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DIOPHANTINE GEOMETRY OVER GROUPS II: COMPLETIONS, CLOSURES AND FORMAL SOLUTIONS

BY

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ABSTRACT

This paper is the second in a series on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the second paper we generalize Merzlyakov's theorem on the existence of a formal solution associated with a positive sentence [Me]. We first construct a formal solution to a general AE sentence which is known to be true over some variety, and then develop tools that enable us to analyze the collection of all such formal solutions.

Introduction

In the first paper in this series on Diophantine geometry over groups we studied sets of solutions to systems of equations defined over a free group and parametric families of such sets ([Se]). With a given system of equations we associated a canonical Makanin–Razborov diagram. This Makanin–Razborov diagram encodes the entire set of solutions to the system. Later on we studied systems of equations with parameters, and with each such system we associated a (canonical) graded Makanin–Razborov diagram that encodes the Makanin–Razborov diagrams of the systems of equations associated with each specialization of the defining parameters.

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In order to prove the correctness of a positive sentence defined over a free group, Merzlyakov has introduced “formal solutions” [Me]. In the special case of AE sentences, Merzlyakov's theorem implies that if a sentence

$$\forall y \quad \exists x \quad \Sigma(x,y,a)=1$$

is a truth sentence, then there exists a formal solution $x=x(y,a)$, so that if we replace the variables x with their corresponding formal solutions, then each equation in the obtained system $\Sigma(x(y,a),y,a)$ represents the trivial word in the free group $F(y,a)$, i.e., the free group generated by the universal variables y and the coefficients a.

To analyze general sentences and predicates, a generalized version of Merzlyakov's theorem is required. First, we naturally need the sentences and predicates in question to include equalities and inequalities. Second, we need to study AE sentences and predicates in which the universal variables do not belong to an entire power set of the free group in question, but rather to some given variety. To analyze these general AE sentences, we need to associate with the variety to which the universal variables belong a canonical collection of *completions* of the variety, which are built from the (taut) Makanin–Razborov diagram associated with it. The union of the Diophantine sets defined by the completions associated with a variety is precisely the variety itself.

Once we define the completions of a variety, we are able to formulate the required generalization of Merzlyakov's theorem. We show (Theorem 1.18) that if a sentence of the form

$$\forall y\in V \quad \exists x \quad \Sigma(x,y,a)=1 \wedge \Psi(x,y,a)\neq 1$$

is a truth sentence, where V is a variety defined by a system of equations using the variables y and the coefficients a, then there exists a collection of *closures* of the completions of the variety V, and with each closure there is an associated formal solution $x=x(s,z,y,a)$ defined over it, so that each of the words in the system of equations obtained by replacing the variables x with the formal solutions $x=x(s,z,y,a)$ in the system $\Sigma(x,y,a)$: $\Sigma(x(s,z,y,a),y,a)$ represents the trivial element in the limit group associated with the corresponding closure of the variety V. Moreover, we prove that in a “generic” point in that closure the inequalities hold as well.

As we will see in the sequel, to get a quantifier elimination procedure for predicates defined over a free group, we will need tools that encode the entire collection of formal solutions associated with a given sentence, and not only their existence. In the second section we present *formal limit groups* and their associated (canonical) *formal* Makanin–Razborov diagrams which are built precisely for that purpose. Indeed, given a truth sentence of the form given above, the formal limit groups and their associated formal Makanin–Razborov diagrams encode the entire collection of formal solutions associated with the given sentence.

In the first two sections we study general AE sentences. In the third one we generalize our constructions to study AE predicates. With an AE predicate we associate a finite collection of graded completions (i.e., a finite collection of parametrized families of completions). Given a graded completion, we further associate with the given predicate a finite collection of *graded formal limit groups*, and with each of these limit groups we associate a (canonical) *graded formal* Makanin–Razborov diagram. As in our construction of the graded Makanin–Razborov diagram in the previous paper in this series, the formal graded diagram encodes the entire collection of formal Makanin–Razborov diagrams for all possible specializations of the defining parameters.

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1. Formal solutions

To analyze the structure of elementary sets over a free group, we need to look at projections of sets that are in the Boolean algebra of AE sets. Our approach to study these projections is based on a finite "trial and error" procedure, which uses extensively the existence of "formal solutions" suggested by Merzlyakov's theorem. However, unlike the original Merzlyakov's theorem [Me], one cannot define a "formal solution" on a general variety in terms of its defining variables. As we will show in this section, general "formal solutions" are associated with each of the resolutions that appear in the canonical Makanin–Razborov diagram associated with the variety, and they are defined not over the variety itself, but rather on *closures* of its canonical (finite) set of *completions*.

Our general approach to proving the existence of formal solutions associated with a given truth sentence uses the theory of actions of groups on real trees and the shortening argument. We start this section with (a special case of) Merzlyakov's original theorem, which we prove combinatorially, in a similar way to Merzlyakov's original argument.

THEOREM 1.1 ([Me]): *Let $F_k =< a_1, \dots, a_k >$ be a free group; let*

$$y = (y_1, \dots, y_\ell) \quad \textit{and} \quad x = (x_1, \dots, x_q).$$

Let $w_1(x,y,a)=1,\dots,w_s(x,y,a)=1$ be a system of equations over F_k, and suppose that the sentence

$$\forall y \quad \exists x \quad w_1(x,y,a)=1,\dots,w_s(x,y,a)=1$$

is a truth sentence. Then there exists a formal solution $x=x(y,a)$ so that each of the words $w_j(x(y,a),y,a)$ is the trivial word in the free group $F=<y,a>$.

Proof: For a given set of tuples of integers $\{(\alpha_i,\beta_i)\}$ we define

$$y_i([\alpha_i,\beta_i])=a_1a_2^{\alpha_i}a_1a_2^{\alpha_i+1}a_1\cdots a_1a_2^{\beta_i}a_1$$

for $i=1,\dots,\ell$.

Each specialization of y and x defines a *tree of cancellations* for each of the equations $w_1(x,y,a)=1,\dots,w_s(x,y,a)=1$. On each tree of cancellations we add notation for the placements of the x-segments, y-segments and a-segments (the a segments are the coefficients that appear in the equations $w_1,\dots,w_s$).

Clearly, given the words $w_1(x,y,a),\dots,w_s(x,y,a)$ the combinatorics of the trees of cancellations for any possible specialization are bounded. Setting the *weight* of a vertex v in a cancellation tree to be the total number of x, y and a segments passing through v, then the bounded combinatorics of the trees imply the existence of a global bound τ on the sum of the weights of the vertices in the cancellation trees corresponding to $w_1(x,y,a),\dots,w_s(x,y,a)$ for all possible specializations of x and y.

Let $\alpha_i^0=3(i+1)\cdot\tau$ and $\beta_i^0=3(i+2)\cdot\tau-1$, and let $y_i^0=y_i([\alpha_i^0,\beta_i^0])$ for $i=1,\dots,\ell$. Let $y^0=(y_1^0,\dots,y_\ell^0)$. Since the sentence that appears in the formulation of the theorem is true, there exists some $x^0=(x_1^0,\dots,x_q^0)$ so that $w_1(x^0,y^0,a)=1,\dots,w_s(x^0,y^0,a)=1$. Let $T_1,\dots,T_s$ be the cancellation trees corresponding to these last equalities.

By a simple pigeon-hole argument, for each y_i^0 there exists some index pl_i so that the segment labeled by $a_1a_2^{pl_i}a_1$ in y_i^0 does not cut any of the vertices in all the cancellation trees $T_1,\dots,T_s$.

At this stage for each $i=1,\dots,q$ we replace each of the segments $a_1a_2^{pl_i}a_1$ in all the appearances of y_i^0 on the cancellation trees $T_1,\dots,T_s$ by a label z_i. This replacement allows us to write each of the variables y_i as $y_i=c_i^1z_ic_i^2$ for some constants c_i^1 and c_i^2, and each of the variables x_j as a word $x_j=x_j(z_1,\dots,z_q,a)$. By the way the segments $a_1a_2^{pl_i}a_1$ are placed on the cancellation trees $T_1,\dots,T_s$; the words $w_1(x(z,a),y(z,a),a),\dots,w_s(x(z,a),y(z,a),a)$ are the trivial words in the free group $F=<z,a>$. Since we can set $z_i=(c_1^i)^{-1}y_i(c_2^i)^{-1}$ for $i=1,\dots,q$ the theorem follows. ∎

As we have already indicated we will need a generalization of Merzlyakov's theorem to a truth sentence defined over an arbitrary (given) variety. We start with a generalization of Merzlyakov's theorem to a sentence containing inequalities, which basically follows from Merzlyakov's proof of his theorem.

THEOREM 1.2: *Let $F_k =< a_1,\dots,a_k >$ be a free group, let $w_1(x,y,a) = 1,\dots,w_s(x,y,a) = 1$ be a system of equations over F_k, and let $v_1(x,y,a),\dots,v_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence*

$$\forall y \quad \exists x \quad w_1(x,y,a)=1,\dots,w_s(x,y,a)=1 \wedge v_1(x,y,a)\neq 1,\dots,v_r(x,y,a)\neq 1$$

is a truth sentence. Then there exists a formal solution $x = x(y,a)$ so that each of the words $w_j(x(y,a),y,a)$ is the trivial word in the free group $F =< y,a >$, and the sentence

$$\exists y \quad v_1(x(y,a),y,a)\neq 1,\dots,v_r(x(y,a),y,a)\neq 1$$

is a truth sentence in F_k.

Furthermore, if the words $w_1,\dots,w_s$ and $v_1,\dots,v_r$ are coefficient-free (i.e., they are words only in the variables x and y and not in the coefficients a), and there are at least 2 universal variables, then the formal solution $x = x(y,a)$ can be taken to be coefficient-free, i.e., $x = x(y)$.

Proof: The first part of Theorem 1.2 follows immediately from the argument used to prove Theorem 1.1, by first choosing the y_i's as in Theorem 1.1, and then choosing the x_j's to satisfy both the equalities $w_1(x,y,a) = 1,\dots,w_s(x,y,a) = 1$ and the inequalities $v_1(x,y,a)\neq 1,\dots,v_r(x,y,a)\neq 1$ (such a tuple of x_j's exist by the assumption of Theorem 1.2 for any possible y's). The obtained formal solution satisfies the conclusion of the first part of the theorem.

Suppose that the words $w_1,\dots,w_s,v_1,\dots,v_r$ are coefficient-free. Let $F_u = < u_1,\dots,u_k >$ be a free group of rank k. Let $\tilde{x} = \tilde{x}(y,u)$ be words obtained from the formal solution $x = x(y,a)$ by replacing the coefficients $a_1,\dots,a_k$ with the new generators $u_1,\dots,u_k$ in correspondence. Since the words $w_1,\dots,w_s,v_1,\dots,v_r$ are coefficient-free, and the words $w_i(x(y,a),y)$ are trivial in the free group $F(y,a)$, the words $w_i(\tilde{x}(y,u),y)$ are trivial in the free group $F(y,a)*F_u$. Since there exists a specialization of the variables y, denoted y_0, for which $v_j(x(y_0,a),y_0)\neq 1$ in F_k, for all j, $1\leq j\leq r$, $v_j(\tilde{x}(y_0,u),y_0)\neq 1$ in F_k*F_u, for all j, $1\leq j\leq r$, which implies that $v_j(\tilde{x}(y,u),y)\neq 1$ in $F(y,a)*F_u$, for all j, $1\leq j\leq r$.

If F_y is a non-abelian free group, and for every substitution $u = t(y)$ at least one of the words $v_j(\tilde{x}(y,t(y)),y) = 1$ in the free group $F(y,a)$, then at least one of the words $v_j(\tilde{x}(y,t(y)),y) = 1$ in $F(y,a) * F_u$, a contradiction. Hence, there must exist a substitution $u = t(y)$, for which $v_j(\tilde{x}(y,t(y)),y) \neq 1$ in $F(y,a)$, for all j, $1 \leq j \leq r$, which implies that there exists some specialization of the variables y, $\hat{y} \in F_k$, for which $v_j(\tilde{x}(\hat{y},t(\hat{y})),\hat{y}) \neq 1$ in F_k, for all j, $1 \leq j \leq r$. Clearly, since the words $w_i(\tilde{x}(y,u),y,a)$ are trivial in $F(y,a) * F_u$, the corresponding words $w_i(\tilde{x}(y,t(y)),y)$ are trivial in $F(y,a)$, so the coefficient-free formal solution $x = \tilde{x}(y,t(y))$ satisfies the conclusion of the second part of the theorem. ∎

To generalize Merzlyakov's theorem to arbitrary varieties, rather than the affine (free) varieties that appear in Theorems 1.1 and 1.2, we need to start by proving an analogous theorem for surface groups. Unlike AE sentences over general varieties, in the special case of a surface group a straightforward generalization of Merzlyakov's theorem is still valid, and was announced in [Kh-My] (section 5), who call it "implicit function theorem".

THEOREM 1.3: *Let $F_k =< a_1,\dots,a_k >$ be a free group, and let $u(y) = [y_1,y_2]\cdots[y_{2g-1},y_{2g}]$ for $g > 1$ or $u(y) = y_0^2\cdots y_g^2$ for $g > 2$ be given. Let the group $Q =< y|u(y) >$ be the surface group corresponding to the equation $u(y) = 1$.*

Let $w_1(x,y,a) = 1,\dots,w_s(x,y,a) = 1$ be a system of equations over F_k, and let $v_1(x,y,a),\dots,v_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence

$$\forall y \quad (u(y) = 1) \quad \exists x \quad w_1(x,y,a) = 1,\dots,w_s(x,y,a) = 1 \wedge$$
$$v_1(x,y,a) \neq 1,\dots,v_r(x,y,a) \neq 1$$

is a truth sentence. Then there exists a formal solution $x = x(y,a)$ so that each of the words $w_j(x(y,a),y,a)$ is the trivial word in the group $Q * F_k =< y,a >$, and the sentence

$$\exists y \quad u(y) = 1 \wedge v_1(x(y,a),y,a) \neq 1,\dots,v_r(x(y,a),y,a) \neq 1$$

is a truth sentence in F_k.

Furthermore, if the sentence is coefficient-free (i.e., the systems of equalities and inequalities contain no elements from the coefficient group F_k), then the formal solution can be taken to be coefficient-free as well, i.e., $x = x(y,a) = x(y)$.

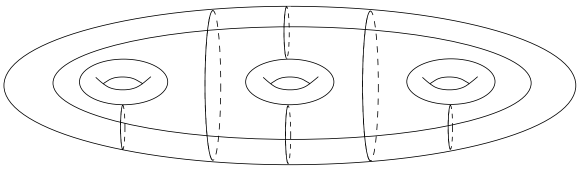
Proof: Let S be a surface with fundamental group $Q =< y|u(y) >$. Let μ: $Q \to F_k$ be a homomorphism with non-abelian image. By recursively applying lemma

5.13 of [Se], we can find a finite set of non-homotopic, non-boundary parallel, disjoint s.c.c. on S, $c_1, \ldots, c_m$, so that each connected component of the surface S obtained by cutting S along this family of s.c.c. has Euler characteristic -1, and the fundamental group of each of these connected components is mapped monomorphically into F_k by the homomorphism μ.

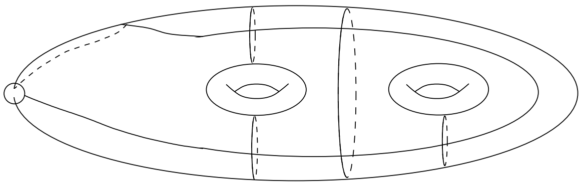
LEMMA 1.4: *There exist two collections of essential, non-homotopic, non-boundary parallel disjoint s.c.c. on the surface S: $b_1, \ldots, b_q$ and $d_1, \ldots, d_t$, and an automorphism $\rho \in Aut(S)$ with the following properties:*

- (i) *Each connected component $\tilde{S}$ obtained by cutting the surface S along the first collection of s.c.c. $b_1, \ldots, b_q$ has Euler charactersitic -1, and the homomorphism $\mu \circ \rho$: $Q \to F_k$ embeds the fundamental group of each of these connected components into F_k.*
- (ii) *Each of the curves d_i intersects non-trivially at least one of the curves b_j.*
- (iii) *The entire collection of s.c.c. $b_1, \ldots, b_q, d_1, \ldots, d_t$ fills the surface S, i.e., $S \setminus \cup\{b_1, \ldots, b_q, d_1, \ldots, d_t\}$ is a disjoint collection of simply connected domains in S.*

Proof: We can choose the collections of curves to be fixed collections of s.c.c.on the surface S. For orientable surface S we choose the collections



For non-orientable surfaces S we choose the collections (we draw it for odd genus, for even genus the picture is similar)



The collections $b_1, \ldots, b_q$ and $d_1, \ldots, d_t$ clearly satisfy properties (ii) and (iii) of the lemma, each connected component obtained by cutting S along the curves $b_1, \ldots, b_q$ has Euler characteristic -1, and the fundamental group of each such connected component is isomorphic to F_2. To complete the proof of part (i), note that if $w \in Q$ is a non-trivial element, and if we set $\rho \in Aut(S)$ to be obtained from large enough powers of the Dehn twists along the curves $c_1, \ldots, c_m$, then

$\mu \circ \rho$ maps w to a non-trivial element in F_k. Hence, for appropriately chosen large powers of the Dehn twists along $c_1, \ldots, c_m$, $\mu \circ \rho$ maps the fundamental group of each of the connected components obtained by cutting S along the collection of s.c.c.$b_1, \ldots, b_q$ isomorphically into F_k. ∎

Let $b_1, \ldots, b_q$ and $d_1, \ldots, d_t$ be the collections of s.c.c. constructed in Lemma 1.4, and let $\rho \in Aut(S)$ be the automorphism chosen so that part (i) of the lemma holds. For the rest of the proof of Theorem 1.3 we replace the homomorphism $\mu: Q \to F_k$ by the composition $\mu \circ \rho$ (and denote this composition μ). We further set $\varphi_1, \ldots, \varphi_q$ to be the automorphisms of Q that correspond to Dehn twists along the s.c.c. $b_1, \ldots, b_q$, and $\psi_1, \ldots, \psi_t$ to be the automorphisms of Q that correspond to Dehn twists along the s.c.c. $d_1, \ldots, d_t$ in correspondence.

In a similar way to the construction of the JSJ decomposition ([Ri-Se2], 4), we define the following sequences of automorphisms of the surface group Q, $\{\nu_n, \tau_n\}$, iteratively. We set $\tau_1 = id.$, and ν_1 to be

$$\nu_1 = \psi_1^{\ell_1^1} \circ \psi_2^{\ell_2^1} \circ \cdots \circ \psi_t^{\ell_t^1}.$$

For every index $n > 1$ we define τ_n to be

$$\tau_n = \varphi_1^{m_1^n} \circ \varphi_2^{m_2^n} \circ \cdots \circ \varphi_q^{m_q^n} \circ \nu_{n-1}$$

and

$$\nu_n = \psi_1^{\ell_1^n} \circ \psi_2^{\ell_2^n} \circ \cdots \circ \psi_t^{\ell_t^n} \circ \tau_n.$$

Given the sequence of automorphisms $\{\nu_n, \tau_n\}$ of the surface group Q, we define the sequence of homomorphisms $\lambda_n: Q \to F_k$ to be a sequence of homomorphisms of the form

$$\lambda_n = \mu \circ \varphi_1^{e_n} \circ \varphi_2^{e_n} \circ \cdots \circ \varphi_q^{e_n} \circ \nu_n.$$

Like in the construction of the JSJ decomposition, our aim in defining the sequence of automorphisms $\{\nu_n, \tau_n\}$ and homomorphisms $\{\lambda_n\}$ is to guarantee that any action of the surface group Q obtained as a limit of a converging subsequence of homomorphisms $\lambda_{n_s}: Q \to F_k$ is a minimal IET action of the surface group Q on the limit real tree. To obtain that goal we need to restrict the sequences of powers $\{\ell_i^n, m_j^n, e_n\}$ used in the iterative definition of the sequences $\{\nu_n, \tau_n, \lambda_n\}$ to satisfy certain combinatorial conditions, similar to the ones presented in section 4 of [Ri-Se2].

Let X be the Cayley graph of the free group $F_k = < a_1, \ldots, a_k >$, let Y be the Cayley graph of the surface group Q (with respect to the generating set $Q = < y_1, \ldots, y_s >$), let (T_b, t_b) be the Bass–Serre tree corresponding to the decomposition of the surface group Q along the collection of s.c.c. $b_1, \ldots, b_q$, and let (T_d, t_d) be the Bass–Serre tree corresponding to the decomposition of the surface group Q along the collection of s.c.c. $d_1, \ldots, d_t$. We denote by d_X, d_Y, d_{T_b}, and d_{T_d} the natural (simplicial) metrics on X, Y, T_b, and T_d in correspondence.

For every element $g \in Q$ we set $\ell_b(g) = d_{T_b}(g(t_b), t_b)$, $\ell_d(g) = d_{T_d}(g(t_d), t_d)$. If g acts hyperbolically on T_b we denote by $tr_b(g)$ the trace of the action of g on T_b, and similarly if g acts hyperbolically on T_d we denote its trace by $tr_d(g)$. For an element $f \in F_k$, let $tr_X(f)$ be the length of a cyclically reduced element that is conjugate to f in F_k, i.e., the "length" of the conjugacy class of f in F_k.

Let $Q = < y_1, \ldots, y_s >$, and suppose that each y_i can be written in a normal form $y_i = a^1_{y_i} a^2_{y_i} \cdots a^{\ell n(y_i)}_{y_i}$ with respect to the graph of groups corresponding to the decomposition of the surface S by the curves $d_1, \ldots, d_t$.

Let PR^{ν_1} be the set of all prefixes of the words $a^1_{y_i} a^2_{y_i} \cdots a^{\ell n(y_i)}_{y_i}$ for all i, $1 \leq i \leq s$. We set $R^{\tau_1} = 1$ and R^{ν_1} to satisfy

$$R^{\nu_1} \geq 2 \cdot \max_{u \in PR^{\nu_1}} d_Y(u, id.)$$

where R^{ν_1} is the size of the ball whose elements are going to be "controlled" by the automorphism ν_1 of Q. Setting R^{ν_1} we define the set HY^{ν_1} to be

$$HY^{\nu_1} = \{g \in Q | d_Y(g, id.) \leq R^{\nu_1} \wedge 0 < tr_d(g)\}$$

and the set NF^{ν_1} to be

$$NF^{\nu_1} = \{g \in Q | d_Y(g, id.) \leq R^{\nu_1} \wedge 0 < \ell_d(g)\}.$$

We define the constants R^{τ_n} and R^{ν_n} iteratively. For each $g \in Q$ for which $d_Y(g, id.) \leq R^{\nu_{n-1}}$ let

$$\nu_{n-1}(g) = a^1_{\nu_{n-1}(g)} a^2_{\nu_{n-1}(g)} \cdots a^{\ell n(\nu_{n-1}(g))}_{\nu_{n-1}(g)}$$

be a normal form of $\nu_{n-1}(g)$ with respect to the graph of groups corresponding to the decomposition of the surface S by the curves $b_1, \ldots, b_q$.

Let PR^{τ_n} be the set of all prefixes of the words $a^1_{\nu_{n-1}(g)} a^2_{\nu_{n-1}(g)} \cdots a^{\ell n(\nu_{n-1}(g))}_{\nu_{n-1}(g)}$ for all $g \in Q$ for which $d_Y(g, id.) \leq R^{\nu_{n-1}}$. We set R^{τ_n} to satisfy

$$R^{\tau_n} \geq 2 \cdot \max_{u \in PR^{\tau_n}} d_Y((\nu_{n-1})^{-1}(u), id.)$$

where R^{τ_n} is the size of the ball whose elements are going to be "controlled" by the automorphism τ_n of Q. Setting R^{τ_n} we define the set HY^{τ_n} to be

$$HY^{\tau_n} = \{g \in Q | d_Y(g, id.) \leq R^{\tau_n} \wedge 0 < tr_b(\nu_{n-1}(g))\}$$

and the set NF^{τ_n} to be

$$NF^{\tau_n} = \{g \in Q | d_Y(g, id.) \leq R^{\tau_n} \wedge 0 < \ell_b(\nu_{n-1}(g))\}.$$

Similarly, for each $g \in Q$ for which $d_Y(g, id.) \leq R^{\tau_n}$ let

$$\tau_n(g) = a^1_{\tau_n(g)} a^2_{\tau_n(g)} \cdots a^{\ell n(\tau_n(g))}_{\tau_n(g)}$$

be a normal form of $\tau_n(g)$ with respect to the graph of groups corresponding to the decomposition of the surface S by the curves $d_1, \ldots, d_q$.

Let PR^{ν_n} be the set of all prefixes of the words $a^1_{\tau_n(g)} a^2_{\tau_n(g)} \cdots a^{\ell n(\tau_n(g))}_{\tau_n(g)}$ for all $g \in Q$ for which $d_Y(g, id.) \leq R^{\tau_n}$. We set R^{ν_n} to satisfy

$$R^{\nu_n} \geq 2 \cdot \max_{u \in PR^{\nu_n}} d_Y((\tau_n)^{-1}(u), id.)$$

where R^{ν_n} is the size of the ball whose elements are going to be "controlled" by the automorphism ν_n of Q. Setting R^{ν_n} we define the set HY^{ν_n} to be

$$HY^{\nu_n} = \{g \in Q | d_Y(g, id.) \leq R^{\nu_n} \wedge 0 < tr_d(\tau_n(g))\}$$

and the set NF^{ν_n} to be

$$NF^{\nu_n} = \{g \in Q | d_Y(g, id.) \leq R^{\nu_n} \wedge 0 < \ell_d(\tau_n(g))\}.$$

Definition 1.5: For every index n and every $g \in NF^{\tau_n}$ let

$$\nu_{n-1}(g) = a^1_{\nu_{n-1}(g)} a^2_{\nu_{n-1}(g)} \cdots a^{\ell n(\nu_{n-1}(g))}_{\nu_{n-1}(g)}$$

be the previously chosen normal form of $\nu_{n-1}(g)$ with respect to the decomposition of Q corresponding to the Bass–Serre tree T_b. For every $h \in NF^{\nu_n}$ let

$$\tau_n(h) = a^1_{\tau_n(h)} a^2_{\tau_n(h)} \cdots a^{\ell n(\tau_n(h))}_{\tau_n(h)}$$

be the previously chosen normal form of $\tau_n(h)$ with respect to the decomposition of Q corresponding to the Bass–Serre tree T_d. We say that a sequence of automorphisms $\{\nu_n, \tau_n\}$ of the surface group Q and homomorphisms $\lambda_n \colon Q \to F_k$ of the form given above is a *quadratic test sequence* if the following conditions hold.

(i) For $n > 1$ and every b_i, $1 \leq i \leq q$:

$$tr_d((b_i)^{m_i^n}) > 100 \cdot 2^n \cdot \max_{1 \leq i \leq q} \ell_d(b_i) \cdot \sum_{d_Y(g, id.) \leq R^{\tau_n}, j \leq \ell n(g)} \ell_d(a^j_{\nu_{n-1}(g)}).$$

(ii) For $n \geq 1$ and every d_i, $1 \leq i \leq t$:

$$tr_b((d_i)^{\ell_i^n}) > 100 \cdot 2^n \cdot \max_{1 \leq i \leq t} \ell_b(d_i) \cdot \sum_{d_Y(h, id.) \leq R^{\nu_n}, j \leq \ell n(h)} \ell_d(a^j_{\tau_n(h)}).$$

(iii) For every $n > 1$ and every $g, g' \in NF^{\tau_n}$:

$$\Big|\frac{\ell_d(\tau_n(g))\ell_b(\nu_{n-1}(g'))}{\ell_d(\tau_n(g'))\ell_b(\nu_{n-1}(g))} - 1\Big| < \frac{1}{100 \cdot q \cdot 2^n}.$$

(iv) For every $n > 1$ and every $g \in HY^{\tau_n}$:

$$\Big|\frac{tr_d(\tau_n(g))\ell_b(\nu_{n-1}(g))}{\ell_d(\tau_n(g))tr_b(\nu_{n-1}(g))} - 1\Big| < \frac{1}{100 \cdot q \cdot 2^n}.$$

(v) For every $n \geq 1$ and every $h, h' \in NF^{\nu_n}$:

$$\Big|\frac{\ell_b(\nu_n(h))\ell_d(\tau_n(h'))}{\ell_b(\nu_n(h'))\ell_d(\tau_n(h))} - 1\Big| < \frac{1}{100 \cdot q \cdot 2^n}.$$

(vi) For every $n \geq 1$ and every $h \in HY^{\nu_n}$:

$$\Big|\frac{tr_b(\nu_n(h))\ell_d(\tau_n(h))}{\ell_b(\nu_n(h))tr_d(\tau_n(h))} - 1\Big| < \frac{1}{100 \cdot q \cdot 2^n}.$$

(vii) There exist constants $c_1, c_2 > 0$ so that for every $n \geq 1$ and every $h, h' \in NF^{\nu_n}$:

$$c_1 < \frac{\ell_b(\nu_n(h))d_X(\lambda_n(h'), id.)}{d_X(\lambda_n(h), id.)\ell_b(\nu_n(h'))} < c_2.$$

(viii) There exist constants $c_3, c_4 > 0$ so that for every $n \geq 1$ and every $h \in HY^{\nu_n}$:

$$c_3 < \frac{tr_b(\nu_n(h))d_X(\lambda_n(h), id.)}{tr_X(\lambda_n(h))\ell_b(\nu_n(h))} < c_4.$$

(ix) For every index n, the homomorphism $\lambda_n: Q \to F_k$ cannot be factored as $\lambda_n = \gamma \circ \pi$, where $\pi: Q \to Q_1$ is an embedding of Q into the fundamental group of a surface S_1 finitely covered by the surface S, and $\gamma: Q_1 \to F_k$ is a homomorphism, i.e., the homomorphism $\lambda_n: Q \to F_k$ cannot be extended to a surface covered by S.

