtained from a sequence of strictly-solid specializations of the original solid limit group, every non-QH, non-abelian vertex group and every edge group in Θ_0 must be mapped by the natural quotient map η, into either a non-QH vertex group or into an edge group in the graph of groups Θ. Furthermore, if a non-QH, non-abelian vertex group or an edge group in Θ_0 is mapped into (a conjugate of) the abelian vertex group V, then the image of that vertex group or edge group in V intersects the subgroup generated by the edge groups connected to V in a subgroup of finite index.

Since the sequence of eventual sculpted and penetrated sculpted resolutions, and their associated test sequences, contain no fenced subsequence, we can further pass to a subsequence of them, for which there is no subsequence of integers $\{m_u\}_{u=1}^{\infty}$, together with a sequence of couples, $\{(u, j_u)\}_{u=1}^{\infty}$, $1 \leq j_u \leq u$, and test sequences that factor through the various penetrated sculpted resolutions, $PenSCRes_{i_{u,j_u}}^{n_{m_u},d_{m_u}}$, that are inherited from subsequences of the given test sequences that factor through the sculpted resolutions, $SCRes_{m_u}^{n_{m_u},d_{m_u}}$, for which there exists a (fixed) proper quotient $WPHG'$ of the stable limit group $WPHG$, so that for every subgroup $\langle q_{i_{u,j_u}}, h_1, w, p, a\rangle$ of a penetrated sculpted resolution $PenSCRes_{i_{u,j_u}}^{n_{m_u},d_{m_u}}$ that is associated with an extra solid specialization, there exists a subgroup $\langle \hat{q}_{i_{u,j_u}}, h_1, w, p, a\rangle$ of a closure of the same penetrated sculpted resolution, $PenSCRes_{i_{u,j_u}}^{n_{m_u},d_{m_u}}$, so that the subgroups $\langle q_{i_{u,j_u}}, h_1, w, p, a\rangle$ and $\langle \hat{q}_{i_{u,j_u}}, h_1, w, p, a\rangle$, satisfy a Diophantine condition that demonstrates that their specializations belong to the same strictly-solid family of the original given solid limit group (of which the stable limit group $WPHG$ is a quotient), and all the subgroups $\langle \hat{q}_{i_{u,j_u}}, h_1, w, p, a \rangle$ factor through the (fixed) proper quotient $WPHG'$ of the stable limit group $WPHG$. We continue with the corresponding subsequence of (fenced) eventual sculpted and penetrated sculpted resolutions, and their associated test sequences.

To save notation, we will denote the eventual penetrated sculpted resolutions that we continue with, $PenSCRes_j^u$, and the eventual sculpted resolutions into which they are embedded, $SCRes_u$, for $1 \leq j \leq u$.

Our approach towards obtaining a global bound on the number of eventual sculpted resolutions of the same width (Theorem 27), is based on the arguments used to obtain a global bound on the number of rigid and families of strictly-solid solutions presented in the first two sections of [S3]. In order to modify the arguments presented in [S3] to prove Theorem 27, we first need to identify *quasi-rooted* and *non-quasi-rooted* (cyclic) edge groups.

Let Θ be the graded abelian decomposition of the stable limit group $WPHG$ with respect to the parameter subgroup $APW = \langle w, p, a \rangle$, and let $E_1, \dots, E_d$ be the edge groups in Θ. The stable limit group $WPHG$ is mapped into each of the penetrated sculpted resolutions $PenSCRes_j^u$, and we denoted its image $\langle q_{u,j}, h_1, w, p, a \rangle$. In order to be able to modify the arguments presented in [S3] to prove Theorem 27, we need to obtain global bounds on the order of roots of images of the maximal abelian subgroups containing the edge groups $E_1, \dots, E_d$ in the penetrated sculpted resolutions $PenSCRes_j^u$, when viewed as subgroups of the sculpted resolution $SCRes_u$. We start by demonstrating our approach for obtaining global bounds on the order of roots, in case the graded abelian decomposition Θ of the stable limit group $WPHG$ is an amalgamated product $WPHG = D *_C E$, where C is cyclic, E is non-abelian, and the parameter subgroup APW is a subgroup of D.

PROPOSITION 29. *Suppose that the graded abelian JSJ decomposition of the stable limit group $WPHG$ with respect to the parameter subgroup $APW = \langle w, p, a \rangle$ is $WPHG = D *_C E$, where C is cyclic and E is non-abelian. Let $C = \langle c \rangle$, and for each couple (u, j), $1 \leq j \leq u$, let c_j^u be the image of the element c in the penetrated sculpted resolution $PenSCRes_j^u$. Let $\{c_j^u(\ell)\}_{\ell=1}^{\infty}$, be the restriction of our given test sequence of specializations that factors through the sculpted resolution $SCRes_u$, to the element c_j^u.*

Then there exists some global bound b_0, and an index u_0, so that for every $u > u_0$, and every j, $1 \leq j \leq u$, there exists some index $\ell(u, j)$ so that for every $\ell > \ell(u, j)$, the order of a root of $c_j^m(\ell)$ is bounded by b_0.

Proof. Identical to the proof of Lemma 4.2 of [S3]. In case there is no such bound b_0, the vertex group E admits a non-trivial abelian decomposition in which the edge group C is elliptic. Such an abelian decomposition allows one to further refine the amalgamated decomposition $WPHG = D *_C E$, a contradiction to the canonical properties of the (graded) JSJ decomposition. □

Proposition 29 gives a bound on the order of roots of cyclic edge groups in the graded abelian decomposition Θ of the stable limit group $WPHG$, in case Θ corresponds to a simple amalgamated product. For general graph of groups Θ, we naturally define the notion of a *quasi-rooted* and a *non-quasi-rooted* edge group.

DEFINITION 30. *Let Θ be the graded abelian JSJ decomposition of the stable limit group $WPHG$ with respect to the parameter subgroup APW. Let E be either an edge group or an abelian vertex group in Θ, let A be the maximal abelian subgroup containing E in the stable limit group $WPHG$, and for each couple (u,j), $1 \leq j \leq u$, let A^u_j be the image of the subgroup A in the penetrated sculpted resolution, $PenSCRes^u_j$. Let $\{a^u_j(\ell)\}_{\ell=1}^{\infty}$, be the generators of the cyclic images of A^u_j that correspond to the given test sequence of specializations that factors through the sculpted resolution $SCRes_u$.*

We say that the edge group (abelian vertex group) E, in the graph of groups Θ, is non-quasi-rooted, *if there exists some global bound b_E, and an index u_E, so that for every $u > u_E$, and every j, $1 \leq j \leq u$, there exists some index $\ell(u,j)$ so that for every $\ell > \ell(u,j)$, the order of a root of $a^u_j(\ell)$ is bounded by b_E.*

We say that an edge group (abelian vertex group) E in Θ, is quasi-rooted, if for every positive integer b, there exists an index u^E_b, so that for every $u > u^E_b$, and every j, $1 \leq j \leq u$, there exists some index $\ell(u,j)$ so that for every $\ell > \ell(u,j)$, $a^u_j(\ell)$ has a root of order at least b.

Note that since the limit group $WPHG$ is assumed to be stable, every abelian vertex group V in the graph of groups Θ, for which the subgroup generated by the edge groups connected to V is not of finite index in V, is necessarily a quasi-rooted abelian vertex group.

Note that according to Definition 30, an edge group or an abelian vertex group in Θ can be neither quasi-rooted nor non-quasi-rooted. Hence, before we continue analyzing the sequence of eventual sculpted and penetrated sculpted resolutions and their given test sequences, we replace them by a subsequence, so that with respect to the chosen subsequence every edge group and abelian vertex group in Θ is either quasi-rooted or non-quasi-rooted.

Definition 31. *Let Θ be the graded abelian decomposition of a stable limit group $WPHG$. We define some (arbitrary) order on the (finite) sets of edge groups and abelian vertex groups in Θ.*

We go over the the edge groups (abelian vertex groups) sequentially, according to their prescribed order. By applying a simple diagonal argument and passing to an appropriate subsequence of the sequence of eventual sculpted and penetrated sculpted resolutions and their test sequences, we can assume that the first edge group (according to the prescribed order) is either quasi-rooted or non-quasi-rooted. Continuing inductively, after passing to a further subsequence, we can assume that all the edge groups and abelian vertex groups in Θ are either quasi-rooted or non-quasi-rooted.

We set the non-quasi-rooted *subgraph of Θ, that we denote Θ_{NQR}, to be the subgraph that is obtained from the graph Θ by erasing the following:*

(i) *quasi-rooted edge groups that connect between two non-abelian vertex groups;*
(ii) *quasi-rooted abelian vertex groups, and all the edges (that must all be quasi-rooted as well) connected to it.*

Note that according to (ii)*, every abelian vertex group in Θ, for which the subgroup generated by the edge groups connected to it is not of finite index (such abelian vertex group must be quasi-rooted), is not in Θ_{NQR}. Finally we denote the fundamental group of Θ_{NQR}, $WPHG_{NQR}$.*

The following theorem generalizes Proposition 29 to a general stable limit group.

Theorem 32. *Let Θ be the graded abelian decomposition of a stable limit group $WPHG$, with respect to the parameter subgroup APW. Then Θ_{NQR}, its non-quasi-rooted subgraph, is connected.*

Proof. To construct the non-quasi-rooted subgraph Θ_{NQR}, we start with the distinguished vertex V_D in Θ, i.e. the vertex stabilized by the subgroup APW in the graph of groups Θ. Since $WPHG$ is a limit group, every non-cyclic abelian vertex group in Θ is adjacent only to non-abelian vertex groups in Θ. Hence, if Θ contains no non-abelian vertex groups apart from the distinguished one, Θ_{NQR} is necessarily connected and contains the distinguished vertex V_D.

Suppose that there are at least two non-abelian vertex groups in Θ. We start with the distinguished vertex V_D. We further assume that all the edge groups that are connected to V_D in Θ are quasi-rooted, and are not connected to abelian vertex groups that are not quasi-rooted. In this case we argue that our sequence of test sequences contains a fenced subsequence

(Definition 28), hence, obtain a contradiction to the assumption that the limit group $WPHG$ is stable.

Let Γ be the subgraph of the graph of groups Θ, obtained from Θ by taking out from it the vertex V_D, all the abelian vertex groups A_j adjacent to V_D in Θ (that are all assumed to be quasi-rooted), and all the edges that are connected to the vertices V_D and A_j in Θ. Note that Γ is not empty, since we assumed that Θ contains at least two non-abelian vertex groups. Let Γ_1 be a connected component of Γ, and let U_1 be its fundamental group. By construction, every vertex group in Γ_1 that is adjacent to V_D or one of the vertices A_j in Θ is non-abelian, and an edge group attached to an edge that connects a vertex in Γ_1 to V_D, or to one of the vertices A_j in Θ, is quasi-rooted.

With the graph of groups Γ_1, we naturally associate a modular group of automorphisms of its fundamental group U_1 (that is a subgroup of the modular group associated with the ambient graph of groups Θ), that we denote $Mod(U_1)$. Note that the automorphisms in $Mod(U_1)$ preserve, in particular, the conjugacy classes of the (quasi-rooted) edge groups that connect vertices in Γ_1 to the vertex V_D or one of the (quasi-rooted) abelian groups A_j connected to it.

Given each of the penetrated sculpted resolutions, $PenSCRes_j^u$, a corresponding test sequence of specializations that factor through it, and the associated sequence of specializations, $\{(q_{u,j}, h_1, w, p, a)_\ell\}$, we replace each of the specializations of the limit group U_1 associated with the specialization, $(q_{u,j}, h_1, w, p, a)_\ell$, by a specialization $(\hat{q}_{u,j}, h_1, w, p, a)_\ell$, that is almost shortest with respect to the action of the modular group $Mod(U_1)$. For each tuple (u, j), $1 \leq j \leq m$, we further pick a convergent subsequence (still denoted), $(\hat{q}_{u,j}, h_1, w, p, a)_\ell$, and we denote the associated limit group $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$.

Recall that by our assumptions on the sequence of eventual sculpted and penetrated sculpted resolutions we started with, and their given test sequences, there does not exist a subsequence of couples $\{(u_t, j(u_t)\}_{t=1}^{\infty}$, for which the corresponding subgroups, $\langle \hat{q}_{u_t,j(u_t)}, h_1, w, p, a \rangle$, are all quotients of a proper quotient of the stable limit group $WPHG$.

Each of the limit groups $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$ is obtained from a convergent (test) sequence of specializations $\{(\hat{q}_{u,j}, h_1, w, p, a)_\ell\}$, hence, there is an associated (faithful) action of it on some real tree. Therefore, by possibly further passing to a subsequence, we may assume that the sequence of actions of the limit groups (still denoted) $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$, on their associated real trees, converges into a faithful action of the stable limit group $WPHG$

on a real tree. In particular, the restricted actions of the images of the subgroup U_1 in $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$ converge into a faithful action of U_1 on some limit (real) tree.

