

BERNOULLI ACTIONS OF AMENABLE GROUPS WITH WEAKLY MIXING MAHARAM EXTENSIONS

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ABSTRACT. We provide a simple criterion for a non-singular and conservative Bernoulli action to have a weakly mixing Maharam extension. As an application, we show that every countable amenable group G admits a Bernoulli action $G \curvearrowright (\{0, 1\}^G, \mu)$ with a weakly mixing Maharam extension, answering a recent question by Vaes and Wahl.

1. INTRODUCTION AND THE STATEMENTS OF THE MAIN RESULTS

1.1. The Maharam extension

Let G be a countable group and let (X, μ) be a Borel G -space, that is to say, a Borel probability measure space (X, μ) endowed with an action of G by measure-class preserving measurable maps. It is well-known that there exists a μ -conull G -invariant subset $X' \subset X$ such that the *Radon-Nikodym cocycle*

$$r_\mu(g, x) = -\log \frac{dg^{-1}\mu}{d\mu}(x),$$

is defined for all $(g, x) \in G \times X'$. The *Maharam extension* $G \curvearrowright (\tilde{X}, \tilde{\mu})$ associated to (X, μ) is the G -action on the set $\tilde{X} = X' \times \mathbb{R}$ defined by

$$g(x, t) = (gx, t + r_\mu(g, x)), \quad \text{for } (x, t) \in X' \times \mathbb{R},$$

which can be readily checked to preserve the *infinite* measure $\tilde{\mu} = \mu \otimes e^t dt$.

We say that the Maharam extension of (X, μ) is *weakly mixing* if for every probability measure-preserving ergodic G -space (Y, ν) , the diagonal action $G \curvearrowright (\tilde{X} \times Y, \tilde{\mu} \otimes \nu)$ is ergodic, or equivalently, if the Maharam extension of $G \curvearrowright (X \times Y, \mu \otimes \nu)$ is ergodic. It is not hard to see that if the Maharam extension of (X, μ) is ergodic, then there is no G -invariant σ -finite measure equivalent to μ . Borel G -spaces with weakly mixing Maharam extensions are often referred to as being of *stable type III₁* in the literature.

1.2. Non-singular Bernoulli actions

We note that $G \curvearrowright \{0, 1\}^G$ by $(gx)_h = x_{g^{-1}h}$ for $g, h \in G$ and $x \in \{0, 1\}^G$. Let (μ_g) be a family of probability measures on $\{0, 1\}$ indexed by $g \in G$, with the property that there exists $\delta > 0$ such that $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$. A classical result of Kakutani [9, Corollary 1] tells us that the product measure $\mu = \prod_{g \in G} \mu_g$ is non-singular, i.e. its measure-class is preserved by the G -action (and thus $(\{0, 1\}^G, \mu)$ is a Borel G -space), if and only if

$$\sum_{h \in G} (\mu_{gh}(0) - \mu_h(0))^2 < \infty, \quad \text{for all } g \in G. \tag{1.1}$$

If (1.1) holds, we say that $G \curvearrowright (\{0, 1\}^G, \mu)$ is a *non-singular Bernoulli action*.

In this paper we shall primarily be interested in the case when the action is conservative, ergodic and there is no G -invariant σ -finite measure equivalent to μ . Such systems are often

said to be of type III, and the first examples of type III-Bernoulli actions for $G = (\mathbb{Z}, +)$ were constructed by Hamachi in [8]. As it turns out, ergodic type III-actions of any group decompose further into a one-parameter family of orbit equivalence classes III_λ , where $0 \leq \lambda \leq 1$, and being of type III_1 is equivalent to having an *ergodic* Maharam extension (see for instance [10] for more details). Ulrich Krengel and Benjamin Weiss asked early on:

Which III_λ -types can occur among non-singular Bernoulli actions?