PROPOSITION 1.6: *There exist quadratic test sequences associated with the surface group Q.*

Proof: Inequalities (i)–(vi) are essentially identical to the inequalities that appear in claim 4.7 of [Ri-Se2]. Let Λ_b^Q be the graph of groups obtained by decomposing the surface group Q along the cyclic groups corresponding to the s.c.c. $b_1, \ldots, b_q$. By construction, the homomorphism $\mu: Q \to F_k$ maps every vertex group and every edge group in Λ_b monomorphically into F_k. The homomorphism $\lambda_n: Q \to F_k$ is defined to be

$$\lambda_n = \mu \circ \varphi_1^{e_n} \circ \cdots \circ \varphi_q^{e_n} \circ \nu_n.$$

Hence, given any finite set of elements $M \subset Q$, there exists some constant e so that if the exponent $e_n > e$, then for every $m_1, m_2 \in M$ for which both $\nu_n(m_1)$ and $\nu_n(m_2)$ do not fix the point $t_b \in T_b$,

$$c_1 < \frac{\ell_b(\nu_n(m_1))d_X(\lambda_n(m_2), id.)}{d_X(\lambda_n(m_1), id.)\ell_b(\nu_n(m_2))} < c_2,$$

where the constants c_1, c_2 are independent of the finite set M.

Similarly, for every element $m \in M$ for which $tr_b(\nu_n(m)) > 0$,

$$c_3 < \frac{tr_b(\nu_n(m))d_X(\lambda_n(m), id.)}{tr_X(\lambda_n(m))\ell_b(\nu_n(m))} < c_4,$$

where the constants c_3, c_4 are independent of the finite set M, and the last two inequalities imply the inequalities (vii) and (viii) for high enough exponent e_n in the definition of the homomorphisms λ_n.

Fixing the index n, to construct a homomorphism λ_n that satisfies properties (i)–(viii) and does not factor through the fundamental group of any surface covered by the surface S (with fundamental group Q), we fix the automorphisms (μ_n, τ_n) of the surface group Q, and suppose that there exists an increasing sequence of exponents $\{e^j\}_{j=1}^{\infty}$ for which the homomorphisms

$$\lambda^j = \mu \circ \varphi_1^{e^j} \circ \cdots \circ \varphi_q^{e^j} \circ \nu_n$$

do factor through the fundamental group of surfaces covered by the surface S. Since up to isomorphism there are only finitely many possibilities for a couple (Q, Q_1), where $Q < Q_1$ and Q_1 is the fundamental group of a surface covered by the surface S, w.l.o.g. we may assume that all the homomorphisms $\{\lambda^j\}$ factor through a (fixed) surface group Q_1 which is the fundamental group of a surface S_1 finitely covered by the surface S. So each homomorphism $\lambda^j\colon Q \to F_k$ factors as $\lambda^j = \gamma^j \circ \pi$ where $\pi\colon Q \to Q_1$ is an embedding, and $\gamma^j\colon Q_1 \to F_k$ is a homomorphism.

From the sequence of homomorphisms $\gamma^j\colon Q_1 \to F_k$ we can extract a subsequence converging into a faithful action of the surface group Q_1 on some real tree Y_∞. Since each homomorphism γ^j is an extension of the homomorphism $\lambda^j\colon Q \to F_k$, and since by the structure of the homomorphisms $\{\lambda^j\colon Q \to F_k\}$ the finite index subgroup $Q < Q_1$ acts discretely on the limit tree Y_∞, the surface group Q_1 acts discretely on the limit tree Y_∞. Furthermore, since every edge stabilizer in the action of Q on Y_∞ is conjugate to a cyclic subgroup generated by one of the s.c.c. b_i, each edge stabilizer in the action of Q_1 on Y_∞ is conjugate to a maximal cyclic subgroup containing the element b_i in the surface group Q_1.

Let Λ_b^Q be the decomposition of the surface group Q along the s.c.c. $b_1, \dots, b_q$, and let $\Lambda_{Y_\infty}^{Q_1}$ be the graph of groups corresponding to the action of Q_1 on the limit tree Y_∞. By the structure of the homomorphisms $\lambda^j \colon Q \to F_k$ every vertex group in Λ_b^Q fixes a point in Y_∞, so it is contained in a vertex group in the graph of groups $\Lambda_{Y_\infty}^{Q_1}$.

The map $\pi \colon S \to S_1$ corresponding to the embedding $\pi \colon Q \to Q_1$ is a covering map. Since Q_1 is the fundamental group of a surface S_1, and since $\Lambda_{Y_\infty}^{Q_1}$ is a graph of groups with cyclic edge stabilizers, each vertex group in $\Lambda_{Y_\infty}^{Q_1}$ is a subsurface of S_1. Let $\hat{S}$ be a connected component of the set $S \setminus \bigcup\{b_1, \dots, b_q\}$. By construction $\chi(\hat{S}) = -1$, and $\hat{Q} = \pi_1(\hat{S})$ is conjugate to a vertex stabilizer in Λ_b^Q. $\hat{Q}$ fixes a point in Y_∞ so it can be conjugated into a vertex group in $\Lambda_{Y_\infty}^{Q_1}$. Let $\hat{Q}_1$ be that vertex stabilizer, and let $\hat{S}_1$ be the subsurface of S_1 for which $\hat{Q}_1 = \pi_1(\hat{S}_1)$. The covering map $\pi \colon S \to S_1$ maps the subsurface $\hat{S}$ into the subsurface $\hat{S}_1$, and the boundary of $\hat{S}$ is mapped to the boundary of $\hat{S}_1$. Since in addition $\chi(\hat{S}) = -1$, then, necessarily, the covering map π maps the subsurface $\hat{S}$ homeomorphically onto the subsurface $\hat{S}_1$.

Since the restriction of the covering map π to each subsurface $\hat{S}$ is a homeomorphism, and since every edge stabilizer in Λ_b^Q is also contained in some edge stabilizer in $\Lambda_{Y_\infty}^{Q_1}$, the closed surface S must be a subsurface of the closed surface S_1, which clearly implies that $S = S_1$. Hence $Q = Q_1$, a contradiction to the construction of the sequence λ^j. Therefore, we can choose an exponent e_n as large as we like so that the homomorphism

$$\lambda_n = \mu \circ \varphi_1^{e_n} \circ \cdots \circ \varphi_q^{e_n} \circ \nu_n$$

does not factor through the fundamental group of any cover of the surface group S, and we get part (ix) in Definition 1.5. ∎

Proposition 1.7: *Let the sequence of automorphisms of Q, $\{\nu_n, \tau_n\}$ and homomorphisms $\{\lambda_n : Q \to F_k\}$ be a quadratic test sequence. Let $g, g' \in Q$ be non-trivial elements, and suppose* $\max(d_Y(g, id.), d_Y(g', id.)) \le R^{\nu_s}$. *Then there exist constants $const^1_{g,g'}, const^2_{g,g'}, const^3_g, const^4_g > 0$ (depending on g and g') so that for every index $n > s+1$*

$$const^1_{g,g'} < \frac{d_X(\lambda_n(g), id.)}{d_X(\lambda_n(g'), id.)} < const^2_{g,g'},$$
$$const^3_g < \frac{tr_X(\lambda_n(g))}{d_X(\lambda_n(g), id.)} < const^4_g$$

and

$$const^1_{g,g'} < \frac{tr_X(\lambda_n(g))}{tr_X(\lambda_n(g'))} < const^2_{g,g'}.$$

Proof: If $\max(d_Y(g,id.),d_Y(g',id.)) \leq R^{\nu_s}$ then $\max(d_Y(g,id.),d_Y(g',id.)) \leq \min(R^{\tau_n},R^{\nu_n})$ for every $n > s$. Since the sequence $\{\nu_n,\tau_n,\lambda_n\}$ is a quadratic test sequence, the last inequality implies that the inequalities listed in Definition 1.5 apply to $\nu(g)$ and $\nu(g')$ for every $n > s$.

Let S be the surface with fundamental group Q. Since the curves $b_1,\ldots,b_q,d_1,\ldots,d_t$ were chosen to fill the surface S, every element $u \in Q$, $u \neq 1$, acts hyperbolically on either the Bass–Serre tree T_b or the Bass–Serre tree T_d. Since in addition $\max(d_Y(g,id.),d_Y(g',id.)) \leq \min(R^{\tau_n},R^{\nu_n})$ for every $n > s$, each of the elements g,g' belongs to either $HY^{\tau_{s+1}}$ or $HY^{\nu_{s+1}}$. Therefore, inequalities (iii)–(viii) that appear in Definition 1.5 hold for the couple (g,g') for every $n > s+1$. Now, from the combination of inequalities (iii), (iv), and (vii) of Definition 1.5, we get the inequality that relates the lengths of the elements $\lambda_n(g),\lambda_n(g')$:

$$const^1_{g,g'} < \frac{d_X(\lambda_n(g),id.)}{d_X(\lambda_n(g'),id.)} < const^2_{g,g'}.$$

From the inequalities (v), (vi), and (viii) of Proposition 1.5 we get the inequality that relates the trace and length of a single element g:

$$const^3_g < \frac{tr_X(\lambda_n(g))}{d_X(\lambda_n(g),id.)} < const^4_g,$$

and the combination of the above two inequalities gives us the third inequality on the ratio of the traces of two given elements g,g':

$$const^1_{g,g'} < \frac{tr_X(\lambda_n(g))}{tr_X(\lambda_n(g'))} < const^2_{g,g'}. \qquad \blacksquare$$

Once we have constructed quadratic test sequences we are ready to complete the proof of Theorem 1.3. With the notation of Theorem 1.3, suppose that the sentence

$$\forall y \quad (u(y)=1) \quad \exists x \quad w_1(x,y,a)=1,\ldots,w_s(x,y,a)=1 \wedge v_1(x,y,a) \neq 1, \ldots, v_r(x,y,a) \neq 1$$

is a truth sentence.

By Proposition 1.6 there exists a quadratic test sequence associated with the surface group Q, so let $\{\nu_n,\tau_n,\lambda_n\}$ be such a quadratic test sequence. Since by the assumptions of the theorem, for every possible specialization of the variables y obtained from a homomorphism from Q to the free group F_k there exists a specialization for the variables x, so that the given equalities $w_1(x,y,a) = 1, \ldots, w_s(x,y,a) = 1$ and inequalities $v_1(x,y,a) \neq 1, \ldots, v_r(x,y,a) \neq 1$ are fulfilled, given the specializations $\lambda_n(y)$ we can choose x_n to be the shortest possible elements, with respect to the metric on the Cayley graph of the coefficient free group F_k, for which

$$w_1(x_n, \lambda_n(y), a) = 1, \ldots, w_s(x_n, \lambda_n(y), a) = 1 \wedge$$
$$\wedge v_1(x_n, \lambda_n(y), a) \neq 1, \ldots, v_r(x_n, \lambda_n(y), a) \neq 1.$$

By possibly passing to a subsequence (still denoted $\{(x_n, \lambda_n(y), a)\}$), we may assume that the sequence of specializations $\{(x_n, \lambda_n(y), a)\}$ converges to an action of a limit group on some real tree Y_∞. We call the limit group into which the sequence converge, a *quadratic test limit group* and denote it $QTL(x,y,a)$.

Since the quadratic test limit group $QTL(x,y,a)$ was constructed using a sequence of elements $\{(x_n, \lambda_n(y), a)\}$ for which the equalities $w_1(x_n, \lambda_n(y), a) = 1, \ldots, w_s(x_n, \lambda_n(y), a) = 1$ are fulfilled, the words $w_1(x,y,a), \ldots, w_s(x,y,a)$ represent the trivial words in the quadratic test limit group $QTL(x,y,a)$. Similarly, the words $v_1(x,y,a), \ldots, v_r(x,y,a)$ represent non-trivial elements in $QTL(x,y,a)$.

Since the sequence $\{\nu_n, \tau_n, \lambda_n\}$ is a quadratic test sequence, Proposition 1.7 implies that the sequence of specializations $\{\lambda_n(y)\}$ converges to the surface group Q, hence $Q < QTL(x,y,a)$. By the same proposition, either the subgroup Q fixes a point in the limit tree Y_∞, or it acts freely on Y_∞. Since Q is a surface group, if Q acts freely on Y_∞ then Q acts freely on some IET component of the limit tree T_∞. Since by property (ix) of a quadratic test sequence (Definition 1.5) the homomorphisms λ_n: $Q \to F_k$ do not factor through the fundamental group of a surface that is finitely covered by the surface S_Q corresponding to the subgroup Q, if Q is not elliptic in the real tree Y, then the subgroup (stabilizer) of the IET component of Y_∞ on which Q acts in $QTL(x,y,a)$ is Q itself.

Suppose that the surface subgroup Q fixes a point in the limit tree Y_∞. If the real tree Y_∞ contains either an axial component, an IET component, or an edge with non-trivial (hence, abelian) stabilizer in its discrete part, then the shortening argument presented in [Se], [Ri-Se1] and [Be] implies that there exists an automorphism τ of the limit group $QTL(x,y,a)$ for which:

(i) τ fixes (elementwise) the subgroup Q and the coefficient group F_k,

(ii) for large enough n, $w_i(\tau(x_n), \lambda_n(y), a) = 1$ for all $i = 1, \ldots, s$ and $v_j(\tau(x_n), \lambda_n(y), a) \neq 1$ for all $j = 1, \ldots, r$,

(iii) $\tau(x_n)$ is strictly shorter than x_n,
which clearly contradicts our choice of the specializations x_n to be the shortest possible. Hence, the limit tree Y_∞ contains no axial components, no IET components and all the edges in the discrete part of Y_∞ have trivial stabilizers.

Let Λ_{Y_∞} be the graph of groups associated with the action of QTL on the limit tree Y_∞. By construction, in case the surface group Q fixes a point in the limit tree Y_∞, both subgroups Q and F_k fix the same point in the real tree Y_∞, hence they are contained in the same vertex stabilizer in the graph of groups Λ_{L_∞}. Since the limit group QTL acts non-trivially on the real tree Y_∞, and since Y_∞ contains no axial components, no IET components, and all the edges in the discrete part of Y_∞ have trivial stabilizers, the graph of groups Λ_{Y_∞} corresponds to a non-trivial free decomposition of the limit group QTL. Since both Q and F_k fixes the same vertex in Λ_{Y_∞}, both Q and F_k are contained in the same factor in the free decomposition of QTL associated with Λ_{Y_∞}.

Let $QTL = H * B_1 * \cdots * B_t * F$ be the most refined (Grushko's) free decomposition of the limit group QTL in which $< Q, F_k >< H$, the factor F is a free group, and the factors $B_1, \dots, B_t$ are freely indecomposable and non-cyclic. By the argument presented above this free decomposition is non-trivial. Suppose that $t \geq 1$, i.e., that there exists a factor in this free decomposition that is not free and does not contain the subgroup $< Q, F_k >$. In this case we modify the sequence of specializations $\{(x_n, \lambda_n(y), a)\}$ so that it factors through a proper quotient of QTL.

For each index i, $1 \leq i \leq t$, let b_i be a set of generators of the factor B_i, let $h_1, \dots, h_m$ be a set of generators of H, and let f be a free basis of the free factor F (if this factor is non-trivial). With each specialization $(x_n, \lambda_n(y), a)$ from our given sequence, we associate the corresponding specialization of the generators $h_1, \dots, h_m$, which we denote h^n. With the specializations h^n and $\lambda_n(y)$ we further associate specializations of the generating sets $b^1, \dots, b^t$ in (new) free groups $F_1, \dots, F_t$ in correspondence, where each such free group is isomorphic to the coefficient free group F_k. We denote these specializations b^n and choose them to satisfy

$$w_1(x(h^n, b^n, f), \lambda_n(y), a) = 1, \dots, w_s(x(h^n, b^n, f), \lambda_n(y), a) = 1 \wedge$$
$$\wedge v_1(x(h^n, b^n, f), \lambda_n(y), a) \neq 1, \dots, v_r(x(h^n, b^n, f), \lambda_n(y), a) \neq 1$$

in the free group $F_k * F_1 * \cdots * F_t * F$, and among all such specializations of the generating set $b^1, \dots, b^t$ we choose specializations for which the specialization of the generating set b^t is the shortest possible (with respect to the metric on the free group F_t). Note that since the words $w_i(x,y,a)$ are trivial in QTL, and for every index n there exists a specialization $(x_n,\lambda_n(y),a)$ in the original free group F_k that factors through QTL and satisfies the inequalities $v_j(x,y,a)\neq 1$, such specializations of the generating sets $b^1,\dots,b^t$ in the limit group $F_k*F_1*\cdots*F_t*F$ do exist.

By possibly passing to a subsequence, we may assume that the sequence of specializations of the generating sets $b^1,\dots,b^t$ converges. In particular, the sequence of specializations of the generating set b^t of the factor B_t converges into either the trivial group or it converges into an action of a limit group $\tilde{B}_t$ on some real tree T_∞, and since all the specializations of b^t are specializations of the factor B_t, $\tilde{B}_t$ is a quotient of B_t. Furthermore, since the specializations of b^t were chosen to be shortest possible, the real tree T_∞ contains no axial components, no IET components, and the stabilizer of all the edges in its discrete part are trivial. Hence, $\tilde{B}_t$ is either trivial or cyclic or free or it inherits a non-trivial free decomposition from its action on T_∞. Therefore, if we set QTL_1 to be the group $D*B_1*\cdots*\tilde{B}_t*F$, then QTL_1 is a proper quotient of QTL.

From the descending chain condition for limit groups ([Se], 5.1), after repeating this construction finitely many times we may assume that the sequence of specializations $\{(x_n,\lambda_n(y),a)\}$ converges into an action of a quadratic test limit group (still denoted) QTL on some real tree Y_∞, and either the surface group Q is not elliptic when acting on Y_∞ or the Grushko's free decomposition of QTL in which $<Q,F_k>$ is elliptic is of the form $H*F$, where F is a free group.

Suppose the subgroup $Q<QTL$ is still elliptic when acting on Y_∞. In this case we modify the sequence of specializations of QTL. Let $h_1,\dots,h_m$ be a generating set of the factor H, and let f be a generating set of the free factor F. With each specialization $(x_n,\lambda_n(y),a)$ from our given sequence, we associate a specialization of the generators $h_1,\dots,h_m$, which we denote h^n, which restricts to $\lambda_n(y)$ on the subgroup $Q<QTL$, that satisfies

$$w_1(x(h^n,f),\lambda_n(y),a)=1,\dots,w_s(x(h^n,f),\lambda_n(y),a)=1\wedge$$
$$\wedge v_1(x(h^n,f),\lambda_n(y),a)\neq 1,\dots,v_r(x(h^n,f),\lambda_n(y),a)\neq 1$$

in the free group F_k*F, and among all such specializations of the generating set $h_1,\dots,h_m$ we choose specializations h^n which are the shortest possible with respect to the metric on the coefficient group F_k.

By possibly passing to a subsequence, we may assume that the sequence of specializations $\{h^n\}$ converges into an action of a limit group $\hat{H}$ on a real tree T_∞. By construction, F_k and Q are subgroups of $\hat{H}$, and $\hat{H}$ is a quotient of H.

If Q fixes a point when acting on T_∞, then by the argument used for the analysis of QTL on Y_∞, $\hat{H}$ admits a non-trivial free decomposition in which the subgroup $< Q, F_k >$ is contained in a factor, so $\hat{H}$ is necessarily a proper quotient of H. Hence, if we set the limit group $QTL_1 = \hat{H} * F$, then QTL_1 is a proper quotient of QTL.

Therefore, from the descending chain condition for limit groups ([Se], 5.1), after repeating this construction finitely many times, we may assume that the sequence of specializations $\{(x_n, \lambda_n(y), a)\}$ converges into an action of a quadratic test limit group (still denoted) QTL on some real tree Y_∞, and the action of the surface group Q is not elliptic when acting on Y_∞.

By Proposition 1.7, if Q is not elliptic when acting on the limit tree Y_∞, then Q acts freely on Y_∞. Since Q is a surface group, and the homomorphisms λ_n: $Q \to F_k$ do not factor through the fundamental group of a surface finitely covered by Q, Q must be the stabilizer of an IET component in the limit tree Y_∞. Hence, the quadratic test limit group, QTL, admits a free decomposition of the form

$$QTL = Q * U * B_1 * \cdots * B_t * F$$

where $F_k < U$, the B_i's are non-cyclic freely indecomposable, and F is a (possibly trivial) free group. By iteratively modifying the specializations of the subgroup $U * B_1 * \cdots * B_t * F$, precisely as we modified the specializations of QTL in case Q is elliptic when acting on Y_∞, we end up with a quadratic test limit group (still denoted) QTL of the form $QTL = Q * F_k * F$.

By the construction of a quadratic test limit group, the words $w_i(x, y, a) = 1$ in $QTL = Q * F_k * F$ for all i, $1 \leq i \leq s$, and the words $v_j(x, y, a) \neq 1$ in $QTL = Q * F_k * F$ for all j, $1 \leq j \leq r$. Hence, there must exist a retract η: $QTL \to Q * F_k$, for which $w_i(\eta(x), y, a) = 1$ and $v_j(\eta(x), y, a) \neq 1$ in $Q * F_k$, for all possible indices i and j. Hence, we have found a formal solution for our given sentence, $\eta(x) = u(y, a)$, in the free product $Q * F_k$, which proves the first part of Theorem 1.3.

Suppose the given sentence is coefficient-free. Repeating our modification of the quadratic test limit group QTL in the coefficient-free case, we end up with a quadratic test limit group of the form $QTL = Q * F$, where F is a free group. In this case there must exist a retract ν: $QTL \to Q$, for which $w_i(\eta(x), y, a) = 1$ and $v_j(\eta(x), y, a) \neq 1$ in Q, for all possible indices i and j. Hence, we have found a coefficient-free formal solution for our given coefficient-free sentence, $\nu(x) = u(y)$, in the surface group Q, and we finally conclude the proof of Theorem 1.3. ∎

Theorem 1.3 generalizes Merzlyakov's theorem to quadratic equations. To get a general form of Merzlyakov's theorem we will also need a generalization to a free abelian group.

PROPOSITION 1.8: *Let $F_k =< a_1, \dots, a_k >$ be a free group, and for some $n > 1$ let $u_{(i,j)}(y) = [y_i, y_j]$ for $1 \leq i < j \leq n$. Let the group $Y_n =< y|u(y) >$ be the corresponding free abelian group on n generators.*

Let $w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1$ be a system of equations over F_k, and let $v_1(x, y, a), \dots, v_r(x, y, a)$ be a collection of words in the alphabet $\{x, y, a\}$. Suppose that the sentence

$$\forall y \quad (u(y) = 1)\ \exists x \quad w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1 \wedge v_1(x, y, a) \neq 1, \\ \dots, v_r(x, y, a) \neq 1$$

is a truth sentence.

Then there exist finitely many free abelian groups of rank n, $Z_n^1, \dots, Z_n^\ell$, where $Z_n^i =< z_1^i, \dots, z_n^i >$ for $i = 1, \dots, \ell$, together with ℓ monomorphisms $\nu_i \colon Y_n \to Z_n^i$, for which $\nu_i(Y_n)$ is a finite index subgroup in Z_n^i, and ℓ formal solutions $\{x_i = x_i(z^i, a)\}$ with the following properties:

(i) *Each of the words $w_j(x_i(z^i, a), y, a)$ is the trivial word in the group $Z_n^i * F_k$, and the sentence*

$$\exists z^i \quad ([z_j^i, z_{j'}^i] = 1, 0 \leq j < j' \leq n) \wedge v_1(x(z^i, a), y, a) \neq 1, \\ \dots, v_r(x(z^i, a), y, a) \neq 1$$

is a truth sentence in F_k.

(ii) *With each monomorphism $\nu_i \colon Y_n \to Z_n^i$ one can naturally associate a Diophantine system Σ_i of n equations in n variables, setting each of the y_j's to be equal to a linear combination of the elements $(z_1^i, \dots, z_n^i)$ corresponding to $\nu_i(y_j)$, where we view $z_1^i, \dots, z_n^i$ as variables.*

With each system Σ_i we can associate the set of (integer) tuples $(y_1, \dots, y_n)$ that are obtained as combinations of tuples of integers $z_1^i, \dots, z_n^i$. Since the homomorphisms ν_i map the group Y_n into a finite index subgroup of Z_n^i, the determinants of the systems Σ_i are non-zero, hence for each system Σ_i this collection of the obtained (integer) tuples $(y_1, \dots, y_n)$ is a finite index subgroup of the free abelian group of rank n, Z^n. We denote the corresponding subgroup C_i. Then the union of these subgroups $C_1, \dots, C_\ell$ covers the entire free abelian group of rank n, Z^n.

Equivalently, as the referee has pointed out, the images from the dual spaces of the various groups Z_n^i to the dual space of Y_n cover the dual space of Y_n.

Proof: As in the proof of Theorem 1.1, for each couple of positive integers (α, β) let
$$\lambda([\alpha,\beta]) = a_1 a_2^{\alpha} a_1 a_2^{\alpha+1} \cdots a_1 a_2^{\beta}.$$
For each positive integer m, we set γ_m to be $\gamma_m = \lambda([1,m])$.

We say that a sequence of $n+1$-tuples of positive integers

$$\{(q_0(j), q_1(j), \ldots, q_n(j))\}_{j=1}^{\infty}$$

is an *abelian test sequence*, if the sequence $\{q_0(j)\}_{j=1}^{\infty}$ is a strictly increasing sequence, and for every index j

$$j \cdot q_0(j) < q_1(j), j \cdot q_1(j) < q_2(j), \ldots, j \cdot q_{n-1}(j) < q_n(j).$$

Given an abelian test sequence we set

$$y_1(j) = \gamma_{q_0(j)}^{q_1(m_j)}, \ldots, y_n(j) = \gamma_{q_0(j)}^{q_n(j)}.$$

Since by the assumptions of the proposition, for every possible specialization of commuting y's there exist specializations for the variables x, so that the given equalities $w_1(x,y,a) = 1, \ldots, w_s(x,y,a) = 1$ and inequalities $v_1(x,y,a) \neq 1, \ldots, v_r(x,y,a) \neq 1$ are fulfilled, given the specializations $y_1(j), \ldots, y_n(j)$ we can choose $x(j)$ to be the shortest possible element for which

$$\begin{aligned} &w_1(x(j),y(j),a) = 1, \ldots, w_s(x(j),y(j),a) = 1 \wedge \\ &\wedge v_1(x(j),y(j),a) \neq 1, \ldots, v_r(x(j),y(j),a) \neq 1. \end{aligned}$$

If the sequence of specializations $\{(x(j),y(j),a)\}$ corresponding to an abelian test sequence converges, we call the obtained limit group an *abelian test limit group*. On the collection of abelian test limit groups we define a natural partial order. We say that an abelian test limit group $ATL_1(x,y,a)$ is bigger than an abelian test limit group $ATL_2(x,y,a)$, if there exists a proper epimorphism η: $ATL_1(x,y,a) \to ATL_2(x,y,a)$ that maps the generators $\{x,y,a\}$ of ATL_1 to the corresponding generators of ATL_2.

By the arguments used in constructing the Makanin–Razborov diagram of a limit group (see lemmas 5.4 and 5.5 in [Se]) there exist *maximal abelian test limit groups* with respect to the collection of all possible abelian test sequences, and in fact there is a canonical finite collection of maximal abelian test limit groups with respect to the entire collection of all possible abelian test sequences, which we denote $MATL_1(x,y,a), \ldots, MATL_\ell(x,y,a)$.

Since the maximal abelian test limit groups

$$MATL_1(x,y,a),\dots,MATL_\ell(x,y,a)$$

were constructed using sequences of elements $\{(x(j),y_1(j),\dots,y_n(j))\}$ for which the equalities $w_1(x,y,a)=1,\dots,w_s(x,y,a)=1$ are fulfilled, the words

$$w_1(x,y,a),\dots,w_s(x,y,a)$$

represent the trivial words in the maximal abelian test limit groups

$$MATL_1(x,y,a),\dots,MATL_\ell(x,y,a)$$

and similarly the words $v_1(x,y,a),\dots,v_r(x,y,a)$ represent non-trivial elements in the maximal abelian test limit groups. Since the sequences $\{y(j)\}$ are abelian test sequences, for each maximal abelian test limit group, $<y>< MATL_i$ is a free abelian group of rank n.