Let $E_1, \ldots, E_s$ be the edges in the graph of groups Θ that connect vertices in Γ_1 to either the vertex V_D, or to one of the (quasi-rooted) abelian vertex groups A_j connected to it. We set the group RU_1 to be $RU_1 = U_1 * \langle r_1 \rangle * \cdots * \langle r_s \rangle$ For each index u, the eventual penetrated sculpted resolution $PenSCRes_j^u$ is naturally embedded into the eventual sculpted resolution, $SCRes_j^u$. Hence, for each couple (u,j), the specializations of the limit group U_1 associated with the sequence, $\{(\hat{q}_{u,j}, h_1, w, p, a)_\ell\}$, can be extended to homomorphisms from the limit group RU_1, by sending each of the generators r_i of RU_1 into a primitive root of the specialization of its associated (quasi-rooted) edge group. By passing to corresponding subsequences, we may assume that for each couple (u,j), the corresponding sequence of homomorphisms of RU_1 converges into an action of RU_1 on some limit tree, and by further passing to a subsequence of couples (u,j), we may assume that this sequence of actions converges into an action of RU_1 on some real tree Y. We denote the limit group corresponding to this limit action $RootU_1$. Clearly, $RootU_1$ is a (proper) quotient of RU_1, and the generators r_i commute with their associated quasi-rooted edge groups in $RootU_1$. Also, by construction, the limit group U_1 is embedded in $RootU_1$.

The limit group $RootU_1$ inherits a (non-trivial) abelian decomposition Δ from its action on the real tree Y, and abelian decomposition in which every non-cyclic abelian vertex group is elliptic. Since the generators r_i commute with the quasi-rooted edge groups associated with them, and since the elements r_i are mapped to primitive roots of the specializations of the quasi-rooted edge groups, the generators r_i and the quasi-rooted edge groups associated with them, are elliptic in Δ.

Since U_1 is embedded in $RootU_1$, and the quasi-rooted edge groups that connect U_1 to the vertex groups V_D, and to the quasi-rooted abelian vertex groups A_j connected to it in Θ, are elliptic in Δ, if the graph of groups Δ is not compatible with the graph of groups Γ_1 of U_1, the graded abelian JSJ decomposition Θ of the stable limit group $WPHG$ can be further refined, a contradiction to its canonical properties. Hence, Δ is compatible with the graph of groups Γ_1 of U_1. But this implies that for large enough u, and for large enough ℓ, the specializations $(q_{u,j}, h_1, w, p, a)_\ell$ are not almost shortest with respect to the action of the modular group $Mod(U_1)$, a contradiction. Hence, at least one of the edge groups connected to V_D is non-quasi-rooted, or V_D is connected to a non-quasi-rooted abelian vertex group.

Let Λ_1 be the subgraph of Θ that contains the distinguished vertex V_D, the non-quasi-rooted edges and non-quasi-rooted abelian vertex groups connected to V_D (and the edges connected to these abelian vertex groups), and the vertices adjacent to those non-quasi-rooted edges and abelian vertex groups. Clearly, $\Lambda_1 \subset \Theta_{NQR}$ and Λ_1 is connected. If there is no non-abelian vertex group in the graph of groups Θ that is not in Λ_1, the theorem follows. Suppose there is a non-abelian vertex group in Θ that is not contained in Λ_1. Then the same argument used for the edge groups connected to the distinguished vertex group V_D, implies that there must exist a non-quasi-rooted edge group E in Θ, or a non-quasi-rooted abelian vertex group in Θ, so that E or A is adjacent to Λ_1 and E is not in Λ_1. We set Λ_2 to be the union of Λ_1 with all the non-quasi-rooted edge groups and abelian vertex groups that are adjacent to Λ_1. By construction, Λ_2 is connected and contained in Θ_{NQR}. Repeating the construction iteratively, we can still enlarge the obtained subgraph Λ_j as long as there exists a non-abelian vertex group in Θ that is not contained in Λ_j. Hence, for some k, $\Theta_{NQR} = \Lambda_k$, Θ_{NQR} is connected, and contains all the non-abelian vertex groups in Θ. □

The subgroups $\langle q_{u,j}, h_1, w, p, a\rangle$ were chosen as limits of almost shortest sequences of specializations. In defining almost shortest specializations [S3, Definition 2.8], we have fixed an order on the edge groups and the QH vertex groups of the graded abelian decomposition Θ of the stable limit group $WPHG$. Fixing a non-quasi-rooted subgraph of the graph of groups Θ, for each index u and each j, $1 \leq j \leq u$, we modify the subgroup $\langle q_{u,j}, h_1, w, p, a\rangle$ by replacing it with a corresponding subgroup that is a limit of a sequence of almost shortest specializations of the eventual penetrated sculpted resolution, $PenSCRes_j^u$, where the order on the edges and QH vertex groups of Θ starts with the edges and QH vertex groups in the non-quasi-rooted subgraph Θ_{NQR}, and then the quasi-rooted edge and abelian vertex groups. We still denote the obtained subgroups of the corresponding sculpted resolutions, $\langle q_{u,j}, h_1, w, p, a\rangle$.

DEFINITION 33. *Let $E_1, \dots, E_q$ be the quasi-rooted edge groups of edges that were taken out from the graph of groups Θ, to obtain the non-quasi-rooted graph of groups Θ_{NQR} (if an edge that was taken out connects two vertices in Θ_{NQR}, then we take the two embeddings of the edge group to be among the quasi-rooted edge groups $E_1, \dots, E_q$). Recall (Definition 31) that we denote the fundamental group of Θ_{NQR}, $WPHG_{NQR}$. We set $G^{NQR} = WPHG_{NQR} * \langle s_1\rangle * \cdots * \langle s_q\rangle$. With each couple of indices (u, j),*

$1 \leq j \leq u$, *and each specialization of the eventual sculpted resolution* $SCRes_u$, *we associate a natural specialization of the group* G^{NQR}, *by mapping its subgroup* $WPHG_{NQR}$ *to the specialization it inherits from the corresponding specialization of the subgroup* $\langle q_{u,j}, h_1, w, p, a \rangle$, *and mapping the elements,* $s_1, \ldots, s_q$, *to the generators in the coefficient (free) group* F_k *of the images of the quasi-rooted edge groups,* $E_1, \ldots, E_q$, *in correspondence.*

By our standard arguments presented in section 5 of [S1], the entire set of specializations of the group G^{NQR}, *factors through a canonical finite collection of (maximal) limit groups* $G_1^{NQR}, \ldots, G_t^{NQR}$ *(that correspond to the Zariski closure of the collection). Hence, for some limit group* G_j^{NQR} *there is a subsequence of couples (still denoted)* (u, j), $1 \leq j \leq u$, *and test sequences that factor through the corresponding eventual sculpted and penetrated penetrated sculpted resolutions (still denoted)* $SCRes_u$ *and* $PenSCRes_j^u$, *so that the corresponding specializations of the subgroup* G^{NQR} *all factor through the limit group* G_j^{NQR}.

If there exists a subsequence of the couples (u, j), *still denoted* (u, j), $1 \leq j \leq u$, *and associated test sequences of specializations that factor through the eventual sculpted resolutions* $SCRes_u$, *so that the corresponding specializations of the subgroup* G_j^{NQR} *do all factor through a proper quotient of* G_j^{NQR}, *we pass to this subsequence (*a fenced subsequence*), and replace the limit group* G_j^{NQR} *by its corresponding proper quotient. We iteratively replace the obtained subsequence by fenced subsequence, as long as a fenced subsequence exists. By the descending chain condition for limit groups, such an iterative process terminates after finitely many steps. We call the obtained quotient of the limit group* G_j^{NQR}, a stable rooted limit group, *and denote it* $RootWPHG_{NQR}$. *Since the group* $WPHG$ *is stable, the non-quasi-rooted subgroup* $WPHG_{NQR}$ *that is naturally mapped into* $RootWPHG_{NQR}$ *is naturally embedded into it.*

Since the sequence of test sequences contain no fenced subsequence, we can further pass to a subsequence of test sequences, for which there is no subsequence of integers u, *with a sequence of integers,* $\{j(u)\}_{u=1}^{\infty}$, $1 \leq j(u) \leq u$, *and subsequences of the test sequences that factor through the various eventual sculpted resolutions* $SCRes_u$, *for which there exists a (fixed) proper quotient of the stable rooted limit group,* $RootWPHG_{NQR}$, *so that the all the specializations of* $RootWPHG_{NQR}$ *that correspond to the given test sequences factor through that fixed proper quotient of* $RootWPHG_{NQR}$.

In the formulation of the following theorem, recall (Definition 1.1 in [S3]) that if $G(x,p,a)$ is a graded limit group with respect to the parameter subgroup $P = \langle p \rangle$, then a specialization (x_0, p_0, a) of the subgroup $G(x,p,a)$ is said to be *R-AP-covered*, if the specialization of the (fixed) set of generators of $G(x,p,a)$ is covered by the union of translates of the fixed set of generators of the subgroup $AP = \langle p, a \rangle$, by elements in the R-ball of the graded limit group $G(x,p,a)$.

Theorem 34 (cf. [Se3, 1.7]). *There exist a (global) constant R, and an index u_0, for which given any positive integer $u > u_0$, and the test sequence that factors through the eventual sculpted resolution $SCRes_u$, there exists an integer $\ell(u)$, so that the sets of specializations of the stable rooted group $RootWPHG_{NQR}$ corresponding to the specializations $(q_{u,j}, h_1, w, p, a)_\ell$, $1 \leq j \leq u$, are R-APW-covered for all $\ell > \ell(u)$.*

Proof. Suppose that there is no such global constant R. Then for every positive integer t, there exists an index $u(t) > t$, an index $j(t)$, $1 \leq j(t) \leq u(t)$, and a test sequence that factors through the penetrated sculpted resolution $SCRes_{u(t)}$, for which the sequence of specializations of the stable rooted group $RootWPHG_{NQR}$, corresponding to the sequence of specializations, $\{(q_{u(t),j(t)}, h_1, w, p, a)_\ell\}_{\ell=1}^{\infty}$, are (all) not *t-APW-covered.* By Theorem 1.3 in [S3], for every t, we can pass to a subsequence that converges into an action of the limit group $RootWPHG_{NQR}$ on a limit $R^{s(t)}$-tree Y_t (by Theorem 1.3 in [S3], the action of $RootWPHG_{NQR}$ is in fact a free action of a limit quotient of $RootWPHG_{NQR}$). Since each specialization in this sequence is not *t-APW*-covered, the action of $RootWPHG_{NQR}$ on the limit $R^{s(t)}$-tree Y_t is not *t-APW*-covered.

By applying Theorem 1.3 in [S3] to a sequence of specializations, $\{(q_{u(t),j(t)}, h_1, w, p, a)_{\ell(t)}\}_{t=1}^{\infty}$, that approximate the actions on the $R^{s(t)}$-trees Y_t, from the sequence of actions of the limit group $RootWPHG_{NQR}$ on the $R^{s(t)}$-trees Y_t, it is possible to extract a subsequence (still denoted Y_t), that converges into a free action of the stable rooted limit group $RootWPHG_{NQR}$ on a limit R^{μ}-tree Y. Since for each index t the action of $RootWPHG_{NQR}$ on the $R^{s(t)}$-tree Y_t is not *t-APW* covered, the action of the stable limit group $RootWPHG_{NQR}$ on the limit R^{μ}-tree Y satisfies the properties of the action obtained in proving Theorem 1.7 of [S3], i.e. the orbit of the sub-R^{μ}-tree Y_{APW} in the limit R^{μ}-tree Y misses either a non-degenerate segment or a germ of a branching point in Y.

The continuation of our argument is a modification of the argument used to prove Theorem 1.7 of [S3]. By the proof of Theorem 1.2 of [S3], from the free action of the stable rooted limit group $RootWPHG_{NQR}$ on the real R^{μ}-tree Y, $RootWPHG_{NQR}$ inherits an abelian decomposition Γ, a decomposition in which the image of the subgroup $APW = \langle w,p,a \rangle$, is contained in a proper subgraph of groups Γ'. Furthermore, since we have added elements that represent (primitive) roots of each of the quasi-rooted subgroups that are connected to (non-abelian) vertex groups in the graph of groups Θ_{NQR}, each quasi-rooted subgroup of $WPHG_{NQR}$ is elliptic in the graph of groups Γ, as well as the subgroups corresponding to their primitive roots. Hence, in the graph of groups inherited by the non-quasi-rooted subgroup $WPHG_{NQR}$ from the graph of groups Γ, every quasi-rooted subgroup is elliptic as well.

Let Δ be the graph of groups obtained from Γ by collapsing Γ' to a vertex. In Δ the (parameter) subgroup APW is contained in a vertex stabilizer, each quasi-rooted subgroup is elliptic, hence, every non-QH vertex group, and every edge group in the non-quasi-rooted decomposition Θ_{NQR} of the non-quasi-rooted subgroup $WPGH_{NQR}$ of $RootWPHG_{NQR}$, is contained in a non-QH vertex group or an edge group in Δ. Furthermore, since the graph of groups Δ is non-trivial, and since $RootWPHG_{NQR}$ is obtained from $WPHG_{NQR}$ by adding roots, the abelian decomposition inherited by $WPHG_{NQR}$ from Δ is non-trivial as well.

Therefore, the abelian decomposition inherited by $WPHG_{NQR}$ is non-trivial and compatible with Θ_{NQR}, so for large enough t, the specializations, $\{(q_{u(t),j(t)},h_1,w,p,a)_{\ell(t)}\}_{t=1}^{\infty}$, are not almost shortest in their strictly-solid families for large enough t, which clearly contradicts our assumptions. □

To prove Theorem 34 we modified the argument used to prove Theorem 1.7 in [S3]. Given Theorem 34, to obtain a bound on the number of eventual sculpted resolutions, i.e. to complete the proof of Theorem 27, we modify the argument used to prove Theorems 2.9 and 2.13 of [S3]. In the sequel, we continue with a subsequence of the eventual sculpted and penetrated sculpted resolutions and their associated test sequences, for which the associated specializations of the stable rooted limit group $WPHG_{NQR}$ satisfy the conclusion of Theorem 34, i.e. are R'-APW-covered for some positive integer R'. These are still denoted $SCRes_u$ and $PenSCRes_j^u$.

