ON MANIFOLDS ADMITTING STABLE TYPE III_1 ANOSOV DIFFEOMORPHISMS

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Dedicated to the memory of Roy Adler whose work has been and still is a great inspiration for me.

ABSTRACT. We prove that for every $d \neq 3$, there is an Anosov diffeomorphism of $\mathbb{T}^d$ which is of stable Krieger type III_1 (its Maharam extension is weakly mixing). This is done by a construction of stable type III_1 Markov measures on the golden mean shift which can be smoothly realized as a C^1 Anosov diffeomorphism of $\mathbb{T}^2$ via the construction in our earlier paper.

1. Introduction

A *topological Markov shift* (TMS) on S is the shift on a shift invariant subset $\Sigma \subset S^{\mathbb{Z}}$ of the form

$$\Sigma_A := \left\{ x \in S^{\mathbb{Z}} : A_{x_i, x_{i+1}} = 1 \right\},$$

where $A = \{A_{s,t}\}_{s,t \in S}$ is a $\{0,1\}$ valued matrix on S. A TMS is (topologically) mixing if there exists $n \in \mathbb{N}$ such that $A^n_{s,t} > 0$ for every $s, t \in S$. TMS appear in ergodic theory as a symbolic model for $C^{1+\alpha}$ Anosov and Axiom A diffeomorphisms via the construction of Markov partition of the manifold M [AW, Sin, Bow, Adl]. In this work we will restrict our attention to measures (fully) supported on the Golden Mean $\Sigma \subset \{1,2,3\}^{\mathbb{Z}}$, which is defined by its adjacency matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The TMS Σ is the symbolic space arising from a Markov partition of the automorphism of $\mathbb{T}^2$ given by $f(x,y) = (x+y, x) \mod 1$, See Section 4. The connection between Σ and f was used in [Kos1] for the construction of conservative, ergodic C^1 Anosov diffeomorphisms of $\mathbb{T}^2$ without a Lebesgue absolutely continuous invariant measure (a.c.i.m.). Krieger in [Kri] has classified the nonsingular transformations up to orbit equivalence according to the associated flow on the ergodic decomposition of the Maharam extension. The Anosov diffeomorphisms in [Kos1] are of type III_1 which means that their Maharam extension is ergodic (the associated flow is a trivial flow). The first step of the construction was to build inhomogeneous Markov measures on Σ such that the shift with respect to (Σ, μ, T) defined by $(Tw)_n = w_{n+1}$ is a type III_1 transformation. After this step, the next step is to utilize the special structure of the μ constructed and realize an ergodic C^1 Anosov diffeomorphism $\mathfrak{g}$ of $\mathbb{T}^2$ which is orbit equivalent to (Σ, μ, T) hence of type III_1.

An ergodic nonsingular transformation T is stable type III_1 if for any ergodic probability preserving transformation S, $T \times S$ is a type III_1 transformation. In this paper we construct inhomogeneous Markov measures which are stable type III_1. In addition the construction is compatible with the second step (smooth realization) of [Kos1]. As a consequence we get the following result.

Theorem. *For every $d \geq 4$ or $d = 2$, there exists a C^1-Anosov diffeomorphism $\mathfrak{g}$ of $\left(\mathbb{T}^d, Lebesgue_{\mathbb{T}^d}\right)$ which is stable type* III_1*. In particular $\mathfrak{g}$ is conservative, ergodic with respect to the Lebesgue measure but it has no Lebesgue a.c.i.m.*

The reason $d \neq 3$ is that these diffeomorphisms are Cartesian products of a stable type III_1 Anosov diffeomorphism of $\mathbb{T}^2$ and an hyperbolic linear toral automorphism of $\mathbb{T}^{d-2}$ (hence we need $d-2 \geq 2$). The construction of the aforementioned Markov measures uses a reformulation of the stable type III_1 property in terms of measurable equivalence relations and then identifying a special class of holonomies (groupoid elements) of the tail relation which change the tail Radon-Nikodym cocycle in a controlled way. Using these holonomies and elementary Markov Chain arguments we provide an inductive construction of a measure μ such that its Maharam extension is an infinite measure preserving K automorphism, hence stable type III_1, see Corollary 2.

The structure of the paper is as follows. Section 2 is an introduction to the relevant material from non-singular ergodic theory, Markov Chains and countable equivalence relations. It provides the definitions and some simple consequences of them. In Section 3 we present the inductive constructions of the aforementioned Markov measures on Σ. Finally in Section 4, we explain how to use the new Markov measures construction and the smooth realization of [Kos1] to obtain a stable type III_1 Anosov diffeomorphism of $\mathbb{T}^2$.

2. Preliminaries on non-singular ergodic theory and Inhomogeneous Markov Shifts

A measurable transformation T of a standard (Polish) measure space $(X, \mathcal{B}, m)$ is non-singular if $T_*m = m \circ T^{-1}$ and m are equivalent measures. In the case where T is invertible, we denote by $(T^n)' = \frac{dm\circ T^n}{dm}$ the sequence of Radon-Nikodym derivatives of T. $\mathscr{X} = (X, \mathcal{B}, m, T)$ is ergodic if $m\left(A \triangle T^{-1}A\right) = 0$ implies $m(A) = 0$ or $m(X \backslash A) = 0$. If T is invertible and the measure m is non atomic then a necessary condition for ergodicity is that the system $\mathscr{X}$ is conservative, meaning that there exists no set $W \in \mathcal{B}$ of positive m measure such that $\{T^{-n}W\}_{n\in\mathbb{N}}$ are pairwise disjoint. Sets with the latter property are called wandering sets. By Halmos recurrence theorem, the system satisfies Poincare recurrence if and only if it is conservative and by Hopf's criteria $\mathscr{X}$ is conservative if and only if for m-almost every $x \in X$

$$\sum_{k=0}^{\infty}\left(T^k\right)'(x) = \infty.$$

The system $\mathscr{X}$ is a K-system if there exists $\mathscr{F} \subset \mathcal{B}$ such that

- $T^{-1}\mathscr{F} \subset \mathscr{F}$. In words $\mathscr{F}$ is a factor of $\mathscr{X}$.
- $\mathscr{F}$ is exact, $\bigcap_{n=0}^{\infty} T^{-n}\mathscr{F} = \{\emptyset, X\}$ modulo measure zero sets.
- Minimality condition: $\bigvee_{n=-\infty}^{\infty} T^n\mathscr{F} = \mathcal{B}$.
- T' is $\mathcal{F}$ measurable.

The first three conditions are as in the classical setting of ergodic theory (probability preserving maps) whereas the fourth condition comes in order to ensure that $\mathscr{X}$ is the unique natural extension of $(X, \mathscr{F}, m, T)$ up to measure theoretic isomorphism [SiT]. A transformation $\mathscr{X}$ is weakly mixing if for every ergodic probability preserving transformation $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, S)$, $T \times S$ is an ergodic transformation of $(X \times \Omega, \mathcal{B}_X \otimes \mathcal{B}_\Omega, m \times \mathbb{P})$. If $T \times S$ is ergodic for all ergodic, invertible ,nonsingular transformation, then $\mathscr{X}$ is mildly mixing. If an invertible transformation is mildly mixing then m is equivalent to a T

invariant probability [FW] while exact (hence non invertible) non singular transformations are mildly mixing [ALW].

The Maharam extension of an invertible transformation $\mathscr{X}$ is a transformation $\tilde{T}$ of $(X \times \mathbb{R}, \mathcal{B}_X \otimes \mathcal{B}_{\mathbb{R}})$ defined by $\tilde{T}(x,y) = (Tx, y - \log T'(x))$. It preserves the σ-finite measure defined by for all $A \in \mathcal{B}_X$, $B \in \mathcal{B}_{\mathbb{R}}$, $\mu(A \times B) = m(A)\int_B e^t dt$. $\tilde{T}$ is conservative (w.r.t. μ) if and only if T is conservative (w.r.t. m) and (the associated flow on) its ergodic decomposition determines the orbit equivalence class of T [Kri]. In particular for an ergodic T, $\tilde{T}$ is ergodic if and only if T is of type III_1, meaning that its Krieger ratio set is $\mathbb{R}$. See subsection 2.2 for the definition of the ratio set for a cocycle, the Krieger ratio set is the ratio set for $\varphi = \log T'$. A transformation $\mathscr{X}$ is stable type III_1 if for every ergodic probability preserving transformation $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, S)$, $T \times S$ is type III_1. For every probability preserving map S, $\frac{d(m\times\mathbb{P})\circ(T\times S)}{d(m\times\mathbb{P})} = T'(x)$, the Maharam extension of $T \times S$ is isomorphic to $\tilde{T} \times S$ via the map $\pi(x,w,y) = (x,y,w) : (X \times \Omega) \times \mathbb{R} \to (X \times \mathbb{R}) \times \Omega$. This observation leads to the following.

Proposition 1. *Let $\mathscr{X} = (X, \mathcal{B}, m, T)$ be an ergodic non-singular transformation. Then the following are equivalent:*
(i) $\tilde{T}$ is weak mixing.
(ii) T is stable type III_1.

Corollary 2. *If $\tilde{T}$ is a conservative K-authomorphism then T is stable type* III_1 *and for every weakly mixing probability preserving transformation* $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, S)$*, $T \times S$ is stable type* III_1.

Proof. The first part follows from [ALW, Par], see also [Aar1, Cor 3.1.8.], since a cartesian product of a conservative transformation with a probability preserving transformation is always conservative and a K-automorphism is the natural extension of an exact (hence mildly mixing) factor. For the second part, let R be an invertible, ergodic probability preserving transformation of $(Y, \mathcal{C}, \nu)$. Since S is weak mixing, then $S \times R$ is an ergodic probability preserving transformation. By the stable type III_1 property of T, $T \times (S \times R) = (T \times S) \times R$ is type III_1, showing that $T \times S$ is stable type III_1. □

The group of automorphisms of a probability space $(X, \mathcal{B}, m)$ is the set of all invertible $(\mathrm{mod}\, m)$ nonsingular transformations. An action of a countable group Γ on the space $(X, \mathcal{B}, m)$ is a map $\phi : \Gamma \to Aut(X, \mathcal{B}, m)$ which respects the group structure. The notation $\Gamma \curvearrowright (X, \mathcal{B}, m)$ will denote the group action of Γ on X and for $\gamma \in \Gamma$, $x \in X$, $\gamma x = \phi_\gamma(x)$.