We continue by looking at a sequence of specializations $\{(x(j),y(j),a)\}$ that converges into an action of one of the maximal abelian test limit groups, which we denote $MATL(x,y,a)$, on some real tree T_∞. Suppose that the subgroup $<y>$ $< MATL(x,y,a)$ fixes a point in the limit tree T_∞. If the real tree T_∞ contains either an axial component, an IET component, or an edge with non-trivial (hence, abelian) stabilizer in its discrete part, then the shortening argument presented in [Se], [Ri-Se1] and [Be] implies that there exists an automorphism τ of the limit group $MATL(x,y,a)$ for which

(i) τ fixes (elementwise) the subgroup $<y>$ and the coefficient group F_k,

(ii) for large enough j, $w_i(\tau(x(j)),y(j),a)=1$ for all $i=1,\dots,s$ and $v_j(\tau(x(j)),y(j),a)\neq 1$ for all $j=1,\dots,r$,

(iii) $\tau(x(j))$ is strictly shorter than $x(j)$,

which clearly contradicts our choice of the specializations $x(j)$ to be the shortest possible. Hence the limit tree T_∞ contains no axial components, no IET components and all the edges in the discrete part of T_∞ have trivial stabilizers, which implies that the maximal abelian test limit group $MATL(x,y,a)$ admits a non-trivial free decomposition in which the subgroup $<y,a>$ is contained in one of the factors.

In this case we continue as in the proof of Theorem 1.3. Let $MATL = H*B_1*\cdots*B_t*F$ be the most refined (Grushko's) free decomposition of the limit group $MATL$ in which $<y,a>< H$, the factor F is a free group, and the factors $B_1,\dots,B_t$ are freely indecomposable and non-cyclic. Suppose that $t\geq 1$, i.e., that there exists a factor in this free decomposition that is not free and does not contain the subgroup $< y, a >$. In this case we modify the sequences of specializations $\{(x(j), y(j), a)\}$ that factor through $MATL(x, y, a)$ so that they all factor through a finite collection of proper quotients of $MATL(x, y, a)$.

For each index i, $1 \leq i \leq t$, let b_i be a set of generators of the factor B_i, let $h_1, \ldots, h_m$ be a set of generators of H, and let f be a free basis of the free factor F (if this factor is non-trivial). Given a sequence $\{(x(j), y(j), a)\}$ that factors through $MATL(x, y, a)$, with each specialization $(x(j), y(j), a)$ from the given sequence we associate the corresponding specialization of the generators $h_1, \ldots, h_m$, which we denote h^j. With the specializations h^j we further associate specializations of the generating sets $b^1, \ldots, b^t$ in (new) free groups $F_1, \ldots, F_t$ in correspondence, where each such free group is isomorphic to the coefficient free group F_k. We denote these specializations b^j, and choose them to satisfy

$$w_1(x(h^j, b^j, f), y(j), a) = 1, \ldots, w_s(x(h^j, b^j, f), y(j), a) = 1 \wedge$$
$$\wedge v_1(x(h^j, b^j, f), y(j), a) \neq 1, \ldots, v_r(x(h^j, b^j, f), y(j), a) \neq 1$$

in the free group $F_k * F_1 * \cdots * F_t * F$, and among all such specializations of the generating set $b^1, \ldots, b^t$ we choose specializations for which the specialization of the generating set b^t is the shortest possible (with respect to the metric on the free group F_t).

By our standard arguments (lemmas 5.4 and 5.5 in [Se]), the collection of all convergent sequences of these modified specializations $\{(x(h^j, b^j, f), y(j), a)\}$ factor through a finite collection of maximal limit groups which we (still) denote $MATL_1(x, y, a), \ldots, MATL_q(x, y, a)$. Let $\{(x(h^j, b^j, f), y(j), a)\}$ be a sequence that converges into one of these maximal limit groups $MATL_i(x, y, a)$. In particular, the sequence of specializations of the generating set b^t of the factor B_t converges into either the trivial group or it converges into an action of a limit group $\tilde{B}_t$ on some real tree R_∞, and since all the specializations of b^t are specializations of the factor B_t, $\tilde{B}_t$ is a quotient of B_t. Furthermore, since the specializations of b^t were chosen to be shortest possible, the real tree R_∞ contains no axial components, no IET components, and the stabilizer of all the edges in its discrete part is trivial. Hence $\tilde{B}_t$ is either trivial or cyclic or free or it inherits a non-trivial free decomposition from its action on R_∞. Therefore, each of the limit groups $MATL_i(x, y, a)$ is a proper quotient of the original maximal abelian test limit group $MATL(x, y, a)$.

From the descending chain condition for limit groups ([Se], 5.1), after repeating this construction finitely many times we may assume that the sequences of specializations $\{(x(j), y(j), a)\}$ factor through a finite collection of maximal abelian test limit groups (still denoted) $MATL_1(x,y,a),\dots,MATL_\ell(x,y,a)$, and the Grushko's free decomposition of each of these limit groups in which the subgroup $<y,a>$ is elliptic is either trivial, or it is of the form $H*F$, where F is a free group, and $<y,a>< H$.

We continue by modifying the sequences of specializations that factor through the maximal abelian test limit groups $MATL_1(x,y,a),\dots,MATL_\ell(x,y,a)$. Let $h_1,\dots,h_m$ be a set of generators of H, and let f be a free basis of the free factor F (if this factor is non-trivial). Given a sequence $\{(x(j),y(j),a)\}$ that factors through $MATL_i(x,y,a)$, with each specialization $(x(j),y(j),a)$ from the given sequence, we associate the corresponding specialization of the generators $h_1,\dots,h_m$, which satisfies

$$w_1(x(h^j,f),y(j),a)=1,\dots,w_s(x(h^j,f),y(j),a)=1\wedge$$
$$\wedge v_1(x(h^j,f),y(j),a)\neq 1,\dots,v_r(x(h^j,f),y(j),a)\neq 1$$

in the free group F_k*F, and among all such specializations of $h_1,\dots,h_m$ we choose specializations that are the shortest possible (with respect to the metric on the free group F_k), which we denote h^j.

The collection of all convergent sequences of these modified specializations factors through a finite collection of maximal abelian test limit groups that are quotients of $MATL_i(x,y,a)$. If they are proper quotients we continue with these modifications. By the descending chain condition for limit groups ([Se], 5.1), after finitely many steps we obtain finitely many maximal abelian test limit groups (still denoted) $MATL_1(x,y,a),\dots,MATL_\ell(x,y,a)$, each of the form $MATL_i(x,y,a)=H_i*F_i$, where F_i is a (possibly trivial) free group, $<y,a>< H_i$ and H_i admits no (non-trivial) free decomposition in which the subgroup $<y,a>$ is contained in a factor. Furthermore, for each sequence of specializations $\{(x(h^j,f),y(j),a)\}$ that factor through a maximal test limit group $MATL_i(x,y,a)$, the specializations h^j of the generators of the factor H_i are the shortest possible among the specializations that satisfy the equalities and inequalities

$$w_1(x(h^j,f),y(j),a)=1,\dots,w_s(x(h^j,f),y(j),a)=1\wedge$$
$$\wedge v_1(x(h^j,f),y(j),a)\neq 1,\dots,v_r(x(h^j,f),y(j),a)\neq 1.$$

At this point we look at a sequence of specializations $\{(x(j),y(j),a)\}$ that converges into a faithful action of a maximal test abelian limit groups $MATL_i(x,y,a)$ $=H_i*F_i$ on a (pointed) real tree (T_∞,t_0). Since the factor H_i is assumed to have no (non-trivial) free decomposition in which the subgroup $<y,a>$ is elliptic, and for every index j the specialization h^j of a (fixed) generating set of the factor H_i was chosen to be shortest possible, the subgroup $< y, a >$ does not fix a point when acting on the limit tree T_∞. Furthermore, by the structure of an abelian test sequence, this implies that the subgroup $< y_1, \dots, y_{n-1}, a >$ fixes the base point $t_0 \in T_\infty$ when acting on the real tree T_∞, y_n acts hyperbolically on the real tree T_∞, and the (free abelian) subgroup $< y_1, \dots, y_{n-1} >$ fixes the segment $[t_0, y_n(t_0)] \subset T_\infty$, where t_0 is the base point of the limit tree T_∞.

Since the stabilizer of the segment $[t_0, y_n(t_0)] \subset T_\infty$ is non-trivial, the segment $[t_0, y_n(t_0)]$ is either contained in an axial component of the tree T_∞ or it is contained in the discrete part of T_∞, and all the edges that are contained in the segment $[t_0, y_n(t_0)]$ have a non-trivial stabilizer that contains the free abelian subgroup $< y_1, \dots, y_{n-1} >$, and these edge stabilizers stabilize (pointwise) the entire axis of y_n in T_∞.

In case the segment $[t_0, y_n(t_0)]$ is contained in an axial component of T_∞, the factor $H_i < MATL_i$ admits the amalgamated product $H_i = V *_{Ab_1} Ab$, where $< y_1, \dots, y_{n-1} >< Ab_1$, and $y_n \in Ab$ but $y_n \notin Ab_1$.

In case the segment $[t_0, y_n(t_0)]$ is contained in the discrete part of T_∞, the graph of groups associated with the action of H_i on the real tree T_∞ contains a circle in which all the edge groups are some abelian subgroup Ab_1, $< y_1, \dots, y_{n-1} >< Ab_1$, and all the edges in the segment $[t_0, y_n(t_0)]$ are in the orbit of edges contained in this circle. Since y_n acts hyperbolically on T_∞, $y_n \notin Ab_1$. In this case, the Bass–Serre generator, bs, associated with the circle in the graph of groups associated with the action of H_i on T_∞ can be chosen to commute with the subgroup Ab_1, $y_n \in< Ab_1, bs >= Ab$, and H_i admits an amalgamated product of the form $H_i = V *_{Ab_1} Ab$.

Therefore, in both cases $H_i = V *_{Ab_1} Ab$, $y_1, \dots, y_{n-1} \in Ab_1$, $y_n \notin Ab_1$, and the abelian group Ab is the direct sum $Ab = Ab_1 + U$ for some abelian subgroup $U < Ab$.

We continue by fixing a generating set for V, $v_1, \dots, v_m$. We further modify the sequences of specializations that factor through the maximal abelian test limit groups $MATL_i(x, y, a)$. Recall that $MATL_i = H_i * F_i$. Let f be a free basis of the free factor F_i (if this factor is non-trivial). Given a sequence $\{(x(j), y(j), a)\}$ that factors through $MATL_i(x, y, a)$, with each specialization $(x(j), y(j), a)$ from the given sequence we associate the corresponding specialization of the generators $v_1, \dots, v_m$, which we denote v^j, and a specialization of the element z_n, denoted z_n^j, which satisfies

$$w_1(x(v^j, z_n^j, f), y(j), a) = 1, \dots, w_s(x(h^j, z_n^j, f), y(j), a) = 1 \wedge$$

$$\wedge v_1(x(h^j, z_n^j, f), y(j), a) \neq 1, \ldots, v_r(x(h^j, z_n^j, f), y(j), a) \neq 1$$

in the free group $F_k * F_i$, and among all such specializations of $v_1, \ldots, v_m$ we choose specializations that are the shortest possible (with respect to the metric on the free group F_k), which we denote v^j.

The collection of all convergent sequences of these modified specializations factor through a finite collection of maximal abelian test limit groups that are quotients of $MATL_i(x, y, a)$. By the descending chain condition for limit groups ([Se], 5.1), after finitely many steps of this "uncovering" procedure, we obtain finitely many maximal abelian test limit groups (still denoted) $MATL_1(x, y, a), \ldots, MATL_\ell(x, y, a)$, each of the form

$$MATL_i(x, y, a) = F_k * Ab_i * F_i$$

where F_i is a (possibly trivial) free group, and $< y_1, \ldots, y_n >$ is a subgroup (of rank n) of the free abelian group Ab_i.

By basic properties of f.g.free abelian groups, since for every index i the subgroup $< y_1, \ldots, y_n >$ is a subgroup of rank n of the abelian group Ab_i, Ab_i can be written as a (possibly trivial) direct sum

$$Ab_i =< z_1^i, \ldots, z_n^i > + < u_1^i, \ldots, u_{s_i}^i >$$

where $< y_1, \ldots, y_n >$ is a finite index subgroup in $< z_1^i, \ldots, z_n^i >$ and the (possibly trivial) subgroup $< u_1^i, \ldots, u_{s_i}^i >$ is a direct summand of rank s.

By the construction of the maximal test abelian groups, $MATL_i(x, y, a)$, the words $w_1(x, y, a), \ldots, w_s(x, y, a)$ represent trivial words in $MATL_i(x, y, a)$, and each of the words $v_1(x, y, a), \ldots, v_r(x, y, a)$ represents a non-trivial element in $MATL_i(x, y, a)$. Hence, if for each index i, we denote the direct summand $< z_1^i, \ldots, z_n^i >$ by rAb_i, then for each i there must exist a retract

$$\eta_i : MATL_i(x, y, a) = F_k * Ab_i * F_i \to F_k * rAb_i * F_i$$

obtained by mapping each of the basis elements $u_1^i, \ldots, u_s^i$ into multiples of the element y_1, for which the elements $\eta_i(v_1(x, y, a)), \ldots, \eta_i(v_r(x, y, a))$ are non-trivial elements in $\eta_i(MTL_i)$.

We continue by replacing each of the maximal abelian test limit groups $MATL_i(x, y, a)$ by its retract $\eta_i(MTL_i(x, y, a)$, and for brevity we (still) denote each of the obtained retracts $MATL_i(x, y, a)$.

Since the maximal abelian test limit groups

$$MATL_1(x, y, a), \ldots, MATL_\ell(x, y, a)$$

were constructed using sequences of elements $\{(x(j), y_1(j), \ldots, y_n(j), a)\}$ for which the inequalities

$$v_1(x(j), y(j), a) \neq 1, \ldots, v_r(x(j), y(j), a) \neq 1$$

are fulfilled, for each i, $1 \leq i \leq \ell$, there exists a specialization of the elements $f_1, \ldots, f_{d_i}$ for which there exists a specialization of the generators of the free abelian group $< z_1^i, \ldots, z_n^i >$ so that all the inequalities

$$v_1(x, y, a) \neq 1, \ldots, v_r(x, y, a) \neq 1$$

are fulfilled. Therefore, setting $Z_n^i = rAb_i$, we have found epimorphisms

$$\tau_i \colon MATL_i(x, y, a) \to F_k * Z_n^i$$

for which

(1) τ_i embeds the subgroup Y_n into a finite index subgroup of Z_n^i,
(2) the words $w_1(x, y, a), \ldots, w_s(x, y, a)$ are trivial in $MATL_i(x, y, a)$ for every i, hence, these words are mapped by τ_i to the trivial element in $F_k * Z_n^i$,
(3) for each i there exists some specialization of z^i so that the inequalities $v_1(x, y, a) \neq 1, \ldots, v_r(x, y, a) \neq 1$ are fulfilled.

The properties of the rank n abelian groups Z_n^i stated above prove part (i) of the proposition. With each monomorphism $\nu_i \colon Y_n \to Z_n^i$ one can naturally associate a Diophantine system Σ_i of n equations in n variables, setting each of the y_j's to be equal to a linear combination of the elements $(z_1^i, \ldots, z_n^i)$ corresponding to $\nu_i(y_j)$, where we view $z_1^i, \ldots, z_n^i$ as variables. With each system Σ_i we associate the set of (integer) tuples $(y_1, \ldots, y_n)$ that are obtained as combinations of tuples of integers $z_1^i, \ldots, z_n^i$, which is a finite index subgroup of the free abelian group of rank n that we denote C_i.

Suppose that the union of the subgroups C_i does not cover the whole free abelian group of rank n, Z^n. Under this last assumption there must exist an element $\hat{y} \in Z^n$ so that $\hat{y} \notin C_i$ for $i = 1, \ldots, \ell$. If for each index i the subgroup C_i is a subgroup of index ind_i of Z^n, then for every element $v \in Z^n$ the element

$$ind_1 \cdot ind_2 \cdot \ldots \cdot ind_\ell \cdot v + \hat{y} \notin C_i$$

for every index i, $1 \leq i \leq \ell$. Therefore, there exists an abelian test sequence of integers $\{(q_0(j), q_1(j), \ldots, q_n(j))\}$ for which there is no convergent subsequence of corresponding elements $\{(x(j), y(j), a)\}$ that factors through any of the maximal abelian test limit groups $MATL_1(x, y, a), \ldots, MATL_\ell(x, y, a)$, which contradicts their universal property. ∎

For completeness, we bring a formulation of Proposition 1.8 for free abelian groups with constants, i.e., when the free abelian group Y_n is assumed to commute with a fixed element $c_1 \in F_k$. The proof is essentially identical to the proof of Proposition 1.8.

PROPOSITION 1.9: *Let $F_k =< a_1, \dots, a_k >$ be a free group, and let $u_1(y) = [c_1, y_1], \dots, u_n(y) = [c_1, y_n]$ for some $n > 1$, $c_1 \in F_k$, $c_1 \neq 1$, and $u_{(i,j)} = [y_i, y_j]$ for $1 \leq i < j \leq n$. W.l.o.g. we may assume that c_1 has no non-trivial roots in F_k. Let the group $CY_{n+1} =< c_1, y | u(y) >$ be the corresponding free abelian group on $n+1$ generators.*

Let $w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1$ be a system of equations over F_k, and let $v_1(x, y, a), \dots, v_r(x, y, a)$ be a collection of words in the alphabet $\{x, y, a\}$. Suppose that the sentence

$$\forall y \quad (u(y) = 1) \quad \exists x \quad w_1(x, y, a) = 1, \dots, w_s(x, y, a) = 1 \wedge v_1(x, y, a) \neq 1, \dots, v_r(x, y, a) \neq 1$$

is a truth sentence.

Then there exist finitely many free abelian groups of rank $n+1$, $CZ^1_{n+1}, \dots, CZ^\ell_{n+1}$, where $CZ^i_{n+1} =< c_1, z^i_1, \dots, z^i_n >$ for $i = 1, \dots, \ell$, together with ℓ monomorphisms ν_i: $CY_{n+1} \to CZ^i_{n+1}$ so that $\nu_i(c_1) = c_1$, and ℓ formal solutions $\{x_i = x_i(z^i, a)\}$ with the following properties:

(i) *Each of the words $w_j(x_i(z^i, a), y, a)$ is the trivial word in the group $CZ^i_{n+1} *_{<c_1>} F_k$, and the sentence*

$$\exists z^i \quad u(z^i) = 1 \wedge v_1(x(z^i, a), y, a) \neq 1, \dots, v_r(x(z^i, a), y, a) \neq 1$$

is a truth sentence in F_k.

(ii) *With each monomorphism $\nu_i : CY_{n+1} \to CZ^i_{n+1}$ one can naturally associate a Diophantine system of equations, setting each of the y_j's to be equal to a linear combination of the elements $(c_1, z^i_1, \dots, z^i_n)$. Since the determinant of this system is non-zero, there exists a (finite index) subgroup $U^i_n < Y_n = < y_1, \dots, y_n >$ and a constant vector $a_i \in Y_n$ for which the solutions of the corresponding Diophantine system are all integers if and only if $y = u + a_i$ where $u \in U^i_n$. Then the union of the cosets $a_1 + U^1_n, \dots, a_\ell + U^\ell_n$ cover the entire abelian group Y_n.*

In the above theorems the sentences considered are defined over a free group (Theorems 1.1 and 1.2), a surface group (Theorem 1.3), and a free abelian group (Propositions 1.8 and 1.9). To state a similar theorem for a general limit group we need to present the *completion* of a limit group associated with a given resolution of it. To define the completion of a limit group associated with a resolution, we first replace the canonical Makanin–Razborov diagram by the canonical *strict Makanin–Razborov diagram.*

Let $Rlim(y,a)$ be a restricted limit group. The (canonical) Makanin–Razborov diagram of $Rlim(y,a)$ gives a canonical collection of (Makanin–Razborov) resolutions, so that any specialization of the restricted limit group $Rlim(y,a)$ factors through (at least) one of the Makanin–Razborov resolutions. Our first step in constructing the *completion* of a restricted limit group is to replace the canonical collection of Makanin–Razborov resolutions associated with a restricted limit group with a canonical collection of strict MR resolutions (strict MR resolutions are defined in ([Se], 5.11)).

Proposition 1.10: *Let $Rlim(y,a)$ be a restricted limit group. There exists a (canonical) collection of strict MR resolutions $Res_1(y,a),\dots,Res_s(y,a)$, so that the limit groups associated with the resolutions $Res_i(y,a)$ are either $Rlim(y,a)$ itself or a quotient of it, for which every specialization of $Rlim(y,a)$ factors through (at least) one of the resolutions $Res_i(y,a)$.*

Proof: To get such a (canonical) collection of strict MR resolutions, we start with the canonical collection of Makanin–Razborov resolutions (the ones that appear in the Makanin–Razborov diagram of the restricted limit group $Rlim(y,a)$). Each resolution in the Makanin–Razborov diagram of $Rlim(y,a)$ which is a strict resolution is taken to be one of the resolutions in our new collection. Each non-strict resolution in the Makanin–Razborov diagram is replaced by a finite collection of strict resolutions either of the limit group $Rlim(y.a)$ or of a quotient of it.

Let $Res(y,a)$ be a resolution in the Makanin–Razborov diagram of $Rlim(y,a)$ that is not a strict MR resolution. Let $\{Rlim_j(y,a)\}$ be the restricted limit groups that appear along the resolution $Res(y,a)$. For each level j, let $ARlim_j(y,a)$ be the restricted limit group obtained from the collection of specializations of $Rlim_j(y,a)$ that factor through (the relevant part of) the resolution $Res(y,a)$. Since $Res(y,a)$ is not a strict MR resolution, at least for some level j, $ARlim_j(y,a)$ is a proper quotient of $Rlim_j(y,a)$. Let j_h be the highest level for which $ARlim_j(y,a)$ is a proper quotient of $Rlim_j(y,a)$. We replace the resolution $Res(y,a)$ by finitely many resolutions obtained in the following way:

(i) The top part of the obtained resolutions is identical with the top part of the resolution $Res(y,a)$ starting at level j_h-1 and above.

(ii) Each of the new resolutions is obtained by starting with the top part of

$Res(y,a)$ and continuing along one of the resolutions that appear in the Makanin–Razborov diagram of the restricted limit group $ARlim_{j_h}(y,a)$.

Clearly, every specialization that factors through the resolution $Res(y,a)$ factors through (at least) one of the obtained resolutions.

We continue the construction of the strict Makanin–Razborov diagram with the obtained resolutions.

Any resolution from the obtained collection which is a strict resolution is taken to be a resolution in our collection of strict resolutions. If an obtained resolution is not strict, we replace it by finitely many resolutions obtained by the same procedure applied before for the resolution $Res(y,a)$. Since a restricted limit group that appears along a resolution obtained by this procedure is a proper quotient of the previous limit group that appears along the same resolution, and since the procedure replaces a restricted limit group along a resolution by its proper quotient, the ascending chain condition for (restricted) limit groups ([Se], 5.1) implies that the procedure described above terminates in a finite time. By construction, every resolution obtained after the termination of the procedure is a strict resolution, and every specialization that factors through the restricted limit group $Rlim(y,a)$ factors through at least one of the collection of strict resolutions we end up with. ∎

We call the canonical collection of strict MR resolutions constructed in Proposition 1.10 the (canonical) *strict Makanin–Razborov diagram* of the restricted limit group $Rlim(y,a)$.

For the purposes of our "trial and error" procedure for quantifier elimination we need to construct *completion* of resolutions which are strict MR resolutions ([Se], 5.11), and are more general than the resolutions that appear in the strict Makanin–Razborov diagram of a limit group. To allow "economical" construction of the completion, we need to restrict the construction of the *completion* to *well-structured* resolutions.

Definition 1.11: Let $F_k =< a_1,\dots,a_k >$ be a free group, let $Rlim(y,a)$ be a restricted limit group defined over F_k, and let $Res(y,a)$ be a strict MR resolution of $Rlim(y,a)$ ([Se], 5.11).

Suppose that the resolution $Res(y,a)$ is given by a decreasing sequence of restricted limit groups:

$$Rlim(y,a) = Rlim_0(y,a) \to Rlim_1(y,a) \to \cdots \to Rlim_i(y,a) \to \cdots \to \\ \cdots \to Rlim_\ell(y,a) =< f,a > *H^\ell$$

where $< f, a >$ is a free group generated by $< f, a >$ and H^ℓ is a free group. Let η_i: $Rlim_i(y,a) \to Rlim_{i+1}(y,a)$ be the canonical quotient maps.

With each restricted limit group $Rlim_i(y,a)$ that appears along the given restricted resolution $Res(y,a)$, there is an associated (restricted) free decomposition

$$Rlim_i(y,a) = R_1^i * \cdots * R_{q(i)}^i * F_{rk(i)}^i * H^i$$

where $F_{rk(i)}^i$ is a (possibly trivial) free group of rank $rk(i)$, H^i is a (possibly trivial) free group, and the coefficient group F_k is contained in one of the factors R_j^i. With each factor R_j^i there is an associated (restricted, possibly trivial) abelian decomposition (graph of groups) and a corresponding restricted modular group.

The associated restricted modular groups of each of the factors R_j^i is generated by the following families of automorphisms of R_j^i (cf.[Se], 8.4):

(i) Dehn twists along edges of the restricted abelian decomposition of R_j^i. If the coefficient group F_k is a subgroup of a factor R_j^i, then the Dehn twists are assumed to fix (elementwise) the vertex stabilized by the coefficient group F_k in the graph of groups associated with the factor R_j^i.

(ii) Dehn twists along essential s.c.c. in QH (quadratically hanging) vertex groups in the restricted abelian decomposition of R_j^i. Again, these Dehn twists are assume to fix (elementwise) the vertex stabilized by the coefficient group F_k if it is contained in a factor R_j^i.

(iii) Let A be an abelian vertex group in the restricted abelian decomposition of R_j^i. Let $A_1 < A$ be the subgroup generated by all edge groups connected to the vertex stabilized by A in the abelian decomposition of R_j^i. Every automorphism of A that fixes A_1 (elementwise) can be naturally extended to an automorphism of the ambient limit group R_j^i, and this automorphism of the factor R_j^i can be assumed to fix the coefficient group F_k if it is contained in the factor R_j^i.

We say that the strict MR resolution $Res(y,a)$ is a *well-structured* resolution if the following conditions hold:

(1) The quotient map η_i: $Rlim_i(y,a) \to Rlim_{i+1}(y,a)$ maps $F_{rk(i)}^i$ monomorphically onto a free factor of $F_{rk(i+1)}^{i+1}$ and H^i onto a free factor of H^{i+1}.

(2) If the factor R_j^i is neither a closed surface group nor a free abelian group, then $\eta_i(R_j^i)$ is a free product of (possibly) some non-free factors in the free decomposition of $Rlim_{i+1}(y,a)$ and (possibly trivial) factors in some free decomposition of the free groups $F_{rk(i+1)}^{i+1}$ and H^{i+1}. In the free decompositions of $Rlim_{i+1}(y,a)$, $F_{rk(i+1)}^{i+1}$ and H^{i+1}, the factors onto which a

non-abelian, non-surface factor R^i_j is mapped are distinct from the factors onto which $R^i_{j'}$ is mapped for $j \neq j'$, and distinct from the factors $\eta_i(H^i)$ and $\eta_i(F^i_{rk(i)})$.

(3) Since we assume that the resolution $Res(y,a)$ is a strict MR resolution, the quotient map η_i maps every non-abelian, non-QH vertex group and every (abelian) edge group in the abelian decompositions associated with a factor R^j_i monomorphically into $Rlim_{i+1}(y,a)$, for all possible tuples (i,j). The image of a QH vertex group under a quotient map is non-abelian.

(4) Let $Ab^i_1, \dots, Ab^i_{q(i)}$ be the (non-cyclic) abelian factors among the factors R^i_j. Then $\eta_i(Ab^i_u) = C^i_u$, C^i_u is a cyclic subgroup of H^{i+1}, and $C^i_1 * \cdots * C^i_{q(i)} * \eta_i(H^i)$ is a free factor in some free decomposition of the free group H^{i+1}.

(5) Let $R^i_{j_1}, \dots, R^i_{j_{d(i)}}$ be the subset of the factors R^i_j that are isomorphic to (non-abelian) closed surface groups. Then the image of such a factor, $\eta_i(R^i_{j_v}) = H^i_v$, H^i_v is a subgroup of H^{i+1}, and

$$H^i_1 * \cdots * H^i_{d(i)} * C^i_1 * \cdots * C^i_{q(i)} * \eta_i(H^i)$$

is a free factor in some free decomposition of the free group H^{i+1}.