Let B be a ball in the Cayley graph of the rooted stable limit group $RootWPHG_{NQR}$ that contains all the defining relations of $RootWPHG_{NQR}$. By Theorem 34 there exists some constant R', so that for any positive integer u, and the associated test sequence that factors through the eventual sculpted resolution $SCRes_u$, there exists an index $\ell(u)$, so that for every $\ell > \ell(u)$, and every j, $1 \leq j \leq u$, the image of the ball B of $RootWPHG_{NQR}$ corresponding to the specialization: $(q_{u,j}, h_1, w, p)_\ell$ is R'-APW-covered.

Let $R = 2 \cdot R'$. Recall ([Se3, 2.3]) that we called the finite subtree T, which is the union of the images of the edges in the ball B_R, that correspond to a homomorphism from the stable limit group, $RootWPHG_{NQR}$, to the coefficient group, an *R-state*. The *combinatorial type* of an R-state is the combinatorial (not metric) finite tree associated with an R-state, including notation for the corresponding image of the vertices and edges in the ball B_R. Clearly, there exist only finitely many combinatorial types of R-states. Hence, by possibly passing to a subsequence and changing the indexing, we may further assume that for every index u, and every j, $1 \leq j \leq u$, and every index ℓ, the combinatorial types of the R-states associated with the specializations of $RootWPHG_{NQR}$, that are associated with the specializations $(q_{u,j}, h_1, w, p, a)_\ell$, are identical.

For each index u, and each j, $1 \leq j \leq u$, the R-state of the rooted stable limit group $RootWPHG_{NQR}$ corresponding to the sequences of specializations $\{(q_{u,j}, h_1, w, p, a)_\ell\}$ converge, after rescaling the metric on them by dilatation constants that depend only on the specializations of the defining parameters $(w,p)_\ell$, into an R-state, which we denote $Rstate_\infty^{u,j}(q_{u,j},h_1,w,p,a)$. Note that the R-state, $Rstate_\infty^{u,j}$, is an $R^{\mu(u,j)}$-tree. By passing to a subsequence and changing the indexing, we may further assume that any sequence of R-states $\{Rstate_\infty^{u,j(u)}(q_{u,j(u)}, h_1, w, p, a)\}_{u=1}^\infty$, where $1 \leq j \leq u$, converges in the Gromov–Hausdorff topology on metric spaces, and the limit (finite) R^μ-tree Y is independent of the particular choice of indices.

Given a specialization of $RootWPHG_{NQR}$ corresponding to a specialization $(q_{u,j}, h_1, w, p, a)_\ell$, for some j, $1 \leq j \leq u$, we set $p_r(\ell, j, u)$ to be a segment in the R-state, $Rstate(q_{u,j}, h_1, w, p, a)_\ell$, that is mapped to the edge e_r in the upper level of the R^μ-tree Y (i.e. the length of the edge e_r in Y is not infinitesimal, and $p_r(\ell, j, u)$ is a subword (fraction) of a specialization of one of the defining parameters). Recall (Definition 2.6 in [S3]) that we say that $p_r(\ell, j, u)$ is *pseudo-periodic* if:

(i) $p_r(\ell, j, u) = v_1 \alpha^s v_2$ and the equality is graphical (i.e. with no cancellations);

(ii) $s > 10$ and the length of α^s is at least three quarters of the length of $p_r(\ell, j, u)$.

If $p_r(\ell, j, u)$ is pseudo-periodic, we set $period(p_r(\ell, j, u))$ to be the shortest length of an element $\alpha \in F_k$ that satisfies condition (i) above. We call *period*$(p_r(\ell,j,u))$ the *pseudo-period* of the pseudo-periodic segment $p_r(\ell,j,u)$. We call the maximal possible s that appears in condition (i) above, the *pseudo-periodicity* of $p_r(\ell,j,u)$.

We continue by going over the segments in the R^μ-tree Y. If for such segment e_r, there exists a subsequence of indices $\{(j,u)\}$, $1 \leq j \leq u$, for which there exist subsequences of the given test sequences, for which $\{p_r(\ell,j,u)\}_{u,\ell=1}^{\infty}$ is either not pseudo-periodic, or with bounded pseudo-periodicity, or if for each ℓ,u and j, $p_r(\ell,j,u) = v_1(\ell,j,u)\alpha(\ell,j,u)^{s(\ell,j,u)}v_2(\ell,j,u)$, where the equation is graphical, $\alpha(\ell,j,u)$ is the pseudo-period of $p_r(\ell,j,u)$ and $s(\ell,j,u)$ is the pseudo-periodicity of $p_r(\ell,j,u)$, and there exists some $\epsilon > 0$ for which $\frac{|v_1(\ell,f,u)|+|v_2(\ell,f,u)|}{|p_r(\ell,f,u)|} > \epsilon$, we restrict our attention to this subsequence and declare the particular segment in the finite tree Y *not truly periodic*. After possibly restricting our attention to some subsequence, we may assume that the sequence of specializations corresponding to each segment is either not truly periodic, or it is pseudo-periodic with no bounded subsequence of pseudo-periodicities. We call these last segments *truly periodic*.

We have chosen the test sequences of specializations to have the same combinatorial type of R-states, $\{Rstate_{\infty}^{u,j}\}$. Given indices u and j, and an index ℓ in a test sequence that factors through and associated with $SCRes_u$, we set $Mper^{u,\ell}$ to be the maximum among all pseudo-periods of truly periodic segments in the R-state corresponding to the specialization of $RootWPHG_{NQR}$ associated with $(q_{u,j},h_1,w,p,a)_\ell$, after rescaling the metric on the corresponding R-state (note that this maximum is independent of the index j, $1 \leq j \leq u$). We denote by $Mp_{j,u}^{\ell}$ the maximum between $Mper^{u,\ell}$ and the maximum length (after rescaling the metric) of a segment between branching points in the R-state corresponding to the specialization of $RootWPHG_{NQR}$ that is associated with $(q_{u,j},h_1,w,p,a)_\ell$, which is mapped to an infinitesimal segment in the limit R^μ-tree Y. For each two specializations $(q_{u,j_1},h_1,w,p,a)_\ell, (q_{u,j_2},h_1,w,p,a)_\ell$ we have associated R-states $RState(\ell,j_1,u)$ and $RState(\ell,j_2,u)$ in correspondence. Given a branching point in these states and a copy of one of the defining parameters that passes through this branching point, we can measure the difference (after rescaling) in the placement of the branching point on the copy of the defining parameter in the two different R-states, $RState(\ell,j_1,u)$ and $RState(\ell,j_2,u)$. We denote by $Mfluct(\ell,j_1,j_2,u)$ the maximum among all these (finite set of) differences. Note that by construction both $Mp_{j,u}^{\ell}$ and $Mfluct(\ell,j_1,j_2,u)$ are asymptotically $o(1)$.

Following Definition 2.2 in [S3], we say that a sequence of indices $\{u_t\}_{t=1}^{\infty}$, together with tuples of integers $1 \leq j_1(t) < j_2(t) < \ldots < j_t(t) \leq u_t$, and test sequences that factor through the various eventual sculpted resolutions $SCRes_{u(t)}$ are *tame* with respect to segments in the R-state that correspond to edges in the upper level of the R^{μ}-tree Y, if there exists a constant c for which $c \cdot Mp^{\ell}_{j_{b_1}(t),u_t} > Mfluct\big(\ell, j_{b_2}(t), j_{b_3}(t), u_t\big)$, for all possible indices t, ℓ, b_1, b_2, b_3, $1 \leq b_1, b_2, b_3 \leq u_t$.

If the test sequences of specializations we started with contain a tame subsequence, we continue with the tame subsequence of specializations. By possibly passing to a further subsequence, we may assume that the maximum $Mp^{\ell}_{j,u}$ is always obtained for the same segment in the corresponding R-state or for a pseudo-period of the same truly periodic segment, that the maximal fluctuation $Mfluct(\ell, j_1, j_1, u)$ is asymptotically $o(Mp^{\ell}_{j,u})$, and that the ratios between the lengths of segments that are mapped to infinitesimal segments in the limit R^{μ}-tree Y, and their length is $O(Mp^{\ell}_{j,u})$, and pseudo-periods of truly periodic segments in the R-state that are asymptotically $O(Mp^{\ell}_{j,u})$, converge, where the limit is taken first on the corresponding test sequences (the index ℓ) and then on the indices u of the sculpted resolutions.

If the original sequence of specializations is tame with respect to the segments in R-state corresponding to edges in the upper level of the R^{μ}-tree Y (i.e. the non-infinitesimal edges in Y), we declare each of the original segments in the upper level of the R^{μ}-tree Y that are truly periodic, and for which their pseudo-period is $O(Mp^{\ell}_{j,u})$ as *not truly periodic* for the continuation. We also check which of the "newly uncovered" segments (i.e. those that are mapped to infinitesimal segments in Y, and are of lengths $O(Mp^{\ell}_{j,u})$ are truly periodic, and pass to an appropriate subsequence. We check if the obtained subsequence is tame with respect to segments in the R-state corresponding to edges in the top two levels in the R^{μ}-tree Y. Since there are only finitely many segments in the R-state, and since there are u distinct specializations for each step u in our sequence, after finitely many "uncoverings" we obtain a sequence which is not tame with respect to the previously "uncovered" segments, and pseudo-periods of truly periodic segments in the R-state.

Once we obtain a sequence of specializations with no tame subsequences with respect to the previously "uncovered" segments and pseudo-periods of truly periodic segments in the corresponding R-states, we can choose a subsequence of indices (u, j), $1 \leq j \leq u$, together with test sequences that factor through the various eventual sculpted resolutions $SCRes_u$, so that for every u and every index ℓ of the subsequence, $u \cdot Mp^{\ell}_{j',u} < Mfluct(\ell, j_2, j_3, u)$, for all possible indices j_2, j_3, and an index j' for which the minimal $Mp^{\ell}_{j',u}$ is obtained. We call such a sequence of specializations a *perturbed* sequence.

Given a perturbed sequence, for each index u we construct the R-state corresponding to the specialization of $RootWPHG_{NQR}$ that is associated with $(q_{u,j'}, h_1, w, p, a)_\ell$, for which $Mp^{\ell}_{j',u}$ is minimal among all possible indices j, $1 \leq j \leq u$. Clearly, given such an R-state, we can present each of the specializations of $RootWPHG_{NQR}$ that are associated with the variables $q_{u,j'}$ in terms of *fractions* of specializations of few of the defining parameters $\{h_1, w, p, a\}$. Hence, for every integer u we can represent the specialization of $RootWPHG_{NQR}$ that is associated with $(q_{u,j'}, h_1, w, p, a)_\ell$ in terms of fractions of specializations of the parameters $\{h_1, w, p, a\}$ that appear in the R-state corresponding to this specialization; and for any other possible index j, we can present the specialization of $RootWPHG_{NQR}$ associated with $(q_{u,j}, h_1, w, p, a)_\ell$ in terms of the same fractions of the specializations of the parameters that appear in the R-state associated with the specialization $(q_{u,j'}, h_1, w, p, a)_\ell$, together with elements $f_{u,\ell}(j)$ that encode the fluctuations in the corresponding branching points in the R-state associated with the specialization $(q_{u,j}, h_1, w, p, a)_\ell$ and the "reference" R-state associated with the specialization $(q_{u,j'}, h_1, w, p, a)$.

At this point we continue as we did in the proofs of Theorems 2.5, 2.9 and 2.13 of [S3]. Each relation from the defining relations of the limit group $RootWPHG_{NQR}$ corresponds to a closed loop on the R-state, and since the combinatorial type of the R-states associated with all specializations from the perturbed sequence are assumed to be identical, the "combinatorial type" of the loop corresponding to a given relation is identical for all the specializations from the perturbed sequence.

Furthermore, by our "uncovering" procedure, every part of a loop that starts at some point on any of the "uncovered" segments of the defining parameters, which are not truly periodic, and get back to the same point, represent the identity element in all R-states corresponding to specializations from our perturbed sequence. Also, every part of a loop that starts at some point on any of the "uncovered" segments of the defining parameters, which are truly periodic, and get back to the same point, represent an element that commutes with a pseudo-period of that truly periodic segment in all R-states corresponding to the specializations from our perturbed sequence.

Since for every index u such part of a loop of a defining relation represent either the identity element or an element that commutes with a pseudo-period of a truly periodic "uncovered" segment in the R-state associated with the (distinguished) specialization of $RootWPHG_{NQR}$ that is associated with $(q_{u,j'},h_1,w,p,a)_\ell$, and the same holds for every other R-state associated with the specialization $(q_{u,j'},h_1,w,p,a)_\ell$, every such part of a loop of one of the defining relations corresponds to a (parametric) equation involving (possibly) coefficients from the free group $F_k=\langle a\rangle$, fractions of parameters corresponding to segments in the R-state associated with the (distinguished) specialization $(q_{u,j'},h_1,w,p,a)_\ell$ that have not been uncovered, and pseudo-periods of truly periodic "uncovered" segments, that we denote $\hat{p}$, and variables corresponding to the "fluctuations" $f_{u,\ell}(j)$. Since the combinatorics of the R-states associated with all the specializations from our perturbed sequence is identical, the set of parametric equations corresponding to different parts of (closed) partial loops corresponding to the defining relations is identical for all specializations from the perturbed sequence. We denote this system of equations by $\Sigma_1(f,\hat{p},a)=1$.