2.1. **Markov chains.**

2.1.1. *Basics of Stationary (homogenous) Chains.* Let S be a finite set which we regard as the state space of the chain, $\pi = \{\pi(s)\}_{s\in S}$ a probability vector on S and $\mathrm{P} = (P(s,t))_{s,t\in S}$ a stochastic matrix. The vector π and P define a Markov chain $\{X_n\}$ on S by

$$\mathbb{P}_{\pi,P}(X_0 = t) = \pi(t) \text{ and } \mathbb{P}_{\pi,P}(X_n = s \,|X_1, .., X_{n-1}) := P(X_{n-1}, s).$$

P is *irreducible* if for every $s, t \in S$, there exists $n \in \mathbb{N}$ such that $P^n(s,t) > 0$ and P is *aperiodic* if for every $s \in S$, $\gcd\{n : P^n(s,s) > 0\} = 1$. Given an irreducible and aperiodic P, there exists a unique stationary ($\pi_{\mathrm{P}}\mathrm{P} = \pi_{\mathrm{P}}$) probability vector π_{P}.

2.1.2. *Inhomogeneous Markov shifts.* An inhomogeneous Markov Chain is a stochastic process $\{X_n\}_{n\in\mathbb{Z}}$ such that for each times $t_1,..,t_l\in\mathbb{Z}$, and $s_1,\ldots,s_l\in S$,

$$\mathbb{P}\left(X_{t_1}=s_1,X_{t_2}=s_2,\ldots,X_{t_l}=s_l\right)=\mathbb{P}\left(X_{t_1}=s_1\right)\prod_{k=1}^{l-1}\mathbb{P}\left(X_{t_{k+1}}=s_{k+1}\middle|\,X_{t_k}=s_k\right).$$

Note that unlike in the classical setting of Markov Chains $\mathbb{P}\left(X_{t_{k+1}}=s_{k+1}\middle|\,X_{t_k}=s_k\right)$ can depend on t_k. The ergodic theoretical formulation is as follows. Let $\{P_n\}_{n=-\infty}^{\infty}\subset M_{S\times S}$ be a sequence of stochastic matrices on S. In addition let $\{\pi_n\}_{n=-\infty}^{\infty}$ be a sequence of probability distributions on S so that for every $s\in S$ and $n\in\mathbb{Z}$,

$$\sum_{t\in S}\pi_{n-1}(t)\cdot P_n\left(t,s\right)=\pi_n(s). \tag{2.1}$$

Then one can define a measure on the collection of cylinder sets,

$$[b]_k^l:=\left\{x\in S^{\mathbb{Z}}:\ x_j=b_j\ \forall j\in[k,l]\cap\mathbb{Z}\right\}$$

(here $b\in S^{\mathbb{Z}}$) by

$$\mu\left([b]_k^l\right):=\pi_k\left(b_k\right)\prod_{j=k}^{l-1}P_j\left(b_j,b_{j+1}\right).$$

Since the equation (2.1) is satisfied, μ satisfies the consistency condition. Therefore by Kolmogorov's extension theorem μ defines a measure on $S^{\mathbb{Z}}$. In this case we say that μ is the Markov measure generated by $\{\pi_n,P_n\}_{n\in\mathbb{Z}}$ and denote $\mu=\mathrm{M}\left\{\pi_n,P_n:\ n\in\mathbb{Z}\right\}$. By $\mathrm{M}\left\{\pi,P\right\}$ we mean the measure generated by $P_n\equiv P$ and $\pi_n\equiv\pi$. We say that μ is non singular for the shift T on $S^{\mathbb{Z}}$ if $T_*\mu\sim\mu$. In order to check if a measure is shift non singular we apply the following reasoning of [Shi], see also [LM].

Definition 3. A filtration $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of a measure space $(X.\mathcal{B})$ is an increasing sequence of sub σ-algebras such that the minimal σ-algebra which contains $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is $\mathcal{B}$. Given a filtration $\{\mathcal{F}_n\}$, we say that $\nu\ll^{loc}\mu$ (ν is locally absolutely continuous with respect to μ) with respect to $\mathcal{F}_n$ if for every $n\in\mathbb{N}$ $\nu_n\ll\mu_n$ where $\nu_n=\nu|_{\mathcal{F}_n}$.

The question is when $\nu\ll^{loc}\mu$ implies $\nu\ll\mu$. Suppose that $\nu\ll^{loc}\mu$ w.r.t $\{\mathcal{F}_n\}$, set $z_n:=\frac{d\nu_n}{d\mu_n}$ and $\alpha_n(x):=z_n(x)\cdot z_{n-1}^{\oplus}(x)$ where $z_{n-1}^{\oplus}=\frac{1}{z_{n-1}}\cdot\mathbf{1}_{[z_{n-1}\neq0]}$. The sequence $\{z_n\}$ is a $\{\mathcal{F}_n\}$ martingale and thus $\nu\ll\mu$ if and only if $\{z_n\}$ converges in L^1 (satisfies the uniform integrability condition).

Theorem 4. [Shi, Thm. 4, p. 528]. *If $\nu\ll^{loc}\mu$ then $\nu\ll\mu$ if and only if*

$$\sum_{k=1}^{\infty}\left[1-E_\mu\left(\sqrt{\alpha_n}\middle|\,\mathcal{F}_{n-1}\right)\right]<\infty\quad\nu\ a.s.$$

If $\nu\ll\mu$ then $\frac{d\nu}{d\mu}=\lim_{n\to\infty}z_n$.

Given a a Markov measure $\mu=\mathrm{M}\left\{\pi_n,P_n:\ n\in\mathbb{Z}\right\}$ on $S^{\mathbb{Z}}$, we want to know when $\mu\sim\mu\circ T$. The natural filtration on the product space is the sequence of algebras $\mathcal{F}_n:=\sigma\left\{[b]_{-n}^n;\ b\in S^{\mathbb{Z}\cap[-n,n]}\right\}$.

The measures which we will construct are fully supported on a TMS $\Sigma_{\mathbf{A}}$ meaning that for every $n\in\mathbb{N}$,

$$\mathrm{supp}P_n:=\left\{(s,t)\in S\times S:\ P_n(s,t)>0\right\}=\mathrm{supp}\mathbf{A}.$$

This implies that $\mu \circ T \ll^{loc} \mu$. In addition the measure $\mathrm{M}\{P_n, \pi_n\}$ will be half stationary in the sense that there exists an irreducible and aperiodic stochastic matrix $\mathbf{Q}$ such that for every $j \leq 0$, $P_j := \mathbf{Q}$ and $\pi_j = \pi_{\mathbf{Q}}$ is the unique stationary distribution for $\mathbf{Q}$. This implies for all $n \in \mathbb{N}$ and $x \in \Sigma$, $\alpha_n(x) = \frac{P_{n-1}(x_{n-1},x_n)}{P_n(x_{n-1},x_n)}$.

By Theorem 4, in this setting, μ is non-singular if and only if

$$\sum_{n=-\infty}^{\infty} \left[1 - E_\nu\left(\sqrt{\alpha_n} \middle| \mathcal{F}_{n-1}\right)(x)\right] = \sum_{n=0}^{\infty} \left[1 - \sum_{s \in S} \sqrt{P_{n-1}(x_{n-1}, s) P_n(x_{n-1}, s)}\right] < \infty$$

for [1] $\mu \circ T$ *a.e.* x. The following corollary concludes our discussion.

Corollary 5. *Let $\nu = M\{\pi_n, P_n : n \in \mathbb{Z}\}$, where $\{P_n\}$ are fully supported on a TMS Σ_A and there exists an aperiodic and irreducible $P \in M_{\mathcal{S}\times\mathcal{S}}$ such that for all $n \leq 0$, $P_n \equiv P$*

- *$\nu \circ T \sim \nu$ if and only if*

$$\sum_{n=0}^{\infty} \sum_{s \in S} \left(\sqrt{P_n(x_n, s)} - \sqrt{P_{n-1}(x_n, s)}\right)^2 < \infty, \ \nu \circ T \ a.s. \ x. \tag{2.2}$$

- *If $\nu \circ T \sim \nu$ then for all $n \in \mathbb{N}$,*

$$\left(T^{\mathbf{n}}\right)'(x) = \prod_{k=0}^{\infty} \frac{P_{k-\mathbf{n}}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})}. \tag{2.3}$$

2.1.3. *A condition for exactness of the one sided shift.* Let S be a countable set and $\{(\pi_n, P_n)\}_{n=1}^{\infty} \subset \mathcal{P}(S) \times \mathcal{M}_{S\times S}$. Denote the one sided shift on $S^{\mathbb{N}}$ by σ and by $\mathcal{F}$ the Borel $\sigma-$algebra of $S^{\mathbb{N}}$. The following is a sufficient condition for exactness (trivial tail σ-field) of the one sided shift which is well known in the theory of non homogenous Markov chains. We include a simple proof.

Proposition 6. *Let S be a finite set and μ be a Markov measure on $S^{\mathbb{N}}$ which is defined by $\{\pi_k, P_k : k \in \mathbb{N} \cup \{0\}\}$. If there exists $C > 0$ and $N_0 \in \mathbb{N}$ so that for every $s, t \in S$, and $k \in \mathbb{N}$,*

$$\left(P_k P_{k+1} \cdots P_{k+N_0-1}\right)(s,t) \geq C \tag{2.4}$$

then the one sided shift $\left(S^{\mathbb{N}\cup\{0\}}, \mathcal{F}, \mu, \sigma\right)$ is exact.