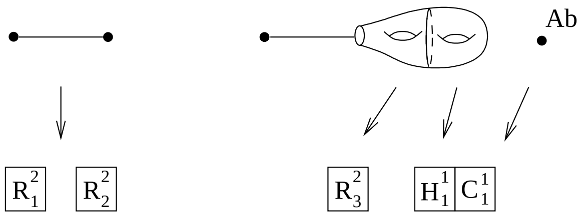
(6) Let $Q^i_1, \dots, Q^i_{s(i)}$ be the QH vertex groups in the abelian decompositions of the factors R^i_j that are non-abelian and not isomorphic to a closed surface group. Recall that $\eta_i(Ab^i_u) = C^i_u$. For each QH subgroup Q^i_t let $\Lambda_{Q^i_t}$ be the cyclic decomposition obtained from the abelian decomposition associated with the factor R^i_j containing the QH subgroup Q^i_t, by collapsing all the edges that are not connected to the QH subgroup Q^i_t. We say that the *circumference* of the QH subgroup Q^i_t, denoted $Circum(Q^i_t)$, is the subgroup generated by Q^i_t and all the Bass–Serre generators connecting (some of) its boundary components to vertex groups in $\Lambda_{Q^i_t}$.

For each QH subgroup Q^i_t let $bd_{t,1}, \dots, bd_{t,c(t)}$ be its boundary components. Let $\eta_i(Circum(Q^i_t)) = V_1 * \cdots * V_{m(t)} * H^i_t$ be the maximal (most refined) free decomposition inherited by $\eta(Circum(Q^i_t))$ from the given free decomposition of $Rlim_i(y,a)$ in which each of the images of the boundary elements $\eta_i(bd_{t,n})$ can be conjugated into one of the free factors V_r, and for each factor V_r there exists at least one image of a boundary element $\eta_i(bd_{t,n})$ that can be conjugated into it. Then H^i_t is a free group and

$$H^{i+1} = H^i_1 * \cdots * H^i_{s(i)} * C^i_1 * \cdots * C^i_{q(i)} * \eta_i(H^i)$$

and

$$V_1 * \cdots * V_m(t) < R^{i+1}_1 * \cdots * R^{i+1}_{q(i+1)} * F^{i+1}_{rk(i+1)}.$$



Note that, by definition, every well-structured resolution is a strict resolution but, in general, a strict resolution is not necessarily well-structured. However, since the (canonical) Makanin–Razborov diagram is constructed from an descending chain of maximal shortening quotients, every resolution in the strict Makanin–Razborov diagram, presented in Proposition 1.10, is well-structured. Hence, the strict Makanin–Razborov diagram allows us to restrict the construction of the *completion* to well-structured resolutions.

Definition 1.12: Let $Res(y,a)$ be a well-structured resolution of a limit group $Rlim(y,a)$. We construct the *completion* of the resolution $Res(y,a)$, denoted $Comp(Res)(z,y,a)$, iteratively from bottom to top. Keeping the notation of Definition 1.11, suppose that the resolution $Res(y,a)$ is given by a decreasing sequence of restricted limit groups

$$Rlim(y,a) = Rlim_0(y,a) \to Rlim_1(y,a) \to \cdots \to Rlim_i(y,a) \to \cdots \to$$
$$\cdots \to Rlim_\ell(y,a) =< f,a > *H^\ell$$

where $< f,a >$ is a free group generated by $< f,a >$, and H^ℓ is a free group. Let η_i: $Rlim_i(y,a) \to Rlim_{i+1}(y,a)$ be the canonical quotient maps. $Res(y,a)$ is a well-structured resolution, so with each restricted limit group $Rlim_i(y,a)$ that appears along the given restricted resolution $Res(y,a)$ there is an associated (restricted) free decomposition

$$Rlim_i(y,a) = R_1^i * \cdots * R_{q(i)}^i * F_{rk(i)}^i * H^i$$

where $F_{rk(i)}^i$ is a (possibly trivial) free group of rank $rk(i)$, H^i is a (possibly trivial) free group, and the coefficient group F_k is contained in one of the factors R_j^i. With each factor R_j^i there is an associated (restricted, possibly trivial) abelian decomposition (graph of groups) and a corresponding restricted modular group.

For presentation purposes we start by describing the construction of the completion, assuming the resolution $Res(y,a)$ is a minimal rank well-structured resolution, i.e., assuming that the terminal free group $Rlim_\ell = F_{rk(\ell)}^\ell * H^\ell$ is the coefficient group F_k (i.e., that $F^{\ell}_{rk(\ell)} = F_k$ and H^{ℓ} is trivial), and then generalize the construction for arbitrary well-structured resolutions.

Suppose that $Res(y,a)$ is a minimal rank well-structured resolution. Since $Res(y,a)$ is of minimal rank, the free products associated with the various levels are trivial, i.e., $Rlim_i(y,a) = R^i_1$ for $0 \leq i \leq \ell - 1$, none of these limit groups are abelian or a closed surface group, and with $Rlim_i(y,a)$ there is an associated graph of groups with (non-trivial) abelian edge groups.

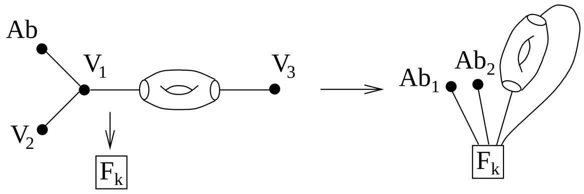
We construct the completion of $Res(y,a)$ iteratively from bottom to top. We start by changing the $\ell-1$-th limit group, $Rlim_{\ell-1}(y,a)$, to a limit group $Comp(Rlim)_{\ell-1}$ by modifying the abelian decomposition associated with $Rlim_{\ell-1}$.

Let $\Lambda^{\ell-1}$ be the abelian decomposition associated with the $\ell-1$-th limit group $Rlim_{\ell-1}(y,a)$. To modify the limit group $Rlim_{\ell-1}(y,a)$ and its abelian decomposition, we start with the terminal (coefficient) free group $Rlim_{\ell}(y,a) = F_k$. With each edge e in the graph of groups $\Lambda^{\ell-1}$ that connects two non-abelian, non-QH vertex groups we associate a generator g_e. Let G_e be the edge group associated with such an edge in $\Lambda^{\ell-1}$. We set the group G_1 to be the group generated by the terminal group $Rlim_{\ell}(y,a)$ together with the additional elements $\{g_e\}$. To define G_1 we add relations that force each of the additional generators g_e to commute with the maximal cyclic subgroup in F_k that contains the image of G_e under $\eta_{\ell-1}$, $\eta_{\ell-1}(G_e)$. For any pair of edges e_1, e_2, for which $\eta_{\ell-1}(G_{e_1})$ commutes with $\eta_{\ell-1}(G_{e_2})$, we further add the relation $[g_{e_1}, g_{e_2}] = 1$. Hence, G_1 is obtained from $Rlim_{\ell} = F_k$ by adding some free abelian groups amalgamated along (maximal) cyclic edge groups.

We continue by modifying G_1 by adding new generators associated with (non-cyclic) abelian vertex groups in $\Lambda^{\ell-1}$.

With each non-cyclic abelian vertex group Ab_n of rank n in the cyclic decomposition $\Lambda^{\ell-1}$ that is connected to some non-abelian vertex group by an edge with (maximal) cyclic stabilizer G_{Ab_n}, we associate new generators $ab_1, \ldots, ab_n$. We modify G_1 by adding all the new generators to it, together with relations that force ab_1 to be equal to the corresponding generator of $\eta_{\ell-1}(Ab_n) < F_k < G_1$, and commutant relations that force the other new generators to commute with the centralizer in G_1 of the image of Ab_n in F_k. Therefore, we end this second modification with a group G_2 that admits a canonical graph of groups composed of one vertex stabilized by the coefficient group F_k and (possibly) several additional vertices, each stabilized by a non-cyclic free abelian group and connected to the vertex stabilized by F_k with a single edge with (maximal) cyclic stabilizer.

We continue by adding QH vertex groups corresponding to the QH vertex groups in $\Lambda^{\ell-1}$ to the group G_2. Let $Q_1,\dots,Q_u$ be the QH vertex groups in $\Lambda^{\ell-1}$. Let $Q'_1,\dots,Q'_u$ be QH vertex group isomorphic to $Q_1,\dots,Q_u$ in correspondence. With G_2 and the QH subgroups $Q'_1,\dots,Q'_u$ we associate a graph of groups, starting with the graph of groups associated with the group G_2, and adding new vertices for each of the vertex groups $Q'_1,\dots,Q'_u$. For each boundary component bd' of a QH vertex group Q'_j, we add an edge connecting the vertex stabilized by Q'_j to the vertex stabilized by $Rlim_\ell(y,a)=F_k$, and identifying the cyclic subgroup generating by bd' to the image in F_k of the corresponding boundary component bd of the QH vertex group Q_j in $\Lambda^{\ell-1}$, $\eta_{\ell-1}(bd)$.



Performing all these operations, we end up with one graph of groups with cyclic edge stabilizers, one vertex stabilized by $Rlim_\ell(y,a)=F_k$, (possibly) a few vertices with free abelian vertex groups connected to the (distinguished) vertex stabilized by F_k with an edge with (maximal) cyclic stabilizer, and (possibly) a few vertices with QH vertex groups, whose boundary components are all connected to the vertex stabilized by F_k. We call the fundamental group of this graph of groups the *completion* of $Rlim_{\ell-1}(y,a)$, and denote it $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$. We call the graph of groups the *completed decomposition* of $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$, and with it we naturally associate the *completed modular group* (inherited from the modular group of $Rlim_{\ell-1}(y,a)$).

LEMMA 1.13: *The $\ell-1$-th limit group $Rlim_{\ell-1}(y,a)$ (canonically) embeds into the completed limit group $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$:*

$$\nu_{\ell-1}\colon Rlim_{\ell-1}(y,a)\to Comp(Rlim)_{\ell-1}(z_{\ell-1},a).$$

Furthermore, the quotient map $\eta_{\ell-1}\colon Rlim_{\ell-1}(y,a)\to Rlim_\ell(y,a)$ naturally extends to a quotient map:

$$Comp(\eta)_{\ell-1}\colon Comp(Rlim)_{\ell-1}(z_{\ell-1},a)\to Rlim_\ell(y,a)=F_k.$$

Proof: The extension of the quotient map $\eta_{\ell-1}\colon Rlim_{\ell-1}(y,a)\to Rlim_\ell(y,a)$ to a quotient map

$$Comp(\eta)_{\ell-1}\colon Comp(Rlim)_{\ell-1}(z_{\ell-1},a)\to Rlim_\ell(y,a)=F_k$$

is natural from the construction of the completion. To construct the natural embedding of $Rlim_{\ell-1}$ into the completion, $Comp(Rlim)_{\ell-1}$, we look at the following example.

Let $Rlim_\ell = F_k$, and let the abelian decomposition of $Rlim_{\ell-1}$ be $Rlim_{\ell-1} = A *_C B$, where C is cyclic, and A and B are some free groups. In this case the quotient map $\eta_{\ell-1}$ maps A, B and C isomorphically into F_k. Let $< c_1 >< F_k$ be the maximal cyclic subgroup in F_k that contains $\eta_{\ell-1}(C)$. In this case the completion has the form

$$Comp(Rlim)_{\ell-1} = F_k *_{<c_1>} < c_1, t >$$

and the natural embedding $\nu_{\ell-1}$ maps $Rlim_{\ell-1}$ isomorphically onto the subgroup

$$\eta_{\ell-1}(A) *_{<c_1>} t\eta_{\ell-1}(B)t^{-1}$$

of the completion, $Comp(Rlim)_{\ell-1}$. The generalization of the natural embedding $\nu_{\ell-1}$ to an arbitrary $\ell-1$ limit group is rather straightforward. ∎

Although the image of $Rlim_{\ell-1}(y,a)$ in the completion

$$Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$$

under the canonical embedding $\nu_{\ell-1}$ can be expressed as words in the generators $(z_{\ell-1},a)$, we prefer to specifically note this image by changing the notation of the completion to $Comp(Rlim)_{\ell-1}(z_{\ell-1},y,a)$.

We continue the construction of the completed resolution iteratively (bottom to top). At each step, i, we start with the completed limit group constructed at the lower level, $Comp(Rlim)_{i+1}(z_{i+1},y,a)$, and its associated *completed* abelian decomposition, and construct $Comp(Rlim)_i(z_i,y,a)$, its associated *completed decomposition* and *completed modular groups*, the natural embedding

$$\nu_i\colon Rlim_i(y,a) \to Comp(Rlim)_i(z_i,y,a)$$

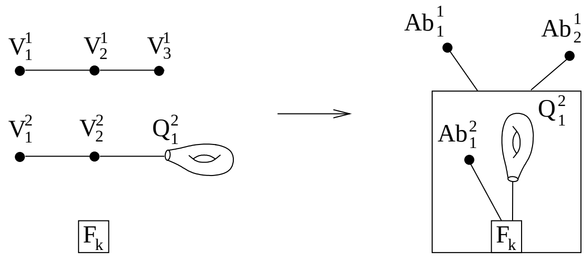
and the completed quotient map $Comp(\eta)_i\colon Comp(Rlim)_i \to Comp(Rlim)_{i+1}$.

We start the construction of the completed limit group $Comp(Rlim)_i(z_i,y,a)$ with the completed limit group $Comp(Rlim)_{i+1}(z_{i+1},y,a)$. Let Λ^i be the abelian decomposition associated with the i-th limit group $Rlim_i(y,a)$ in the well-structured resolution $Res(y,a)$. With each edge with abelian stabilizer that connects two non-abelian, non-QH vertex groups in the abelian decomposition Λ_i we associate a new generator g_e. We set G_1 to be a group generated by $Comp(Rlim)_{i+1}(z_{i+1},y,a)$ and all the new generators $\{g_e\}$ corresponding to these edges in Λ^i. To define G_1 we further add commutant relations that force each of the new variables g_e to commute with the centralizer of the image under $\nu_{i+1} \circ \eta_i$ of the abelian stabilizer of the edge it is associated with in $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$, and with other generators $g_{e'}$ for which the image of the edge group associated with them in $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$ can be conjugated to commute with the image of the edge group associated with g_e in $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$. Hence, G_1 is the fundamental group of a graph of groups having one (distinguished) vertex stabilized by $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$, and (possibly) a few other vertices with free abelian stabilizers, each connected to the distinguished vertex with a unique edge, and the stabilizer of that edge is the centralizer of the image of the corresponding edge group in Λ^i in $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$.

We continue by modifying G_1 by adding new generators associated with (non-cyclic) abelian vertex groups in Λ^i. With each non-cyclic abelian vertex group Ab_n of rank n in the cyclic decomposition Λ^i, we associate new generators $ab_1, \dots, ab_n$. We modify G_1 by adding all the new generators to it, together with relations that force the subgroup of Ab_n generated by the edge groups connected to Ab_n in Λ^i to be equal to its image under $\nu_{i+1} \circ \eta_i$ in $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$, and relations that force $ab_1, \dots, ab_n$ to commute and to commute with the centralizer of the image of the subgroup of Ab_n generated by the edge groups connected to Ab_n in Λ^i in G_1. Therefore, we end this second modification with a group G_2 that admits a canonical graph of groups composed of one vertex stabilized by $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$, and (possibly) several additional vertices, each stabilized by a non-cyclic free abelian group and connected to the vertex stabilized by $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$ by a single edge with abelian stabilizer.

We continue by adding QH vertex groups corresponding to the QH vertex groups in Λ^i to the group G_2. Let $Q_1, \dots, Q_u$ be the QH vertex groups in Λ^i. Let $Q'_1, \dots, Q'_u$ be the QH vertex group isomorphic to $Q_1, \dots, Q_u$ in correspondence. With G_2 and the QH subgroups $Q'_1, \dots, Q'_u$ we associate a graph of groups, starting with the graph of groups associated with the group G_2, and adding new vertices for each of the vertex groups $Q'_1, \dots, Q'_u$. For each boundary component bd' of a QH vertex group Q'_j, we add an edge connecting the vertex stabilized by Q'_j to the vertex stabilized by $Comp(Rlim)_{i+1}(z_{i+1}y, a)$, and identifying the cyclic subgroup generated by bd' to the image in $Comp(Rlim)_{i+1}(z_{i+1}, y, a)$ under $\nu_{i+1} \circ \eta_i$, of the corresponding boundary component bd of the QH vertex group

Q_j in Λ^i.



Performing all these operations, we end up with one graph of groups with abelian edge stabilizers, one (distinguished) vertex stabilized by $Comp(Rlim)_{i+1}(z_{i+1}y,a)$, (possibly) a few vertices with free abelian vertex groups connected to the (distinguished) vertex with an edge with (maximal) cyclic stabilizer, and (possibly) a few vertices with QH vertex groups whose boundary components are all connected to the distinguished vertex. We call the fundamental group of this graph of groups, the *completion* of $Rlim_i(y,a)$, and denote it $Comp(Rlim)_i(z_i,a)$. We call the graph of groups, the *completed decomposition* of $Comp(Rlim)_i(z_i,a)$, and with it we naturally associate the *completed modular group* (inherited from the modular group of $Rlim_i(y,a)$). In a similar way to Lemma 1.13, with the completion of $Rlim_i(y,a)$ we associate two natural maps, a canonical embedding

$$\nu_i\colon Rlim_i(y,a) \to Comp(Rlim)_i(z_i,a)$$

and a quotient map

$$Comp(\eta)_i\colon Comp(Rlim)_i(z_i,a) \to Comp(Rlim)_{i+1}(z_{i+1}y,a).$$

Although the image of $Rlim_i(y,a)$ in the completion $Comp(Rlim)_i(z_i,a)$ under the canonical embedding ν_i can be expressed as words in the generators (z_i,a), we prefer to specifically note this image by changing the notation of the completion to $Comp(Rlim)_i(z_i,y,a)$.

Finally, we say that $Comp(Rlim)_0(z_0,y,a)$ is the *completed limit group* of the minimal rank well-structured resolution $Res(y,a)$, and the sequence of completed limit groups $Comp(Rlim)_i(z_i,y,a)$, $1 \leq i \leq \ell$, together with their associated *completed* decompositions is the *completed resolution* of $Res(y,a)$, which we denote $Comp(Res)(z_0,y,a)$.

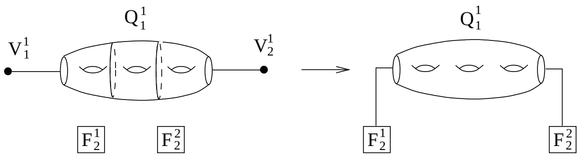
So far we have presented the construction of the completion for minimal rank well-structured resolutions $Res(y,a)$. At this point we generalize the construction to an arbitrary well-structured resolution.

The terminal limit group of the well-structured resolution $Res(y,a)$ is $Rlim_\ell(y,a) = F^\ell_{rk(\ell)} * H^\ell$. We start the construction of the completion of $Res(y,a)$ by setting $Comp(Rlim)_\ell(z_\ell,a)$ to be $Comp(Rlim)_\ell(z_\ell,a) = F^\ell_{rk(\ell)}$. We set G_1 to be the free product of $Comp(Rlim)_\ell(z_\ell,a)$ with those factors among the factors $R^{\ell-1}_j$ in the given free decomposition of $Rlim_{\ell-1}(y,a)$ that are isomorphic to either non-abelian closed surface groups $S^{ell-1}_1,\dots,S^{\ell-1}_{d(\ell-1)}$ or non-cyclic free abelian groups $Ab^{\ell-1}_1,\dots,AB^{\ell-1}_{q(\ell-1)}$.

We continue the construction of the completion by considering the abelian decompositions associated with the various factors $R^{\ell-1}_j$ that are neither abelian nor closed surface groups. Let $\Lambda^{\ell-1}_j$ be the abelian decomposition associated with such a factor $R^{\ell-1}_j$. With each edge e in one of the graph of groups $\Lambda^{\ell-1}_j$ that connects two non-abelian, non-QH vertex groups we associate a generator g_e. Let G_e be the edge group associated with such an edge in $\Lambda^{\ell-1}_j$. We set the group G_2 to be the group generated by G_1 together with the additional elements $\{g_e\}$. To define G_2 we add relations which force each of the additional generators g_e to commute with the maximal cyclic subgroup in $Comp(Rlim)_\ell(z_\ell,a)$ that contain the image of G_e under $\eta_{\ell-1}$, $\eta_{\ell-1}(G_e)$. For any pair of edges e_1,e_2, for which $\eta_{\ell-1}(G_{e_1})$ commutes with $\eta_{\ell-1}(G_{e_2})$, we further add the relation $[g_{e_1},g_{e_2}]=1$. Hence, G_2 is obtained from G_1 by adding some free abelian groups amalgamated along (maximal) cyclic edge groups.

We continue by modifying G_2 by adding new generators associated with (non-cyclic) abelian vertex groups in the decompositions $\Lambda^{\ell-1}_j$. With each non-cyclic abelian vertex group Ab_n of rank n in a cyclic decomposition $\Lambda^{\ell-1}_j$ that is connected to some non-abelian vertex group by an edge with (maximal) cyclic stabilizer G_{Ab_n}, we associate new generators $ab_1,\dots,ab_n$. We modify G_2 by adding all the new generators to it, together with relations that force ab_1 to be equal to the corresponding generator of $\eta_{\ell-1}(Ab_n) < Comp(Rlim)_\ell(z_\ell,a) < G_2$, and commutant relations that force the other new generators to commute with the centralizer in G_2 of the image of Ab_n in $Comp(Rlim)_\ell(z_\ell,a)$. Therefore, we end this second modification with a group G_3 that admits a canonical free decomposition in which, with a single factor, we associate a graph of groups composed of one vertex stabilized by a factor of $Comp(Rlim)_\ell(z_\ell,a)$, and (possibly) several additional vertices, each stabilized by a non-cyclic free abelian group and connected to the vertex stabilized by the factor of $Comp(Rlim)_\ell(z_\ell,a)$ by a single edge with (maximal) cyclic stabilizer.

We continue by adding QH vertex groups corresponding to the QH vertex groups in the decompositions $\Lambda^{\ell-1}_j$ to the group G_3. Let $U_1*\cdots*U_b$ be the most refined free decomposition of G_3 in which the images under $\eta_{\ell-1}$ of all the edge groups and all the non-QH vertex groups in the abelian decompositions $\Lambda_j^{\ell-1}$ are elliptic. Let $Q_1,\dots,Q_u$ be the QH vertex groups in the various graph of groups $\Lambda_j^{\ell-1}$. Let $Q'_1,\dots,Q'_u$ be the QH vertex group isomorphic to $Q_1,\dots,Q_u$ in correspondence. With G_3 and the QH subgroups $Q'_1,\dots,Q'_u$ we associate a graph of groups, starting with the free decomposition $U_1 * \cdots * U_b$ of G_3, and adding new vertices for each of the vertex groups $Q'_1,\dots,Q'_u$. For each boundary component bd' of a QH vertex group Q'_t, we add an edge connecting the vertex stabilized by Q'_t to the vertex stabilized by the vertex stabilized by the factor U_s into which the corresponding boundary component bd of Q_t is mapped by $\eta_{\ell-1}$. We further identify the cyclic subgroup generated by bd' with the image in U_b of the corresponding boundary component bd under η_i.



We call the fundamental group of this graph of groups, the *completion* of $Rlim_{\ell-1}(y,a)$, and denote it $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$. We call the graph of groups, the *completed decomposition* of $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$, and with it we naturally associate the *completed modular group* (inherited from the modular group of $Rlim_{\ell-1}(y,a)$). In a similar way to Lemma 1.13, the $\ell-1$-th limit group $Rlim_{\ell-1}(y,a)$ (canonically) embeds into the completed limit group $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$:

$$\nu_{\ell-1}\colon Rlim_{\ell-1}(y,a) \to Comp(Rlim)_{\ell-1}(z_{\ell-1},a).$$

Furthermore, the quotient map $\eta_{\ell-1}\colon\ Rlim_{\ell-1}(y,a) \to Rlim_{\ell}(y,a)$ naturally extends to a quotient map:

$$Comp(\eta)_{\ell-1}\colon Comp(Rlim)_{\ell-1}(z_{\ell-1},a) \to Rlim_{\ell}(y,a) = F_k.$$

Also, we prefer to note the image of $Rlim_{\ell-1}(y,a)$ in the completion $Comp(Rlim)_{\ell-1}(z_{\ell-1},a)$ under the canonical embedding $\nu_{\ell-1}$, so we change the notation of the completion to $Comp(Rlim)_{\ell-1}(z_{\ell-1},y,a)$.

As in the minimal rank case, we continue the construction of the completed resolution iteratively (bottom to top). At each step, i, we start with the completed limit group constructed at the lower level, $Comp(Rlim)_{i+1}(z_{i+1},y,a)$, and its associated *completed* abelian decomposition, and construct $Comp(Rlim)_i(z_i,y,a)$, its associated *completed decomposition* and *completed modular groups*, the natural embedding $\nu_i\colon Rlim_i(y,a) \to Comp(Rlim)_i(z_i,y,a)$, and the completed quotient map $Comp(\eta)_i\colon Comp(Rlim)_i \to Comp(Rlim)_{i+1}$.

We start the construction of the completed limit group $Comp(Rlim)_i(z_i,y,a)$ with the completed limit group $Comp(Rlim)_{i+1}(z_{i+1},y,a)$. We set G_1 to be the free product of $Comp(Rlim)_{i+1}(z_{i+1},a)$ with those factors among the factors R^i_j in the given free decomposition of $Rlim_i(y,a)$ that are isomorphic to either non-abelian closed surface groups $S^i_1,\dots,S^i_{d(i)}$, or to abelian groups $Ab^i_1,\dots,Ab^i_{q(i)}$.

We continue the construction of the completion by considering the abelian decompositions associated with the various factors R^i_j that are neither abelian nor closed surface groups. Let Λ^i_j be the abelian decomposition associated with such a factor R^i_j. With each edge e in one of the graph of groups Λ^i_j that connects two non-abelian, non-QH vertex groups we associate a generator g_e. Let G_e be the edge group associated with such an edge in Λ^i_j. We set the group G_2 to be the group generated by G_1 together with the additional elements $\{g_e\}$. To define G_2 we add relations that force each of the additional generators g_e to commute with the centralizer of the image (under $\nu_{i+1}\circ\eta_i$) of the associated edge group G_e in $Comp(Rlim)_{+1i}(z_{i+1},y,a)$, $\nu_{i+1}\circ\eta_i(G_e)$. For any pair of edges e_1,e_2, for which $\nu_{i+1}\circ\eta_i(G_{e_1})$ (can be conjugated to) commute with $\nu_{1+1}\circ\eta_i(G_{e_2})$, we further add the relation $[g_{e_1},g_{e_2}]=1$. Hence, G_2 is obtained from G_1 by adding some free abelian groups amalgamated along (non-trivial) subgroups of smaller rank.

We continue by modifying G_2 by adding new generators associated with (non-cyclic) abelian vertex groups in the various abelian decompositions Λ^i_j. With each non-cyclic abelian vertex group Ab_n of rank n in one of the abelian decompositions Λ^{i_j} that is connected to some non-abelian vertex group, we associate new generators $ab_1,\dots,ab_n$. We modify G_2 by adding all the new generators to it, together with relations that force the subgroup of Ab_n generated by the edge groups connected to Ab_n in Λ^i_j to be equal to its image under $\nu_{i+1}\circ\eta_i$ in $Comp(Rlim)_{i+1}(z_{i+1},y,a)$, and relations that force $ab_1,\dots,ab_n$ to commute and to commute with the centralizer of the image of the subgroup of Ab_n generated by the edge groups connected to Ab_n in Λ^i_j in G_2. Therefore, we end this part with a group G_3 that admits a canonical free decomposition in which with a single factor we associate a graph of groups composed of one vertex stabilized by a factor of $Comp(Rlim)_{i+1}(z_{i+1},y,a)$, and (possibly) several additional vertices, each stabilized by a non-cyclic free abelian group and connected to the vertex stabilized by the factor of $Comp(Rlim)_{i+1}(z_{i+1},y,a)$ by a single edge with (non-trivial) abelian stabilizer.

We continue by adding QH vertex groups corresponding to the QH vertex groups in the decompositions Λ_j^i to the group G_3. Let $U_1 * \cdots * U_b$ be the most refined free decomposition of G_3 in which the images under $\nu_{i+1} \circ \eta_i$ of all the edge groups and all the non-QH vertex groups in the abelian decompositions Λ_j^i are elliptic. Let $Q_1, \ldots, Q_u$ be the QH vertex groups in the various graph of groups Λ_j^i. Let $Q'_1, \ldots, Q'_u$ be the QH vertex group isomorphic to $Q_1, \ldots, Q_u$ in correspondence. With G_3 and the QH subgroups $Q'_1, \ldots, Q'_u$ we associate a graph of groups, starting with the free decomposition $U_1 * \cdots * U_b$ of G_3, and adding new vertices for each of the vertex groups $Q'_1, \ldots, Q'_u$. For each boundary component bd' of a QH vertex group Q'_t, we add an edge connecting the vertex stabilized by Q'_t to the vertex stabilized by the vertex stabilized by the factor U_s into which the corresponding boundary component bd of Q_t is mapped by $\nu_{i+1} \circ \eta_i$. We further identify the cyclic subgroup generated by bd' with the image in U_b of the corresponding boundary component bd under $\nu_{i+1} \circ \eta_i$.