To the system of equations $\Sigma_1(f,\hat{p},a)$ that involves the loops around each of the branching points, we add equations that guarantee partial equivariance, i.e. the additional equations guarantee that the specializations of each of the generators of $RootWPHG_{NQR}$ in all places they appear in the image of the ball of radius $R'=R/2$, $B_{R'}$, are identical. These equations can also be written in terms of the same fractions of the defining parameters $\hat{p}$, and the variables corresponding to the various fluctuations f that were used in formulating the equations that involve the loops around the branching points. We denote the system of equations that combine the equations corresponding to the loops around branching points and the equations coming from the partial equivariance condition by $\Sigma(f,\hat{p},a)=1$.

By our standard arguments presented in section 5 of [S1], the specializations of the fluctuations and the "uncovered" fractions of parameters $\{f,\hat{p},a\}$ corresponding to the test sequences of specializations $(q_{u,j},h_1,w,p,a)_\ell$ factor through a (canonical) finite collection of maximal limit groups $\{MLim_t(f,\hat{p},a)\}$. Hence, by passing to a further subsequence we may assume that they factor through a unique maximal limit group $MLim(f,\hat{p},a)$. By Theorem 2.5 of [S3] we may assume that this maximal limit group is not rigid with respect to the parameter subgroup $\langle\hat{p}\rangle$.

With the limit group $MLim(f,\hat{p},a)$ we associate a limit group $MPLim(f,\hat{p},p',a)$, generated by the limit group $MLim(f,\hat{p},a)$, and the fractions p' that were "uncovered" through the process. By construction, there exists a natural map $\eta: RootWPHG_{NQR}\to MPLim(f,\hat{p},p',a)$.

The limit group $MLim(f,\hat{p},a)$ was obtained as a limit group from a sequence of specializations of the eventual sculpted resolutions $SCRes_u$, for which the corresponding specializations of the penetrated sculpted resolutions $PenSCRes_j^u$ were assumed to be almost shortest strictly-solid specializations (with respect to the given covering closure) that factor through the stable limit group $WPHG$. If the limit group $MLim(f,\hat{p},a)$ admits a non-trivial graded free decomposition, then its associated limit group $MPLim(u,\hat{p},p',a)$ admits a non-trivial graded free decomposition (with respect to the parameter subgroup $\langle \hat{p},p',a\rangle$), which implies that the rooted stable limit group $RootWPHG_{NQR}$ admits a non-trivial graded free decomposition in which all the quasi-rooted subgroups can be conjugated into the various factors, which finally implies that the subgroup $WPHG_{NQR}$ admits a non-trivial graded free decomposition in which all the quasi-rooted subgroups can be conjugated into the various factors. Hence, the stable limit group $WPHG$ admits a graded free decomposition, through which the corresponding specializations of the eventual penetrated sculpted resolutions $PenSCRes_j^u$ factor, a contradiction to our assumption that they represent extra strictly-solid families. Hence, we may assume that the limit group $MLim(f,\hat{p},a)$ admits no non-trivial graded free decomposition.

The rooted stable limit group $RootWPHG_{NQR}$ is mapped into the limit group $MPLim(f,\hat{p},p',a)$ by the homomorphism $\eta : RootWPHG_{NQR} \to MPLim(f,\hat{p},p',a)$. Recall that Θ_{NQR} is the (graded) non-quasi-rooted abelian decomposition associated with the non-quasi-rooted subgroup $WPHG_{NQR}$, and let Γ be the graded abelian JSJ decomposition associated with $MPLim(f,\hat{p},p',a)$. By the construction of the limit group $MPLim(f,\hat{p},p',a)$, the image of the stable rooted limit group $RootWPHG_{NQR}$, $\eta(RootWPHG_{NQR})$, cannot be conjugated into the fundamental group of a proper subgraph of the graph of groups Γ, and every quasi-rooted subgroup of $WPHG_{NQR}$ (which is naturally a subgroup of $RootWPHG_{NQR}$) is elliptic in Γ (simply since every non-cyclic abelian subgroup is elliptic). Since Θ_{NQR} is the graded non-quasi-rooted abelian decomposition of the non-quasi-rooted subgroup $WPHG_{NQR}$, the subgroup generated by the edge groups that are connected to an abelian vertex group in Θ_{NQR} is of finite index in the abelian vertex group, and every quasi-rooted subgroup of $WPHG_{NQR}$ is elliptic in Γ, every non-QH vertex group and every edge group in Θ_{NQR} is mapped by η into either a non-abelian, non-QH vertex group or into an edge group in Γ, or it is mapped into an abelian vertex group and intersects the subgroup generated by the edge groups connected to it in a subgroup of finite index.

By Theorem 1.3 in [S3], we can pass to subsequences of the eventual sculpted and penetrated sculpted resolutions and subsequences of their associated test sequences, so that the specializations of the stable rooted limit group $RootWPHG_{NQR}$ associated with the specializations, $\{(q_{u,j},h_1,w,p,a)_\ell\}$, converge into an action of the stable rooted limit group $RootWPHG_{NQR}$ on an R^μ-tree T. Along our "uncovering" procedure, we have uncovered fractions p' of the defining parameters $\langle w,p\rangle$, until we reached a stage in which the size of the fluctuations are much bigger than the size of the fractions $\hat{p}$ of the defining parameters that are not yet uncovered.

From the action of the stable rooted limit group $RootWPHG_{NQR}$ on the upper level of the R^μ-tree T, it inherits an abelian decomposition Δ_1. Each non-QH, non-abelian vertex group V in Δ_1, stabilizes a point in the upper level of the R^μ-tree T, hence, it acts on $R^{\mu-1}$-tree T_V. The vertex group V inherits an abelian decomposition Δ_2 from its action on the upper level of the $R^{\mu-1}$-tree T_V. By iteratively continuing with non-abelian, non-QH vertex groups in the newly obtained abelian decompositions, we finally reach a level n for which the fluctuations are elliptic in the abelian decompositions Δ_n constructed at that level, and they are not all elliptic in the various abelian decompositions Δ_{n+1} constructed at the $n+1$ level. Since there are only finitely many orbits of vertex stabilizers in each of the abelian decompositions Δ_i, and the abelian decompositions for conjugate vertices are identical, with each level i, we associate only finitely many abelian decompositions Δ_i.

DEFINITION 35. *Let V be a non-QH, non-abelian vertex group in one of the abelian decompositions Δ_n. We say that an abelian decomposition Γ_V of the vertex group V is a* multi-graded compatible abelian decomposition *of V, if the following conditions hold:*

(i) *Every edge group connected to V in Δ_n is elliptic in Γ_V.*

(ii) *By condition* (i), *we can use the abelian decomposition Γ_V to refine the decomposition Δ_n, and obtain an abelian decomposition Δ'_n. Let Γ_n be the abelian decomposition obtained from Δ'_n by collapsing all the edges in Δ'_n that correspond to edges in Δ_n, and let V_n be the fundamental group of Γ_n (which is the fundamental group of Δ_n). Note that the structure of Γ_n is similar to the structure of Γ_V, i.e. there is a one-to-one correspondence between the edges and the non-QH, non-abelian vertex groups. Then every edge group connected to V_n in Δ_{n-1} is elliptic in Γ_n.*

(iii) *We continue by induction (decreasing i, from n to 1). Given the abelian decomposition Γ_i of a vertex group V_i in the abelian decomposition Δ_{i-1}, we refine the abelian decomposition Δ_{i-1}, to obtain an abelian decomposition Δ'_{i-1}. We further collapse all the edges in Δ'_{i-1} that correspond to edges in Δ_i, to obtain an abelian decomposition Γ_{i-1} of the fundamental group of Δ_{i-1}, which we denote V_{i-1}. Then every edge group connected to V_{i-1} in Δ_{i-2} is elliptic in Γ_{i-2}.*

(iv) *By condition* (iii) *we finally obtain an abelian decomposition Γ_1 of $RootWPHG_{NQR}$. We further assume that the parameter subgroup $\langle w,p,a\rangle$ is elliptic in the abelian decomposition Γ_1, that every non-cyclic abelian subgroup of $RootWPHG_{NQR}$ is elliptic in the abelian decomposition Γ_1, and that every quasi-rooted subgroup of $WPHG_{NQR}$, which is a subgroup of $RootWPHG_{NQR}$, is elliptic in Γ_1.*

Given a hierarchy of decompositions of vertex stabilizers in a tower of graphs of groups associated with the rooted stable limit group $RootWPHG_{NQR}$, the collection of multi-graded decompositions of $RootWPHG_{NQR}$ that are compatible with the given hierarchy of abelian decompositions is encoded in a *multi-graded compatible abelian decomposition* of the (graded) stable rooted limit group $RootWPHG_{NQR}$.

Theorem 36. *Let $RootWPHG_{NQR}$ be a multi-graded limit group, where the parameter subgroups is taken to be $\langle w,p\rangle$ and $E_1,\dots,E_q$, where $E_1,\dots,E_q$ are its quasi-rooted subgroups. Suppose that $RootWPHG_{NQR}$ admits no multi-graded free decompositions, i.e. $RootWPHG_{NQR}$ cannot be decomposed into a free product in which the subgroups $\langle w,p,a\rangle, E_1,\dots,E_q$ can be conjugated into the factors.*

Let Δ_1 be an abelian decomposition of the limit group $RootWPHG_{NQR}$, viewed as an ungraded limit group. With each non-QH, non-abelian vertex group V in Δ_1 we further associate an abelian decomposition Δ_2, with each non-QH, non-abelian vertex group in one of the abelian decompositions Δ_2 we associate an abelian decomposition Δ_3 and so on, until we associate an abelian decomposition Δ_n with each non-QH, non-abelian vertex group in each of the abelian decompositions Δ_{n-1}.

Then there exists a reduced (perhaps trivial) splitting of $RootWPHG_{NQR}$ with abelian edge groups, which we call a multi-graded compatible abelian JSJ decomposition *of $RootWPHG_{NQR}$ (with respect to the above sequence of hierarchical decompositions) with the following properties:*

(i) *The multi-graded compatible abelian JSJ decomposition is compatible with the given sequence of hierarchical decompositions of* $RootWPHG_{NQR}$.

(ii) *Every (multi-graded compatible) canonical maximal QH subgroup (CMQ) of* $RootWPHG_{NQR}$ *is conjugate to a vertex group in the multi-graded compatible JSJ decomposition. Every (multi-graded compatible)* QH *subgroup of* $RootWPHG_{NQR}$ *can be conjugated into one of the (multi-graded compatible) CMQ subgroups of* $RootWPHG_{NQR}$. *Every vertex group in the multi-graded compatible JSJ decomposition which is not a CMQ subgroup of* $RootWPHG_{NQR}$ *is elliptic in any multi-graded compatible abelian splitting of* $RootWPHG_{NQR}$.

(iii) *A one edge (multi-graded compatible) abelian splitting* $RootWPHG_{NQR} = D *_A E$ *or* $RootWPHG_{NQR} = D*_A$, *which is hyperbolic in another such elementary multi-graded compatible abelian splitting, is obtained from the multi-graded compatible abelian JSJ decomposition of* $RootWPHG_{NQR}$ *by cutting a 2-orbifold corresponding to a (multi-graded compatible) CMQ subgroup of* $RootWPHG_{NQR}$ *along a weakly essential s.c.c.*

(iv) *Let* Δ *be a one edge (multi-graded compatible) splitting along an abelian subgroup* $RootWPHG_{NQR}{=}D{*_A}E$ *or* $RootWPHG_{NQR}{=}D{*_A}$, *which is elliptic with respect to any other one edge (multi-graded compatible) abelian splitting of* $RootWPHG_{NQR}$. *Then* Δ *is obtained from the multi-graded compatible JSJ decomposition of* $RootWPHG_{NQR}$ *by a sequence of collapsings, foldings, and conjugations.*

(v) *If* JSJ_1 *is another multi-graded compatible abelian JSJ decomposition of* $RootWPHG_{NQR}$, *then* JSJ_1 *is obtained from the multi-graded compatible abelian JSJ decomposition by a sequence of slidings, conjugations and modifying boundary monomorphisms by conjugations (see section 1 of [RS] for these notions).*

Proof. Given two multi-graded abelian decompositions Λ_1 and Λ_2 of the multi-graded limit group $RootWPHG_{NQR}$, the "machine" for the construction of a JSJ decomposition, presented in [RS], constructs a multi-graded abelian decompositions, Λ_C, which is a "common refinement" of Λ_1 and Λ_2. If Λ_1 and Λ_2 are compatible with the given sequence of hierarchical decompositions of $RootWPHG_{NQR}$, so is the common refinement produced by the "JSJ machine". Therefore, the (canonical) multi-graded abelian decomposition produced by the "JSJ machine" from the entire collection of multi-graded compatible abelian decompositions of $RootWPHG_{NQR}$, is a multi-graded compatible abelian decomposition of $RootWPHG_{NQR}$. As in the construction of the (multi-graded) abelian JSJ decomposition of a limit group, the (canonical) multi-graded abelian decomposition produced from the entire collection of graded compatible abelian decompositions of $RootWPHG_{NQR}$, satisfies properties (i)–(v) listed in the theorem. $\square$

As we have already observed, from the action of the stable rooted limit group $RootWPHG_{NQR}$ on the upper level of the R^{μ}-tree T, $RootWPHG_{NQR}$ inherits an abelian decomposition Δ_1. Each non-QH, non-abelian vertex group V in Δ_1, that stabilizes a point in the upper level of T, acts on an $R^{\mu-1}$-tree T_V. The vertex group V inherits an abelian decomposition Δ_2 from this action. By iteratively continuing with non-abelian, non-QH vertex groups in the newly obtained abelian decompositions, we finally reach a level n for which the fluctuations are contained in non-QH, non-abelian vertex groups in the abelian decompositions Δ_n constructed at that level, and they are not all elliptic in the various abelian decompositions Δ_{n+1} constructed at the $n+1$ level.