Despite the classical flair of this question, the first examples of non-singular Bernoulli actions of $G = (\mathbb{Z}, +)$ with *ergodic* Maharam extensions, and thus type III_1 , were constructed by the second author [11] as late as 2009. A few years later, the second author [12] further showed that for a large class of Bernoulli actions of $G = (\mathbb{Z}, +)$, being of type III *implies* that the Maharam extensions are weakly mixing (in fact, have the K-property). In particular, this shows that within this class of examples, only type III_1 is possible. Danilenko and Lemańczyk extended this in [5] to a more general class of Bernoulli actions. However, all of these examples are rather special to the additive group of integers as they rely on the notion of a "past" in the group.

More recently, Vaes and Wahl in [14] studied the question above for general countable groups. In their paper, they formulate and make substantial progress on, the following conjecture:

Conjecture. [14] Let G be a countable group with non-vanishing first ℓ^2 -cohomology, or equivalently, suppose that there is a function $f : G \rightarrow \mathbb{C}$ with $f(0) = 0$, which is not in $\ell^2(G)$, but satisfies

$$\sum_{h \in G} (f(gh) - f(h))^2 < \infty, \quad \text{for all } g \in G.$$

Then there is a non-singular Bernoulli action $G \curvearrowright (\{0, 1\}^G, \mu)$ with an ergodic (or even weakly mixing) Maharam extension.

If G does not admit a function as above, for instance, if G has property (T), then they show [14, Theorem 3.1] that every non-singular and conservative Bernoulli action of G is equivalent to a probability-measure preserving one, so in particular it cannot have an ergodic Maharam extension. On the other hand, amenable groups do admit plenty of such functions, and in this case they show:

Theorem 1.1. [14, Theorem 6.1] *Let G be a countable amenable group, and suppose that either:*

- G has an infinite order element, or
- G admits an infinite subgroup of infinite index.

Then there is a non-singular Bernoulli action $G \curvearrowright (\{0, 1\}^G, \mu)$ with a weakly mixing Maharam extension.

They further show that if one is willing to replace the set $\{0, 1\}$ with a Cantor set Z , then every countable amenable group admits a non-singular Bernoulli action $G \curvearrowright (Z^G, \mu)$ with a weakly mixing Maharam extension. They explicitly ask whether the passage to a Cantor set is necessary, or whether one can keep the base $\{0, 1\}$ in this generality. One of the aims of this paper is to prove that one always can.

1.3. Main results

Crucial to the approach of Vaes and Wahl is a two-part criterion [14, Proposition 6.6] for when a non-singular Bernoulli action of a countable amenable group G has a weakly mixing Maharam extension, which involves a simple asymptotic condition on the family (μ_g) on $\{0, 1\}$

and a rather technical assumption that G can be exhausted by so called " λ -inessential" subsets. The conditions on G in Theorem 1.1 are there to guarantee that such an exhaustion is possible.

We show in this paper that the latter technical condition is not needed, and that the asymptotic condition on (μ_g) stated by Vaes and Wahl, together with the conservativity of the resulting Bernoulli action, is enough to guarantee that the Maharam extension is weakly mixing. More precisely, we shall show:

Theorem 1.2. *Let G be a countable amenable group and suppose that $G \curvearrowright (\{0, 1\}^G, \mu)$ is a non-singular conservative Bernoulli action with the property that there exist $\delta > 0$ such that $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$ and a probability measure λ on $\{0, 1\}$ such that*

$$\lim_{g \rightarrow \infty} \mu_g(0) = \lambda(0) \quad \text{and} \quad \sum_{g \in G} (\mu_g(0) - \lambda(0))^2 = \infty. \quad (1.2)$$

Then the Maharam extension of $G \curvearrowright (\{0, 1\}^G, \mu)$ is weakly mixing.

Remark 1.3. Conversely, we prove in the appendix that if a countable group G admits a non-singular Bernoulli action which satisfies the conditions (1.2), then G must be amenable.