Remark. In the setting of Markov maps, exactness was proved under various distortion properties [see [Tha] and [Aar1, Chapter 4]]. Their conditions guarantees the existence of an absolutely continuous σ-finite invariant measure. For stationary Markov chains, triviality of the tail algebra (exactness of the shift) was proven by Blackwell and Freedman.

Proof. The measure $\mu \circ T^{-n}$ is the Markov measure generated by $Q_k := P_{k+n}$ and $\tilde{\pi}_k := \pi_{k+n}$. Let α_n be the collection of n cylinders of the form $[d]_1^n$ and $\alpha^* = \cup_n \alpha_n$.

[1]Here T denotes the two sided (invertible) shift

For every $D=[a]_0^n\in\alpha_n$ and $B=[b]_0^{n(B)}\in\alpha^*$,

$$\begin{aligned}\mu\left(D\cap T^{-(n+N_0)}B\right) &= \mu(D)\left(P_nP_{n+1}\cdots P_{n+N_0-1}\right)_{a_n,b_0}\prod_{j=0}^{n(B)-1}P_{N_0+n+j}\left(b_j,b_{j+1}\right)\\ &\geq C\mu(D)\pi_{n+N_0}\left(b_0\right)\prod_{j=0}^{n(B)-1}P_{N_0+n+j}\left(b_j,b_{j+1}\right)=C\cdot\mu(D)\mu\circ T^{-(n+N_0)}(B).\end{aligned}$$

Consequently for all $B\in\mathcal{F}$ and $D\in\alpha_n$,

$$\mu\left(D\cap T^{-(n+N_0)}B\right)\geq C\cdot\mu(D)\mu\circ T^{-(n+N_0)}(B).$$

Let $B\in\cap_{n=1}^{\infty}\sigma^{-n}\mathcal{F}$ and $D\in\alpha_n$. Writing $B_n\in\mathcal{F}$ for a set such that $B=T^{-n}B_n$,

$$\begin{aligned}\mu\left(D\cap B\right) &= \mu\left(D\cap T^{-(n+N_0)}B_{n+N_0}\right)\\ &\geq C\cdot\mu(D)\mu\circ T^{-(n+N_0)}\left(B_{n+N_0}\right)\\ &= C\mu(D)\mu(B)\end{aligned}$$

and thus for every $n\in\mathbb{N}$, $\mu\left(B\left|\alpha_n\right.\right)\geq C\mu(B)$. Since $\alpha_n\uparrow\alpha^*$ and α^* generates $\mathcal{F}$,

$$\mu\left(B\left|\alpha_n\right.\right)(x)\xrightarrow[n\to\infty]{}1_B(x)\ \mu-a.s.$$

by the Martingale convergence theorem. It follows that if $\mu(B)>0$ then

$$1_B(x)\geq C\mu(B)>0\ \mu-a.s.$$

This shows that for every $B\in\cap_{n=1}^{\infty}\sigma^{-n}\mathcal{F}$, $\mu(B)\in\{0,1\}$ (the shift is exact). □

2.2. **Cocycles and Measurable Equivalence relations .** Let $(X,\mathcal{B},m)$ be a standard measure space. A measurable countable equivalence relation on X is a measurable set $\mathcal{R}\subset X\times X$ such that the relation $x\sim y$ if and only if $(x,y)\in\mathcal{R}$ is an equivalence relation and for all $x\in X$, $\mathcal{R}_x=\{y\in X:\ (x,y)\in\mathcal{R}\}$ is countable. The saturation of a set $A\in\mathcal{B}$ by the equivalence relation is the set $\mathcal{R}(A):=\bigcup_{x\in A}R_x$. We say that $\mathcal{R}$ *is $m-$ergodic* if for each $A\in\mathcal{B}_X$, $m(\mathcal{R}(A))=0$ or $m\left(X\backslash\mathcal{R}(A)\right)=0$ and $\mathcal{R}$ is non-singular if for all $A\in\mathcal{B}$, $m(A)=0$ if and only if $m(\mathcal{R}(A))=0$. An equivalence relation is finite (respectively countable) if for all $x\in X$, $\mathcal{R}_x$ is a finite (countable) set. The full group of an equivalence relation $\mathcal{R}$, denoted by $[\mathcal{R}]$, is a subgroup of all $V\in Aut(X,\mathcal{B})$ such that for m-almost all $x\in X$, $(x,Vx)\in\mathcal{R}$. A partial transformation of $\mathcal{R}$ is a one to one transformation V from $B\in\mathcal{B}$ to $V(B)\in\mathcal{B}$. Denote by $Dom(V)=B$ the domain of V and $Ran(V)=V(B)$ the range of V. The groupoid of R, denoted by $[[\mathcal{R}]]$, is then defined as the set of all partial transformations of $\mathcal{R}$. Both the full group and the groupoid of $\mathcal{R}$ appear in the study of the ergodic decomposition of cocyclic extensions of equivalence relations (and countable group actions).

Given an action of a countable group $\Gamma\curvearrowright(X,\mathcal{B},m)$ one can define the orbit equivalence relation $\mathcal{R}_\Gamma=\{(x,\gamma x):\ \gamma\in\Gamma\}$. By the Feldman-Moore Theorem [Fe-Mo], for every countable equivalence relation $\mathcal{R}$, there exists a nonsingular action on $(X,\mathcal{B},m)$ of countable group Γ for which $\mathcal{R}=\mathcal{R}_\Gamma$. Notice that this action is not uniquely defined (there are many such countable group actions). A function $f:\Gamma\times X\to\mathbb{R}$ is an $\mathbb{R}$-valued cocycle of the Γ-action if for all $\gamma_1,\gamma_2\in\Gamma$,

$$f(\gamma_1\gamma_2,x)=f(\gamma_1,\gamma_2x)+f(\gamma_2,x).$$

A $\mathbb{R}$-valued orbital cocycle is a function $\varphi:\mathcal{R}\to\mathbb{R}$ such that for all $(x,y),(y,z)\in\mathcal{R}$,

$$\varphi(x,z)=\varphi(x,y)+\varphi(y,z).$$

Two $\mathcal{R}$ cocycles φ_1,φ_2 are (measurably) cohomologous if there exists and a measurable function $b:X\to\mathbb{R}$ and a measurable set $X'\subset X$ with $\mu(X\setminus X')=0$ such that for all $(x,y)\in(X'\times X')\cap\mathcal{R}$,

$$\varphi_1(x,y)=\varphi_2(x,y)+b(x)-b(y).$$

An equivalence relation $\mathcal{R}$ with a $\mathbb{R}$ valued cocycle φ define a φ-extension of $\mathcal{R}$, which is an equivalence relation $\mathcal{R}^{\varphi}\subset(X\times\mathbb{R})\times(X\times\mathbb{R})$, defined by $((x_1,y_1),(x_2,y_2))\in\mathcal{R}^{\varphi}$ if and only if $(x_1,x_2)\in\mathcal{R}$ and $y_1+\varphi(x_1,x_2)=y_2$. If $\Gamma\curvearrowright(X,\mathcal{B},m)$ is such that $\mathcal{R}=\mathcal{R}_\Gamma$ then every orbital cocycle $\varphi:\mathcal{R}\to\mathbb{R}$ corresponds to a $\Gamma\curvearrowright(X,\mathcal{B},m)$ cocycle defined (up to set of m measure zero) by $f(\gamma,x)=\varphi(\gamma x,x)$ [Sch, Prop. 2.3]. The Radon-Nikodym cocycle of a countable equivalence relation $\mathcal{R}$ is thus defined by $\psi_{RN}(x,\gamma x)=\log\frac{dm\circ\gamma}{dm}(x)$ where $\Gamma\curvearrowright(X,\mathcal{B},m)$ is any group acting on X such that $\mathcal{R}=\mathcal{R}_\Gamma$. This cocycle defines an orbital cocycle which does not depend (up to a set of measure zero) on our choice of action $\Gamma\curvearrowright(X,\mathcal{B},m)$ for which $\mathcal{R}_\Gamma=\mathcal{R}$.

A number $r\in\mathbb{R}$ is an essential value for an orbital cocycle $\varphi:\mathcal{R}\to\mathbb{R}$ if for all $A\in\mathcal{B}$ with $m(A)>0$ and $\epsilon>0$, there exists $V\in[\mathcal{R}]$ such that

$$m\left(A\cap V^{-1}A\cap\{x\in X:|\varphi(x,Vx)-r|<\epsilon\}\right)>0.$$

Write $e(\mathcal{R},\varphi)$ for the set of all essential values of $(\mathcal{R},\varphi)$. The Krieger ratio set of a nonsingular transformation $\mathscr{X}$ is $R(T)=e(\mathcal{R}_T,\psi_{RN})$. The set $e(\mathcal{R},\varphi)$ is a closed subgroup of $\mathbb{R}$ and $\mathcal{R}^{\varphi}$ is ergodic if and only if $\mathcal{R}$ is ergodic and $e(\mathcal{R},\varphi)=\mathbb{R}$. Since in the definition of an essential value one is only concerned with points in A, one can change/relax the definition of the essential values to requiring that $V\in[[\mathcal{R}]]$ with $Dom(V)\subset A$. This relaxation, although merely formal, gives the following useful criteria for r to be an essential value for $(\mathcal{R},\varphi)$.

Lemma 7. [CHP, Lemma 2.1.][2] *Let $\mathcal{R}$ be an ergodic, non-singular, countable equivalence relation of a measure space $(X,\mathcal{B},m)$, φ an $\mathcal{R}$ cocycle and $\mathcal{C}\subset\mathcal{B}$ a (countable) algebra of sets which generates $\mathcal{B}$. If there exists $\delta>0$ such that for all $\epsilon>0$ and $C\in\mathcal{C}$, there exists $B\subset C$ and $V\in[[\mathcal{R}]]$ with:*
(a) $Dom(V)=B$ and $V(B)\subset C$,
(b) $m(B)>\delta m(C)$
(c) for all $x\in B$, $|\varphi(x,Vx)-r|<\epsilon$
then $r\in e(\mathcal{R},\varphi)$.