The stable rooted limit group $RootWPHG_{NQR}$ is embedded by η into the limit group $MPLim(f,\hat{p},p',a)$. By construction, the graded abelian JSJ decomposition of the limit group $MPLim(f,\hat{p},p',a)$ is non-trivial, hence, the abelian decomposition inherited by the stable rooted limit group $RootWPHG_{NQR}$ from its embedding into $MPLim(f,\hat{p},p',a)$, is non-trivial, every quasi-rooted subgroup of the non-quasi-rooted subgroup $WPHG_{NQR}$ (that is embedded in $RootWPHG_{NQR}$) is elliptic in it, and by construction this abelian decomposition of $RootWPHG_{NQR}$ is multi-graded (with respect to the parameter subgroup $\langle w,p\rangle$, and the divisible subgroups), and compatible with the hierarchical sequence of abelian decompositions the Rooted stable limit group $RootWPHG_{NQR}$ inherits from its action on the top n levels of the R^{μ}-tree T. Hence, the multi-graded compatible JSJ decomposition of the rooted stable limit group $RootWPHG_{NQR}$, with respect to the sequence of abelian decompositions it inherits from its action on the top n levels of the R^{μ}-tree T is non-trivial, which implies that the multi-graded compatible JSJ decomposition of the non-quasi-rooted subgroup $WPHG_{NQR}$ (with respect to the hierarchical sequence of abelian decompositions it inherits from its action on the top n levels of the R^{μ}-tree T) is non-trivial. Let $\Gamma_{WPHG_{NQR}}$ be the multi-graded compatible abelian JSJ decomposition of the non-quasi-rooted limit group $WPHG_{NQR}$, associated with its action on the top n levels of the R^{μ}-tree T.

As we did in the proof of Theorems 2.9 and 2.13 in [S3], the properties of the restriction of the embedding η, from the non-quasi-rooted limit group $WPHG_{NQR}$ we started with, into the graded limit group $MPLim(f,\hat{p},p',a)$, allow us to bound the *reduced complexity* of the multi-graded compatible abelian JSJ decomposition $\Gamma_{WPHG_{NQR}}$, in terms of the reduced complexity of the non-quasi-rooted graded abelian decomposition Θ_{NQR} of $WPHG_{NQR}$. We start by recalling the definition of the *reduced complexity* of a graded limit group (Definition 2.10 in [S3]).

Let $Glim(y,p,a)$ be a (multi-) graded limit group, with respect to the subgroup $P=\langle p\rangle$, and suppose that $Glim(y,p,a)$ admits no non-trivial graded free decomposition. Let Γ be a graded abelian decomposition of $Glim(y,p,a)$.

Let $Q_1,\dots,Q_n$ be the QH vertex groups in the graded abelian decomposition Γ, and let $S_1,\dots,S_n$ be their corresponding (punctured) surfaces. We set the complexity of a QH vertex group Q_j, to be the ordered couple $(|\chi(S_j)|,genus(S_j))$. We set the *reduced complexity* of the graded abelian decomposition Γ, denoted $Rcomp(\Gamma)$, to be the tuple

$$\big((|\chi(S_1)|,genus(S_1)),\dots,(|\chi(S_n)|,genus(S_n)),d\big)\,,$$

where the sequence of couples $(|\chi(S_1)|,genus(S_1)),\dots,(|\chi(S_n)|,genus(S_n))$ is ordered in a decreasing lexicographical order, and the integer d is set to be $d=e-a$, where e is the number of edges between non-QH vertex groups in Γ, and a is the number of abelian vertex groups adjacent to the collection of these edges.

Let Γ_1 and Γ_2 be two graded abelian decompositions of the graded limit groups $Glim_1(y,p,a)$ and $Glim_2(y,p,a)$ in correspondence, and $Glim_1(y,p,a)$, $Glim_2(y,p,a)$ admit no non-trivial graded free decompositions. We say that $Rcomp(\Gamma_1)\leq Rcomp(\Gamma_2)$ if the sequence associated with Γ_1 is not greater than the sequence associated with Γ_2 in the natural lexicographical order on the above sequences. Clearly, the reduced complexity of any two such graded decompositions can be compared.

PROPOSITION 37. *Let $\Gamma_{WPHG_{NQR}}$ be the multi-graded compatible abelian JSJ decompositions of the non-quasi-rooted limit group $WPHG_{NQR}$, with respect to the sequence of abelian decompositions it inherits from its action on the top n levels of the R^{μ}-tree T, let Θ be the graded abelian decomposition of the stable limit group $WPHG$, and let Θ_{NQR} be the (original) non-quasi-rooted multi-graded abelian decomposition of $WPHG_{NQR}$. Then,*

$$Rcomp(\Gamma_{WPHG_{NQR}})<Rcomp(\Theta_{NQR})\leq Rcomp(\Theta)\,.$$

Proof. The second inequality is immediate from the construction of the non-quasi-rooted subgroup $WPHG_{NQR}$. Since Θ_{NQR} is the multi-graded abelian JSJ decomposition of $WPHG_{NQR}$, where all the quasi-rooted subgroups are supposed to be elliptic, and $\Gamma_{WPHG_{NQR}}$, being the compatible abelian JSJ decomposition, is a multi-graded abelian decomposition of $WPHG_{NQR}$ in which all the quasi-rooted subgroups are elliptic, Θ_{NQR} has to be a refinement (in the JSJ sense) of $\Gamma_{WPHG_{NQR}}$, hence, $Rcomp(\Gamma_{WPHG_{NQR}}) \leq Rcomp(\Theta_{NQR})$.

Suppose that $Rcomp(\Gamma_{WPHG_{NQR}}) = Rcomp(\Theta_{NQR})$. Then the graph of groups $\Gamma_{WPHG_{NQR}}$ is identical to Θ_{NQR}, which implies that all the edge groups, and all the QH vertex groups in the graded abelian JSJ decomposition Θ_{NQR}, are contained in non-QH, non-abelian vertex groups, and edge groups, in the abelian decompositions Δ_n, inherited by (subgroups of) $WPHG_{NQR}$ from the n-th level of its action on the R^{μ}-tree T.

Since Θ_{NQR} is the multi-graded abelian JSJ decomposition of $WPHG_{NQR}$, all the edge groups, and the boundary components of QH vertex groups in Θ_{NQR} are mapped by $\eta : WPHG_{NQR} \to RootWPHG_{NQR}$ into non-QH, non-abelian vertex groups or into edge groups in the graded abelian JSJ decomposition of the limit group $MPLim(u, \hat{p}, p', a)$. This implies that for large enough u, $1 \leq j \leq u$, and large enough ℓ, for the specializations, $(q_{u,j}, h_1, w, p, a)_{\ell}$, the sizes of the corresponding specializations of edge groups, a (fixed) generating set of a non-distinguished, non-QH vertex group, and the the boundary elements of QH vertex groups in Θ_{NQR} (after appropriate conjugation), are much smaller than either the sizes of a corresponding specialization of a (fixed) generating set of at least one of the QH vertex groups in Θ_{NQR}, or than the size of an element that conjugates the specialization of some non-abelian vertex group in Θ_{NQR} (and commutes with the corresponding edge group). This clearly contradicts the assumption that the specializations $(q_{u,j}, h_1, w, p, a)_{\ell}$ are almost shortest. Hence, $Rcomp(\Gamma_{WPHG_{NQR}}) < Rcomp(\Theta_{NQR})$. □

By Proposition 37, the reduced complexity of $\Gamma_{WPHG_{NQR}}$ is strictly smaller than the reduced complexity of the non-quasi-rooted abelian decomposition Θ_{NQR}, which is bounded by the reduced complexity of the graded abelian JSJ decomposition Θ of the stable limit group $WPHG$.

Since every quasi-rooted edge group in Θ is elliptic in $\Gamma_{WPHG_{NQR}}$, the graded abelian decomposition $\Gamma_{WPHG_{NQR}}$ can be naturally extended to a graded abelian decomposition of the stable limit group $WPHG$, by adding the edges and the abelian vertex groups that were removed from the abelian decomposition Θ to obtain the abelian decomposition Θ_{NQR}. We denote this graded abelian decomposition of the stable limit group $WPHG$, that is obtained from $\Gamma_{WPHG_{NQR}}$, Θ^2. Since, $Rcomp(\Gamma_{WPHG_{NQR}}) < Rcomp(\Theta_{NQR})$, by Proposition 37, $Rcomp(\Theta^2) < Rcomp(\Theta)$.

With the graded abelian decomposition Θ^2 we associate its non-quasi-rooted subgraph (of groups) according to Definition 31, which we denote Θ^2_{NQR}. According to Definition 31, with the graph of groups Θ^2_{NQR} we also associate a collection of quasi-rooted edge groups and abelian vertex groups, a collection that includes all the quasi-rooted edge groups and abelian vertex groups in the (original) graded abelian decomposition Θ. With Θ^2_{NQR} we associate its fundamental group, which we call the (second) non-quasi-rooted subgroup, and denote $WPHG^2_{NQR}$. By construction, the non-quasi-rooted subgroup, $WPHG^2_{NQR}$, is a subgroup of the non-quasi-rooted subgroup $WPHG_{NQR}$ associated with the graded abelian decomposition Θ.

According to Definition 31, along the construction of the (second) non-quasi-rooted subgraph and subgroup, we pass to a subsequence of our given sequence of eventual sculpted and penetrated sculpted resolutions and their associated test sequences. In the sequel we continue to denote them, $SCRes_u$ and $PenSCRes^u_j$.

Suppose that the number of (conjugacy classes of) quasi-rooted edge groups, associated with the obtained decomposition Θ^2, is strictly bigger than the number of quasi-rooted edge groups associated with the graded abelian decomposition of the stable limit group $WPHG$, Θ, we started with. Then we replace the graded abelian decomposition Θ, by the multi-graded abelian decomposition of the stable limit group $WPHG$ with respect to the parameter subgroup $\langle w,p\rangle$, and the obtained collection of quasi-rooted edge groups, which we denote Θ_2. By construction, the number of quasi-rooted edge groups associated with Θ_2 is strictly bigger than the number of quasi-rooted edge groups associated with Θ.

In this case, of larger number of quasi-rooted edge groups, we start the entire procedure with the graph of groups Θ_2 instead of the graph of groups Θ. With Θ_2 we associate a non-quasi-rooted subgraph, and a non-quasi-rooted group, and a stable rooted limit group, for which the corresponding specializations are $\hat{R}$-APW-covered (Theorem 34), for some positive integer $\hat{R}$, that may be different than the integer R needed for the original stable rooted limit group.

Therefore, we can analyze the action of the associated non-quasi-rooted subgroup on a corresponding $R^{\mu'}$-tree. Note that in case the number of quasi-rooted edge groups is larger, the number of Bass–Serre generators corresponding to the edges that are in Θ_2 but not in its associated non-quasi-rooted subgraph, is strictly bigger than the number of Bass–Serre generators corresponding to such edges in the original graph of groups Θ.

Suppose that the number of (conjugacy classes of) quasi-rooted edge groups and abelian vertex groups in Θ^2 is identical to their number in Θ. For each couple of indices u and j, where $1 \leq j \leq u$, and a specialization $(q_{u,j}, h_1, w, p, a)_\ell$ of the penetrated sculpted resolution $PenSCRes_j^u$, we associate with the specialization $(q_{u,j}, h_1, w, p, a)_\ell$ a specialization $(\hat{q}_{u,j}, h_1, w, p, a)_\ell$ which is an almost shortest specialization in the same solid family of the specialization $(q_{u,j}, h_1, w, p, a)_\ell$, with respect to the modular groups associated with the graded abelian decomposition Θ^2, where the order on the modular groups according to which the specializations are almost shortest, starts with the modular groups associated with the non-quasi-rooted subgraph Θ^2_{NQR}, and then the modular groups associated with the quasi-rooted edge and abelian vertex groups in Θ^2.

For each u and j, $1 \leq j \leq u$, we replace the subgroup $\langle q_{u,j}, h_1, w, p, a \rangle$ of the penetrated sculpted resolution $PenSCRes_j^u$, by a subgroup $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$, where the subgroup $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$ is obtained as a limit group from a subsequence of specializations, $(\hat{q}_{u,j}, h_1, w, p, a)_\ell$, that are almost shortest with respect to the action of the modular groups associated with Θ^2, according to their prescribed order. To the subgroup $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$ of the (corresponding closure of the) penetrated sculpted resolution $PenSCRes_j^u$, we add elements that demonstrate the Diophantine condition that implies that specializations of the subgroups, $\langle q_{u,j}, h_1, w, p, a \rangle$ and $\langle \hat{q}_{u,j}, h_1, w, p, a \rangle$, belong to the same family of specializations of the stable limit group $WPHG$, that is associated with the abelian decomposition Θ^2 (see Definition 1.5 in [S3] for the Diophantine condition that defines such family).