We call the fundamental group of this graph of groups, the *completion* of $Rlim_i(y,a)$, and denote it $Comp(Rlim)_i(z_i,a)$. We call the graph of groups, the *completed decomposition* of $Comp(Rlim)_i(z_i,a)$, and with it we naturally associate the *completed modular group* (inherited from the modular group of $Rlim_i(y,a)$). In a similar way to Lemma 1.13, the i-th limit group $Rlim_i(y,a)$ (canonically) embeds into the completed limit group $Comp(Rlim)_i(z_i,a)$:

$$\nu_i \colon Rlim_i(y,a) \to Comp(Rlim)_i(z_i,a).$$

Furthermore, the quotient map $\eta_i\colon\ Rlim_i(y,a) \to Rlim_{i+1}(y,a)$ naturally extends to a quotient map

$$Comp(\eta)_i \colon Comp(Rlim)_i(z_i,a) \to Comp(Rlim)_{i+1}(z_{i+1},y,a).$$

Also, we prefer to note the image of $Rlim_i(y,a)$ in the completion $Comp(Rlim)_i(z_i,a)$ under the canonical embedding ν_i, so we change the notation of the completion to $Comp(Rlim)_i(z_i,y,a)$.

Finally, we say that $Comp(Rlim)_0(z_0,y,a)$ is the *completed limit group* of the minimal rank well-structured resolution $Res(y,a)$, and the sequence of completed limit groups $Comp(Rlim)_i(z_i,y,a)$, $1 \le i \le \ell$, together with their associated *completed* decompositions is the *completed resolution* of $Res(y,a)$, which we denote $Comp(Res)(z_0,y,a)$.

The following are some basic properties of the completion of a well-structured resolution. All follow in a rather straightforward way from the construction of the completion, so we omit their proof. Still, they are crucial in obtaining formal solutions defined over the completion of a well-structured resolution of a restricted limit group.

LEMMA 1.14: *Let $Res(y,a)$ be a well-structured resolution of a restricted limit group $Rlim(y,a)$, and let $Comp(Res)(z,y,a)$ be its completed resolution. Then:*

(i) *If we replace each of the completed limit groups $Comp(Rlim)_i(z,y,a)$ with the group $Comp(Rlim)_i(z,y,a) * H^i$, and change the completed quotient map $Comp(\eta)_i$ accordingly (i.e., $Comp(\eta)_i$ will map the additional free factor H^i isomorphically onto a factor of H^{i+1}, and the additional surface and abelian factors onto factors of H^{i+1}), then the completed resolution $Comp(Res)(z,y,a)$ is a well-structured resolution.*

(ii) *A generator of a cyclic edge group connecting two non-QH vertex groups in the completed decomposition of one of the factors of the completed limit group $Comp(Rlim)_i(z,y,a)$ for some i is projected by some composition of maps*

$$Comp(\eta)_i \circ \cdots \circ Comp(\eta)_{i'} \colon Comp(Rlim)_i(z,y,a) \to Comp(Rlim)_{i'-1}(z,y,a)$$

to either an element with no root that is not contained in one of the factors of $Comp(Rlim)_{i'}(z,y,a)$, a hyperbolic element with no root in the completed decomposition of $Comp(Rlim)_{i'}(z,y,a)$, or to a non-boundary element with no root in a QH vertex in a completed decomposition.

(iii) *The subgroup $<y,a>\,<\,Comp(Rlim)(z,y,a)$ is mapped onto $Comp(Rlim)_\ell = F^\ell_{rk(\ell)}$ by the composition of the quotient maps. The rank of the completed resolution, $Comp(Res(z,y,a))$, is at most the rank of the well-structured resolution $Res(y,a)$.*

(iv) *For any specialization (y_0,a) of the restricted limit group $Rlim(y,a)$ that factors through the resolution $Res(y,a)$, there exists some specialization z_0 so that (z_0,y_0,a) is a specialization of the completed limit group $Comp(Rlim)(z,y,a)$ that factors through the completed resolution $Comp(Res)(z,y,a)$.*

Completed resolutions are defined in order to enable one to construct *formal solutions*. If all the decompositions associated with the various levels of a well-structured resolution contain no (non-cyclic) abelian vertex groups, then it is indeed possible to construct formal solutions defined over the completed resolution. However, in the presence of (non-cyclic) abelian vertex groups we still need to define a *closure* of a resolution, and a finite collection of such closures which forms a *covering closure*.

Definition 1.15: Let $Res(y,a)$ be a well-structured resolution and let $Comp(Res)(z,y,a)$ be its completion.

Let $Ab_1,\dots,Ab_d$ be the non-conjugate, non-cyclic, maximal abelian subgroups that appear along the completion, $Comp(Res)(z,y,a)$, and are mapped onto a non-cyclic abelian factor in a free decomposition associated with one of the levels of the completion.

Let $PAb_1,\dots,PAb_{pd}$ be the non-conjugate, non-cyclic, maximal *pegged* abelian groups that appear along the completed resolution, i.e., maximal non-cyclic abelian subgroups in $Comp(Rlim)(z,y,a)$, that are mapped onto a non-cyclic abelian vertex group in some abelian decomposition associated with some level of the completed resolution $Comp(Res)(z,y,a)$, and this abelian vertex is connected to the other vertices of the completed decomposition of that level by an edge with (maximal) cyclic stabilizer. We call the maximal cyclic subgroup of a pegged abelian group connecting it to the other vertices of the corresponding completed decomposition, the *peg* of the *pegged* abelian group PAb.

Let $S_1,\dots,S_d$ be free abelian groups so that $Ab_1 < S_1,\dots,Ab_d < S_d$ are subgroups of finite index. Let $PS_1,\dots,PS_{pd}$ be free abelian groups so that $PAb_1 < PS_1,\dots,PAb_{pd} < PS_{pd}$ are subgroups of finite index, and the pegs $peg_1,\dots,peg_{pd}$ are primitive elements in the ambient free abelian groups $PS_1,\dots,PS_{pd}$.

A *closure* of the completed resolution $Comp(Res)(z,y,a)$ is obtained by replacing the free abelian groups $Ab_1,\dots,Ab_d$ by the free abelian groups $S_1,\dots,S_d$, and the pegged abelian groups $PAb_1,\dots,PAb_{pd}$ by the free abelian groups $PS_1,\dots,PS_{pd}$ in correspondence, along the entire completed resolution, i.e., from the top level through the bottom level in which a subgroup of the pegged abelian group appears along the completed resolution. We say that the free abelian groups $S_1,\dots,S_d$ and $PS_1,\dots,PS_{pd}$ are the *extension* of the pegged abelian groups $PAb_1,\dots,PAb_{pd}$ in correspondence. We denote a closure of the completed resolution by $Cl(Res)(s,z,y,a)$, and the corresponding limit group by $Cl(Rlim)(s,z,y,a)$. Naturally, $Comp(Rlim)(z,y,a)$ is embedded in $Cl(Rlim)(s,z,y,a)$.

By construction, properties (i)–(iii) of Lemma 1.14 which are valid for the completion remain valid for a closure. However, in general only a subset of the specializations that factor through a resolution can be extended to specializations that factor through a closure of it. Therefore, we need to generalize what we did in Propositions 1.8 and 1.9 for free abelian groups and define a *covering closure*.

Definition 1.16: We will keep the notation of Definition 1.15. Let $Res(y,a)$ be a well-structured resolution and let $Comp(Res)(z,y,a)$ be its completion. With each closure, $Cl(Res)(s,z,y,a)$, there are associated embeddings of abelian and pegged abelian groups into their extensions:

$$Ab_1 < S_1, \dots, Ab_d < S_d, PAb_1 < PS_1, \dots, PAb_{pd} < PS_{pd}.$$

Like in Proposition 1.8, with each embedding $Ab_j < S_j$ we can associate a subgroup C_j of the free abelian group of rank $rk(Ab_j)$, and like in Proposition 1.9, with each embedding $PAB_j < PS_j$ we can associate a Diophantine system of $rk(PAB_j)-1$ equations in $rk(PAB_j)-1$ variables (with coefficients), with non-zero determinant. With this system we can associate a coset Co_j of a finite index subgroup U_j of the free abelian group $Z^{rk(PAB_j)-1}$ (see Propositions 1.8 and 1.9). Therefore, with each closure $Cl(Res)(s,z,y,a)$ we can associate a tuple $(C_1,\dots,C_d,Co_1,\dots,Co_{pd})$ of cosets of finite index subgroups in the corresponding free abelian groups. We call this collection of cosets, the *closure domain*, and denote it by $Dom(Cl(Res))$.

We say that a collection of closures of a resolution $Res(y,a)$,

$$\{Cl(Res)_1, \dots, Cl(Res)_q\},$$

is a *covering closure* if the union of the closures domains

$$Dom(Cl(Res)_1), \dots, Dom(Cl(Res)_q)$$

covers the entire cross product of the abelian and pegged abelian groups

$$Ab_1, \dots, Ab_d, PAb_1, \dots, PAb_{pd}.$$

The main importance of a covering closure is the following simple observation.

LEMMA 1.17: *Let $Cl(Res)_1(s,z,y,a), \dots, Cl(Res)_q(s,z,y,a)$ be a covering closure of a well-structured resolution $Res(y,a)$. For any specialization (y_0,a) that factors through the resolution $Res(y,a)$ there exists an index i and elements s_0, z_0, so that (s_0,z_0,y_0,a) is a specialization of (at least) one of the closures $Cl(Res)_i(s,z,y,a)$.*

Proof: By Lemma 1.14, for any specialization (y_0,a) that factors through the resolution $Res(y,a)$ there exists some element z_0 so that (z_0,y_0,a) is a specialization that factors through the completed resolution $Comp(Res)(z,y,a)$. By definition, every specialization (z_0, y_0, a) that factors through the completion can be extended to a specialization (s_0, z_0, y_0, a) that factors through one of the closures $Cl(Res)_1(s,z,y,a)_1, \ldots, Cl(Res)_q(s,z,y,a)$, and the lemma follows. ∎

The completion of a resolution $Comp(Res)(z,y,a)$, its closures $Cl(Res)(s,y,a)$, and the notion of a *covering closure* finally allow us to generalize Merzlyakov's theorem for free groups and Propositions 1.3, 1.8 and 1.9 for surface and free abelian groups, to present *formal solutions* associated with a well-structured resolution of a restricted limit group.

THEOREM 1.18: *Let $F_k =< a_1, \ldots, a_k >$ be a free group, and let $u_1(y,a), \ldots, u_m(y,a)$ be a collection of words in the alphabet $\{y,a\}$ for which the group $Rlim(y,a) =< y, a | u_1(y,a), \ldots, u_m(y,a) >$ is a restricted limit group. Let $Res(y,a)$ be a well-structured resolution of the restricted limit group $Rlim(y,a)$, and let $Comp(Res)(z,y,a)$ be the completion of the resolution $Res(y,a)$ with a corresponding completed limit group $Comp(Rlim)(z,y,a)$.*

Let $w_1(x,y,a) = 1, \ldots, w_s(x,y,a) = 1$ be a system of equations over F_k and let $v_1(x,y,a), \ldots, v_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence

$$\forall y \quad (u_1(y,a)=1, \ldots, u_m(y,a)=1) \quad \exists x \quad w_1(x,y,a)=1, \ldots, w_s(x,y,a)=1 \wedge$$
$$\wedge v_1(x,y,a) \neq 1, \ldots, v_r(x,y,a) \neq 1$$

is a truth sentence.

Then there exists a covering closure

$$Cl(Res)_1(s,z,y,a), \ldots, Cl(Res)_q(s,z,y,a),$$

and for each index $1 \leq i \leq q$ there exists a formal solution $x_i(s,z,y,a)$, so that each of the words $w_j(x_i(s,z,y,a),y,a)$ is the trivial word in the restricted limit group corresponding to the i-th closure $Cl(Rlim)_i(s,z,y,a)$.

In addition, for each index i there exists a specialization (s_0^i, z_0^i, y_0^i, a) that factors through the i-th closure $Cl(Res)_i(s,z,y,a)$, so that for every index j

$$v_j(x_i(s_0^i, z_0^i, y_0^i, a), y_0^i, a) \neq 1.$$

Furthermore, if the limit group $Rlim(y,a)$ is not abelian, and the words

$$w_1(x,y,a), \ldots, w_s(x,y,a), v_1(x,y,a), \ldots, v_r(x,y,a)$$

are coefficient-free, then the formal solutions $x = x_i(s,z,y,a)$ can be taken to be coefficient-free, i.e., $x = x_i(s,z,y)$.

Proof: Our approach to proving the existence of formal solutions defined over a closure of a well-structured resolution of a general limit group is basically a combination of our approach to proving the existence of formal solutions for free groups (Theorems 1.1 and 1.2), surface groups (Theorem 1.3) and free abelian groups (Propositions 1.8 and 1.9). Like in these theorems we start with defining test sequences associated with the completion $Comp(Res)(z,y,a)$ of the given resolution $Res(y,a)$. We start the construction of test sequences with a generalization of Lemma 1.4 to punctured surfaces (rather than closed ones).

LEMMA 1.19: *Let S be a punctured surface with fundamental group Q, suppose that $\chi(S) \leq -2$ or S is a punctured torus, and let $br_1,\dots,br_w$ be its boundary components. Let $\mu\colon Q \to F_k$ be a homomorphism with non-abelian image, and suppose that for each i, $1 \leq i \leq w$, $\mu(br_i) \neq 1$.*

There exist two collections of essential, non-homotopic, non-boundary parallel disjoint s.c.c. on the surface S: $b_1,\dots,b_q$ and $d_1,\dots,d_t$, and an automorphism $\rho \in Aut(S)$ with the following properties:

- (i) *Each connected component $\tilde{S}$ obtained by cutting the surface S along the first collection of s.c.c. $b_1,\dots,b_q$ has Euler charactersitic -1, and the homomorphism $\mu \circ \rho\colon Q \to F_k$ embeds the fundamental group of each of these connected components into F_k.*
- (ii) *Each of the curves d_i intersects non-trivially at least one of the curves b_j.*
- (iii) *The entire collection of s.c.c. $b_1,\dots,b_q,d_1,\dots,d_t$ fill the punctured surface S, i.e., $S \setminus \bigcup\{b_1,\dots,b_q,d_1,\dots,d_t\}$ is a disjoint collection of connected components, where each connected component is either homeomorphic to a disk or to an annulus. If such a component is homeomorphic to an annulus, then one of its boundary components is one of the boundary components of the surface S, br_i.*

Proof: Similar to the proof of Lemma 1.4. ∎

Suppose that the completed resolution $Comp(Res)(z,y,a)$ is given by the sequence of epimorphisms

$$Comp(Rlim)(z,y,a) = Comp(Rlim)_0(z,y,a) \to Comp(Rlim)_1(z,y,a) \to \cdots$$
$$\cdots \to Comp(Rlim)_\ell(z,y,a) = F =< f,a >$$

where $\eta_i\colon Comp(Rlim)_i(z,y,a) \to Comp(Rlim)_{i+1}(z,y,a)$, $1 \leq i \leq \ell$, are the associated quotient maps, and suppose that the terminal free group in the completed resolution $Comp(Res)(z,y,a)$ is $F =< a_1,\dots,a_k,f_1,\dots,f_c >$.

To construct test sequences associated with the completed resolution $Comp(Res)(z,y,a)$ we fix a (bottom to top) order of the (punctured and closed) QH subgroups that appear in the completed abelian decompositions associated with the various levels of the completed resolution $Comp(Res)(z,y,a)$, and a (bottom to top) order of the non-cyclic abelian factors and pegged abelian vertex groups that appear in these abelian decompositions.

For each QH-vertex group QH_i in the completed abelian decomposition of one of the completed limit groups $Comp(Rlim)_i(z,y,a)$, we fix a finite set of essential, non-boundary parallel s.c.c. $b^i_1,\dots,b^i_{q_i},d^i_1,\dots,d^i_{t_i}$ that satisfy the topological properties of Lemma 1.19. With each of the completed restricted limit groups $Comp(Rlim)_i(z,y,a)$ we associate a preferred system of generators, the one inherited from a fixed system of generators of the completed limit group, $Comp(Rlim)(z,y,a)$. We further fix a set of generators for each of the (pegged) abelian groups and each of the QH vertex groups that appear in the completed abelian splittings associated with the various levels of the completed restricted resolution $Comp(Res)(z,y,a)$.

For each QH subgroup QH_i that appear in one of the abelian decompositions along the completed resolution $Comp(Res)(z,y,a)$, let $\varphi^i_1,\dots,\varphi^i_{q_i}$ be the automorphisms of QH_i that correspond to Dehn twists along the (pre-chosen) s.c.c. $b^i_1,\dots,b^i_{q_i}$, and let $\psi^i_1,\dots,\psi^i_{t_i}$ be the automorphisms of QH_i that correspond to Dehn twists along the (pre-chosen) s.c.c. $d^i_1,\dots,d^i_{t_i}$ in correspondence. In a similar way to the construction of quadratic test sequences (Definition 1.5), we define the following sequences of automorphisms of the surface group QH_i, $\{\nu^i_n,\tau^i_n\}$, iteratively. We set $\tau^i_1=id.$, and ν^i_1 to be

$$\nu^i_1=(\psi^i_1)^{\ell^{i,1}_1}\circ(\psi^i_2)^{\ell^{i,1}_2}\circ\cdots\circ(\psi^i_{t_i})^{\ell^{i,1}_{t_i}}.$$

For every index $n>1$ we define τ^i_n to be

$$\tau^i_n=(\varphi^i_1)^{m^{i,n}_1}\circ(\varphi^i_2)^{m^{i,n}_2}\circ\cdots\circ(\varphi^i_{q_i})^{m^{i,n}_{q_i}}\circ\nu^i_{n-1}$$

and

$$\nu^i_n=(\psi^i_1)^{\ell^{i,n}_1}\circ(\psi^i_2)^{\ell^{i,n}_2}\circ\cdots\circ(\psi^i_{t_i})^{\ell^{i,n}_{t_i}}\circ\tau^i_n.$$

Like in the case of a quadratic limit group (Theorem 1.3), our aim in defining the sequence of automorphisms $\{\nu^i_n,\tau^i_n\}$ is to guarantee that any action of the QH subgroup QH_i, obtained as a limit of a converging subsequence of a test sequence of the ambient completed limit group $Comp(Rlim)(z,y,a)$, is a minimal IET action of the subgroup QH_i on the limit real tree. To obtain that goal we need to restrict the sequences of powers $\{\ell_j^{i,n}, m_j^{i,n}\}$ used in the iterative definition of the sequences $\{\nu_n^i, \tau_n^i\}$ to satisfy certain combinatorial conditions.

Let X be the Cayley graph of the free group $F_k =< a_1, \ldots, a_k >$, let Y be the Cayley graph of the completed limit group $Comp(Rlim)(z, y, a)$, let (T_b^i, t_b^i) be the Bass–Serre tree corresponding to the decomposition of the QH subgroup QH_i along the collection of s.c.c. $b_1^i, \ldots, b_{q_i}^i$, and let (T_d^i, t_d^i) be the Bass-Serre tree corresponding to the decomposition of the QH subgroup QH_i along the collection of s.c.c. $d_1^i, \ldots, d_{t_i}^i$. We denote by d_X, d_Y, $d_{T_b^i}$, and $d_{T_{d_i}^i}$ the natural (simplicial) metrics on X, Y, T_b^i, and T_d^i in correspondence. For every element $g \in QH_i$ we set $\ell_b^i(g) = d_{T_b^i}(g(t_b^i), t_b^i)$, $\ell_d^i(g) = d_{T_d^i}(g(t_d^i), t_d^i)$. If g acts hyperbolically on T_b^i we denote by $tr_b^i(g)$ the trace of the action of g on T_b^i, and similarly if g acts hyperbolically on T_d^i we denote its trace by $tr_d^i(g)$. For an element $f \in F_k$, let $tr_X(f)$ be the length of a cyclically reduced element that is conjugate to f in F_k, i.e., the "length" of the conjugacy class of f in F_k.

Let $QH_i =< y_1^i, \ldots, y_{s_i}^i >$, and suppose that each y_j can be written in a normal form $y_j = a^1_{y_j^i} a^2_{y_j^i} \cdots a^{\ell n(y_j^i)}_{y_j^i}$ with respect to the graph of groups corresponding to the decomposition of the (punctured or closed) surface S_i (with fundamental group QH_i) associated with the curves $d_1^i, \ldots, d_{t_i}^i$.

Let $PR^{\nu_1^i}$ be the set of all prefixes of the words $a^1_{y_j^i} a^2_{y_j^i} \cdots a^{\ell n(y_j^i)}_{y_j^i}$ for all j, $1 \leq j \leq s_i$. We set $R^{\tau_1^i} = 1$ and $R^{\nu_1^i}$ to satisfy

$$R^{\nu_1^i} \geq 2 \cdot \max_{u \in PR^{\nu_1^i}} d_Y(u, id.)$$

where $R^{\nu_1^i}$ is the size of the ball whose elements are going to be "controlled" by the automorphism ν_1^i of QH_i. Setting $R^{\nu_1^i}$, we define the set $HY^{\nu_1^i}$ to be

$$HY^{\nu_1^i} = \{g \in QH_i | d_Y(g, id.) \leq R^{\nu_1^i} \wedge 0 < tr_d^i(g)\}$$

and the set $NF^{\nu_1^i}$ to be

$$NF^{\nu_1^i} = \{g \in QH_i | d_Y(g, id.) \leq R^{\nu_1^i} \wedge 0 < \ell_d^i(g)\}.$$

We define the constants $R^{\tau_n^i}$ and $R^{\nu_n^i}$ iteratively. For each $g \in QH_i$ for which $d_Y(g, id.) \leq R^{\nu_{n-1}^i}$ let

$$\nu_{n-1}^i(g) = a^1_{\nu_{n-1}^i(g)} a^2_{\nu_{n-1}^i(g)} \ldots a^{\ell n(\nu_{n-1}^i(g))}_{\nu_{n-1}^i(g)}$$

be a normal form of $\nu_{n-1}^i(g)$ with respect to the graph of groups corresponding to the decomposition of the surface S_i by the curves $b_1^i, \ldots, b_q^i$.

Let $PR^{\tau_n^i}$ be the set of all prefixes of the words $a^1_{\nu^i_{n-1}(g)} a^2_{\nu^i_{n-1}(g)} \cdots a^{\ell n(\nu^i_{n-1}(g))}_{\nu^i_{n-1}(g)}$ for all $g \in QH_i$ for which $d_Y(g, id.) \leq R^{\nu^i_{n-1}}$. We set $R^{\tau_n^i}$ to satisfy

$$R^{\tau_n^i} \geq 2 \cdot \max_{u \in PR^{\tau_n^i}} d_Y((\nu^i_{n-1})^{-1}(u), id.)$$

where $R^{\tau_n^i}$ is the size of the ball whose elements are going to be "controlled" by the automorphism τ_n^i of QH_i. Setting $R^{\tau_n^i}$, we define the set $HY^{\tau_n^i}$ to be

$$HY^{\tau_n^i} = \{g \in QH_i | d_Y(g, id.) \leq R^{\tau_n^i} \wedge 0 < tr^i_b(\nu^i_{n-1}(g))\}$$

and the set $NF^{\tau_n^i}$ to be

$$NF^{\tau_n^i} = \{g \in QH_i | d_Y(g, id.) \leq R^{\tau_n^i} \wedge 0 < \ell^i_b(\nu^i_{n-1}(g))\}.$$

Similarly, for each $g \in QH_i$ for which $d_Y(g, id.) \leq R^{\tau_n^i}$ let

$$\tau_n^i(g) = a^1_{\tau_n^i(g)} a^2_{\tau_n^i(g)} \cdots a^{\ell n(\tau_n^i(g))}_{\tau_n^i(g)}$$

be a normal form of $\tau_n^i(g)$ with respect to the graph of groups corresponding to the decomposition of the surface S_i by the curves $d^i_1, \ldots, d^i_q$.

Let $PR^{\nu_n^i}$ be the set of all prefixes of the words $a^1_{\tau_n^i(g)} a^2_{\tau_n^i(g)} \cdots a^{\ell n(\tau_n^i(g))}_{\tau_n^i(g)}$ for all $g \in QH_i$ for which $d_Y(g, id.) \leq R^{\tau_n^i}$. We set $R^{\nu_n^i}$ to satisfy

$$R^{\nu_n^i} \geq 2 \cdot \max_{u \in PR^{\nu_n^i}} d_Y((\tau_n^i)^{-1}(u), id.)$$

where $R^{\nu_n^i}$ is the size of the ball whose elements are going to be "controlled" by the automorphism ν_n^i of QH_i. Setting $R^{\nu_n^i}$, we define the set $HY^{\nu_n^i}$ to be

$$HY^{\nu_n^i} = \{g \in QH_i | d_Y(g, id.) \leq R^{\nu_n^i} \wedge 0 < tr^i_d(\tau_n^i(g))\}$$

and the set $NF^{\nu_n^i}$ to be

$$NF^{\nu_n^i} = \{g \in QH_i | d_Y(g, id.) \leq R^{\nu_n^i} \wedge 0 < \ell^i_d(\tau_n^i(g))\}.$$

Definition 1.20: Let $\{\nu_n^i, \tau_n^i\}$ be sequences of automorphisms of each of the QH subgroups QH_i that appear in the abelian decompositions associated with the various levels of the completed resolution $Comp(Res)(z, y, a)$. Let λ_n: $Comp(Rlim)(z, y, a) \to F_k$ be a sequence of homomorphisms that factor through the completed resolution $Comp(Res)(z, y, a)$. Let

$$F = < a_1, \ldots, a_k, f_1, \ldots, f_c >$$

be the terminal free group of the completed resolution $Comp(Res)(z,y,a)$.

For every QH vertex group QH_i, every index n and every $g \in NF^{\tau_n^i}$ let

$$\nu_{n-1}^i(g) = a^1_{\nu_{n-1}^i(g)} a^2_{\nu_{n-1}^i(g)} \cdots a^{\ell n(\nu_{n-1}^i(g))}_{\nu_{n-1}^i(g)}$$

be the previously chosen normal form of $\nu_{n-1}^i(g)$ with respect to the decomposition of QH_i corresponding to the Bass–Serre tree T_b. For every $h \in NF^{\nu_n^i}$ let

$$\tau_n^i(h) = a^1_{\tau_n^i(h)} a^2_{\tau_n^i(h)} \cdots a^{\ell n(\tau_n^i(h))}_{\tau_n^i(h)}$$

be the previously chosen normal form of $\tau_n^i(h)$ with respect to the decomposition of QH_i corresponding to the Bass–Serre tree T_d^i. For each positive integer m we set

$$f_1(m) = a_1a_2a_1a_2^{b+1}a_1a_1^{2b+1}a_1 \cdots a_1a_2^{mb+1}a_1, \ldots,$$
$$f_b(m) = a_1a_2^ba_1a_2^{2b}a_1 \cdots a_1a_2^{(m+1)b}a_1.$$

We say that a sequence of automorphisms $\{\nu_n^i, \tau_n^i\}$ of the QH vertex groups QH_i together with the sequence of homomorphisms $\lambda_n\colon Comp(Rlim)(z,y,a) \to F_k$ is a *test sequence* for the completed resolution $Comp(Res)(z,y,a)$ if the following conditions hold.