Remark 8. The tail relation of non-singular noninvertible transformation $(X,\mathcal{B}_X,\nu,S)$ is defined by

$$\mathcal{T}_S=\{(y_1,y_2):\ \exists n\in\mathbb{N},\ S^ny_1=S^ny_2\}.$$

[2]See also the formulation in [DaL, Lemma 1.1.] which is similar to the formulation here. Notice that in [CHP], $\delta=0.9$ yet a similar proof works for a general $\delta>0$.

If S is a countable-to-one transformation, then $\mathcal{T}_S$ is a hyperfinite equivalence relation. In this case the trasnformation S is *exact* if $\cap_n S^{-n}\mathcal{B} = \{\emptyset, X\} \mod m$ or equivalently $\mathcal{T}_S$ is ergodic. For a proof of this statement and the definition of hyperfinite equivalence relations see [Haw].

To every function $\varphi : X \to \mathbb{R}$ corresponds an orbital cocycle $\hat{\varphi}$ on $\mathcal{T}_S$ defined by for $(y_1, y_2) \in \mathcal{T}(S)$,

$$\hat{\varphi}(y_1, y_2) := \sum_{n=0}^{\infty} \{\varphi(S^n y_1) - \varphi(S^n y_2)\} = \varphi(N, y_1) - \varphi(N, y_2), \tag{2.5}$$

where $N \in \mathbb{N}$ is any number so that $S^N y_1 = S^N y_2$.

Proposition 9. *Given a countable to one non-singular noninvertible transformation $(X, \mathcal{B}_X, \nu, S)$ and a function $\varphi : X \to \mathbb{R}$, $\mathcal{T}(S)^{\hat{\varphi}} = \mathcal{T}(S_\varphi)$ where S_φ is the skew product transformation on $X \times \mathbb{R}$ defined by $S_\varphi(x, y) = (Sx, y + \varphi(x))$.*

Proof. Follows from the definitions. □

If φ_1 is cohomologous to φ_2 then $e(S, \varphi_1) = e(S, \varphi_2)$ [Sch]. If $(X, \mathcal{B}_X, \mu, S)$ is a nonsingular system and ν is a μ equivalent measure then for all $\gamma \in [\mathcal{R}_S]$, for almost all $x \in X$

$$\begin{aligned}\psi_{RN,\nu}(x, \gamma x) &:= \log \frac{d\nu \circ \gamma}{d\nu}(x) \\ &= \log \frac{d\mu \circ \gamma}{d\mu}(x) + \log \frac{d\nu}{d\mu}(\gamma x) - \log \frac{d\nu}{d\mu}(x) \\ &= \psi_{RN,\mu}(x, \gamma x) + b(\gamma x) - b(x)\end{aligned}$$

with $b(x) = \log \frac{d\nu}{d\mu}(x)$. This shows that $\psi_{RN,\nu}$ and $\psi_{RN,\mu}$ are cohomologous cocycles and implies the following fact.

Fact 10. *If $(X, \mathcal{B}_X, \mu, T)$ is stable type* III_1 *and $\nu \sim \mu$ then $(X, \mathcal{B}_X, \nu, T)$ is stable type* III_1.

3. Stable type III_1 Markov measures

For $\lambda > 1$ let

$$\mathbf{Q}_\lambda = \begin{pmatrix} \frac{\lambda G}{1+\lambda G} & 0 & \frac{1}{1+\lambda G} \\ \frac{1}{G} & 0 & \frac{1}{G^2} \\ 0 & 1 & 0 \end{pmatrix}$$

where $G = \frac{1+\sqrt{5}}{2}$ is the golden mean. The following relation is the main observation for the inductive construction.

$$\frac{\mathbf{Q}_\lambda(3,2)\,\mathbf{Q}_\lambda(2,3)\,\mathbf{Q}_\lambda(3,2)\,\mathbf{Q}_\lambda(2,1)\,\mathbf{Q}_\lambda(1,1)}{\mathbf{Q}_\lambda(3,2)\,\mathbf{Q}_\lambda(2,1)\,\mathbf{Q}_\lambda(1,3)\,\mathbf{Q}_\lambda(3,2)\,\mathbf{Q}_\lambda(2,1)} = \lambda.$$

This equality has a more compact interpretation on (time homogeneous) Markovian measure $\mathbb{P}_{\delta_3,\mathbf{Q}_\lambda}$ on $\{1,2,3\}^{\mathbb{N}}$ where $\delta_3 = (0,0,1)$ as

$$\mathbb{P}_{\delta_3,\mathbf{Q}_\lambda}\left([323211]_1^6\right) = \lambda \mathbb{P}_{\delta_3,\mathbf{Q}_\lambda}\left([321321]_1^6\right). \tag{3.1}$$

Write $a = 323211 \in \{1,2,3\}^6$ and $b = 321321 \in \{1,2,3\}^6$ (words of length 6). The construction will rely on using elements in the groupoid of the tail relation which changes between the sequences $[a]_k^{k+6}$ and $[b]_k^{k+6}$ for some of the k's according to the cylinder set. The following Lemma will play a role in the inductive construction, to shorten notation write for $z \in \{a, b\}$,

$$L_{z,N}(x) = \sum_{k=0}^{\lfloor N/6 \rfloor} 1_{[z]_1^6} \circ T^{6k}(x) = \#\left\{0 \le k \le \lfloor N/6 \rfloor :\ x_{6k+1}x_{6k+2}..x_{6k+5} = z\right\}.$$

The function $L_{z,n}(x)$ is locally constant on cylinders of the form $\left\{[x]_1^N :\ x \in \{1,2,3\}^N\right\}$ and hence will be regarded as $L_{z,n} : \{1,2,3\}^N \to \mathbb{N}$.

Lemma 11. *Let $\epsilon > 0$ and $\lambda > 1$. For any initial distribution π on $\{1,2,3\}$ and $p \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $n > N$,*

$$\mathbb{P}_{\pi,\mathbf{Q}_\lambda}\left(x \in \Sigma :\ L_{a,n}(x) - L_{b,n}(x) > p\right) > 1 - \epsilon.$$

Proof. The measure $\mathbb{P}_{\pi,\mathbf{Q}_\lambda}$ defines a measure on $\left(\{1,2,3\}^6\right)^{\mathbb{N}}$ which is a distribution of an aperiodic and irreducible Markov chain with state space $S \subset \{1,2,3\}^6$. The result follows from the Mean ergodic theorem for Markov chains, see for example [LPW, Th. 4.16], and the fact that, $\mathbb{P}_{\pi_{\mathbf{Q}_\lambda},\mathbf{Q}_\lambda}\left([a]_1^6\right) = \lambda\mathbb{P}_{\pi_{\mathbf{Q}_\lambda},\mathbf{Q}_\lambda}\left([b]_1^6\right) > \mathbb{P}_{\pi_{\mathbf{Q}_\lambda},\mathbf{Q}_\lambda}\left([b]_1^6\right)$. □

3.1. **The inductive construction.**

3.1.1. *A short explanation of the idea behind the construction.* The inductive construction inputs 3 sequences $\{\lambda_j, M_j, N_j\}_{j=1}^{\infty}$ with $M_0 = 2 < N_1$, $\lambda_k \downarrow 1$ and $M_k \ll N_{k+1} \ll M_{k+1}$ where $a_n \ll b_n$ means $a_n < b_n$ for all n and $a_n/b_n \to 0$. Given $\{\lambda_j, M_j, N_j\}_{j=1}^{K}$ we first choose λ_{K+1} then N_{K+1} and then M_{k+1}. This choice of parameters defines a Markovian measure $\nu = \mathrm{M}\{\pi_n, P_n :\ n \in \mathbb{Z}\}$ as follows: First for all $n \le 1$, $P_n = \mathbf{Q}_1 = \mathbf{Q}$ and $\pi_n = \pi_{\mathbf{Q}}$. The sequences $\{M_k, N_k\}_{k\in\mathbb{N}}$ gives rise to a partition of $\mathbb{N}$ into two types of intervals $I_k := [M_{k-1}, N_k)$ and $J_k := [N_k, M_k)$. We set the transition matrices to be

$$P_n = \begin{cases} \mathbf{Q}_{\lambda_\mathbf{k}}, & n \in I_k \\ \mathbf{Q}, & n \in (\mathbb{Z}\backslash\mathbb{N}) \cup (\cup_k J_k). \end{cases} \tag{3.2}$$

The measures π_n are defined according to the consistency criteria (2.1). By the structure of the measure, the two sided shift is a K-automorphism with the one sided shift as its exact factor and the non singularity of the shift with respect to ν is only concerned with the P_n where $P_n \neq P_{n+1}$, that is $n \in \cup_{k=1}^{\infty}\{M_k, N_k\}$. Non-singularity of the shift is thus equivalent to $\sum_{k=1}^{\infty}(\lambda_k - 1)^2 < \infty$ which follows from condition 1 on λ_k. Intuitively the first condition on λ_k enables one to approximate the Radon-Nikodym derivatives with respect to the shift by a finite product. The growth condition on $M_k - N_k$ then implies by the Hopf criteria that the shift is conservative since $\sum (T^n)'(x) = \infty$ almost everywhere. The condition on N_k imply that ν is not equivalent to the stationary Markov measure $\mathrm{M}\{\pi_{\mathbf{Q}}, \mathbf{Q}\}$. This condition together with the second condition on λ_k enables us to show that the tail relation of the natural (noninvertible) factor of the Maharam extension is exact (by a reasoning which uses the EVC Lemma and the switching of $[a]$'s and $[b]$'s).