With the non-quasi-rooted subgroup Θ^2_{NQR}, and the given sequence of eventual sculpted and penetrated sculpted resolutions, and their associated test sequences, we further associate a stable rooted limit group according to Definition 33, which we denote, $RootWPHG^2_{NQR}$. Note that in defining the (second) rooted limit group we may need to pass a subsequence of our given sequence of eventual sculpted and penetrated sculpted resolutions, and their associated test sequences.

In Theorem 34 we have shown that there exist some global constant R, and an index u_0, and for any positive integer $u > u_0$, an index $\ell(u)$, so that for every $u > u_0$, and j, $1 \leq j \leq u$, and every $\ell > \ell(u)$, the specialization of the (first) rooted stable limit group $RootWPHG_{NQR}$, associated with the specialization $(q_{u,j}, h_1, w, p, a)_\ell$, is R-APW-covered. A modified argument shows that a similar statement holds for the sequences of specializations of the (second) rooted stable limit group $RootWPHG^2_{NQR}$, associated with a specializatio $\{(\hat{q}_{u,j}, h_1, w, p, a)_\ell\}$ (which are almost shortest with respect to the modular groups associated with Θ^2 according to their prescribed order).

Theorem 38. *There exist a (global) constant R, and an index u_0, for which given any positive integers $u > u_0$ and j, $1 \leq j \leq u$, and a test sequence that factors through the eventual sculpted resolution $SCRes_u$ and associated with it, there exists an integer $\ell(u)$, so that the specialization of the (second) stable rooted limit group $RootWPHG^2_{NQR}$, corresponding to the specialization, $(\hat{q}_{u,j}, h_1, w, p, a)_\ell$, is R-APW-covered for all $\ell > \ell(u)$.*

Proof. The argument we use is similar to the one used to prove Theorem 34. Suppose that there is no such global constant R. Then for every positive integer t, there exists an index $u(t) > t$, an index $j(t)$, $1 \leq j(t) \leq u(t)$, and a test sequence that factors through the penetrated sculpted resolution $SCRes_{u(t)}$, for which the sequence of specializations of the (second) stable rooted limit group $RootWPHG^2_{NQR}$, corresponding to the sequence of specializations $\{(\hat{q}_{u(t),j(t)}, h_1, w, p, a)_\ell\}_{\ell=1}^{\infty}$ are not t-APW-covered. By Theorem 1.3 in [S3], for every t, we can pass to a subsequence that converges into an action of the stable rooted limit group $RootWPHG^2_{NQR}$ on a limit $R^{\mu(t)}$-tree Y_t. Since each specialization in this sequence is not t-APW-covered, the action of $RootWPHG^2_{NQR}$ on the limit tree Y_t is not t-APW-covered.

Applying Theorem 1.3 in [S3] once again, from the sequence of actions of $RootWPHG^2_{NQR}$ on the $R^{\mu(t)}$-trees Y_t, it is possible to extract a subsequence that converges into a free action of $RootWPHG^2_{NQR}$ on some R^{μ}-tree Y. Since for each index t the action of $RootWPHG^2_{NQR}$ on the $R^{\mu(t)}$-tree Y_t is not t-APW covered, the action of the limit group $RootWPHG^2_{NQR}$ on the limit R^{μ}-tree Y satisfies the properties of the action obtained in proving Theorem 1.7 of [S3], i.e. the orbit of the subtree Y_{APW}, which is the union of the orbits of the edges associated with the given set of generators of the parameter subgroup $\langle w, p, a\rangle$ in the R^{μ}-tree Y, misses either a non-degenerate segment or a germ of a branching point in Y.

The continuation of our argument is a modification of the argument that was used in proving Theorem 34. The non-quasi-rooted limit group $WPHG^2_{NQR}$ (that is assumed to be isomorphic to $WPHG_{NQR}$) is embedded into the rooted limit group $RootWPHG^2_{NQR}$. By the proof of Theorem 1.2 in [S3], from the action of the rooted limit group $RootWPHG^2_{NQR}$ on the R^{μ}-tree Y, $RootWPHG^2_{NQR}$ inherits an abelian decomposition in which all the quasi-rooted edge groups in $RootWPHG^2_{NQR}$ are elliptic. Hence, its subgroup, $WPHG^2_{NQR}$, that is isomorphic to $WPHG^2_{NQR}$, inherits an abelian decomposition, $\Lambda^2_{WPHG^2_{NQR}}$, a decomposition in which the image of the parameter subgroup $APW = \langle w, p, a \rangle$ is contained in a proper subgraph of groups, Λ', and every quasi-rooted edge group is elliptic.

Let $\Delta^2_{WPHG^2_{NQR}}$ be the graph of groups obtained from $\Lambda^2_{WPHG^2_{NQR}}$ by collapsing Λ' to a vertex. In $\Delta^2_{WPHG^2_{NQR}}$ the subgroup APW is contained in a vertex stabilizer, each quasi-rooted subgroup is elliptic, and each edge group is a non-trivial abelian subgroup of $WPHG^2_{NQR}$.

Since the limit group $WPHG$ is assumed to be stable, and the quasi-rooted edge groups in $WPHG_{NQR}$ and $WPHG^2_{NQR}$ are assumed to be identical, $WPHG_{NQR}$ is naturally isomorphic to $WPHG^2_{NQR}$. The abelian decomposition $\Delta^2_{WPHG^2_{NQR}}$ is a multi-graded abelian decomposition of $WPHG^2_{NQR}$ (hence, of $WPHG_{NQR}$) with respect to the parameter subgroup $\langle w, p \rangle$, and the quasi-rooted edge groups in the graph of groups Θ_{NQR}. Therefore, every edge group and every non-QH vertex group in Θ_{NQR} (the graded abelian JSJ decomposition of the stable limit group $WPHG$), is elliptic in $\Delta^2_{WPHG^2_{NQR}}$.

Furthermore, the second rooted limit group, $WPHG^2_{NQR}$, was obtained as a limit from a sequence of specializations, $\{(\hat{q}_{u(t),j(t)}, h_1, w, p, a)_{\ell(t)}\}$, of the stable limit group $WPHG$, that were chosen to be almost shortest with respect to the abelian decomposition Θ^2, which was obtained from the compatible JSJ decomposition of the non-quasi-rooted limit group $WPHG_{NQR}$. The compatible JSJ decomposition was obtained with respect to the action of the non-quasi-rooted limit group $WPHG_{NQR}$ on the top n levels of the limit R^{μ}-tree T.

Suppose first that the specializations of the defining parameters $\{w, p\}$ have bounded periodicity. In this case, of bounded periodicity, since the shortenings are performed with respect to Θ^2 that is obtained from the compatible JSJ decomposition with respect to the top n levels in T, changes in the action of the (second) rooted limit group $RootWPHG^2_{NQR}$ on the R^μ-tree Y, in comparison with the action of the (first) rooted limit group $RootWPHG_{NQR}$ on the R^μ-tree T can occur only in levels $n+1$ and below. Hence, the top n levels in the R^μ-trees T and Y have the same structure. In particular, since all these levels are APW-covered in T, they are APW-covered in Y. Therefore, the multi-graded abelian decomposition $\Delta^2_{WPHG^2_{NQR}}$ has to be compatible with the multi-graded compatible abelian JSJ decomposition $\Gamma_{WPHG_{NQR}}$, so it is compatible with the multi-graded abelian decomposition Θ^2 of the stable limit group $WPHG$ (according to which the shortenings are taken).

Suppose that the periodicity of the specializations of the defining parameters $\{w,p\}$ are not necessarily bounded. In this case the actions of the rooted limit groups $RootWPHG_{NQR}$ and $RootWPHG^2_{NQR}$ on the top n levels in the R^μ-trees T and Y, may differ, but the difference is caused by shifting the points stabilized by (non-distinguished) vertex groups in the compatible abelian JSJ decomposition $\Gamma_{WPHG_{NQR}}$. Therefore, even with these possible changes in the actions on the top n levels, the abelian decomposition $\Delta^2_{WPHG^2_{NQR}}$ has to be compatible with the multi-graded compatible abelian JSJ decomposition $\Gamma_{WPHG_{NQR}}$, so it is compatible with the multi-graded abelian decomposition Θ^2 of the stable limit group $WPHG$ (according to which the shortenings are taken).

Hence, for large enough t, the specializations $\{(\hat{q}_{u(t),j(t)},h_1,w,p,a)_{\ell(t)}\}$ are not almost shortest with respect to the action of the modular group associated with the graded abelian decomposition Θ^2, which clearly contradicts our assumptions. □

We continue with the sequences of specializations $\{(\hat{q}_{u,j},h_1,w,p,a)_\ell\}$ as we continued with the specializations $\{(q_{u,j},h_1,w,p,a)_\ell\}$. We sequentially "uncover" fractions p' of the defining parameters $\{w,p\}$ until we obtain a perturbed subsequence of specializations. With the perturbed subsequence we associate limit groups, $MLim(f,\hat{p},a)$ and $MPLim(f,\hat{p},p',a)$, where f are variables representing fluctuations and $\hat{p}$ represent the parts of the defining parameters which are not yet "uncovered". With the limit group $MPLim(f,\hat{p},p',a)$ there is a natural embedding, $\eta_2 : RootWPHG^2_{NQR} \to MPLim(f,\hat{p},p',a)$, which maps the distinguished vertex group in Θ_{NQR} into the distinguished vertex group in the multi-graded abelian JSJ decomposition of $MPLim$, and every quasi-rooted subgroup in $RootWPHG_{NQR}$ is mapped into a non-QH vertex group or into an edge group in the abelian

JSJ decomposition of $MPLim$. Note that since the group $MPLim$ was obtained from a perturbed sequence, its multi-graded abelian JSJ decomposition is necessarily non-trivial.

As we did with the sequences of specializations $\{(q_{u,j}, h_1, w, p, a)_\ell\}$, the (finite) sequential "uncovering" process associates a sequence of (hierarchical) abelian decompositions with the stable rooted limit group $RootWPHG^2_{NQR}$. Since the graded abelian JSJ decomposition of $MPLim(f, \hat{p}, p', a)$ is non-trivial, and η_2 embeds the rooted stable limit group $RootWPHG^2_{NQR}$ into $MPLim(f, \hat{p}, p', a)$, the multi-graded compatible abelian JSJ decomposition of the rooted graded limit group $RootWPHG^2_{NQR}$ is non-trivial, and so is the multi-graded compatible abelian JSJ decomposition obtained by the (second) non-quasi-rooted subgroup $WPHG^2_{NQR}$ from its action on the top $n' > n$ levels in the R^μ-tree Y (i.e. from the levels in Y that lie above the fluctuations level). We denote the obtained compatible abelian JSJ decomposition, $\Gamma^2_{WPHG^2_{NQR}}$.

By the argument that was used in proving Theorem 36, since the sequences of specializations $\{(\hat{q}_{u,j}, h_1, w, p, a)_\ell\}$ were obtained by shortening the sequences of specializations $\{(q_{u,j}, h_1, w, p, a)_\ell\}$ with respect to the multi-graded compatible abelian JSJ decomposition, $\Gamma_{WPHG_{NQR}}$, that is obtained with respect to the action of $WPHG_{NQR}$ on the top n levels in the R^μ-tree T, the (new) abelian decomposition, $\Gamma^2_{WPHG^2_{NQR}}$, is a refinement of the compatible abelian JSJ decomposition $\Gamma_{WPHG_{NQR}}$ (i.e. it is compatible with it).

Therefore, every abelian edge group and every non-QH vertex group in $\Gamma_{WPHG_{NQR}}$ is elliptic in $\Gamma^2_{WPHG^2_{NQR}}$. Hence, $Rcomp(\Gamma^2_{WPHG^2_{NQR}}) \leq Rcomp(\Gamma_{WPHG_{NQR}})$, and by the same argument used to prove Proposition 37, $Rcomp(\Gamma^2_{WPHG^2_{NQR}}) < Rcomp(\Gamma_{WPHG_{NQR}})$.