In a similar way to quadratic test sequences (Definition 1.5), for every QH subgroup QH_i that appear in one of the abelian decompositions associated with the completed resolution $Comp(Res)(z,y,a)$:

(i) For $n > 1$ and every b_j^i, $1 \leq j \leq q_i$:

$$tr_d^i((b_j^i)^{m_j^{i,n}}) > 100 \cdot 2^n \cdot \max_{1 \leq j \leq q_i} \ell_d^i(b_j^i) \cdot \sum_{d_Y(g,id.) \leq R^{\tau_n^i}, p \leq \ell n(g)} \ell_d^i(a^p_{\nu_{n-1}^i(g)}).$$

(ii) For $n \geq 1$ and every d_j^i, $1 \leq j \leq t_i$:

$$tr_b^i((d_j^i)^{\ell_j^{i,n}}) > 100 \cdot 2^n \cdot \max_{1 \leq j \leq t_i} \ell_b^i(d_j^i) \cdot \sum_{d_Y(h,id.) \leq R^{\nu_n^i}, j \leq \ell n(h)} \ell_d(a^p_{\tau_n^i(h)}).$$

(iii) For every $n > 1$ and every $g, g' \in NF^{\tau_n^i}$:

$$\left| \frac{\ell_d^i(\tau_n^i(g))\ell_b^i(\nu_{n-1}^i(g'))}{\ell_d^i(\tau_n^i(g'))\ell_b^i(\nu_{n-1}^i(g))} - 1 \right| < \frac{1}{100 \cdot q_i \cdot 2^n}.$$

(iv) For every $n > 1$ and every $g \in HY^{\tau_n^i}$:

$$\left| \frac{tr_d^i(\tau_n^i(g))\ell_b^i(\nu_{n-1}^i(g))}{\ell_d^i(\tau_n^i(g))tr_b^i(\nu_{n-1}^i(g))} - 1 \right| < \frac{1}{100 \cdot q_i \cdot 2^n}.$$

(v) For every $n \geq 1$ and every $h, h' \in NF^{\nu_n^i}$:

$$\left|\frac{\ell_b^i(\nu_n^i(h))\ell_d^i(\tau_n^i(h'))}{\ell_b^i(\nu_n^i(h'))\ell_d^i(\tau_n^i(h))} - 1\right| < \frac{1}{100 \cdot q_i \cdot 2^n}.$$

(vi) For every $n \geq 1$ and every $h \in HY^{\nu_n^i}$:

$$\left|\frac{tr_b^i(\nu_n^i(h))\ell_d^i(\tau_n^i(h))}{\ell_b^i(\nu_n^i(h))tr_d(\tau_n^i(h))} - 1\right| < \frac{1}{100 \cdot q_i \cdot 2^n}.$$

(vii) There exist constants $c_1, c_2 > 0$ so that for every $n \geq 1$ and every $h, h' \in NF^{\nu_n^i}$:

$$c_1 < \frac{\ell_b^i(\nu_n^i(h))d_X(\lambda_n(h'), id.)}{d_X(\lambda_n(h), id.)\ell_b^i(\nu_n^i(h'))} < c_2.$$

(viii) There exist constants $c_3, c_4 > 0$ so that for every $n \geq 1$ and every $h \in HY^{\nu_n^i}$:

$$c_3 < \frac{tr_b^i(\nu_n^i(h))d_X(\lambda_n(h), id.)}{tr_X(\lambda_n(h))\ell_b^i(\nu_n^i(h))} < c_4.$$

(ix) For every index n, the restriction of the homomorphism λ_n to the QH subgroup QH_i, $\lambda_n\colon QH_i \to F_k$, cannot be factored as $\lambda_n = \gamma \circ \pi$, where $\pi\colon QH_i \to Q$ is an embedding of QH_i into the fundamental group of a surface S finitely covered by the surface S_i (with fundamental group QH_i), and $\gamma\colon Q \to F_k$ is a homomorphism, i.e., the homomorphism $\lambda_n\colon Q \to F_k$ cannot be extended to a surface covered by S.

In addition to properties (i)–(ix) we need to restrict the images of the generators $\{f_1, \ldots, f_b\}$ of the terminal free group F under the homomorphisms λ_n, and to further restrict the ratios of the sizes of elements from different QH and abelian vertex groups to be in accordance with the (fixed) order previously defined on these vertex groups.

(x) For each index n, $\lambda_n(f_1) = f_1(m_n), \ldots, \lambda_n(f_b) = f_b(m_n)$ where $n < m_n < m_{n+1}$.

(xi) We have already fixed an order on the non-cyclic abelian factors and pegged abelian vertex groups in the completed abelian splittings, so let $Ab_1, \ldots, Ab_d$ be the non-cyclic abelian factors that a pear in the free decompositions associated with the various levels of the completion, $Comp(Res)$, and $PAb_1, \ldots, PAb_{pd}$ be the pegged abelian groups that appear in the completed abelian splittings associated with the various levels of the completed resolution $Comp(Res)(z, y, a)$. Let $\ell(i)$ be the level of the completed resolution $Comp(Res)(z, y, a)$ in which an abelian factor Ab_i appears, or a pegged abelian group PAb_i appears as a vertex group. Let $B_i(n)$ be the set of all elements $g \in Comp(Rlim)(z,y,a)$, so that $d_Y(g, id.) \leq n$, and either $g \in Comp(Rlim)_{\ell(i)+1}(z,y,a)$ or $g \in Ab_j$ or $g \in PAb_j$ for some abelian or pegged abelian groups, Ab_j or PAb_j, which are lower in the order defined on the abelian and pegged abelian groups in the completed resolution $Comp(Res)(z,y,a)$.

We have also fixed bases for the non-cyclic abelian factors and pegged abelian groups that appear in the completed abelian splittings, so let $q^i_1, \ldots, q^i_{d_i}$ be a pre-chosen basis of the abelian factor Ab_i, or $q^i_0, q^i_1, \ldots, q^i_{d_i}$ be the pre-chosen basis of PAb_i, where q^i_0 is the peg of the pegged abelian group PAb_i.

Let $dist_i(n)$ be $dist_i(n) = \max\{d_X(\lambda_n(g), id.) | g \in B_i(n)\}$. In case the abelian group in question is a pegged abelian group PAb_i, we choose $\lambda_n(q^i_j)$ to commute with $\lambda_n(q^i_0)$ and

$$n \cdot dist_i(n) < d_X(\lambda_n(q^i_1), id.), n \cdot d_X(\lambda_n(q^i_1), id.) < d_X(\lambda_n(q^i_2), id.), \ldots$$
$$\ldots, n \cdot d_X(\lambda_n(q^i_{d_i-1}), id.) < d_X(\lambda_n(q^i_{d_i}), id.).$$

If the abelian group in question is a non-cyclic abelian factor Ab_i, we set $h_i(n) = f_1(n \cdot dist_i(n))$. We further set each of the elements $\lambda_n(q^i_1), \ldots, \lambda_n(q^i_{d_i})$ to commute with $h_i(n)$, and

$$n \cdot |h_i(n)| < d_X(\lambda_n(q^i_1), id.), n \cdot d_X(\lambda_n(q^i_1), id.) < d_X(\lambda_n(q^i_2), id.), \ldots$$
$$\ldots, n \cdot d_X(\lambda_n(q^i_{d_i-1}), id.) < d_X(\lambda_n(q^i_{d_i}), id.).$$

(xii) Let QH_i be a QH vertex group in the completed abelian splitting of

$$Comp(Rlim)_{\ell(i)}(z,y,a).$$

We have already fixed an order of the QH vertex groups in the completed abelian splittings of $Comp(Rlim)_{\ell(i)}(z,y,a)$. Let $QB_i(n)$ be the set of all elements $g \in Comp(Rlim)(z,y,a)$, so that $d_Y(g, id.) \leq n$, and either $g \in Comp(Rlim)_{\ell(i)+1}(z,y,a)$ or $g \in Ab_j$ or $g \in PAb_j$ for some non-cyclic abelian factor Ab_j, or a pegged abelian group PAb_j in the abelian decomposition associated with $Comp(Rlim)_{\ell(i)}(z,y,a)$ or $g \in QH_j$ for some QH subgroup QH_j which is lower in the order defined on QH vertex groups in the completed resolution $Comp(Res)(z,y,a)$. Let $Qdist_i(n)$ be $Qdist_i(n) = \max\{d_X(\lambda_n(g), id.) | g \in QB_i(n)\}$. Then for each element $c \in QH_i$ so that $d_Y(c, id.) \leq n$ and c corresponds to a non-boundary parallel curve on S_{QH_i}, the (punctured) surface corresponding to QH_i:

$$n \cdot Qdist_i(n) < d_X(\lambda_n(c), id.).$$

(xiii) Let $c \in QH_i$ and suppose $d_Y(c, id.) \leq n$. If c corresponds to a non-boundary parallel curve in S_{QH_i}, the (punctured) surface corresponding to QH_i, and c has no roots in QH_i; then $\lambda_n(c)$ generates a maximal cyclic subgroup in F_k.

(xiv) For every index n and every peg q_0^i of a pegged abelian group PAb_i in one of the levels of the completed resolution, $Comp(Res)(z, y, a)$, $\lambda_n(q_0^i)$ is a primitive element in F_k.

LEMMA 1.21: *There exist test sequences for every given completed resolution $Comp(Res)(z, y, a)$. Furthermore, given any two integers $s_1 < s_2$ we can choose the n-th specialization of a basis element of some (pegged) abelian group Ab_t or PAb_t that appears in one of the completed abelian decompositions associated with the various levels of completed resolution $Comp(Res)(z, y, a)$, $\lambda_n(q_r^t)$, for some $r \geq 1$, to be of the form $\lambda_n(q_r^t(n)) = u^{ms_2+s_1}$ for some integer m, where u has no non-trivial roots in F_k.*

Proof: The construction of a test sequence for a general completed resolution is a combination of the constructions of free, quadratic, and abelian test sequences presented in the proofs of Theorems 1.1, 1.3 and 1.8 in correspondence. We start by applying Lemma 1.6 to construct a sequence of automorphisms $\{(\nu_n^i, \tau_n^i)\}$ that satisfy properties (i)–(vi) for each of the QH subgroups QH_i that appear in the abelian decompositions associated with the various levels of the completed resolutions $Comp(Res)(z, y, a)$.

We construct the homomorphisms λ_n: $Comp(Rlim)(z, y, a) \rightarrow F_k$ iteratively from bottom to top. First we set $\lambda_n(f_1) = f_1(m_n), \ldots, \lambda_n(f_b) = f_b(m_n)$, for some $m_n > m_{n-1}$ that will be specified in the sequel. Hence, property (x) is fulfilled. If $Comp(Rlim)_{\ell-1}(z, y, a)$ is the completed limit group that lies above the bottom one, then if we choose m_n to be large enough, the image under the (completed) map $Comp(\eta)_{\ell-1}$: $Comp(Rlim)_{\ell-1}(z, y, a) \rightarrow F_k$ of all the edge groups that appear in the abelian decomposition associated with $Comp(Rlim)_{\ell-1}(z, y, a)$ is non-trivial and primitive, and the image of all the QH vertex groups that appear in this abelian decomposition is non-abelian.

We set λ_n: $Comp(Rlim)_{\ell-1}(z, y, a) \rightarrow F_k$ as follows. On $Comp(Rlim)_\ell(z, y, a)$ $=< f_1, \ldots, f_b >$ we have already defined the homomorphism λ_n, and we denote this restriction of λ_n as λ_n^F. We continue by defining λ_n iteratively on each of the abelian vertex groups that appear in the abelian decomposition associated with $Comp(Rlim)_{\ell-1}(z, y, a)$, in accordance with their given order, to satisfy condition (xi). Clearly, if q_r^t, $r \geq 1$, is a basis element of any of the abelian or pegged abelian groups that appear in this abelian decomposition and $s_1 < s_2$ are given integers, we can choose $\lambda_n(q_r^t) = u^{ms_2+s_1}$ where m is an integer, and u has no roots in F_k.

Next, we define the homomorphism λ_n iteratively on each of the QH subgroups QH_i that appear in the abelian decomposition associated with $Comp(Rlim)_{\ell-1}(z,y,a)$ in accordance with their given order. On each subgroup QH_i we set the homomorphism λ_n to be of the form

$$\lambda_n^{QH_i} = \lambda_n^F \circ Comp(\eta)_{\ell-1} \circ \alpha_n^i \circ \nu_n^i,$$

where α_n^i is an automorphism of the subgroup QH_i of the form

$$\alpha_n^i = \rho_n^i \circ (\varphi_1^i)^{e_n^i} \circ (\varphi_2^i)^{e_n^i} \circ \cdots \circ (\varphi_{q_i}^i)^{e_n^i}$$

and ρ_n^i is a (modular) automorphism of QH_i chosen according to Lemma 1.19.

We have chosen the homomorphism λ_n^F: $< f_1,\ldots,f_b > \to F_k$ to guarantee that $\lambda_n^F \circ Comp(\eta)_{\ell-1}$: $QH_i \to F_k$ has a non-abelian image. Hence, by Lemma 1.19 and the same argument used to prove the existence of a quadratic test sequence (Lemma 1.6), we choose the exponents e_n^i to be large enough; properties (vii)–(ix) and properties (xii) and (xiii) hold for the homomorphisms λ_n: $Comp(Rlim)_{\ell-1}(z,y,a) \to F_k$. By possibly further increasing the integer m_n used to define the homomorphism λ_n^F: $< f_1,\ldots,f_b > \to F_k$, and further increasing the exponents e_n^i used to define the homomorphisms $\lambda_n^{QH_i}$, and the exponents used in setting the image of basis elements of the abelian vertex groups, we get property (xiv) to hold as well.

So far we have defined the homomorphism λ_n only on the $\ell-1$-th completed limit group $Comp(Rlim)_{\ell-1}(z,y,a)$. Repeating the construction of the homomorphism λ_n for each of the abelian vertex groups and each of the QH vertex groups that appear in the abelian decompositions associated with each of the upper levels iteratively and according to the pre-fixed order, we obtain a homomorphism λ_n: $Comp(Rlim)(z,y,a) \to F_k$ that satisfies properties (i)–(xiv) of the lemma. ∎

Having constructed test sequences, we continue the proof of Theorem 1.18 by essentially combining the arguments used to prove Theorems 1.1, 1.3 and 1.8 and 1.9. Also, we may assume that our given completed limit group $Comp(Rlim)(z,y,a)$ is not the free group $< f,a >$, since in that case Theorem 1.18 follows from Theorem 1.2.

By the assumptions of Theorem 1.18, for every possible specialization of (y,a) that factors through the restricted limit group $Rlim(y,a)$, there exists a specialization for the variables x, so that the given equalities $w_1(x,y,a) = 1, \dots, w_s(x,y,a) = 1$ and inequalities $v_1(x,y,a) \neq 1, \dots, v_r(x,y,a) \neq 1$ are fulfilled. Given a well-structured resolution $Res(y,a)$ of $Rlim(y,a)$, its completion $Comp(Res)(z,y,a)$, and a test sequence $\{(\nu_n^i, \tau_n^i, \lambda_n)\}$ associated with the completed resolution $Comp(Res)(z,y,a)$, for each index n, we can choose a specialization x_n to be a shortest possible specialization (in the word metric on F_k) for which

$$w_1(x_n, \lambda_n(y), a) = 1, \dots, w_s(x_n, \lambda_n(y), a) = 1 \wedge$$
$$\wedge v_1(x_n, \lambda_n(y), a) \neq 1, \dots, v_r(x_n, \lambda_n(y), a) \neq 1.$$

If the sequence of specializations $\{(x_n, \lambda_n(z), \lambda_n(y), a)\}$ corresponding to a test sequence and some shortest possible specializations $\{x_n\}$ converges, we call the obtained limit group a *test limit group*. On the set of test limit groups we define a natural partial order, by setting $TL_1(x,z,y,a) \geq TL_2(x,z,y,a)$ if $TL_2(x,z,y,a)$ is a quotient of $TL_1(x,z,y,a)$. By the arguments used in constructing the Makanin–Razborov diagram of a limit group (lemmas 5.4 and 5.5 in [Se]), there exist *maximal test limit groups* and in fact there are finitely many equivalence classes of maximal test limit groups that we denote

$$MTL_1(x,z,y,a), \dots, MTL_m(x,z,y,a).$$

Since the maximal test limit groups $MTL_1(x,z,y,a), \dots, MTL_m(x,z,y,a)$ were constructed using sequences of specializations $\{(x_n, \lambda_n(z), \lambda(y), a)\}$ for which the equalities $w_1(x_n, \lambda_n(y), a) = 1, \dots, w_s(x_n, \lambda_n(y), a) = 1$ are fulfilled, the words $w_1(x,y,a), \dots, w_s(x,y,a)$ represent the trivial words in the maximal test limit groups

$$MTL_1(x,z,y,a), \dots, MTL_m(x,z,y,a).$$

Similarly, the words $v_1(x,y,a), \dots, v_r(x,y,a)$ represent non-trivial words in these maximal test limit groups.

By repeating the iterative modifications of the specializations $\{x_n\}$, applied in the proofs of Theorems 1.3 and 1.8, we may further replace the maximal test limit groups $MTL_1(x,z,y,a), \dots, MTL_m(x,z,y,a)$ by quotients of them (still denoted $MTL_j(x,z,y,a)$), so that each of the maximal test limit groups $MTL_j(x,z,y,a)$ admits a free decomposition

$$MTL_j(x,z,y,a) = G_j(g,z,y,a) * < e_1, \dots, e_{d_j} >$$

where $< e_1,\ldots,e_{d_j} >$ is a (possibly trivial) free group on the set $e_1,\ldots,e_{d_j}$, $Comp(Rlim)(z,y,a) < G_j(g,z,y,a)$, and $G_j(g,z,y,a)$ admits no free decomposition in which $Comp(Rlim)(z,y,a)$ is contained in one of the factors.

At this point we continue with each of the maximal test limit groups $MTL_j(x,z,y,a)$ in parallel, by combining the arguments used to prove Theorems 1.3 and 1.8. Let $MTL(x,z,y,a)$ be one of the maximal test limit groups. We fix a generating system of the factor $G(g,z,y,a)$ of $MTL(x,z,y,a)$, $G(g,z,y,a) = < g_1,\ldots,g_c >$. We continue with all the sequences $\{(x_n,\lambda_n(z),\lambda_n(y),a)\}$ for which the corresponding sequence $\{(\lambda_n(z),a)\}$ is a test sequence, and for every index n:

(1) The specialization x_n is in the free group $F_k * F_d$, where

$$MTL(x,y,z,a) = G(g,z,y,a) * F_d.$$

(2) The tuple $(x_n,\lambda_n(z),\lambda_n(y),a)$ factors through the maximal test limit group $MTL(x,z,y,a)$.

(3) $v_1(x_n,\lambda_n(y),a) \neq 1,\ldots,v_r(x_n,\lambda_n(y),a) \neq 1$.

(4) The specializations $g_1(n),\ldots,g_c(n) \in F_k$ obtained from the specialization x_n have the shortest length (in the simplicial metric on the Cayley graph of F_k) among all possible specializations x_n that satisfy properties (1)–(3).

We look at such a sequence that converges to a faithful action of $MTL(x,z,y,a)$ on a real tree Y_1. By the shortening argument used in the proofs of Theorems 1.3 and 1.8, since the specializations $g_1(n),\ldots,g_c(n)$ were chosen to be shortest possible, and since the factor $G(g,z,y,a)$ admits no free decomposition in which the completion $Comp(z,y,a)$ is elliptic, the factor $G(g,z,y,a)$ does not fix a point in the real tree Y_1. Furthermore, the action of $MTL(x,z,y,a)$ on the real tree Y_1 must be of one of the following types:

(1) $MTL(x,z,y,a)$ (and $G(g,z,y,a)$) inherits from its action on the real tree Y_1 a graph of groups with one QH subgroup $Q < Comp(Rlim)(z,y,a)$, which is the highest order QH subgroup in the completed resolution $Comp(Res)(z,y,a)$, and several vertex groups $V_j(v,a)$, corresponding to the various orbits of point stabilizers in the action of $MTL(x,z,y,a)$ on Y_1. The decomposition $Comp(Rlim)(z,y,a)$ inherited from this decomposition of $MTL(x,z,y,a)$ is exactly of the same form, and is compatible with the (top level of the) completed resolution $Comp(Res)(z,y,a)$.

(2) $MTL(x,z,y,a)$ (and the factor $G(g,z,y,a)$) acts discretely on Y_1, and it inherits a graph of groups with several vertex groups located on a loop with abelian stabilizer, where the Bass–Serre generator bs correspond-

ing to the loop can be chosen to commute with the (abelian) loop stabilizer. Hence, the maximal test limit group MTL admits an amalgamated product of the form $MTL = V *_{Ab_1} Ab$, where the subgroup Ab is abelian, and contains the (pegged) abelian subgroup of highest order in $Comp(Res)(z,y,a)$. Furthermore, the (pegged) abelian group of highest order in $Comp(Res)(z,y,a)$ is not elliptic in this amalgamated product. The decomposition $Comp(Rlim)(z,y,a)$ inherited from this decomposition of MTL is exactly of the same form, and is compatible with the (top level of the) completed resolution $Comp(Res)(z,y,a)$.

(3) The action of the maximal test limit group $MTL(x,z,y,a)$ (and the factor $G(g,z,y,a)$) on the real tree Y_1 corresponds to a unique axial component and several point stabilizers located on that axial component. In this case MTL (and the factor $G(g,z,y,a)$) admits an amalgamated product of the form $MTL = V *_{Ab_1} Ab$, where the subgroup Ab is abelian, and contains the (pegged) abelian subgroup of highest order in $Comp(Res)(z,y,a)$. Furthermore, the (pegged) abelian group of highest order in $Comp(Res)(z,y,a)$ is not elliptic in this amalgamated product. The decomposition $Comp(Rlim)(z,y,a)$ inherited from this decomposition of MTL is exactly of the same form, and is compatible with the (top level of the) completed resolution $Comp(Res)(z,y,a)$.

If the action of the maximal test limit group $MTL(x,z,y,a)$ satisfies the properties of case (1), we continue the analysis of $MTL(x,z,y,a)$ by analyzing each of the vertex groups $V_j(v,a)$ in parallel. If the action of MTL satisfies the properties of cases (2) or (3), we continue with the vertex group V, where MTL inherits the amalgamated product $MTL = V *_{Ab_1} Ab$ from its action on the real tree Y_1. By iteratively repeating this "uncovering" process (as we did in the proof of Theorem 1.8), we are finally able to replace each of the maximal test limit groups, $MTL(x,z,y,a)$, with finitely many quotients (still denoted $MTL_j(x,z,y,a)$) of the form

$$MTL_j(x,z,y,a) = Cl_j(s,z,y,a) * E_j(e_1,\dots,e_{d_j})$$

where $E_j(e_1,\dots,e_{d_j})$ is a free group, freely generated by $< e_1,\dots,e_{d_j} >$, and $Cl_j(s,z,y,a)$ is a closure of the resolution $Res(y,a)$.

The maximal test limit groups $MTL_j(x,z,y,a)$ were obtained as the limit of sequences of the form $\{(x_n,\lambda_n(z),\lambda_n(y),a)\}$, where for every index n the tuple $(x_n,\lambda_n(z),\lambda_n(y),a)$ factors through $MTL_j(x,z,y,a)$ and, in addition, $v_1(x_n,\lambda_n(y),a) \neq 1,\dots,v_r(x_n,\lambda_n(y),a) \neq 1$. Hence, there must exist a retraction

$$\eta: MTL_j(x,z,y,a) \to Cl_j(s,z,y,a)$$

so that the words $\eta(v_1(x,y,a)),\dots,\eta(v_r(x,y,a))$ are all non-trivial in the closure $Cl_j(s,z,y,a)$. Therefore, each generator x of $\eta(MTL_j(x,z,y,a))$ can be naturally presented as a word in the closure $Cl_i(Res)(s,z,y,a)$, i.e., $x=\eta(x)(s,z,y,a)$, each of the words $w_1(x(s,z,y,a),y,a),\dots,w_s(x(s,z,y,a),y,a)$ represents the trivial word in the closure $Cl_j(Res)(s,z,y,a)$, and there exists some specialization (s_0^i,z_0^i,y_0^i,a) for which

$$v_1(x(s_0^i,z_0^i,y_0^i,a),y_0^i,a)\neq 1,\dots,v_r(x(s_0^i,z_0^i,y_0^i,a),y_0^i,a)\neq 1.$$

Since every test sequence associated with the completed resolution $Comp(Res)(z,y,a)$ can be adjoined by a sequence $\{x_n\}$ to form a sequence that factors through at least one of the maximal test limit groups $MTL_j(x,z,y,a)$, the same argument used in proving Proposition 1.8 shows that the collection of closures $Cl_j(Res)(s,y,a)$ associated with the maximal test limit groups $MTL_j(x,z,y,a)$ has to be a covering closure.

If the graded resolution $Res(y,a)$ is not abelian, and the words $w_1(x,y,a),\dots,w_s(x,y,a)$ and $v_1(x,y,a),\dots,v_r(x,y,a)$ are coefficient-free, then each of the maximal test limit groups constructed by our iterative procedure has the form

$$MTL_j(x,z,y,a)=F_k * Cl_j(s,z,y) * E_j(e_1,\dots,e_{d_j})$$

where F_k is the coefficient group, and $E_j(e_1,\dots,e_{d_j})$ is a (possibly trivial) free group.

In this coefficient-free case, there must exist a retraction

$$\eta: MTL_j(x,z,y,a) \to Cl_j(s,z,y)$$

for which the elements $\eta(v_1(x,y)),\dots,\eta(v_r(x,y))$ are mapped to non-trivial elements in $Cl_j(s,z,y)$, so that the formal solutions can be taken to be coefficient-free, i.e., $x=\eta(x)(s,z,y)$. ∎

For the purposes of our "trial and error" procedure for quantifier elimination, a slight generalization of Theorem 1.18 is required. The proof of the more general form that appears below is identical to the proof of Theorem 1.18.

THEOREM 1.22: *Let $F_k=<a_1,\dots,a_k>$ be a free group, and let $u_1(y,a),\dots,u_m(y,a)$ be a collection of words in the alphabet $\{y,a\}$ for which the group $Rlim(y,a)=<y,a|u_1(y,a),\dots,u_m(y,a)>$ is a restricted limit group. Let*

$Res(y,a)$ be a well-structured resolution of the restricted limit group $Rlim(y,a)$, and let $Comp(Res)(z,y,a)$ be the completion of the resolution $Res(y,a)$ with a corresponding completed limit group $Comp(Rlim)(z,y,a)$.

Let $\Sigma_1(x,y,a)=1,\dots,\Sigma_r(x,y,a)=1$ be a collection of systems of equations over F_k, and let $\Psi_1(x,y,a),\dots,\Psi_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence

$$\forall y \quad (u_1(y,a)=1,\dots,u_m(y,a)=1) \quad \exists x$$
$$(\Sigma_1(x,y,a)=1 \wedge \Psi_1(x,y,a)\neq 1) \vee \dots \vee (\Sigma_r(x,y,a)=1 \wedge \Psi_r(x,y,a)\neq 1)$$

is a truth sentence.

Then there exists a covering closure

$$Cl(Res)_1(s,z,y,a)_1,\dots,Cl(Res)_q(s,z,y,a),$$

and for each index i, $1\leq i\leq q$, there exists a formal solution $x_i(s,z,y,a)$ and an index $j(i)$, $1\leq j(i)\leq r$, with the following properties:

(i) *For each index i, $1 \leq i \leq q$, all the words in the system $\Sigma_{j(i)}(x_i(s,z,y,a),y,a)$ represent the trivial word in the restricted limit group corresponding to the i-th closure $Cl(Rlim)_i(s,y,a)$.*

(ii) *For each index i, $1\leq i\leq q$, there exists a specialization (s_0^i,z_0^i,y_0^i,a) that factors through the i-th closure $Cl(Res)_i(s,z,y,a)$, so that all the words in the collection $\Psi_{j(i)}(x_i(s_0^i,z_0^i,y_0^i,a),y_0^i,a)$ are not the trivial element in F_k.*

Furthermore, if the limit group $Rlim(y,a)$ is not abelian, and the words $w_1(x,y,a),\dots,w_s(x,y,a)$, $v_1(x,y,a),\dots,v_r(x,y,a)$ are coefficient-free, then the formal solutions $x=x_i(s,z,y,a)$ can be taken to be coefficient-free, i.e., $x=x_i(s,z,y)$.

2. Formal limit groups

Theorems 1.18 and 1.22 show that given a restricted limit group and a well-structured resolution of that limit group, the validity of an AE sentence over the given limit group implies the existence of a finite collection of formal solutions defined over a covering closure of the resolution of that limit group. The finite collection of formal solutions constructed in Theorems 1.18 and 1.22 are guaranteed to satisfy the equalities specified in the sentence, and at the same time to admit some particular specialization for which all the inequalities specified in the sentence are valid.

Our approach to quantifier elimination of predicates is based on the existence of formal solutions satisfying the properties listed in Theorems 1.18 and 1.22.

However, instead of distinguishing formal solutions which admit specializations that satisfy the inequalities specified in the sentence, our strategy is to analyze the entire family of formal solutions that satisfy only the equalities specified in the sentence, so that if the sentence is indeed a truth sentence, Theorems 1.18 and 1.22 guarantee that the entire family contains a finite collection of formal solutions defined over a covering closure of the given (well-structured) resolution and admitting specializations that satisfy also the inequalities specified in the sentence.

Given a restricted limit group $Rlim(y,a)$ and a well-structured resolution $Res(y,a)$ of it, to analyze the family of all formal solutions of a given system of equations $\Sigma(x,y,a)=1$, defined over some closure of the completion of our given resolution, we need to present *formal limit groups* and their canonically associated *formal Makanin–Razborov diagrams*.

We start by defining *formal limit groups*.

Definition 2.1: Let $Res(y,a)$ be a well-structured resolution of a restricted limit group $Rlim(y,a)$, let $Comp(Res)(z,y,a)$ be the completed resolution of $Res(y,a)$, and let $\Sigma(x,y,a)=1$ be a system of equations.

Let the sequence of automorphisms $\{(\nu_n^i,\tau_n^i)\}$ of the QH subgroups QH_i that appear in the abelian decompositions associated with the various levels in the completed resolution $Comp(Res)(z,y,a)$, together with the homomorphisms $\{\lambda_n\colon Comp(Rlim)\to F_k\}$, be a test sequence associated with the completed resolution $Comp(Res)(z,y,a)$. For each index n suppose that there exists some specialization x_n for which $\Sigma(x_n,\lambda_n(y),a)=1$.

For each index n, the homomorphism $\lambda_n\colon Comp(Rlim)(z,y,a)\to F_k$ together with the specialization x_n defines a homomorphism from the free group with generating set $<x,z,y,a>$, denoted $F(x,z,y,a)$, to the coefficient group F_k, $\alpha_n\colon F(x,z,y,a)\to F_k$.