3.1.2. *The construction.* Induction base: Let λ_1 be any number greater than 1, N_1 be any natural number larger than $M_0 = 2$ and M_1 be any number greater than N_1.

Induction step: Assume we are given $\{\lambda_k, N_k, M_k\}_{k=1}^{K} \subset \{(1,\infty)\times\mathbb{N}\times\mathbb{N}\}^k$. Notice that this defines π_n and P_n for all $n < M_K$ by (3.2).

Choose $\lambda_{K+1} > 1$ so that

(1) Finite approximation of the Radon-Nikodym derivatives condition:

$$(\lambda_{K+1})^{2M_K} < e^{\frac{1}{2^K}}. \tag{3.3}$$

(2) Lattice condition:

$$\lambda_K \in (\lambda_{K+1})^{\mathbb{N}}, \tag{3.4}$$

where for $x \in \mathbb{R}$, $x^{\mathbb{N}} = \{x^n : n \in \mathbb{N}\}$. Write $\zeta_{K+1} := \log_{\lambda_{K+1}}(\lambda_1) \in \mathbb{N}$.

Remark 12. By the Lattice condition for all $k \le K+1$ there exists $\mathbb{N} \ni q \le \zeta_{K+1}$ so that $\lambda_{K+1}^q = \lambda_k$.

Now that λ_{K+1} is chosen, **choose n_{K+1} large enough so that $n_{K+1} > M_k$ and**

$$\mathbb{P}_{\pi_{M_K},\mathbf{Q}_{\lambda_{K+1}}}\left(x \in \Sigma : L_{a,n_{K+1}}(x) - L_{b,n_{K+1}}(x) > \zeta_{K+1}\right) > 0.99. \tag{3.5}$$

Notice that π_{M_K} is defined by $P_{M_K} = \mathbf{Q}_{\lambda_{K+1}}$ and the consistency criteria 2.1 and that the existence of n_{K+1} follows from Lemma 11. **Set** $N_{K+1} = M_K + 6n_K$.

Finally, choose M_{K+1} so that

$$(M_{K+1} - 2N_{K+1})\,\lambda_1^{-2N_{K+1}} \ge 1. \tag{3.6}$$

Theorem 13. *Let $\{\lambda_k, N_k, M_k\}_{k=1}^{\infty}$ be chosen according to the inductive construction and $\nu = M\{\pi_n, P_n : n \in \mathbb{Z}\}$ where π_n and P_n are defined by (3.2). The shift $(\Sigma, \mathcal{B}, \nu, T)$ is a non-singular, conservative K-automorphism hence ergodic. Furthermore its Maharam extension is a K automorphism, hence it is stable type* III_1.

The proof of Theorem 13 will be separated into two parts. We first show that the shift is a conservative K-automorphism. Then in Subsubsection 3.1.3 we prove the K-property of the Maharam extension.

Proof. **[Non-Singularity,K property and conservativity]**

Since $\nu \circ T$ is the Markov measure generated by $\tilde{P}_j = P_{j-1}$ and $\tilde{\pi}_j = \pi_{j-1}$, it follows from (2.2) and (3.2) that the shift is non-singular if and only if

$$\sum_{k=1}^{\infty}\sum_{s\in S}\left\{\left(\sqrt{P_{N_k}(x_{N_k},s)} - \sqrt{P_{N_k-1}(x_{N_k},s)}\right)^2 + \left(\sqrt{P_{M_k}(x_{M_k},s)} - \sqrt{P_{M_k-1}(x_{M_k},s)}\right)^2\right\} < \infty,\ \mu\circ T\ a.s.\ x.$$

Since for all $j \in \mathbb{Z}$, $P_j(3,2) \equiv 1$, $P_j(2,1) = 1 - P_j(2,3) \equiv \frac{G}{1+G}$, the sum is dominated by

$$\sum_{k=1}^{\infty}\sum_{s\in S}\left\{4\left(\sqrt{P_{N_k}(1,s)} - \sqrt{P_{N_k-1}(1,s)}\right)^2\right\} = 4\sum_{k=1}^{\infty}\left\{\left(\sqrt{\frac{\lambda_k G}{1+\lambda_k G}} - \sqrt{\frac{G}{1+G}}\right)^2 + \left(\sqrt{\frac{1}{1+\lambda_k G}} - \sqrt{\frac{1}{1+G}}\right)^2\right\}.$$

This sum converges or diverges together with $\sum_{k=1}^{\infty}|\lambda_k-1|^2$. As a consequence of condition (3.3) on $\{\lambda_j\}$, this sum is finite hence the shift is non-singular. Since $P_j\equiv\mathbf{Q}$ for all $j\leq 1$,

$$T'(x)=\frac{d\mu\circ T}{d\mu}(x)=\prod_{k=1}^{\infty}\frac{P_{k-1}(x_k,x_{k+1})}{P_k(x_k,x_{k+1})}$$

is a $\mathcal{B}_{\Sigma_+}$ measurable function. The sequence $\{P_j\}_{j=1}^{\infty}$ satisfies (2.4) and hence the one sided shift $(\Sigma_+,\mathcal{F},\mu_+,\sigma)$ is an exact factor such that T' is $\mathcal{F}$ measurable. Here μ_+ denotes the measure on the one sided shift space defined by $\{\pi_j,P_j\}_{j\geq 1}$. This shows that the two sided shift is a K automorphism.

In order to show that the shift is conservative we first show that for all $t\in\mathbb{N}$ large enough and $n\in[N_t,M_t-N_t)$, for almost every $x\in\Sigma$,

$$(T^n)'(x)\geq\lambda_1^{-2N_t}/2, \tag{3.7}$$

hence by (3.6),

$$\sum_{n=1}^{\infty}(T^n)'(x)\geq\sum_{t=1}^{\infty}\sum_{n=N_t}^{M_t-N_t}(T^n)'(x)=\infty.$$

By Hopf's criteria, the shift is conservative. Fix $t\in\mathbb{N}$ and $n\in[N_t,M_t-N_t)$. For $x,y>0$, $z=x^{\pm y}$ denotes $x^{-y}\leq z\leq x^y$. By (2.3),

$$\frac{d\mu\circ T^n}{d\mu}(x)=\prod_{\substack{k\in\mathbb{N}\\ P_{k-n}\neq P_k}}\frac{P_{k-n}(x_k,x_{k+1})}{P_k(x_k,x_{k+1})}.$$

First, we characterize the set of $k's$ such that $P_{k-n}\neq P_k$. It follows from $n\in[N_t,M_t-N_t)$ that $\cup_{k=1}^t[M_{k-1},N_k)-n\subset(-\infty,0]$ and $\cup_{k=1}^t[M_{k-1}+n,N_k+n)\subset[n,n+N_t)\subset[N_t,M_t)$. This means that for all $u\leq t$, and $k\in[M_{u-1},N_u)$, $P_k=\mathbf{Q}_{\lambda_u}$ and $P_{k-n}=\mathbf{Q}$ and for all $k\in[M_{u-1}+n,N_u+n)$, $P_k=\mathbf{Q}$ and $P_{k-n}=\mathbf{Q}_{\lambda_u}$. One can check directly that for all other $k\in[1,M_t)$, $P_k=P_{k-n}=\mathbf{Q}$.

Similarly for all $k\geq M_t$, $P_{k-n}=P_k$, unless $k\in\cup_{u=t+1}^{\infty}([N_u,N_u+n)\cup[M_{u-1},M_{u-1}+n))$. For $u>t$, if $k\in[N_u,N_u+n)$ then $P_k=\mathbf{Q}$ and $P_{k-n}=\mathbf{Q}_{\lambda_u}$ and if $k\in[M_{u-1},N_u)$ then $P_k=\mathbf{Q}$ and $P_{k-n}=\mathbf{Q}_{\lambda_u}$. Using this

$$(T^n)'(x) \quad = \quad I_t\cdot\tilde{I}_t$$

where

$$I_t=\prod_{u=1}^{t}\left[\left(\prod_{k=M_{u-1}}^{N_u}\frac{\mathbf{Q}(x_k,x_{k+1})}{\mathbf{Q}_{\lambda_u}(x_k,x_{k+1})}\right)\cdot\left(\prod_{k=M_{u-1}+n}^{N_u+n-1}\frac{\mathbf{Q}_{\lambda_u}(x_k,x_{k+1})}{\mathbf{Q}(x_k,x_{k+1})}\right)\right]$$

and

$$\tilde{I}_t=\prod_{u=t+1}^{\infty}\left[\left(\prod_{k=M_{u-1}}^{M_{u-1}+n-1}\frac{\mathbf{Q}(x_k,x_{k+1})}{\mathbf{Q}_{\lambda_u}(x_k,x_{k+1})}\right)\cdot\left(\prod_{k=N_u}^{N_u+n-1}\frac{\mathbf{Q}_{\lambda_u}(x_k,x_{k+1})}{\mathbf{Q}(x_k,x_{k+1})}\right)\right].$$

For all $u\in\mathbb{N}$ and $s,t\in\{1,2,3\}$ with $\mathbf{Q}(s,t)>0$,

$$\frac{1}{\lambda_u}\leq\frac{\mathbf{Q}(s,t)}{\mathbf{Q}_{\lambda_u}(s,t)}\leq\lambda_u.$$

Therefore

$$\lambda_u^{-2n} \le \left(\prod_{k=M_{u-1}}^{M_{u-1}+n-1} \frac{\mathbf{Q}_{\lambda_u}(x_k, x_{k+1})}{\mathbf{Q}(x_k, x_{k+1})} \right) \cdot \left(\prod_{k=N_u}^{N_u+n-1} \frac{\mathbf{Q}(x_k, x_{k+1})}{\mathbf{Q}_{\lambda_u}(x_k, x_{k+1})} \right) \le \lambda_u^{2n},$$