By iteratively repeating these constructions, we either strictly increase the number of (conjugacy classes of) quasi-rooted edge groups, or we obtain a sequence of multi-graded (compatible) abelian decompositions of the non-quasi-rooted limit group $WPHG_{NQR}$: $\Gamma_{WPHG_{NQR}}, \Gamma^2_{WPHG^2_{NQR}}, \Gamma^3_{WPHG^3_{NQR}}, \dots$, for which

$$Rcomp(\Gamma_{WPHG_{NQR}}) > Rcomp(\Gamma^2_{WPHG^2_{NQR}}) > Rcomp(\Lambda^3_{WPHG^3_{NQR}}) > \dots$$

Any sequence of graded abelian decompositions of the limit group $WPHG_{NQR}$ with strictly decreasing reduced complexity terminates in a finite time. Hence, we sequentially increase the number of (conjugacy classes of) quasi-rooted edge groups in the graphs of groups $\Theta, \Theta_2, \Theta_3, \dots$ But each time we add an additional quasi-rooted edge group, we associate at least one additional Bass–Serre generator corresponding to edges that are in the graphs of groups Θ_i but not in their associated non-quasi-rooted subgraphs. Since the number of Bass–Serre generators in Θ_i is (globally) bounded by the first betti number of the stable limit group $WPHG$, the whole process has to terminate in a finite time. This finally contradicts the existence of an unbounded number of eventual sculpted resolutions of the same width along an infinite path of the sieve procedure, which concludes the proof of Theorem 27. □

The combination of Proposition 26 and Theorem 27 clearly gives a contradiction, that finally proves the termination of the sieve procedure (Theorem 22). □

Recall that $T_2(p) \subset EAE(p)$ is defined to be the set of specializations of the defining parameters p, that have a witness with valid PS statement that terminates after the first step of the procedure for the construction of the tree of stratified sets (cf. Theorem 3.2 in [S5]). At this stage we have all the tools needed for showing that the set $T_2(p)$ is in the Boolean algebra of AE sets. By construction, if $p_0 \in T_2(p)$ then there must exist a valid PS statement of the form $((h_1^2, g_1^1), \dots, (h^2_{\nu(ps)}, g^1_{\nu(ps)}), h_0^1, w_0, p_0, a)$, that factors through one of the PS resolutions $PSHGHRes$ constructed with respect to all proof systems that terminate after the first step of the procedure for the construction of the tree of stratified sets.

By Proposition 7, the sets $TSPS(p)$ associated with the various PS resolutions $PSHGHRes$, i.e. the sets of specializations p_0 of the defining parameters $P = \langle p \rangle$ for which there exists a test sequence of valid PS statements that factor through any of the PS resolutions $PSHGHRes$, is in the Boolean algebra of AE sets. By Lemma 6 if there exists a valid PS statement that can be extended to a specialization that factors through a PS resolution $PSHGHRes$, then either there exists a test sequence of valid PS statements that factors through that PS resolution, or there must exist a valid PS statement that can be extended to a specialization that factors through one the of the Collapse extra PS limit groups associated with the various PS resolutions $PSHGHRes$.

We continue with each of the collapse PS limit groups in parallel. Exactly as we did with each of the PS resolutions $PSHGHRes$, we associate (canonically) with each of the developing resolutions constructed along the sieve procedure, a set $TSPS(p)$, that contains all the specializations of the defining parameters p, that have a test sequence of valid PS statements that factor through the given developing resolution. By Proposition 7 applied to the developing resolutions constructed along the sieve procedure, the sets $TSPS(p)$ associated with these developing resolutions are all in the Boolean algebra of AE sets. By Lemma 6 applied to the various developing resolutions, if there exists a valid PS statement that can be extended to a specialization that factors through a developing resolution, then either there exists a test sequence of valid PS statements that factors through that developing resolution, or there must exist a valid PS statement that factors through one of the Collapse extra PS limit groups associated with that developing resolution.

Hence, by iteratively applying Lemma 6 and Proposition 7 to the various developing resolutions constructed along the sieve procedure, if for some specialization p_0 of the defining parameters p, there exists a valid PS statement that can be extended to a specialization that factors through one of the PS limit groups $PSHGH$ we started the sieve procedure with, then this specialization p_0 is an element of at least one of the sets $TSPS(p)$ associated with the various developing resolutions constructed along the sieve procedure.

The set $T_2(p)$ is the union of the sets $TSPS(p)$, associated with the various developing resolutions constructed along the sieve procedures, corresponding to the finitely many possibilities of proof systems of depth 2 (i.e. proof systems that terminate after the first step of the construction of the tree of stratified sets). Since each of the sets $TSPS(p)$ is in the Boolean algebra of AE sets, and a Boolean algebra is closed under finite unions, we have finally completed the proof of Theorem 3.2 in [S5] in the general case.

Theorem 39 (cf. [S5, 3.2]). *Let $T_2(p) \subset EAE(p)$ be the subset of all specializations $p_0 \in EAE(p)$ of the defining parameters p, that have witnesses with a proof system that terminates after the first step of the procedure for the construction of the tree of stratified sets. Then $T_2(p)$ is in the Boolean algebra of AE sets.*

At this stage, we are finally ready to show that the entire set $EAE(p)$ is in the Boolean algebra of AE sets. The tree of stratified sets has a finite depth, which (by definition) bounds the depth of all possible proof systems associated with the tree of stratified sets. For each integer d, we set $T_d(p)$ to be the set of specializations p_0 of the defining parameters $P = \langle p \rangle$ for which there exists a valid PS statement for some proof system of depth d. Clearly, $EAE(p) = T_1(p) \cup T_2(p) \cup \ldots \cup T_{d_0}(p)$, where d_0 is the depth of tree of stratified sets. Since a Boolean algebra is closed under finite unions, to show that the set $EAE(p)$ is in the Boolean algebra of AE sets, it is enough to show that each of the sets $T_d(p)$ is in the Boolean algebra of AE sets.

By the structure of the tree of stratified sets, and the global bounds on the possible numbers of disjoint rigid and strictly solid families associated with each of the strata in the stratification associated with each rigid and solid limit groups that appears in the tree of stratified sets, given a fixed depth d, there exist finitely many possible proof systems of depth d. Given a fixed proof system of depth d, which we denote PS, we collect all its associated valid PS statements in a canonical collection of finitely many PS limit groups which we denote $PS(HG)^dH$. With each of the PS limit groups $PS(HG)^dH$ we associate a sieve procedure, precisely as we did with the PS limit groups $PSHGH$ that contain all the valid proof statements of depth 2. With each of the developing resolutions constructed along these sieve procedures, we associate a set $TSPS(p)$, that contain the set of specializations of the defining parameters p that have a test sequence of valid PS statements that factor through that developing resolution. By iteratively applying Lemma 6 and Proposition 7, the sets $TSPS(p)$ associated with the various developing resolutions constructed along these sieve procedures are all in the Boolean algebra of AE sets, and if a specialization p_0 has a valid PS statement of depth d, i.e. if $p_0 \in T_d(p)$, then p_0 is contained in at least one of the sets $TSPS(p)$ constructed along the sieve procedures associated with the PS limit groups $PS(HG)^dH$.

The set $T_d(p)$ is the union of the sets $TSPS(p)$ associated with the various developing resolutions constructed along the sieve procedures corresponding to the finitely many possibilities of proof systems of depth d. Since each of the sets $TSPS(p)$ is in the Boolean algebra of AE sets, and a Boolean algebra is closed under finite unions, for each depth d, the set $T_d(p)$ is in the Boolean algebra of AE sets. Since the sets $EAE(p)$ is the union $EAE(p) = T_1(p) \cup T_2(p) \cup \ldots \cup T_{d_0}(p)$, where d_0 is the depth of tree of stratified sets, the set $EAE(p)$ itself is in the Boolean algebra of AE sets, and we have finally completed the proof of Theorem 1.4 in [S5].

Theorem 40 (cf. [S5, 1.4]). *Let $F_k = \langle a_1, \ldots, a_k \rangle$ be a free group, and let $EAE(p)$ be a set defined by the predicate*

$$EAE(p) = \exists w \forall y \exists x \; \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1 \,.$$

Then $EAE(p)$ is in the Boolean algebra of AE sets.

The proof of Theorem 40 shows that a set $EAE(p)$ defined using a conjunction of a system of equalities and a system of inequalities,

$$EAE(p) = \exists w \forall y \exists x \ \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1\,,$$

is in the Boolean algebra of AE sets. The generalization of that proof to a set $EAE(p)$ defined using a (finite) disjunction of conjunctions of a system of equalities and a system of inequalities,

$$EAE(p) = \exists \forall y \exists x \ \big(\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1\big) \vee \dots$$
$$\dots \vee (\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1)\,,$$

is rather straightforward. Indeed, the only change required is in the construction of the tree of stratified sets. In constructing the tree of stratified sets when the predicate defining the set $EAE(p)$ is the conjunction of a system of equalities and a system of inequalities, we have constructed the tree of stratified sets iteratively, where in each step we have first collected all the formal solutions defined over closures of the resolutions of the remaining y's from the previous step, and then applied the collections of formal solutions to the set of the remaining y's from the previous step, and analyzed the set of y's for which (at least) one of the inequalities from our given system is in fact an equality when we substitute the families of formal solutions we have collected, using an iterative procedure for the analysis of quotient resolutions.

When the set $EAE(p)$ is defined using a (finite) disjunction of conjunctions of a system of equalities and a system of inequalities, we work in parallel with each of the conjunctions in each step of the iterative procedure that constructs the tree of stratified sets. In each step of this iterative procedure, we do the following for the indices j, $1 \leq j \leq r$, in parallel. We first collect all the formal solutions of the system of equalities $\Sigma_j(x,y,w,p,a)$, that are defined over closures of the resolutions of the remaining y's from the previous step, and then apply the collections of these formal solutions to the set of the remaining y's from the previous step, and analyze the set of y's for which (at least) one of the inequalities from the system $\Psi_j(x,y,w,p,a)$ is in fact an equality when we substitute the families of formal solutions we have collected, using the iterative procedure for the analysis of quotient resolutions.

The termination of this modified procedure for the construction of the tree of stratified sets follows using the same argument used in proving Theorem 40 (Theorem 2.10 in [S5]). Given the tree of stratified sets, the analysis of the set $EAE(p)$ is identical to the analysis described in proving Theorem 40. This finally shows that the set $EAE(p)$ is in the Boolean algebra of AE sets, also when $EAE(p)$ is defined using a (finite) disjunction of conjunctions of a system of equalities and a system of inequalities, hence, concludes the proof of Theorem 1.3 of [S5] for general EAE sets.

Theorem 41 (cf. [S5, 1.3]). *Let $F_k = \langle a_1, \dots, a_k \rangle$ be a free group, and let the AE set $AE(w,p)$ be defined as*

$$AE(w,p) = \forall y \exists x \ \big(\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1\big) \vee \dots$$
$$\dots \vee \big(\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1\big)\,.$$

Then the projection of the AE set $AE(w,p)$, i.e. the set

$$EAE(p) = \exists w \forall y \exists x \ \big(\Sigma_1(x,y,w,p,a) = 1 \wedge \Psi_1(x,y,w,p,a) \neq 1\big) \vee \dots$$
$$\dots \vee \big(\Sigma_r(x,y,w,p,a) = 1 \wedge \Psi_r(x,y,w,p,a) \neq 1\big)$$

is in the Boolean algebra of AE sets.

Appendix: A Brief Survey of the Sieve Procedure

The sieve procedure presented in this paper concludes our analysis of definable sets, and proves quantifier elimination over a free group. Since the sieve procedure is technically involved, and it uses notions and constructions from previous papers in this sequence, in this appendix we briefly summarize our approach to quantifier elimination and the main principles of the sieve procedure.

Let $EAE(p)$ be a set defined by the predicate

$$EAE(p) = \exists w \forall y \exists x \ \Sigma(x,y,w,p,a) = 1 \wedge \Psi(x,y,w,p,a) \neq 1\,.$$

A specialization w_0 of the variables w is said to be a *witness* for a specialization p_0 of the defining parameters p, if the following sentence:

$$\forall y \ \exists x \ \Sigma(x,y,w_0,p_0,a) = 1 \wedge \Psi(x,y,w_0,p_0,a) \neq 1$$

is a true sentence. Clearly, if there exists a witness for a specialization p_0 then $p_0 \in EAE(p)$, and every $p_0 \in EAE(p)$ has a witness.

By definition, in order to show that a specialization p_0 of the defining parameters p is in the set $EAE(p)$, we need to find a witness w_0 for the specialization p_0. By the construction of the tree of stratified sets, given a witness w_0 for a specialization p_0, it is possible to prove the validity of the AE sentence corresponding to the couple (w_0, p_0), using a proof that is encoded by a subtree of the tree of stratified sets, that is constructed in section 2 of [S5], i.e. a proof built from a finite sequence of (families of) formal solutions, constructed along a (finite) collection of paths in the tree of stratified sets. By the finiteness of the tree of stratified sets there are only finitely many possibilities for such a collection of paths (a subtree). Hence, there are only finitely many possibilities for the structure of a proof encoded by the tree of stratified sets, and these finitely many structures of proofs are sufficient for proving the validity of the AE sentences corresponding to all couples (w_0, p_0), where $p_0 \in EAE(p)$ and w_0 is a witness for p_0. We call each possibility for the structure of a proof encoded by the tree of stratified sets a *proof system*.

Given $p_0 \in EAE(p)$, we are not able to say much about a possible witness for p_0 using the information encoded in the tree of stratified sets. With each proof system encoded by a subtree of the tree of stratified sets associated with the set $EAE(p)$, we can naturally associate a subset of $EAE(p)$. This subset is defined to be all the specializations $p_0 \in EAE(p)$, for which there exists a witness w_0 so that the validity of the AE sentence corresponding to the couple (w_0, p_0) can be proved using a proof with the structure of the given proof system. Our strategy towards proving quantifier elimination is to show that the subset of $EAE(p)$ associated with a given proof system is in the Boolean algebra of AE sets. Since there are only finitely many proof systems encoded by the tree of stratified sets associated with an EAE set, and the EAE set itself is a union of the subsets associated with its (finitely many) proof systems, this implies that the set $EAE(p)$ is in the Boolean algebra of AE sets.