We say that the test sequence $\{\nu_n^i,\tau_n^i,\lambda_n\}$ together with the sequence of specializations $\{x_n\}$ is a *convergent formal sequence*, if the sequence of corresponding homomorphisms $\{\alpha_n\colon F(x,z,y,a)\to F_k\}$ is convergent (in the Gromov–Haussdorff topology, after appropriate rescaling). We call the obtained limit group, a *formal limit group*, and denote it $FL(x,z,y,a)$.

Note that by construction $Comp(Rlim)(z,y,a)<FL(x,z,y,a)$. Since every abelian subgroup of a limit group is contained in a unique maximal abelian subgroup, every abelian subgroup of a limit group is contained in a unique maximal abelian subgroup of the same rank. By replacing the (pegged) abelian groups in $Comp(Rlim)(z,y,a)$ by the maximal abelian groups of the same rank containing

them in $FL(x,z,y,a)$, we obtain a (canonical) closure of $Comp(Rlim)(z,y,a)$ in $FL(x,z,y,a)$, which we call the *formal closure* of $Comp(Rlim)(z,y,a)$ in the formal limit group $FL(x,z,y,a)$. We denote the formal closure, $FCl(s,z,y,a)$.

To analyze formal limit groups we need to construct their associated (canonical) *formal Makanin–Razborov diagrams.* To do that we need to study the algebraic structure of formal limit groups, $FL(x,z,y,a)$, relative to their subgroup, the completed restricted limit group $Comp(Rlim)(z,y,a)$. To do that we start by defining the *formal JSJ decomposition* of a formal limit group.

In the sequel below let $Res(y,a)$ be a well-structured resolution of a restricted limit group $Rlim(y,a)$, and let $Comp(Rlim)(z,y,a)$ be the limit group associated with its completion. Let $\Sigma(x,y,a)=1$ be a system of equations and let $FL(x,z,y,a)$ be a formal limit group corresponding to the system $\Sigma(x,y,a)=1$. Let $FCl(s,z,y,a)$ be the formal closure of $Comp(Rlim)(z,y,a)$ in the formal limit group $FL(x,z,y,a)$.

Definition 2.2: Suppose that the formal limit group $FL(x,z,y,a)$ does not split as a non-trivial free product in which $Comp(Rlim)(z,y,a)$ is contained in one of the factors. An essential one edge abelian splitting, $FL(x,z,y,a)=D*_A E$ or $FL(x,z,y,a)=D*_A$, in which $Comp(Rlim)(z,y,a)$ is elliptic, is called a *formal abelian splitting.*

The arguments used in section 2 of [Se] naturally generalize to construct the formal JSJ decomposition of a formal limit group.

THEOREM 2.3 (cf. ([Se], 9.2)): *Let $FL(x,z,y,a)$ be a formal limit group which does not split as a non-trivial free product in which the completed limit group $Comp(Rlim)(z,y,a)$ is contained in one of the factors.*

There exists an essential (perhaps trivial) abelian splitting of $FL(x,z,y,a)$, which we call the formal JSJ decomposition of $FL(x,z,y,a)$, with the following properties:

(i) *$Comp(Rlim)(z,y,a)$ is elliptic in the formal JSJ decomposition. If the formal JSJ decomposition of $FL(x,z,y,a)$ is not trivial, then any one edge abelian splitting obtained by collapsing all edges but one in the formal JSJ decomposition is a formal splitting of $FL(x,z,y,a)$.*

(ii) *Every (formal) canonical maximal QH subgroup (CMQ) of $FL(x,z,y,a)$ is conjugate to a vertex group in the formal JSJ decomposition. Every formal QH subgroup of $FL(x,z,y,a)$ can be conjugated into one of the formal CMQ subgroups of $FL(x,z,y,a)$. Every vertex group in the formal JSJ* *decomposition which is not a CMQ subgroup of* $FL(x,z,y,a)$ *is elliptic in any formal splitting of* $FL(x,z,y,a)$.

(iii) *A one edge formal splitting* $FL(x,z,y,a) = D*_A E$ *or* $FL(x,z,y,a) = D*_A$ *which is hyperbolic in another such elementary formal splitting is obtained from the formal JSJ decomposition of* $FL(x,z,y,a)$ *by cutting a surface corresponding to a formal CMQ subgroup of* $FL(x,z,y,a)$ *along an essential s.c.c.*

(iv) *Let* Θ *be a one edge formal abelian splitting* $FL(x,z,y,a) = D *_A E$ *or* $FL(x,z,y,a) = D*_A$ *which is elliptic with respect to any other such one edge formal splitting of* $FL(x,z,y,a)$. *Then* Θ *is obtained from the formal JSJ decomposition of* $FL(x,z,y,a)$ *by a sequence of collapsings, conjugations, and modifying boundary monomorphisms by conjugations.*

(v) *Let* Λ *be a general formal abelian splitting of* $FL(x,z,y,a)$. *There exists a formal splitting* Λ_1 *obtained from the formal JSJ decomposition by splitting the formal CMQ subgroups along weakly essential s.c.c. on their corresponding surfaces, so that there exists a* $FL(x,z,y,a)$*-equivariant simplicial map between a subdivision of the Bass–Serre tree* T_{Λ_1} *to* T_Λ.

(vi) *If* JSJ_1 *is another formal JSJ decomposition of* $FL(x,z,y,a)$, *then* JSJ_1 *is obtained from the formal JSJ decomposition by a sequence of slidings, conjugations, and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions)*

(vii) *If* $FL(x,z,y,a)$ *admits a formal abelian splitting, then the formal JSJ decomposition of* $FL(x,z,y,a)$ *is non-trivial.*

In case the formal JSJ decomposition is non-trivial, it allows one to "further simplify" formal limit groups. In order to construct *formal Makanin–Razborov diagrams* we still need to study the structure of formal limit groups with trivial formal JSJ decomposition.

Proposition 2.4: *Let* $FL(x,z,y,a)$ *be a formal limit group which does not split as a non-trivial free product in which the completed limit group* $Comp(Rlim)(z,y,a)$ *is contained in one of the factors. If the formal JSJ decomposition of* $FL(x,z,y,a)$ *is trivial, then* $FL(x,z,y,a)$ *is a formal closure of* $Comp(Rlim)(z,y,a) < FL(x,z,y,a)$, *i.e.,* $FL(x,z,y,a) = FCl(s,z,y,a)$.

Proof: Suppose that $FL(x,z,y,a)$ does not split as a non-trivial free product in which $Comp(Rlim)(z,y,a)$ is contained in one of the factors, and the formal JSJ decomposition of $FL(x,z,y,a)$ is trivial.

The formal limit group $FL(x,z,y,a)$ was constructed from a sequence of homomorphisms $\alpha_n\colon F(x,z,y,a)\to F_k$ that are composed from a test sequence $\{\nu_n^i,\tau_n^i,\lambda_n\}$ and specializations $\{x_n\}$, for which $\Sigma(x_n,\lambda_n(y),a)=1$. From the convergence of the sequence of homomorphisms $\{\alpha_n\}_{n=1}^{\infty}$, we get a (non-trivial) stable action of the formal limit group $FL(x,z,y,a)$ on some real tree Y_1, in which the stabilizer of every non-degenerate segment is either trivial or abelian.

By the classification of stable actions of groups on real trees (see theorem 1.5 in [Se]), there is a graph of groups Λ_1 with abelian edge stabilizers and fundamental group $FL(x,z,y,a)$, corresponding to the action of $FL(x,z,y,a)$ on the real tree Y_1. If the completed limit group $Comp(Rlim)(z,y,a)$ is contained in a vertex group of Λ_1, or more generally, if $Comp(Rlim)(z,y,a)$ is contained in a fundamental group of a proper subgraph of Λ_1, the formal limit group $FL(x,z,y,a)$ admits a formal abelian or a free decomposition in which the completed limit group $Comp(Res)(z,y,a)$ is elliptic, a contradiction to our assumptions.

Hence, by the properties of a test sequence, if $Comp(Res)(z,y,a)$ is equal to its terminal free group $F=<a,f>$, then necessarily $FL(x,z,y,a)=F$, and the theorem follows. Otherwise, note that by the properties of test sequences, the action of $Comp(Res)(z,y,a)$ on the real tree Y_1 either contains a unique IET orbit and finitely many orbits of point stabilizers, or the action is discrete with a unique orbit of non-abelian point stabilizer and a unique edge stabilized by some abelian subgroup of $Comp(Res)(z,y,a)$. Since $FL(x,z,y,a)$ inherits from its action on Y_1 a graph of groups of the same form as the one $Comp(Res)(z,y,a)$ inherits from its action on Y_1, the action of $FL(x,z,y,a)$ on Y_1 must contain also either a unique orbit of an IET component and finitely many orbits of point stabilizers, or its action is discrete with one orbit of non-abelian point stabilizers and one orbit of edges stabilized by some abelian subgroup of $FL(x,z,y,a)$.

In both types of the dynamics of the action of the formal limit group $FL(x,z,y,a)$ on the limit tree Y_1, the generators x of the formal limit group $FL(x,z,y,a)$ can be written as words in elements from the vertex stabilizers V_i and elements from the formal closure $FCl(s,z,y,a)$. Now, if we restrict the specializations of the subgroup $<x,z,y,a>$ to each of the vertex groups V_i in the graph of groups associated with the action of $FL(x,z,y,a)$ on the real tree Y_1, with each vertex group V_i we associate an action of V_i on some real tree V_2^i. Repeating this argument for the action of the vertex stabilizers V_i on a real tree Y_2^i, if V_i is not the free group $F=<f,a>$, the action of V_i on Y_2^i must be one of two types. Either it contains a unique orbit of an IET component and finitely many orbits of point stabilizers, or the action is discrete with one orbit of non-abelian point stabilizers and one orbit of edges stabilized by some abelian subgroup of the closure $FCl(s,z,y,a)$. Hence, the generators x of the formal limit group $FL(x,z,y,a)$ can be written as words in elements from the vertex stabilizers W_i and elements from the formal closure $FCl(s,z,y,a)$. A finite induction argument clearly finishes the proof of the proposition. ∎

Defining formal limit groups and their formal JSJ decompositions, and showing that a (relatively) freely indecomposable formal limit group with trivial formal JSJ decomposition is a formal closure $FCl(s,z,y,a)$ of the completed limit group $Comp(Rlim)(z,y,a)$, we are able to present *formal Makanin–Razborov diagrams.*

Let $Res(y,a)$ be a well-structured resolution of a restricted limit group $Rlim(y,a)$, let $Comp(Res)(z,y,a)$ be the completed resolution of $Res(y,a)$, and let $\Sigma(x,y,a)=1$ be a system of equations.

Recall that a formal limit group $FL(x,z,y,a)$ of the system $\Sigma(x,y,a)=1$ with respect to the resolution $Res(y,a)$ was constructed from a convergent sequence of homomorphisms $\{\alpha_n\colon F(x,z,y,a)\to F_k\}$ composed from a test sequence $\{\nu_n^i,\tau_n^i,\lambda_n\}$, and a sequence of specializations $\{x_n\}$ for which $\Sigma(x_n,\lambda_n(y),a)=1$.

On the set of formal limit groups we can naturally define a partial order, by setting $FL_1(x,z,y,a)\geq FL_2(x,z,y,a)$ if there exists an epimorphism

$$\eta\colon FL_1(x,z,y,a)\to FL_2(x,z,y,a).$$

By the arguments used in the construction of the Makanin–Razborov diagram of a limit group (see lemmas 5.4 and 5.5 in [Se]), given the system of equations $\Sigma(x,y,a)=1$, and the (well-structured) resolution $Res(y,a)$, there exist *maximal formal limit groups* (with respect to the natural partial order), and up to the natural equivalence relation on formal limit groups there are only finitely many equivalence classes of formal limit groups which we denote $MFL_1(x,z,y,a),\dots,MFL_t(x,z,y,a)$.

To construct the *formal Makanin–Razborov diagrams* of the system $\Sigma(x,y,a)=1$ with respect to the restricted well-structured resolution $Res(y,a)$, we proceed as in the construction of the graded Makanin–Razborov diagrams (section 10 in [Se]). We first decompose each of the maximal formal limit groups into the maximal (Grushko's) free decomposition in which the completed limit group $Comp(Rlim)(z,y,a)$ is contained in a factor. We associate the formal JSJ decomposition with the factor containing $Comp(Rlim)(z,y,a)$, and the (standard) abelian JSJ decomposition with each of the other factors. These associated JSJ decompositions naturally define the *formal modular groups* of each of the maximal formal limit groups $MFL_i(x,z,y,a)$. Having these modular groups, and assuming they are not trivial, we are able to define *formal shortening quotients* of each of the maximal formal limit groups. By the argument used to prove claim 5.3 in [Se], every formal shortening quotient of a formal limit group is a proper quotient of it.

Now, by the arguments used to prove lemmas 5.4 and 5.5 in [Se], there are maximal formal shortening quotients, and up to the natural equivalence classes of formal limit groups there are only finitely many equivalence classes of maximal shortening quotients of each of the maximal formal limit groups $MFL_1(x,z,y,a),\dots,MFL_t(x,z,y,a)$.

Since a formal limit group is in particular a limit group, any properly decreasing sequence of formal limit groups terminates. Therefore, if we continue the above construction of maximal formal free decompositions, formal JSJ decompositions and formal modular groups for each of the factors, and then collect the (canonical) family of maximal formal shortening quotients, we are guaranteed that this iterative construction terminates. We call the obtained diagram the *formal Makanin–Razborov diagram* of the system of equations $\Sigma(x,y,a)=1$ with respect to the (well-structured) resolution $Res(y,a)$. We call each path in this (directed) diagram a *formal resolution* and denote it $FRes(x,z,y,a)$. Note that by Proposition 2.4, every formal resolution terminates with a formal limit group of the form $FCl(s,z,y,a)*\hat{F}$, where $FCl(s,z,y,a)$ is a closure of the resolution $Res(y,a)$ and $\hat{F}$ is some (possibly trivial) free group. In particular, every formal resolution in the formal Makanin–Razborov diagram is defined over some closure of the completed limit group $Comp(Rlim)(z,y,a)$.

Theorems 1.18 and 1.22 prove that if an AE sentence defined over some limit group is a truth sentence, then there exist formal solutions that may serve as "witnesses" for the correctness of the sentence in a "generic" point of the variety associated with the given limit group. Along our "trial and error" procedure for quantifier elimination we are not going to look for specific formal solutions that satisfy the conclusions of Theorems 1.18 and 1.22, but rather study the entire collection of formal solutions by studying the maximal formal limit groups associated with systems of equations with respect to some given resolutions, and then use the conclusions of Theorems 1.18 and 1.22 which guarantee that these entire collections of formal solutions contain formal solutions with the properties listed in these theorems. Since this point of view is used extensively in our quantifier elimination procedure, we prefer to state it specifically as a corollary.

Corollary 2.5: *Let $F_k=<a_1,\dots,a_k>$ be a free group, and let $u_1(y,a),\dots,u_m(y,a)$ be a collection of words in the alphabet $\{y,a\}$, for which the* *group $Rlim(y,a) =< y,a|u_1(y,a),\dots,u_m(y,a) >$ is a restricted limit group. Let $Res(y,a)$ be a well-structured resolution of the restricted limit group $Rlim(y,a)$, and let $Comp(Res)(z,y,a)$ be the completion of the resolution $Res(y,a)$ with a corresponding completed limit group $Comp(Rlim)(z,y,a)$.*

Let $\Sigma(x,y,a) = 1$ be a system of equations over F_k, and let $v_1(x,y,a),\dots,v_r(x,y,a)$ be a collection of words in the alphabet $\{x,y,a\}$. Suppose that the sentence

$$\forall y \quad (u_1(y,a)=1,\dots,u_m(y,a)=1) \quad \exists x$$
$$\Sigma(x,y,a)=1 \wedge v_1(x,y,a)\neq 1,\dots,v_r(x,y,a)\neq 1$$

is a truth sentence.

Then there exists a covering closure

$$Cl(Res)_1(s,z,y,a),\dots,Cl(Res)_q(s,z,y,a),$$

and formal resolutions $FRes_1(x,z,y,a),\dots,FRes_q(x,z,y,a)$ in the formal Makanin–Razborov diagram associated with the system of equations $\Sigma(x,y,a)=1$ and the well-structured resolution $Res(y,a)$, with the following properties:

(i) *For each index i, $1\leq i\leq q$, the formal resolution $FRes_i(s,z,y,a)$ terminates with a formal limit group of the form $Cl_i(s,z,y,a)*F_i$ for some (possibly trivial) free group F_i.*

(ii) *For each index i, $1\leq i\leq q$, there exists a* formal solution *$x_i(s,z,y,a)$ that factors through the resolution $FRes_i(x,z,y,a)$, and a specialization (s_0^i,y_0^i,f_0^i,a) that factors through the i-th closure $Cl(Res)_i(s,z,y,a)$, so that for every index j*

$$v_j(x_i(s_0^i,y_0^i,f_0^i,a),y_0^i,a)\neq 1.$$

In particular, in case the above sentence is a truth sentence, the set of closures that appear in the terminal points of the formal resolutions in the formal Makanin–Razborov diagrams of the system $\Sigma(x,y,a)=1$ and the resolution $Res(y,a)$ contains a covering closure of the restricted resolution $Res(y,a)$.

In a similar way, we get the following corollary from Theorem 1.22.

Corollary 2.6: *Let $F_k =< a_1,\dots,a_k >$ be a free group, and let $u_1(y,a),\dots,u_m(y,a)$ be a collection of words in the alphabet $\{y,a\}$, for which the group $Rlim(y,a) =< y,a|u_1(y,a),\dots,u_m(y,a) >$ is a restricted limit group. Let $Res(y,a)$ be a well-structured resolution of the restricted limit group $Rlim(y,a)$,* *and let $Comp(Res)(z, y, a)$ be the completion of the resolution $Res(y, a)$, with a corresponding completed limit group $Comp(Rlim)(z, y, a)$.*

Let $\Sigma_1(x, y, a) = 1, \ldots, \Sigma_r(x, y, a) = 1$ be a collection of systems of equations over F_k, and let $\Psi_1(x, y, a), \ldots, \Psi_r(x, y, a)$ be a set of collections of words in the alphabet $\{x, y, a\}$. Suppose that the sentence:

$$\forall y\ (u_1(y,a) = 1, \ldots, u_m(y,a) = 1)\ \exists x\ (\Sigma_1(x,y,a) = 1 \wedge \Psi_1(x,y,a) \neq 1) \vee \cdots$$
$$\vee \cdots \vee (\Sigma_r(x,y,a) = 1 \wedge \Psi_r(x,y,a) \neq 1)$$

is a truth sentence.

Then there exists a covering closure

$$Cl(Res)_1(s,z,y,a)_1, \ldots, Cl(Res)_q(s,z,y,a),$$

and formal resolutions $FRes_1(x,z,y,a), \ldots, FRes_q(x,z,y,a)$, where for each index i, $1 \leq i \leq q$, there exists some index $j(i)$, $1 \leq j(i) \leq r$, for which $FRes_i(x,z,y,a)$ is a formal resolution in the formal Makanin–Razborov diagram of the system of equations $\Sigma_{j(i)}(x,y,a) = 1$ and the well-structured resolution $Res(y,a)$, with the following properties:

(i) *For each index i, $1 \leq i \leq q$, the formal resolution $FRes_i(s,z,y,a)$ terminates with a formal limit group of the form $Cl_i(s,z,y,a) * F_i$ for some (possibly trivial) free group F_i.*

(ii) *For each index i, $1 \leq i \leq q$, there exists a formal solution $x_i(s,z,y,a)$ that factors through the formal resolution $FRes_i(x,z,y,a)$, so that there exists a specialization (s_0^i, y_0^i, f_0^i, a) that factors through the i-th closure $Cl(Res)_i(s,z,y,a)$ so that all the words in the collection*

$$\Psi_{j(i)}(x_i(s_0^i, y_0^i, f_0^i, a), y_0^i, f_0^i, a)$$

are not the trivial element in F_k.

In particular, in case the above sentence is a truth sentence, the set of closures that appear as the terminal points of the formal resolutions in the collection of formal Makanin–Razborov diagrams of the systems $\Sigma_j(x,y,a) = 1$ and the resolution $Res(y,a)$ contains a covering closure of the restricted resolution $Res(y,a)$.

3. Graded formal limit groups

Theorems 1.18 and 1.22 show that given a restricted limit group and a well-structured resolution of that limit group, the validity of an AE sentence defined over the given restricted limit group implies the existence of a finite collection of formal solutions defined over a covering closure of the given well-structured resolution of that limit group.

In the previous section we studied the entire family of formal solutions of a system of equations $\Sigma(x,y,a)=1$ with respect to a given well-structured resolution $Res(y,a)$. For the purpose of our quantifier elimination procedure, an understanding of the structure of such families of formal solutions is not sufficient. Our quantifier elimination procedure requires a careful analysis of parametric families of formal solutions, i.e., given a graded limit group $Glim(y,p,a)$, a (well-structured) graded resolution of it, $GRes(y,p,a)$, and a parametric system of equations $\Sigma(x,y,p,a)=1$, we need to study how the family of formal solutions of the system $\Sigma(x,y,p,a)$ and the graded resolution $GRes(y,p,a)$ varies with a change of the defining parameters p. To analyze the variation of the set of formal solutions, we combine the techniques used in the previous section with those used in the construction of the graded Makanin–Razborov diagram of a graded limit group presented in sections 9 and 10 in [Se].

Let $Glim(y,p,a)$ be a graded limit group, and let $GRes(y,p,a)$ be a well-structured graded resolution of it. The graded resolution $GRes(y,p,a)$ terminates in either a rigid graded limit group, $Rgd(y,p,a)$, or a solid graded limit group, $Sld(y,p,a)$. By section 10 in [Se], for any given specialization p_0 of the defining parameters p, the graded resolution $GRes(y,p,a)$ "covers" at most finitely many ungraded resolutions of the form $Res(y,p_0,a)$, where each of the resolutions $Res(y,p,a)$ terminates in either a rigid specialization of $Rgd(y,p,a)$, in case the graded resolution $GRes(y,p,a)$ terminates in the rigid graded limit group $Rgd(y,p,a)$, or a solid family of specializations of $Sld(y,p,a)$, in case the graded resolution $GRes(y,p,a)$ terminates in the solid graded limit group $Sld(y,p,a)$. Since the graded resolution $GRes(y,p,a)$ is well-structured, so are all the non-degenerate ungraded resolutions $Res(y,p_0,a)$ covered by it.

In analyzing the entire collection of formal solutions of the system $\Sigma(x,y,p_0,a)=1$ and an ungraded resolution of the form $Res(y,p_0,a)$, which is "covered" by the graded resolution $GRes(y,p,a)$, we will assume that the ungraded resolution $Res(y,p_0,a)$ is non-degenerate. According to section 11 of [Se], there are finitely many ways for the "covered" ungraded resolution $Res(y,p_0,a)$ to be degenerate, and the collection of all formal solutions defined over ungraded resolutions that are degenerated in a particular way (one of the finitely many possibilities) can be analyzed using precisely the same procedure used to analyze the collection of formal solutions defined over non-degenerate ungraded resolutions.

In Section 1 of this paper we defined the completion of a well-structured ungraded resolution. Hence, with each of the (non-degenerate, well-structured) ungraded resolutions $Res(y,p_0,a)$ that is covered by the graded resolution $GRes(y,p,a)$, we can associate its completion $Comp(Res)(z,y,p_0,a)$. Since all the (non-degenerate) ungraded resolutions $Res(y,p_0,a)$ are covered by the graded resolution $GRes(y,p,a)$, they all terminate in the same free group which we denote by $< a,f >$, and the variables z added at the various levels in each of the completions $Comp(Res)(z,y,p_0,a)$, associated with the ungraded resolutions $Res(y,p_0,a)$, are added in a compatible way in accordance with the structure of the graded resolution $GRes(y,p,a)$. Also, the decompositions associated with the different levels of the completed resolutions $Comp(Res)(y,p_0,a)$ are compatible, which implies the compatibility of the QH subgroups appearing in the different levels, as well as the abelian vertex and edge groups that appear in the various levels of those completions.

As in the ungraded setup, given a graded limit group, $Glim(y,p,a)$, and a graded well-structured resolution of it, $GRes(y,p,a)$, to analyze the family of all formal solutions of a given system of parametric equations $\Sigma(x,y,p,a)=1$ and a (non-degenerate) ungraded resolutions $Res(y,p_0,a)$ that is covered by the given graded resolution $GRes(y,p,a)$, we need to present *graded formal limit groups* and their canonically associated *graded formal Makanin–Razborov diagrams.* As we did in the first two sections of this paper, to construct graded formal limit groups we start by defining *graded test sequences* associated with the graded resolution $GRes(y,p,a)$.

The structure of all the non-degenerate ungraded resolutions $GRes(y,p_0,a)$ covered by the graded resolution $GRes(y,p,a)$ is compatible. Hence, as we did in the ungraded case, to define graded test sequences we fix a (bottom to top) order of the QH and abelian vertex groups that appear in the completion of the non-degenerate ungraded resolutions $GRes(y,p_0,a)$ covered by $GRes(y,p,a)$. We also fix an (ordered) bases for the abelian and pegged abelian groups that appear in such (non-degenerate) completed resolutions $Comp(GRes)(z,y,a)$.

Definition 3.1: Let $Glim(y,p,a)$ be a graded limit group, and let $GRes(y,p,a)$ be a well-structured graded resolution of $Glim(y,p,a)$ that terminates in either a rigid graded limit group $Rgd(y,p,a)$ or a solid graded limit group $Sld(y,p,a)$. Let X be the Cayley graph of $F_k=< a_1,\dots,a_k >$ and let Y be the Cayley graph of $Comp(Glim)(z,y,p,a)$. Let d_X and d_Y be the corresponding (simplicial) metrics.

Let $\{(y_n,p_n,a)\}$ be a sequence of either rigid or solid specializations of the terminal rigid or solid graded limit groups of the graded resolution $GRes(y,p,a)$, $Rgd(y,p,a)$ or $Sld(y,p,a)$ in correspondence, and let $\{GRes(y,p_n,a)\}$ be the ungraded resolutions covered by $GRes(y,p,a)$ corresponding to these specializations. We say that the sequence $\{(y_n,p_n,a)\}$, together with a sequence of automorphisms $\{(\nu_n^i,\tau_n^i)\}$ of the QH subgroups that appear in the graded abelian decompositions associated with the various levels in the completed resolutions $Comp(GRes)(z,y,p_n,a)$, and a sequence of homomorphisms

$$\{\lambda_n\colon Comp(Glim)(z,y,p_n,a)\to F_k\},$$

is a *graded test sequence* if the following conditions hold.

For every index n:

(i) The ungraded resolutions $GRes(y,p_n,a)$ are non-degenerate.

(ii) Conditions (i)–(xiv) of Definition 1.20 (defining an ungraded test sequence) hold for the completed ungraded resolution $Comp(GRes)(z,y,p_n,a)$, the automorphisms $\{(\nu_n^i,\tau_n^i)\}$ and the homomorphism

$$\lambda_n\colon Comp(GRes)(z,y,p_n,a)\to F_k.$$

(iii) If $g\in Comp(Glim)(z,y,p,a)$, $d_Y(g,id.)\le n$, and g is not elliptic in at least one of the abelian decompositions associated with the various levels of the (ungraded) completed resolution $Comp(Res)(z,y,p_n,a)$, then

$$n\cdot\max d_X(p_n,id.)<d_X(\lambda_n(g),id.).$$

Lemma 3.2: *Let $GRes(y,p,a)$ be a well-structured graded resolution of the graded limit group $Glim(y,p,a)$. Let $\{(y_n,p_n,a)\}$ be a sequence of either rigid or solid specializations of the terminal rigid or solid limit group, $Rgd(y,p,a)$ or $Sld(y,p,a)$, for which the corresponding ungraded resolutions $GRes(y,p_n,a)$ covered by the graded resolution $GRes(y,p,a)$ are non-degenerate. Then there exists a sequence of automorphisms $\{(\nu_n^i,\tau_n^i)\}$ of the QH subgroups that appear in the abelian decompositions associated with the various levels of the completed resolutions $\{Comp(GRes)(z,y,p_n,a)\}$ and homomorphisms*

$$\{\lambda_n\colon Comp(Glim)(z,y,p_n,a)\to F_k\},$$

so that these sequences together with the given sequence of specializations $\{(y_n,p_n,a)\}$ is a graded test sequence. Furthermore, given any two integers $s_1<s_2$ we can choose the n-th specialization of a basis element q_r^t of some of the (pegged) abelian groups that appear in an abelian decomposition associated with some level of the completed resolution $Comp(GRes)(y,p_n,a)$ to be of the

form $\lambda_n(q_i^t) = h_n^{ms_2+s_1}$ *for some integer* m, *and so that the element* h_n *has no non-trivial roots in* F_k.