For all $u > t,\ n < M_t \le M_{u-1}$, thus

$$\tilde{I}_t = \prod_{u=t+1}^{\infty} \left[\lambda_u^{\pm 2n}\right] = \prod_{u=t+1}^{\infty} \left[\lambda_u^{\pm 2M_{u-1}}\right] \stackrel{(3.3)}{=} e^{\pm \sum_{n=t+1}^{\infty} \frac{1}{2^n}} \xrightarrow[t\to\infty]{} 1.$$

Consequently there exists $t_0 \in \mathbb{N}$ so that for all $x \in \Sigma_{\mathbf{A}}$, $t > t_0$ and $N_t \le n \le M_t - N_t$,

$$(T^n)'(x) \ge I_t/2.$$

A similar analysis shows that (notice that $\lambda_1 = \max_{u\in\mathbb{N}} \lambda_u$ and $\sum_{u=1}^{t} (N_u - M_{u-1}) < N_t$)

$$I_t \ge \prod_{u=1}^{t} \lambda_u^{-2(N_u - M_{u-1})} \ge \lambda_1^{-2N_t},$$

proving (3.7). ☐

3.1.3. *Proof of the K-property of the Maharam extension.* Denote by $S : \Sigma_+ \to \Sigma_+$ the shift, $\pi : \Sigma \to \Sigma_+$ the projection $\pi(x) = (x_1, x_2, \cdots) := x_+$ and $\nu^+ = \nu|_{\mathcal{F}}$ where $\mathcal{F} = \mathcal{B}_{\Sigma_+}$. The shift $(\Sigma, \mathcal{B}, \nu, T)$ is a K-automorphism and $\mathcal{F}$ is its exact factor. Since $T'(x) = \frac{d\nu\circ T}{d\nu}(x)$ is $\mathcal{F}$- measurable, the skew product extension $S_{\log T'}(x_+, y) = (Sx_+, y + \log T'(x)) : \left(\Sigma_+ \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}}, \nu^+ \times \left(e^{-t}dt\right)\right) \circlearrowleft$ is well defined and the factor map $\pi \times id(x, y) = (x_+, y)$ is a factor map from the Maharam extension of T (denoted by $\tilde{T}$) to $S_{\log T'}$. In addition $\mathcal{F} \otimes \mathcal{B}_{\mathbb{R}}$ is a minimal factor, that is $\vee_{n=0}^{\infty} \tilde{T}^n (\mathcal{F} \otimes \mathcal{B}_{\mathbb{R}}) = \mathcal{B}_{\Sigma} \otimes \mathcal{B}_{\mathbb{R}}$. It remains to show that $S_{\log T'}$ is an exact endomorphism. This is done by showing the ergodicity of the tail relation of $S_{\log T'}$ using its identification as $\mathcal{T}(S)^{\hat{\varphi}}$ where $\hat{\varphi}$ is the tail relation cocycle arising from the function $\varphi = \log T'$.[3]

Denote by

$$\mathscr{C}_t := S^{-M_{t-1}} \left\{ w \in \Sigma_+ :\ L_{a,n_t}(w) - L_{b,n_t}(w) > \zeta_t \right\}.$$

(recall that $n_t = (N_t - M_{t-1})/6$). Notice that $\mathcal{C}_t$ is a finite union of cylinder sets of the form $D = [w]_{M_{t-1}}^{N_t}$ and that by (3.5) and the definition of ν (as an inhomogeneous Markov measure),

$$\nu^+(\mathcal{C}_t) = \mathbb{P}_{\pi_{M_{t-1}}, \mathbf{Q}_t} \left(\left\{ w \in \Sigma_+ :\ L_{a,n_t}(w) - L_{b,n_t}(w) > \zeta_t \right\} \right) > 0.99.$$

Denote by $\mathcal{C}_t^*$ the collection of all cylinder sets $D = [d]_{M_{t-1}}^{N_t}$ such that $D \subset \mathcal{C}_t$. The following Fact\Lemma whose proof uses the specification of the SFT Σ_+ is needed for gluing cylinders $[c]_1^{M_{t-1}-3}$ with the cylinders in $\mathcal{C}_t^*$.

Lemma 14. *For any $C = [c]_1^n$ with $n \le M_{t-1} - 3$*

$$\nu^+(C \cap \mathcal{C}_t) \ge G^{-4}(0.99)\nu(C).$$

Proof. The claim uses the specification of Σ and goes by showing that for any cylinders of the form $[c]_1^{M_{t-1}-3}$ and $D = [d]_{M_{t-1}}^{M_t}$ there exists a choice of a word $w(c, d) = w_{M_{t-1}-2} w_{M_{t-1}-1} \in \{1, 2, 3\}^2$ such

[3]Note that $\hat{\varphi}$ is not necessarily the Radon- Nikodym cocycle of $\mathcal{T}(S)$.

$c_{M_{t-1}-3}$	w	$d_{M_{t-1}}$
1 or 2	11	1 or 3
1 or 2	13	2
3	21	1 or 3
3	23	2

TABLE 1.

that

$$P_{M_{t-1}-3}\left(c_{M_{t-1}-3}, w_{M_{t-1}-2}\right) P_{M_{t-1}-2}\left(w_{M_{t-1}-2}, w_{M_{t-1}-1}\right) P_{M_{t-1}-1}\left(w_{M_{t-1}-1}, d_{M_{t-1}}\right) = \mathbf{Q}\left(c_{M_{t-1}-3}, w_{M_{t-1}-2}\right) \mathbf{Q}\left(w_{M_{t-1}-2}, w_{M_{t-1}-1}\right) \mathbf{Q}\left(w_{M_{t-1}-1}, d_{M_{t-1}}\right) \geq G^{-4}$$

(see Table 1 for our choice of w).

Writing $\hat{w}(c,d) := cw(c,d)d$ for the concatenated word,

$$\nu^+\left([\hat{w}(c,d)]_1^{N_t}\right) \geq G^{-4}\nu^+(C)\mathbb{P}_{\delta_{d_{M_{t-1}}},\mathbf{Q}_{\lambda_t}}\left([d]_1^{N_t-M_{t-1}}\right) \geq G^{-4}\nu^+(C)\mathbb{P}_{\pi_{M_{t-1}},\mathbf{Q}_{\lambda_t}}\left([d]_1^{N_t-M_{t-1}}\right).$$

Summing over all the cylinders $D = [d]_{M_{t-1}}^{N_t} \in \mathcal{C}_t^*$,

$$\nu^+\left(C \cap \mathcal{C}_t\right) \geq \sum_{D \in \mathcal{C}_t^*} \nu^+\left([\hat{w}(c,d)]_1^{N_t}\right) \geq G^{-4}\nu^+\left(C\right)\mathbb{P}_{\pi_{M_{t-1}},\mathbf{Q}_{\lambda_t}}\left(\mathcal{C}_t\right) \geq (0.99)G^{-4}\nu^+(C).$$

In the last inequality we used (3.5). If $C = [c]_1^n$ with $n < M_{t-1} - 3$ the result follows by writing $C = \uplus_w [cw]_1^{M_{t-1}-3}$ where w ranges over all elements in $\{1,2,3\}^{\{M_{t-1}-3\}-n}$.

□

Lemma 15. *Let $t_0 \in \mathbb{N}$. For every $t_0 \leq t \in \mathbb{N}$ and a cylinder $C = [c]_1^n$ with $n \leq M_{t-1} - 3$ there exists $V \in [[\mathcal{T}(S)]]$ with $Dom(V), Ran(V) \subset C$, $\nu^+\left(Dom(V)\right) > (0.99)\, G^{-4}\nu(B)$ and for all $w \in Dom(V)$, $\hat{\varphi}(w, Vw) = -\log \lambda_{t_0}$.*

Corollary 16. *The tail relation $\mathcal{T}\left(S_{\log T'}\right) = \mathcal{T}(S)^{\hat{\varphi}}$ is ergodic.*

Proof. As before, let α_n be the collection of n cylinders of the form $[d]_1^n \cap \Sigma_+$ and $\alpha^* = \cup_n \alpha_n$. The sequence α_n is an increasing sequence of partitions of Σ_+ which generates $\mathcal{F}$. We first claim that $\{-\log \lambda_t\}_{t=1}^{\infty} \subset e\left(\mathcal{T}(S), \hat{\varphi}\right)$. Fix $t_0 \in \mathbb{N}$. By Lemma 15, for every $t_0, t \in \mathbb{N}$, $t \geq t_0$ and $C \in \alpha_n$ with $n \leq M_{t-1} - 3$, there exists $V \in [[\mathcal{T}(S)]]$ with (a) $Dom(V), Ran(V) \subset C$. (b) $\nu^+\left(Dom(V)\right) > (0.99)\, G^{-4}\nu(B)$ and (c) for all $w \in Dom(V)$, $\hat{\varphi}(w, Vw) = -\log \lambda_{t_0}$.