Suppose that a specialization $p_0 \in EAE(p)$ has a witness w_0 with a given proof system. The structure of the tree of stratified sets guarantees the existence of a collection of elements (from the coefficient group F_k), that include rigid and strictly solid specializations of the rigid and solid limit groups along the subtree of the tree of stratified sets that corresponds to the given proof system, and satisfies certain conditions that are all specified by the proof system (Definition 1.23 in [S5]). This collection of elements forms a proof for the validity of the AE sentence associated with the couple (w_0, p_0), and we call it a *valid PS statement*.

Note that the structure of a valid proof statement depends only on the proof system, and not on the particular specialization. By the standard arguments presented in section 5 of [S1], with the entire collection of valid PS statements we can naturally associate its Zariski closure, that corresponds to a (canonical) finite collection of maximal limit groups that we call PS (proof system) *limit groups*. We call each specialization of a PS limit groups (that has a structure of a proof, but perhaps is not) a virtual proof.

The PS limit groups are the input for the sieve procedure. The goal of the sieve procedure is to show that the collection of specializations of the defining parameters, for which there exists a valid PS statement that factors through a given PS limit group, is in the Boolean algebra of AE sets.

Let $P = \langle p \rangle$ be the group of defining parameters. We start the first step of the sieve procedure with each of the PS limit groups in parallel. Given a PS limit group $PSHGH$, we associate with it its canonical taut graded Makanin–Razborov diagram [Se4, 2.5], with respect to the parameter subgroup P. We call these resolutions PS resolutions.

We start our analysis of the set of PS resolutions, by associating with each such resolution a finite collection of reference resolutions, that collect all the "generic" points that factor through a given graded PS resolution for which the corresponding virtual proof (associated with the "generic" point) is in fact a fake proof. We do that by collecting all the test sequences that factor through a PS resolution [S2, 1.19], for which the corresponding virtual proofs are fake proofs, and apply the construction of graded formal limit groups [S2, 3], to associate a finite collection of reference bundles with this collection of test sequences.

By construction, every value p_0 of the defining parameters P, for which there exists a "generic" point that does not factor through one of the "bad" reference bundles, is necessarily in the set $EAE(p)$ (since it has a test sequence of valid PS statement). Indeed, by the results of section 3 of [S3], the collection of such values of the defining parameters P is in the Boolean algebra of AE sets, and this is our first approximation for the given EAE set.

The constructed reference bundles collect all the "generic" points for which the associated virtual proofs are in fact fake proofs. For some of these reference bundles, if the virtual proof associated with a generic point in a fiber is fake, the virtual proofs in the whole fiber are fake. For others (called $Extra\ PS$), it may be that a fiber contains valid PS statements, though the virtual proofs associated with a generic point in the fiber is fake. Hence, to continue the analysis of the set $EAE(p)$, we need to collect all such valid PS statements. We do that by collecting all the virtual proofs in such bundles, for which the reason that a generic virtual proof fails evaporates (or rather collapses). Such a collapse can be described by the union of finitely many Diophantine conditions, and the collection of virtual proofs that satisfy one of these Diophantine conditions is a Diophantine set, with which we naturally associate finitely many limit groups, that we call *Collapse extra PS* limit groups. We start the first step of the sieve procedure with these groups.

The first step of the sieve procedure studies the structure of the obtained Collapse extra PS limit groups (and their resolutions) in comparison with the PS resolutions (of the PS limit groups) from which they were constructed. We start by looking at the Zariski closure of all the virtual proofs that can be extended to specializations of a given Collapse extra PS limit group. This Zariski closure is naturally contained in the variety associated with the PS limit group we started with. If it is strictly contained in this variety, we continue by replacing the PS limit group with the limit group that is associated with the Zariski closure (which is in this case a proper quotient of the PS limit group we started with), and apply the descending chain condition (d.c.c.) for limit groups to guarantee that such a replacement can occur only finitely many times along the entire procedure. Otherwise for the rest of the first step we may assume that there is no drop in the Zariski closure of the Diophantine set of virtual proofs.

By construction, there is a natural map from the completion of the PS resolution we started with to a given Collapse extra PS limit group. We proceed by analyzing the multi-graded resolutions of the given Collapse extra PS limit group with respect to the (images of the) non-QH, non-abelian vertex groups in the top level of the completion of the PS resolution we started with (in practice we use *auxiliary* resolutions to do it with respect to the (images of the) non-QH, non-abelian vertex groups and edge groups in the graded abelian JSJ decomposition of the original PS limit group). Using core resolutions that were presented in section 4 in [S5], it is possible to bound the complexity of each such multi-graded resolution by the complexity of the top level of the completion of the PS resolution we started with (which is essentially the complexity of the graded abelian JSJ decomposition of the associated PS limit group).

If there is a drop in the complexity of an obtained multi-graded resolution (in comparison with the complexity of the top level of the completion of the corresponding PS resolution), we continue in a similar way to what we did in this case in the procedure for validation of an AE sentence [Se4, 4]. In this case it is (essentially) guaranteed that the image of the original PS limit group in the terminal limit group of an obtained multi-graded resolution is a proper quotient of the original PS limit group we started with. Hence, in this case, with a given multi-graded resolution we can associate a finite collection of *anvils*, that are certain fiber-products of the completion of the given multi-graded resolution with a completion of a resolution from the (graded) taut Makanin–Razborov diagram of the image of the PS limit group in the terminal level of the given multi-graded resolution along this image. To the anvil we can naturally map a *developing resolution*, which is a resolution of the original PS limit group that is obtained from a resolution induced by the original PS limit group from the given multi-graded resolution, followed by the resolution of the image of the PS limit group in the terminal level of the given multi-graded resolution.

If there is no drop in the complexity of an obtained multi-graded resolution, it is guaranteed that the structure of the core resolution associated with such a multi-graded resolution is identical to the structure of the top level of the completion of the PS resolution we started the first step with. In this case we continue to multi-graded resolutions associated with lower levels in the completion of the PS resolution we started with. If there is a drop in complexity we associate anvils and developing resolutions with the obtained multi-graded resolutions, precisely as we constructed them in case of a complexity drop in the top level.

Suppose that there exists a multi-graded resolution for which there is no drop in the complexity of the core resolutions associated with all its levels. In this case we enlarge the collection of resolutions we consider, and examine all the graded resolutions (with respect to the defining parameters P), for which the formed part of the core resolutions associated with all their levels, have the same structure as the formed part of the abelian decompositions associated with the various levels of the completion of the PS resolution we started the first step with (the formed part of an abelian decomposition is the part containing its QH and abelian vertex groups — see Definition 4.17 in [S5]).

We set the ambient resolution to be the anvil. For each such graded resolution we look at the core resolutions associated with its various parts. If for some part the complexity of a core resolution associated with it is strictly smaller than the complexity of the corresponding level of the completion of the PS resolution, we set the ambient resolution to be a *carrier*, and the resolution induced by the image of the original PS limit group to be the developing resolution. If no reduction of complexity occurs, we set the resolution associated with the image of the original PS limit group to be a *sculpted resolution*, and set the developing resolution to be the ambient resolution.

To conclude the first step, we associate with each developing resolution (that is naturally mapped into the anvil), a finite collection of reference resolutions, in the same way we associated these with the original PS resolutions. With each anvil we further associate a Collapse extra PS limit groups, that collects those specializations of the bad reference fibers, for which the reason for the failure of a "generic" virtual proof collapses, in the same way we associated these with the original PS resolutions.

We continue with the sieve procedure iteratively, taking into account all the constructions that were made along the different branches of the procedure in all the previous steps. We suppose that with each anvil and its associated developing resolution that were constructed in step $n-1$ of the sieve procedure, there is an associated positive integer called *width* that denotes the number of algebraic envelopes associated with the anvil, i.e. the number of (nested) sequences of core resolutions associated with the anvil. $width(n-1)$=1 if and only if no sculpted resolution is associated with the anvil. In case $width(n-1) > 1$, with the (nested) collection of algebraic envelopes (sequences of core resolutions associated with the anvil), there are associated sculpted resolutions and possibly carriers. We start the general step of the sieve procedure with the (finite) collection of anvils constructed at the previous step of the procedure, and their associated Collapse extra PS limit groups.

The general step of the procedure is essentially similar to the first step, though it is technically more involved since it has to take into account the past of the procedure. Like in the first step, we start by looking at the Zariski closure of all the virtual proofs that can be extended to specializations of a given Collapse extra PS limit group. If this Zariski closure is strictly contained in the variety we started the first step with, we continue by replacing the PS limit group with the limit group that is associated with the Zariski closure (which is in this case a proper quotient of the PS limit group we started with), and start again the first step of the sieve procedure with this Zariski closure. As we have already noted, by the d.c.c. for limit groups this can happen at most finitely many times along a branch of the sieve procedure.

By construction, there are natural maps from the completions of the various resolutions constructed in previous steps of the procedure into the Collapse extra PS resolution. As in the first step, we proceed by analyzing the multi-graded resolutions of the given Collapse extra PS limit group with respect to the images of these resolutions. If there is a drop in the complexity of an obtained multi-graded resolution (in comparison with the complexity of the top level of the completion of the corresponding resolution), we continue by associating with the multi-graded resolution a finite collection of anvils and developing resolutions in a similar way to what we did in the first step.

If there is no drop in the complexity of an obtained multi-graded resolution, it is guaranteed that the structure of the core resolution associated with such a multi-graded resolution is identical to the structure of the core resolution associated with the corresponding multi-graded resolution that was constructed in the previous step of the procedure. In this case we continue to multi-graded resolutions associated with lower levels in the anvil we started the general step with. If there is a drop in complexity we associate anvils and developing resolutions with the obtained multi-graded resolutions, precisely as we constructed them in case of a complexity drop in the top level.

Suppose that there exists a multi-graded resolution for which there is no drop in the complexity of the core resolutions associated with all its levels. In this case we proceed to study the resolutions associated with the next sculpted resolution or algebraic envelope, if such are associated with the anvil we started with. If there is a reduction in the complexity of the a resolution associated with one of the algebraic envelopes associated with the anvil we started with, we construct a finite collection of anvils, developing resolutions, sculpted resolutions, and carriers, in a similar way to the construction used in the first step. If no reduction of complexity occurs, we add an algebraic envelope to the collection of algebraic envelopes associated with the anvil we started with, and either declare this new algebraic envelope to be a carrier for a sculpted resolution (in case with this new algebraic envelope the complexity of a resolution associated with this sculpted resolution drops), or we form a new (additional) sculpted resolution. In both cases we construct a finite collection of anvils and developing resolutions, with which we associate the newly constructed algebraic envelopes, carriers, and sculpted resolutions.

As in the first step, to conclude the general step we associate with each developing resolution (that is naturally mapped into the anvil), a finite collection of reference resolutions, in the same way we associated these with the original PS resolutions. With each anvil we further associate Collapse extra PS limit groups, that collect those specializations of the bad reference fibers, for which the reason for the failure of a "generic" virtual proof collapses, in the same way we associated these with the original PS resolutions.

After setting the sieve procedure, we are still required to prove it terminates after finitely many steps. To prove termination of our iterative procedure in the minimal (graded) rank case (section 1 in [S5]), we used the strict decrease in the complexity of the resolutions associated with successive steps of the procedure, a strict decrease that forces termination. In the procedures used to construct the tree of stratified sets and to validate a general AE sentence, we proved termination by combining the decrease in either the Zariski closures or the complexities of the resolutions and decompositions associated with successive steps of these procedures. In presenting the general step of the sieve procedure, we considered the possibility that both the Zariski closures and the complexities of the various core resolutions and developing resolutions associated with an anvil do not decrease. In this case, we have associated additional algebraic envelopes, sculpted resolutions, and carriers with the corresponding anvil. Therefore, in addition to the arguments used to prove the termination of the procedure for the construction of the tree of stratified sets, in order to prove termination of the sieve procedure, we prove the existence of a global bound on the number of sculpted resolutions associated with an anvil. The argument used to prove this bound is based on the argument used to prove a global bound on the number of rigid and strictly-solid families of specializations of rigid and solid limit groups presented in the first two sections of [S3].

At this stage we are finally ready to show that the entire set $EAE(p)$ is in the Boolean algebra of AE sets. By the structure of the tree of stratified sets (constructed in section 2 of [S5]), and the global bounds on the possible numbers of distinct rigid and strictly solid families associated with each of the strata in the stratification associated with each rigid and solid limit groups that appears in the tree of stratified sets, there exist finitely many possible proof systems associated with the set $EAE(p)$. Given a fixed proof system, which we denote PS, we collect all its associated valid PS statements in a canonical collection of finitely many PS limit groups. With each of the PS limit groups we associate a (terminating) sieve procedure. With each of the developing resolutions constructed along these sieve procedures, we associate a set $TSPS(p)$, that contains the set of specializations of the defining parameters P that have a test sequence of valid PS statements that factor through that developing resolution (i.e. a "generic" valid PS statement). The sets $TSPS(p)$ associated with the various developing resolutions constructed along these sieve procedures are all in the Boolean algebra of AE sets, and if a specialization p_0 has a valid PS statement then p_0 is contained in at least one of the sets $TSPS(p)$ constructed along the sieve procedures associated with the PS limit groups.

Hence, the set $EAE(p)$ is the (finite) union of the sets $TSPS(p)$, and since these are all in the Boolean algebra of AE sets, so is the set $EAE(p)$, and quantifier elimination is finally obtained.

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