Proof: Since for every index n, the ungraded resolution $GRes(y, p_n, a)$ is assumed to be non-degenerate, the proof of the lemma is identical to the proof of Lemma 1.21. ∎

Constructing graded test sequences, we are finally able to start analyzing the family of formal solutions of a given system of equations with respect to a given (well-structured) graded resolution. We start this analysis by defining *graded formal limit groups.*

Definition 3.3: Let $\Sigma(x, y, p, a) = 1$ be a system of equations, and let $GRes(y, p, a)$ be a well-structured graded resolution of a graded limit group $Glim(y, p, a)$. Let $\{(y_n, p_n, a)\}$ be a sequence of either rigid or solid specializations of the terminal rigid or solid limit group of the graded resolution $GRes(y, p, a)$, $Rgd(y, p, a)$ or $Sld(y, p, a)$ in correspondence. Let the sequence of specializations $\{(y_n, p_n, a)\}$ together with a sequence of automorphisms $\{(\nu_n^i, \tau_n^i)\}$ of the QH subgroups QH_i that appear in the abelian decompositions associated with the various levels of the completed resolutions $\{Comp(GRes)(z, y, p_n, a)\}$, and the sequence of homomorphisms

$$\{\lambda_n: Comp(GRes)(z, y, p_n, a) \to F_k\}$$

be a graded test sequence. Suppose that for every index n there exists some specialization x_n for which $\Sigma(x_n, \lambda_n(y), p_n, a) = 1$. For each index n, the homomorphism $\lambda_n: Comp(GRes)(z, y, p_n, a) \to F_k$ together with the specialization x_n define a homomorphism from the free group $F(x, z, y, p, a)$ to F_k, $\beta_n: F(x, z, y, p, a) \to F_k$.

We say that the given test sequence together with the sequence of specializations $\{x_n\}$ is a *convergent graded formal sequence*, if the corresponding sequence of homomorphisms $\{\beta_n: F(x, z, y, p, a) \to F_k\}$ is convergent (in the Gromov–Haussdorff topology, after appropriate rescaling). We call the obtained limit group a *graded formal limit group*, and denote it $GFL(x, z, y, p, a)$.

As in analyzing (ungraded) formal limit groups, with a graded formal limit group we can naturally associate a *graded formal closure.*

Definition 3.4: Let $GFL(x, z, y, p, a)$ be a graded formal limit group, and let

$$C(z, y, p, a) =< z, y, p, a >< GFL(x, z, y, p, a).$$

Since every abelian subgroup of a limit group is contained in a unique maximal abelian subgroup, every abelian subgroup of a limit group is contained in a unique maximal abelian subgroup of the same rank. By replacing the (pegged) abelian groups that appear in the abelian decompositions associated with the various levels in the graded completion, $C(z,y,p,a)$, by the maximal abelian groups of the same rank containing them in $GFL(x,z,y,p,a)$, we obtain a (canonical) subgroup (graded closure) in $GFL(x,z,y,p,a)$, which we call the *graded formal closure* of $C(z,y,p,a)$ in the graded formal limit group $GFL(x,z,y,p,a)$. We denote the graded formal closure $GFCl(s,z,y,p,a)$.

To analyze graded formal limit groups we need to construct their associated (canonical) *graded formal Makanin–Razborov diagrams.* To do that we need to study the algebraic structure of graded formal limit groups $GFL(x,z,y,p,a)$ relative to the subgroup $< z,y,p,a >$. To do that we start by defining the *graded formal JSJ decomposition* of a graded formal limit group.

In the sequel below let $GRes(y,p,a)$ be a well-structured graded Makanin–Razborov resolution of a graded limit group $Glim(y,p,a)$, and let $\Sigma(x,y,p,a)=1$ be a system of equations. Let $GFL(x,z,y,p,a)$ be a graded formal limit group with respect to the system of equations $\Sigma(x,y,p,a)=1$ relative to the well-structured graded resolution $GRes(y,p,a)$. We further use the notation

$$C(z,y,p,a) =< z,y,p,a >< GFL(x,z,y,p,a),$$
$$AP =< a,p >< GFL(x,z,y,p,a).$$

Definition 3.5: Suppose the graded formal limit group $GFL(x,z,y,p,a)$ does not split into a non-trivial free product in which $C(z,y,p,a)$ is contained in one of the factors. A one edge essential abelian splitting, $GFL(x,z,y,p,a)=D*_A E$ or $GFL(x,z,y,p,a)=D*_A$, in which $C(z,y,p,a)$ is elliptic, is called a *graded formal abelian splitting.*

The arguments used in section 2 of [Se] to construct the abelian JSJ decomposition of a limit group naturally generalize to construct the graded formal JSJ decomposition of a graded formal limit group.

Theorem 3.6 (cf. ([Se], 9.2)): *Let $GFL(x,z,y,p,a)$ be a graded formal limit group which does not split to a non-trivial free product in which the subgroup $C(z,y,p,a)$ is contained in one of the factors.*

There exists an essential (perhaps trivial) abelian splitting of $GFL(x,z,y,p,a)$, which we call the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$, with the following properties:

(i) *$C(z,y,p,a)$ is elliptic in the graded formal JSJ decomposition. If the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ is non-trivial, then any one edge abelian splitting obtained by collapsing all edges but one in the graded formal JSJ decomposition is a graded formal splitting of $GFL(x,z,y,p,a)$.*

(ii) *Every graded (formal) canonical maximal QH subgroup (CMQ) of $GFL(x,z,y,p,a)$ is conjugate to a vertex group in the graded formal JSJ decomposition. Every graded formal QH subgroup of $GFL(x,z,y,p,a)$ can be conjugated into one of the graded formal CMQ subgroups of $GFL(x,z,y,p,a)$. Every vertex group in the graded formal JSJ decomposition which is not a CMQ subgroup of $GFL(x,z,y,p,a)$ is elliptic in any graded formal splitting of $GFL(x,z,y,p,a)$.*

(iii) *A one edge graded formal splitting*

$$GFL(x,z,y,p,a)=D*_A E \quad \text{or} \quad GFL(x,z,y,p,a)=D*_A,$$

which is hyperbolic in another such elementary graded formal splitting, is obtained from the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ by cutting a surface corresponding to a graded (formal) CMQ subgroup of $GFL(x,z,y,p,a)$ along an essential s.c.c..

(iv) *Let Θ be a one edge graded formal splitting $GFL(x,z,y,p,a)=D*_A E$ or $GFL(x,z,y,p,a)=D*_A$, which is elliptic with respect to any other such one edge graded formal splitting of $GFL(x,z,y,p,a)$. Then Θ is obtained from the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ by a sequence of collapsings, conjugations, and modifying boundary monomorphisms by conjugations.*

(v) *Let Λ be a general graded formal abelian splitting of $GFL(x,z,y,p,a)$. There exists a graded formal splitting Λ_1 obtained from the graded formal JSJ decomposition by splitting the graded (formal) CMQ subgroups along essential s.c.c. on their corresponding surfaces, so that there exists a $GFL(x,z,y,p,a)$-equivariant simplicial map between a subdivision of the Bass–Serre tree T_{Λ_1} to T_Λ.*

(vi) *If JSJ_1 is another graded formal JSJ decomposition of $GFL(x,z,y,p,a)$, then JSJ_1 is obtained from the graded formal JSJ decomposition by a sequence of slidings, conjugations, and modifying boundary monomorphisms by conjugations (see section 1 of [Ri-Se2] for these notions).*

(vii) *If $GFL(x,z,y,p,a)$ admits a formal abelian splitting, then the graded formal JSJ decomposition of $GFL(x,z,y,p,f,a)$ is non-trivial.*

As in analyzing graded limit groups, the graded formal JSJ decomposition allows one to "further simplify" graded formal limit groups in case it is not trivial. In order to construct *graded formal Makanin–Razborov diagrams* we still need to study the structure of *rigid* graded formal limit groups (i.e., graded formal limit groups with trivial graded formal JSJ decomposition), as well as *solid* graded formal limit groups.

In both cases we divide our study into two parts. First we assume that the graded resolution we have started with, $GRes(y,a)$, terminates in a rigid limit group, and then we analyze the case in which the graded resolution $GRes(y,a)$ terminates in a solid limit group.

THEOREM 3.7: *Let $GFL(x,z,y,p,a)$ be a graded formal limit group that does not split as a non-trivial free product in which the subgroup $C(z,y,p,a)$ is contained in one of the factors, and suppose that the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ is trivial.*

(i) *Suppose that the graded resolution we have started with, $GRes(y,a)$, terminates in a rigid limit group. Then there exists a rigid graded (not formal!) limit group $Rgd(b,p,a)$, so that $GFL(x,z,y,p,a)$ can be written as an amalgamated product*

$$GFL(x,z,y,p,a) = Rgd(b,p,a) *_{Term(\hat{s},z,p,a)} GFCl(s,z,y,p,a).$$

(ii) *Suppose that the graded resolution we have started with, $GRes(y,a)$, terminates in a solid limit group. Then there exists a graded (not formal!) limit group $Glim_t(b,p,a)$, which is either rigid or solid (with respect to the defining parameters p), so that $GFL(x,z,y,p,a)$ can be written as an amalgamated product*

$$GFL(x,z,y,p,a) = Glim_t(b,p,a) *_{Term(\hat{s},z,p,a)} GFCl(s,z,y,p,a),$$

where in both cases $GFCl(s,z,y,p,a)$ is the graded formal closure associated with $GFL(x,z,y,p,a)$ and $Term(\hat{s},z,p,a)$ is the terminal subgroup of the graded formal closure $GFCl(s,z,y,p,a)$.

Proof: Suppose that the graded resolution $GRes(y,a)$ terminates in a rigid limit group, and the graded formal limit group $GFL(x,z,y,p,a)$ does not split as a non-trivial free product in which $C(z,y,p,a)$ is contained in one of the factors, and the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ is trivial.

The graded formal limit group $GFL(x,z,y,p,a)$ was constructed from a convergent graded formal sequence, obtained from a test sequence composed of specializations $\{(y_n,p_n,a)\}$, of the terminal rigid graded limit group of the graded resolution $GRes(y,p,a)$, automorphisms $\{(\nu_n^i,\tau_n^i)\}$ of the QH subgroups that appear in the various levels in the completed resolutions $Comp(GRes)(z,y,p_n,a)$, and homomorphisms $\{\lambda_n\colon Comp(GRes)(z,y,p_n,a)\to F_k\}$, and a sequence of specializations $\{x_n\}$, for which $\Sigma(x_n,\lambda_n(y),p_n,a)=1$, so that the sequence of homomorphisms $\alpha_n : F(x,z,y,p,a)\to F_k$ converges.

From the convergence of the sequence of homomorphisms

$$\{\alpha_n\colon F(x,z,y,p,a)\to F_k\},$$

we get a (non-trivial) stable action of the graded formal limit group $GFL(x,z,y,p,a)$ on a real tree Y_1. By the classification of stable actions of f.g. groups on real trees (see theorem 1.5 in [Se]), with the action of $GFL(x,z,y,p,a)$ on the real tree Y_1, there is an associated (non-trivial) graph of groups Λ_1 with abelian edge stabilizers. If the subgroup $C(z,y,p,a)$ is contained in a vertex group of Λ_1, or more generally, if $C(z,y,p,a)$ is contained in the fundamental group of a proper subgraph of Λ_1, the graded formal limit group $GFL(x,z,y,p,a)$ admits a non-trivial graded formal decomposition, a contradiction to our assumptions.

Suppose that the graded limit group $Glim(y,p,a)$, associated with the graded resolution $GRes(y,p,a)$, is of the form $Glim(y,p,a)=Rgd(w,p,a)*<f>$, where $F=<f_1,\dots,f_c>$ is a free group and $Rgd(w,p,a)$ is a rigid graded (not formal!) limit group, and the graded resolution $GRes(y,p,a)$ terminates in the rigid graded limit group $Rgd(w,p,a)$. In this case, since $GFL(x,z,y,p,a)$ admits no free decomposition in which $C(z,y,p,a)$ is contained in one of the factors, since the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ is trivial, and since $GFL(x,z,y,p,a)$ admits a non-trivial action on a real tree, necessarily $GFL(x,z,y,p,a)=Rgd(b,p,a)*F$, so $GFL(x,z,y,p,a)=Rgd(b,p,a)*_{Rgd(w,p,a)}(Rgd(w,p,a)*F)$, which proves the statement of the theorem in case

$$Glim(x,z,y,p,a)=Rgd(w,p,a)*F.$$

Hence, we may assume that $Glim(y,p,a)$ is not of the form $Rgd(w,p,a)*F$ for $Rgd(w,p,a)$ rigid, F free, and $GRes(y,p,a)$ terminates in $Rgd(w,p,a)$. By the properties of a graded formal test sequence (Definition 3.1), the action of $C(z,y,p,a)$ on the real tree Y_1 either contains a unique IET orbit and finitely many orbits of point stabilizers, or the action is discrete with a unique orbit of non-abelian point stabilizer and a unique edge stabilized by some abelian subgroup of $C(z,y,p,a)$. Since $GFL(x,z,y,p,a)$ inherits from its action on Y_1 a graph of groups of the same form as the one $C(z,y,p,f,a)$ inherits from its action on Y_1, the action of $GFL(x,z,y,p,a)$ on Y_1 must contain either a unique orbit of an IET component and finitely many orbits of point stabilizers, or its action is discrete with one orbit of non-abelian point stabilizers and one orbit of edges stabilized by some abelian subgroup of $GFL(x,z,y,p,a)$.

In both types of the dynamics of the action of $GFL(x,z,y,p,a)$ on the limit tree Y_1, the generators x of the graded formal limit group $GFL(x,z,y,p,a)$ can be written as words in elements from the vertex stabilizers V_1^i and elements from the graded formal closure $GFCl(s,z,y,p,a)$. "Uncovering" the top level, we continue with each of the non-abelian, non-QH vertex stabilizers V_1^i in the graph of groups Λ_1, and the actions these vertex stabilizers obtain on corresponding real trees Y_2^i. Repeating the argument used for analyzing the action of $GFL(x,y,z,p,a)$ on the real tree Y_1, for the actions of the groups V_1^i on the real trees Y_2^i, if $C(z,y,p,a) \cap V_1^i$ is not of the form $Term(w,p,a) * F$ where $F =< f_1, \ldots, f_c >$ is free, and $Term(w,p,a)$ is the terminal subgroup of the completion $C(z,y,p,a)$, i.e., if we did not yet "finished" with all the levels of the graded resolution $GRes(y,p,a)$, the action of V_1^i on Y_2^i must have the same dynamics as one of the two possibilities described above, i.e., either it contains a unique orbit of an IET component and finitely many orbits of point stabilizers, or the action is discrete with one orbit of non-abelian point stabilizers and one orbit of edges stabilized by some abelian subgroup of the graded formal closure $GFCl(s,z,y,p,a)$. Hence, the generators x of the formal limit group $GFL(x,z,y,p,a)$ can again be written as words in elements from vertex stabilizers W_2^i and elements from the graded formal closure $GFCl(s,z,y,p,a)$.

This process continues until at some stage ℓ, $C(z,y,p,a) \cap V_\ell^i$ is of the form $Term(w,p,a) * F$, where F is free, and $Term(w,p,a)$ is the terminal group in the graded completion $C(z,y,p,a)$. Therefore, a finite induction argument implies that every element in the graded formal limit group $GFL(x,z,y,p,a)$ can be written as a word in elements from the graded formal closure $GFCl(s,z,y,p,a)$, and elements from the terminal vertex V_ℓ, which is necessarily rigid. We denote this terminal vertex group V_ℓ, $Rgd(b,p,a)$. Clearly, the terminal group of the graded formal closure $GFCl(s,z,p,a)$, which we denote $Term(\hat{s},z,p,a)$, is a subgroup of $Rgd(b,p,a)$, and part (i) of the theorem follows.

The proof of the theorem in case the graded resolution $GRes(y,p,a)$ terminates in a solid limit group (part (ii)) is identical. ∎

Defining graded formal limit groups and their graded formal JSJ decompositions, and showing that a (relatively) freely indecomposable graded formal limit group with trivial formal JSJ decomposition has the "same structure" as a graded formal closure $GFCl(s,z,y,p,a)$, we are able to present *graded formal Makanin–*

Razborov diagrams.

Let $GRes(y,p,a)$ be a graded well-structured resolution of a graded limit group $Glim(y,p,a)$, and let $\Sigma(x,y,p,a)=1$ be a system of equations.

Recall that a graded formal limit group $GFL(x,z,y,p,a)$ of the system $\Sigma(x,y,p,a)=1$ with respect to the graded resolution $GRes(y,p,a)$ was constructed from a graded formal test sequence.

On the set of graded formal limit groups $GFL(x,z,y,p,a)$ associated with the system $\Sigma(x,z,y,p,a)=1$, and the graded formal resolution $GRes(y,p,a)$, we can naturally define a partial order. We say that $GFL_1(x,z,y,p,a) \geq GFL_2(x,z,y,p,a)$, if there exists an epimorphism

$$\eta: GFL_1(x,z,y,p,a) \to GFL_2(x,z,y,p,a).$$

By the arguments used in the construction of the Makanin–Razborov diagram of a limit group (lemmas 5.4 and 5.5 in [Se]), given the system of equations $\Sigma(x,y,p,a)=1$ and the graded resolution $GRes(y,p,a)$, there exist *maximal graded formal limit groups* and up to the natural equivalence relation on graded formal limit groups there are only finitely many equivalence classes of maximal graded formal limit groups which we denote

$$MGFL_1(x,z,y,p,a),\ldots,MGFL_n(x,z,y,p,a).$$

To construct the *graded formal Makanin–Razborov diagram* of the system $\Sigma(x,y,p,a)=1$ with respect to the graded resolution $GRes(y,p,a)$, we proceed as in the construction of the graded Makanin–Razborov diagrams (see sections 9 and 10 in [Se]). We first decompose each of the maximal graded formal limit groups into the maximal (canonical) free decomposition in which the subgroup

$$C(z,y,p,a)=< z,y,p,a >< MGFL_i(x,z,y,p,a)$$

is contained in a factor. We associate the graded formal JSJ decomposition with the factor containing $C(z,y,p,a)$, and the (standard) abelian JSJ decomposition with each of the other factors. These associated JSJ decompositions naturally define the *graded formal modular groups* of each of the maximal graded formal limit groups $MGFL_i(x,z,y,p,a)$. Having these modular groups, we are able to define *graded formal shortening quotients* of each of the maximal graded formal limit groups.

Theorem 3.7 analyzes graded formal limit groups with trivial graded formal JSJ decomposition. In a similar way, one can analyze graded formal limit groups with non-trivial graded formal JSJ decomposition, which are isomorphic to one of their graded formal shortening quotients, which we call *solid formal limit groups*.

THEOREM 3.8: *Let $GFL(x,z,y,p,a)$ be a graded formal limit group which does not split as a non-trivial free product in which the subgroup $C(z,y,p,a)$ is contained in one of the factors. Suppose that the graded formal JSJ decomposition of $GFL(x,z,y,p,a)$ is non-trivial, and there exists a graded formal shortening quotient of $GFL(x,z,y,p,a)$ that is isomorphic to $GFL(x,z,y,p,a)$.*

Then there exists a solid (not formal!) limit group $Sld(b,p,a)$, so that the graded formal limit group $GFL(x,z,y,p,a)$ can be written in the form

$$GFL(x,z,y,p,a) = Sld(b,p,a) *_{Term(\hat{s},z,p,a)} GFCl(s,z,y,p,a),$$

where $GFCl(s,z,y,p,a)$ is the graded formal closure associated with $GFL(x,z,y,p,a)$, and $Term(\hat{s},z,p,a)$ is the terminal subgroup of the graded formal closure $GFCl(s,z,y,p,a)$.

Proof: Identical with the proof of Theorem 3.7. ∎

The combination of Theorems 3.7 and 3.8 allows us to complete the construction of the graded formal Makanin–Razborov diagram. Suppose that the graded formal limit group $GFL(x,z,y,p,a)$ admits a non-trivial graded formal JSJ decomposition, and every graded formal shortening quotient of it is a proper quotient. By the arguments used to prove lemmas 5.4 and 5.5 in [Se], there are maximal graded formal shortening quotients, and up to the natural equivalence classes of graded formal limit groups there are only finitely many equivalence classes of maximal graded formal shortening quotients of each of the maximal graded formal limit groups $MGFSQ_1(x,z,y,p,a),\ldots,MGFSQ_n(x,z,y,p,a)$.

Since a graded formal limit group is in particular a limit group, any properly decreasing sequence of graded formal limit groups terminates. Therefore, if we continue the above construction of maximal graded formal free decompositions, graded formal JSJ decompositions and graded formal modular groups for each of the factors, and then collect the (canonical) family of maximal graded formal shortening quotients and assume each graded formal shortening quotient is indeed a proper quotient, we are guaranteed that this iterative construction terminates. When the process of collecting maximal graded formal shortening quotients which are proper quotients terminates, we are left with either rigid or solid formal limit groups. By Theorems 3.7 and 3.8, a rigid or solid formal limit group has the form

$$GFL(x,z,y,p,a) = Glim_t(b,p,a) *_{Term(\hat{s},z,p,a)} GFCl(s,z,y,p,a)$$

for some rigid or solid (not formal!) limit group $Glim_t(b,p,a)$. We continue the diagram with the graded (not formal!) Makanin–Razborov diagrams of the (graded) rigid or solid limit group $Glim_t(b,p,a)$.

We call the obtained diagram the *graded formal Makanin–Razborov diagram* of the system of equations $\Sigma(x,y,p,a)=1$ with respect to the graded resolution $GRes(y,p,a)$. We call each path in this (directed) diagram a *graded formal* (Makanin–Razborov) *resolution* and denote it $GFRes(x,z,y,p,a)$. Note that with every such graded formal resolution one can naturally associate a graded formal closure $GFCl(s,z,y,p,a)$ of the original graded resolution $Gres(y,p,a)$. Also, note that every non-QH, non-abelian vertex group and every edge group in the abelian decomposition associated with the terminal rigid or solid limit group of the graded resolution $GRes(y,a)$ with which we have started remains elliptic along all the graded formal resolutions of the formal limit group $GFL(x,z,y,p,a)$.

Theorems 1.18 and 1.22 prove that if an AE sentence is a truth sentence over the set of specializations of some (ungraded) limit group, then there exist formal solutions that may serve as "witnesses" for the correctness of the sentence in a "generic" point.

Corollaries 2.5 and 2.6 state the existence of these "witnesses" in terms of the formal Makanin–Razborov diagram associated with a system of equations $\Sigma(x,y,a)=1$ with respect to a given well-structured resolution $Res(y,a)$. Since our "trial and error" procedure for quantifier elimination is based on finding "witnesses" in strata of the parameter set, i.e., graded formal solutions that satisfy the properties of Theorems 1.18 and 1.22 on "nice" subsets of the ambient set of defining parameters, we present generalizations of Corollaries 2.5 and 2.6 in the graded framework.

Corollary 3.9: *Let $F_k=<a_1,\dots,a_k>$ be a free group, let $Glim(y,p,a)$ be a graded limit group defined by the set of relations $u_1(y,p,a)=1,\dots,u_m(y,p,a)=1$, and let $GRes(y,p,a)$ be a well-structured graded resolution of the graded limit group $Glim(y,p,a)$.*

Let $\Sigma(x,y,p,a)=1$ be a system of equations over F_k, and let

$$v_1(x,y,p,a),\dots,v_r(x,y,p,a)$$

be a collection of words in the alphabet $\{x,y,p,a\}$. Let $T(p)$ be defined by the predicate

$$T(p)=\forall y \quad (u_1(y,p,a)=1,\dots,u_m(y,p,a)=1) \quad \exists x \quad \Sigma(x,y,p,a)=1\wedge$$
$$\wedge v_1(x,y,p,a)\neq 1,\dots,v_r(x,y,p,a)\neq 1.$$

If a specialization $p_0\in T(p)$, and (y_0,p_0,a) is a specialization of the terminal *rigid or solid graded limit group of the graded resolution $GRes(y,p,a)$, then there exists a collection of formal resolutions*

$$FRes_1(x,z,y,p_0,a),\ldots,FRes_q(x,z,y,p_0,a)$$

covered by graded formal resolutions

$$GFRes_{i_1}(x,z,y,p,a),\ldots,GFRes_{i_q}(x,z,y,p,a)$$

from the graded formal Makanin–Razborov diagram of the system $\Sigma(x,y,p,a) = 1$ with respect to the graded resolution $GRes(y,p,a)$, for which:

(i) *The collection of formal closures*

$$FCl_1(s,z,y,p_0,a),\ldots,FCl_q(s,z,y,p_0,a)$$

associated with the formal resolutions

$$FRes_1(x,z,y,p_0,a),\ldots,FRes_q(x,z,y,p_0,a)$$

is a covering closure of the (ungraded) resolution $GRes(y,p_0,a)$ corresponding to the specialization (y_0,p_0,a).

(ii) *For each index i, $1 \leq i \leq q$, there exists a formal solution $x_i(s,z,y,p_0,a)$ that factors through the formal resolution $FRes_i(x,z,y,p_0,a)$, and a specialization $(s_0^i,z_0^i,y_0^i,p_0,a)$, so that for every index j*

$$v_j(x_i(s_0^i,z_0^i,y_0^i,p_0,a),y_0^i,p_0,a) \neq 1.$$

In particular, in case the above predicate is truth at p_0, then for any specialization (y_0,p_0,a) of the terminal rigid or solid limit group of the graded resolution $GRes(y,p,a)$, the set of formal closures associated with formal resolutions covered by the graded formal resolutions in the graded formal Makanin–Razborov diagram of the system $\Sigma(x,y,p,a) = 1$ with respect to the graded resolution $GRes(y,p,a)$ contains a covering closure of the (ungraded) resolution $GRes(y,p_0,a)$ corresponding to the specialization (y_0,p_0,a).

In a similar way, we get the following corollary from Theorem 2.6.

Corollary 3.10: *Let $F_k = < a_1,\ldots,a_k >$ be a free group, let $Glim(y,p,a)$ be a graded limit group defined by the set of relations $u_1(y,p,a) = 1,\ldots,u_m(y,p,a) = 1$, and let $GRes(y,p,a)$ be a well-structured graded resolution of the graded limit group $Glim(y,p,a)$.*

Let $\Sigma_1(x,y,p,a)=1,\dots,\Sigma_r(x,y,p,a)=1$ be a collection of systems of equations over F_k, and let $\Psi_1(x,y,p,a),\dots,\Psi_r(x,y,p,a)$ be a set of collections of words in the alphabet $\{x,y,p,a\}$. Let $T(p)$ be defined by the predicate

$$T(p)=\forall y \quad (u_1(y,p,a)=1,\dots,u_m(y,p,a)=1) \quad \exists x$$
$$(\Sigma_1(x,y,p,a)=1\wedge\Psi_1(x,y,p,a)\neq 1)\vee\cdots\vee(\Sigma_r(x,y,p,a)=1\wedge\Psi_r(x,y,p,a)\neq 1).$$

If a specialization $p_0\in T(p)$, and (y_0,p_0,a) is a specialization of the terminal rigid or solid graded limit group of the graded resolution $GRes(y,p,a)$, then there exists a collection of formal resolutions

$$FRes_1(x,z,y,p_0,a),\dots,FRes_q(x,z,y,p_0,a)$$

covered by graded formal resolutions

$$GFRes_{i_1}(x,z,y,p,a),\dots,GFRes_{i_q}(x,z,y,p,a)$$

from the graded formal Makanin–Razborov diagrams of the systems

$$\Sigma_{j(i_1)}(x,y,p,a),\dots,\Sigma_{j(i_q)}(x,y,p,a)$$

in correspondence, with respect to the graded resolution $GRes(y,p,a)$, for which:

(i) *The collection of formal closures*

$$FCl_1(s,z,y,p_0,a),\dots,FCl_q(s,z,y,p_0,a)$$

associated with the formal resolutions

$$FRes_1(x,z,y,p_0,a),\dots,FRes_q(x,z,y,p_0,a)$$

is a covering closure of the (ungraded) resolution $GRes(y,p_0,a)$ corresponding to the specialization (y_0,p_0,a).

(ii) *For each index i, $1\leq i\leq q$, there exists a formal solution $x_i(s,z,y,p_0,a)$ that factors through the formal resolution $FRes_i(x,z,y,p_0,a)$, and a specialization $(s_0^i,z_0^i,y_0^i,p_0,a)$, for which*

$$\Psi_{j(i)}(x_i(s_0^i,z_0^i,y_0^i,p_0,a),y_0^i,p_0,a)\neq 1.$$

In particular, in case the above predicate is truth at p_0, then for any specialization (y_0,p_0,a) of the terminal rigid or solid limit group of the graded resolution $GRes(y,p,a)$, the set of formal closures associated with formal resolutions covered by the graded formal resolutions in the graded formal Makanin–Razborov diagrams of the systems

$$\Sigma_1(x,y,p,a)=1,\dots,\Sigma_r(x,y,p,a)=1$$

with respect to the graded resolution $GRes(y,p,a)$ contains a covering closure of the (ungraded) resolution $GRes(y,p_0,a)$ corresponding to the specialization (y_0,p_0,a).

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