By an application of the criteria from Lemma 7 with $\delta = (0.99)\, G^{-4}$ and $\mathcal{C}$ the algebra generated by α^* , $-\log \lambda_{t_0} \in e\left(\mathcal{T}(S), \hat{\varphi}\right)$. Since $e\left(\mathcal{T}(S), \hat{\varphi}\right)$ is an additive subgroup of $\mathbb{R}$, $-\log \lambda_t \uparrow 0$ and $\{-\log \lambda_t\}_{t \in \mathbb{N}} \subset e\left(\mathcal{T}(S), \hat{\varphi}\right)$, $e\left(\mathcal{T}(S), \hat{\varphi}\right) = \mathbb{R}$ and thus $\mathcal{T}(S)^{\hat{\varphi}}$ is ergodic. □

Proof of Lemma 15. Let $t_0, t \in \mathbb{N}$, $t \geq t_0$ and $C = [c]_1^n$ with $n \leq M_{t-1} - 3$. By Remark 12 there exists $\mathbb{N} \ni q = q((t, t_0) \leq \zeta_t$ such that $\lambda_t^q = \lambda_{t_0}$. For $D = [d]_{N_t}^{M_t} \in \mathcal{C}_t^*$, there exists a minimal $l = l(D) \leq n_t$ such that for all $w \in D$

$$L_{a,l}\left(S^{M_{t-1}}w\right) - L_{b,l}\left(S^{M_{t-1}}w\right) = L_{a,l}\left(S^{M_{t-1}}d\right) - L_{b,l}\left(S^{M_{t-1}}d\right) = q.$$

Write $D(i)=[d]_{M_{t-1}+6i}^{M_{t-1}+6(i+1)}, 0\leq i\leq l(D)$ and for convenience write $D=D(0)D(1)\cdots D(l(D))[d]_{M_{t-1}+6l(D)}^{N_t}$. Let $\hat{D}=\left[\hat{d}\right]_{M_{t-1}}^{N_t}$ be defined by the following rule. Firstly $\left[\hat{d}\right]_{M_{t-1}+6l(D)}^{N_t}=[d]_{M_{t-1}+6l(D)}^{N_t}$. For $1\leq i\leq l(D)$,

$$\hat{D}(i)=\begin{cases}D(i), & D(i)\notin\{a,b\}\\ b & D(i)=a\\ a & D(i)=b.\end{cases}$$

The cylinder $\hat{D}$ satisfies $\hat{d}_{M_{t-1}}=d_{M_{t-1}}$ and

$$\frac{\nu^+(D)}{\nu^+\left(\hat{D}\right)}=\left(\frac{\mathbb{P}_{\delta_3,\mathbf{Q}_{\lambda_t}}(a)}{\mathbb{P}_{\delta_3,\mathbf{Q}_{\lambda_t}}(b)}\right)^{L_{a,l}\left(S^{M_{t-1}}w\right)}\left(\frac{\mathbb{P}_{\delta_3,\mathbf{Q}_{\lambda_t}}(b)}{\mathbb{P}_{\delta_3,\mathbf{Q}_{\lambda_t}}(a)}\right)^{L_{b,l}\left(S^{M_{t-1}}w\right)}\overset{(3.1)}{=}\lambda_t^q=\lambda_{t_0}^{-1}. \tag{3.8}$$

Similarly, since $\mathbf{Q}=\mathbf{Q}_1$,

$$\frac{\mathbb{P}_{\delta_{d_{M_{t-1}}},\mathbf{Q}}(D)}{\mathbb{P}_{\delta_{\hat{d}_{M_{t-1}}},\mathbf{Q}}\left(\hat{D}\right)}=\prod_{k=M_{t-1}}^{N_t-1}\frac{\mathbf{Q}\left(d_k,d_{k+1}\right)}{\mathbf{Q}\left(\hat{d}_k,\hat{d}_{k+1}\right)}=1. \tag{3.9}$$

This defines an element map $V_D:D\to\hat{D}$ in $[[\mathcal{T}(S)]]$ by taking a point $w=\overline{w}d\tilde{w}$ with $\overline{w}\in\{1,2,3\}^{M_{t-1}-1}$ and $\tilde{w}\in\Sigma_+$ to $V_D(w)=\overline{w}\hat{d}\tilde{w}$ (that is changing only the values of w in the coordinates of the cylinder set D). First we claim that for all $w\in D$,

$$\hat{\varphi}\left(w,V_D(w)\right)=-\log\lambda_{t_0}.$$

To see this notice first that $S^{N_t}w=S^{N_t}V_D(w)$ for all $w\in D$, hence by (2.3) and (2.5),

$$\begin{aligned}\hat{\varphi}\left(w,V_D(w)\right)&=\log\frac{d\nu\circ T^{N_t}}{d\nu}(w)-\log\frac{d\nu\circ T^{N_t}}{d\nu}\left(V_D(w)\right)\\&=\log\frac{\nu^+(D)}{\nu^+(\hat{D})}+\log\frac{\mathbb{P}_{\delta_{\hat{d}_{M_{t-1}}},\mathbf{Q}}\left(\hat{D}\right)}{\mathbb{P}_{\delta_{d_{M_{t-1}}},\mathbf{Q}}(D)}=-\log\lambda_{t_0}.\end{aligned}$$

Set $V:\mathcal{C}_t\to\Sigma_+$ by for $D\in\mathcal{C}_t^*$, $V|_D=V_D$. By the previous argument for all $w\in\mathcal{C}_t$, $\hat{\varphi}(w,Vw)=-\log\lambda_{t_0}$. We claim that $V\in[[\mathcal{T}(S)]]$ with $Dom(V)=\mathcal{C}_t$. Since for all $w\in D\in\mathcal{C}_t^*$, $(w,V(w))=(w,V_D(w))\in\mathcal{T}(S)$ it remains to show that V is one to one which follows from the injectivity of the map $D\mapsto\hat{D}:\mathcal{C}_t^*\to\mathcal{F}$. The injectivity of the latter is established by looking at the position of the a's and b's in $\left[\hat{d}\right]_{M_{t-1}}^{M_{t-1}+6\hat{l}(\hat{D})}$ where

$$\hat{l}\left(\hat{D}\right)=\min\left\{k\leq n_t:\ L_{b,k}\left(S^{M_{t-1}}\hat{d}\right)-L_{a,k}\left(S^{M_{t-1}}\hat{d}\right)=q\right\}$$

and noticing that the similar operation of the definition of $\hat{d}$ from D with $l(D)$ replaced by $\hat{l}(D)$ recovers D. By Lemma (14), $\nu^+\left(C\cap\mathcal{C}_t\right)>(0.99)\,G^{-4}\nu^+(C)$ and the element $V|_{C\cap\mathcal{C}_t}\in[[\mathcal{T}(S)]]$ satisfies the conclusion of the Lemma.

□

Remark 17. The countable group $\mathcal{S}_\infty$ of all finite permutations of $\mathbb{N}$ acts on $\{1,2,3\}^{\mathbb{N}}$ via $\pi(w)_n=w_{\pi_n}$ for $\pi\in\mathcal{S}_\infty$ and $w\in\{1,2,3\}^{\mathbb{N}}$. This action generates the exchangeability relation. In the proof of

Lemma 15, for all $D \in \mathcal{C}_t^*$, we specified $\pi \in \mathcal{S}_\infty$ such that $V_D = \pi|_D$ and thus V is locally constant and chosen from the orbit of $\mathcal{S}_\infty$ on Σ_+. Since $\mu^+ = \mathrm{M}\{\pi_{\mathbf{Q}}, \mathbf{Q}\}$ is a Gibbs measure, in fact the measure of maximal entropy on Σ_+, it is an invariant measure for the action of $\mathcal{S}_\infty$, see [ANS]. By this,

$$\hat{\varphi}(w, V_D w) = -\log \frac{d\nu^+ \circ \pi|_D}{d\nu^+}(w).$$

By a more involved argument using these permutations, one can show that if $P_n = \mathbf{Q}_{\lambda_n}$ with $\lambda_n = 1$ for every $n \le 1$ and $\nu = \mathrm{M}\{\pi_n, P_n\}$, if ν is singular with respect to $\mathrm{M}\{\pi_{\mathbf{Q}_1}, \mathbf{Q}_1\}$ and the shift is ν-nonsingular then the shift (Σ, ν, T) is stable type III_1. See [DaL, Kos2] for some recent results of this type in the case of 2 state Markov Chains and half-stationary Bernoulli shifts.

4. Stable type III_1 Anosov C^1 Diffeomorphims of $\mathbb{T}^d$, $d \ge 3$.

Let (M, d) be a Riemannian manifold. A diffeomorphism $f : M \to M$ is an Anosov diffeomorphism if there exists $\lambda > 1$, $C > 0$ and a decomposition $T_x M = E_s^x \oplus E_u^x$, $x \in M$, such that:

- the decomposition is Df-equivariant. That is for all $x \in M$, $D_f(x)E_x^s = E_{f(x)}^s$ and $D_f(x)E_x^u = E_{f(x)}^u$.
- for all $x \in M$, $v \in E_x^s$ and $n \in \mathbb{N}$, $|D_{f^n}(x)v| \le C\lambda^n |v|$ (contraction on the stable manifolds).
- for all $x \in M$, $v \in E_x^u$ and $n \in \mathbb{N}$, $\left|D_{f^{-n}}(x)v\right| \le C\lambda^n |v|$ (expansion on the unstable manifolds).

Example. $f : \mathbb{T}^2 \to \mathbb{T}^2$ the toral automorphism defined by

$$f(x, y) = (\{x + y\}, x) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1,$$

where $\{t\}$ is the fractional part of t. Since $\left|\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right| = 1$, f preserves the Lebesgue measure on $\mathbb{T}^2$. In addition for every $z \in \mathbb{T}^2$, the tangent space can be decomposed as $\mathrm{span}\{v_s\} \oplus \mathrm{span}\{v_u\}$ where $v_u = (1, 1/G)$ and $v_s = (1, -G)$.

For every $w \in V_u := \mathrm{span}\{v_u\}$ and $x \in \mathbb{T}^2$

$$D_f(x)w = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} w = Gw,$$

For every $u \in V_s := \mathrm{span}\{v_s\}$ and $x \in \mathbb{T}^2$, $D_f(x)u = \left(-\frac{1}{G}\right) u$. In this example, for all $x \in \mathbb{T}^2$, $E_x^s = V^s$ and $E_x^u = V^u$. These facts can be used [Adl, AW] to construct the Markov partition for f with three elements $\{R_1, R_2, R_3\}$, see figure 4.1.

The adjacency Matrix of the Markov partition is then defined by $A_{i,j} = 1$ if and only if $R_i \cap f^{-1}(R_j) \neq \emptyset$. Here the adjacency Matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $\Phi : \Sigma = \Sigma_A \to \mathbb{T}^2$ be the map defined by

$$\Phi(x) := \bigcap_{n=-\infty}^{\infty} \overline{f^{-n} R_{x_n}},$$

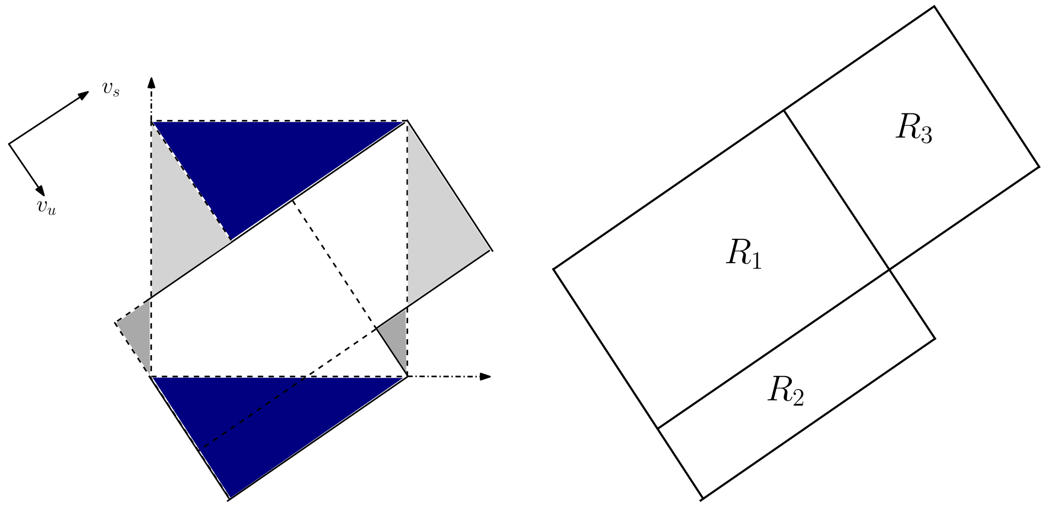

FIGURE 4.1. The construction of the Markov partition

note that $\left\{\cap_{n=-N}^{N}\overline{f^{-n}R_{x_n}}\right\}_{N=1}^{\infty}$ is a decreasing sequence of compact sets hence by the Baire Category Theorem $\Phi(x)$ is well defined. The map $\Phi:\Sigma\to\mathbb{T}^2$ is continuous, finite to one, and for every $x\in\Sigma_{\mathbf{A}}$,

$$\Phi\circ T(x)=f\circ\Phi(x).$$

In addition, for every $x\in\mathbb{T}^2\setminus\cup_{n\in\mathbb{Z}}\cup_{i=1}^{3}f^{-n}\left(\partial R_i\right)$ there exists a unique $w\in\Sigma$ so that $w=\Phi^{-1}(x)$. Thus Φ is a semi-conjugacy (topological factor map) between $(\Sigma_{\mathbf{A}},T)$ to $\left(\mathbb{T}^2,f\right)$. The Lebesgue measure λ on $\mathbb{T}^2$ is conservative and invariant under f and the set $\cup_{n\in\mathbb{Z}}\cup_{i=1}^{3}f^{-n}\left(\partial R_i\right)$ (points with non-unique expansion in Σ) is a Lebesgue-null set. As a consequence Φ is an isomorphism between $\left(\Sigma_{\mathbf{A}},\mu_{\pi_{\mathbf{Q}},\mathbf{Q}},T\right)$ and $\left(\mathbb{T}^2,\lambda,f\right)$ where $\mu_{\pi_{\mathbf{Q}},\mathbf{Q}}$ is the stationary Markov measure with

$$P_j\equiv\mathbf{Q}:=\begin{pmatrix}\frac{G}{1+G} & 0 & \frac{1}{1+G}\\ \frac{G}{1+G} & 0 & \frac{1}{1+G}\\ 0 & 1 & 0\end{pmatrix}\tag{4.1}$$

and

$$\pi_j=\pi_{\mathbf{Q}}:=\begin{pmatrix}1/\sqrt{5}\\ 1/G\sqrt{5}\\ 1/G\sqrt{5}\end{pmatrix}=\begin{pmatrix}\lambda\left(R_1\right)\\ \lambda\left(R_2\right)\\ \lambda\left(R_3\right)\end{pmatrix}.\tag{4.2}$$

Similarly, letting $\nu=\mathrm{M}\left\{\pi_n,P_n:\ n\in\mathbb{Z}\right\}$ be a measure arising from the inductive construction, ν is nonatomic, the shift (Σ,ν,T) is conservative and Φ is one to one on the support of μ. Thus $\Phi:(\Sigma,\mathcal{B}_\Sigma,\nu,T)\to\left(\mathbb{T}^2,\mathcal{B}_{\mathbb{T}^2},\mu=\Phi_*\nu,f\right)$ is an isomorphism. Therefore $\left(\mathbb{T}^2,\mathcal{B}_{\mathbb{T}^2},\mu=\Phi_*\nu,f\right)$ is stable type III_1 .The modified inductive construction of [Kos1] inserts another sequence $\epsilon_k\downarrow 0$ in the inductive construction via the following order (In Subsection (4.1) we specify the modified induction process),

$$\left\{\lambda_j,N_j,,\epsilon_j,M_j\right\}_{j=1}^{K}\to\lambda_{K+1}\to N_{K+1}\to\epsilon_{K+1}\to M_{K+1}.$$

The construction of [Kos1] then defines an homeomorphism $\mathfrak{h}_\epsilon:\mathbb{T}^2\to\mathbb{T}^2$ such that

- $\left(\mathfrak{h}_\epsilon\right)_*\lambda\sim\Phi_*\nu$ where $\nu=\mathrm{M}\left\{\pi_n,P_n:n\in\mathbb{Z}\right\}$ is the measure coming from the inductive construction. By Fact 10, $\left(\mathbb{T}^2,\mathcal{B}_{\mathbb{T}^2},\left(\mathfrak{h}_\epsilon\right)_*\lambda,f\right)$ is stable type III_1.

- $\mathfrak{g} = \mathfrak{h}_\epsilon \circ f \circ \mathfrak{h}_\epsilon^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$ is a C^1 Anosov diffeomorphism.

Theorem 18. *(i)* $\left(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \lambda, \mathfrak{g}\right)$ *is a* C^1 *Anosov diffeomorphism which is stable type* III_1.

(ii) For any hyperbolic linear toral automorphism $Q : \mathbb{T}^d \to \mathbb{T}^d$ *the transformation* $\mathfrak{g} \times Q : \left(\mathbb{T}^{d+2}, \mathcal{B}_{\mathbb{T}^{d+2}}, \lambda_{d+2}\right) \circlearrowleft$ *is stable type* III_1 *where* λ_{d+2} *is the Lebesgue measure on* $\mathbb{T}^{d+2}$ *and for* $x \in \mathbb{T}^2, y \in \mathbb{T}^d$, $\mathfrak{g} \times Q\,(x, y) = (\mathfrak{g}(x), Q(y))$.

Proof. $\left(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \lambda, \mathfrak{g}\right)$ is stable type III_1 since it is isomorphic to $\left(\mathbb{T}^2, , \mathcal{B}_{\mathbb{T}^2}, (\mathfrak{h}_\epsilon)_* \lambda, f\right)$ (via the homeomorphism $\mathfrak{h}_\epsilon$). Every hyperbolic toral automorphism is mixing with respect to the Lebesgue measure on $\mathbb{T}^d$ hence weak mixing. Part *(ii)* follows from the proof of the second part of Corollary 2. □

Remark 19. The class of Riemmanian manifolds M for which there exists an Anosov diffeomorphism $Q : M \to M$ are called Anosov Manifolds. It is a famous open question whether every Anosov Manifold is an infranil manifold, see [Gor] and the references therein. If M is an Anosov manifold such that there exists an Anosov diffeomorphism $Q : M \to M$ which preserves a measure $\mu \ll vol_M$ [4], then $\mathfrak{g} \times Q$ is a stable type III_1 Anosov diffeomorphism of $\mathbb{T}^2 \times M$ with respect to the volume measure on $\mathbb{T}^2 \times M$. Since there are no Anosov diffeomorphims of $\mathbb{T}$, the question whether there are type III_1 Anosov diffeormorphisms of $\mathbb{T}^3$ is still open. The main challenge in this case is that the Markov partitions of hyperbolic toral automorphism of $\mathbb{T}^3$ have fractal boundaries. This presents further difficulties in the smooth realization process (construction of $\mathfrak{h}_\epsilon$).

4.1. **The modified induction process (inserting $\{\epsilon_t\}_{t=1}^\infty$):** We first begin by warning the reader that in [Kos1], $\varphi = G = \frac{1+\sqrt{5}}{2}$, hence in all statements there which involve φ one should swap φ to G.

The modified induction process is as follows. First we demand that $N_1 > 20$ and that $\lambda_1 < e^{2^{-N_1}}$. As this involves only a change in the basis of the inductive construction, such choices are possible. Choose ϵ_1 small enough in order to satisfy the conditions of [Kos1, Proposition 8.2] (with $k = 1$). Finally let M_1 be large enough to satisfy the conditions of [Kos1, Lemma 9.1].

Given $\{\lambda_j, N_j, \epsilon_j.M_j\}_{j=1}^K$ we **first choose** λ_{K+1} to satisfy

$$\lambda_{K+1}^{2M_K} \leq e^{2^{-N_K}},$$

and the lattice condition (3.4). This condition is compatible with the conditions on λ_{K+1} in Section 3.

Secondly, choose N_{K+1} to satisfy (3.5). That is the conditions on λ_{K+1} and N_{K+1} do not depend on ϵ's.

Now **choose ϵ_{K+1} to be small enough** so that the conclusions of [Kos1, Prop. 8.2, Lemmas 9,10,11] hold for $k = K + 1$.

Finally **choose M_{K+1} large enough** with respect to N_{K+1} to satisfy (3.6) and the conclusion of [Kos1, Lemma 9.1]. This finishes the inductive step.

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[4] vol_M is the volume measure on M

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