Vaes and Wahl [14, Proposition 4.1 and 6.8] prove that every countable amenable group admits a non-singular and conservative Bernoulli action satisfying the conditions (1.2) with $\lambda(0) = \lambda(1) = 1/2$ (this is not quite the way it is phrased in [14]; see remark below). In combination with Theorem 1.2, this allows us to conclude that:

Corollary 1.4. *Every countable amenable group admits a non-singular Bernoulli action $G \curvearrowright (\{0, 1\}^G, \mu)$ with a weakly mixing Maharam extension.*

Remark 1.5. Let G be a countable amenable group, and fix $0 < \delta < 1/2$ and pick a proper function $\varphi : G \rightarrow (0, \infty)$ such that $\sum_{g \in G} e^{-c\varphi(g)} = \infty$ for all $c > 0$. Let $F : G \rightarrow [0, \delta)$ be the function from [14, Proposition 6.8] constructed from φ , and set

$$\mu_g(0) = \mu_g(1) = \frac{1}{2} + F(g), \quad \text{for } g \in G,$$

so that $1/2 - \delta \leq \mu_g(0) \leq 1/2 + \delta$. It follows from [14, Proposition 6.8] that $G \curvearrowright (\{0, 1\}^G, \mu)$ is a non-singular Bernoulli action with

$$\lim_{g \rightarrow \infty} \mu_g(0) = \frac{1}{2} \quad \text{and} \quad \sum_{g \in G} (\mu_g(0) - \frac{1}{2})^2 = \infty,$$

and for every $c > 0$

$$\sum_{g \in G} e^{-c \sum_{h \in G} (\mu_{gh}(0) - \mu_h(0))^2} \geq \sum_{g \in G} e^{-c\varphi(g)} = \infty.$$

By [14, Proposition 4.1], the last condition implies that $G \curvearrowright (\{0, 1\}^G, \mu)$ is conservative.

1.4. A few words about the proof of Theorem 1.2

Let G be a countable amenable group and $G \curvearrowright (X, \mu) = (\{0, 1\}^G, \mu)$ a non-singular and conservative Bernoulli action satisfying the conditions of Theorem 1.2. In particular, if we set

$$G^+ = \{g \in G : \mu_g(0) > \lambda(0)\} \quad \text{and} \quad G^- = \{g \in G : \mu_g(0) < \lambda(0)\}, \quad (1.3)$$

then we see that either

$$\sum_{g \in G^+} (\mu_g(0) - \lambda(0))^2 = \infty \quad \text{or} \quad \sum_{g \in G^-} (\mu_g(0) - \lambda(0))^2 = \infty.$$

Since $\mu_g(0) - \lambda(0) = -(\mu_g(1) - \lambda(1))$, for all g , we may without loss of generality assume that the first alternative holds (otherwise we just interchange 0 and 1). So, from now on, the standing assumptions throughout the rest of the paper are:

$$\lim_{g \rightarrow \infty} \mu_g(0) = \lambda(0) \quad \text{and} \quad \sum_{g \in G^+} (\mu_g(0) - \lambda(0))^2 = \infty. \quad (1.4)$$

Let now (Y, ν) be an ergodic probability-measure-preserving G -space. By the recent result of Danilenko [6, Theorem 0.1], the diagonal action $G \curvearrowright (X \times Y, \mu \otimes \nu)$ is ergodic, and we wish to prove that its Maharam extension is ergodic as well. By [13, Corollary 5.4], this is *equivalent* to showing that for every $\varepsilon > 0$ and $t \geq 0$ (or $t \leq 0$) and $\mu \otimes \nu$ -measurable subset $C \subset X \times Y$ with positive measure,

$$\mu \otimes \nu(C \cap \left(\bigcup_{g \in G} g^{-1}C \cap \{|r_{\mu \otimes \nu}(g, \cdot) - t| < \varepsilon\} \right)) > 0. \quad (1.5)$$

Remark 1.6. A word of clarification might be in order. In Corollary 5.4 in [13], to prove ergodicity, one needs to ensure (1.5) for *all* $t \in \mathbb{R}$. However, by [13, Lemma 3.3], the set of t for which (1.5) holds is a closed subgroup of $(\mathbb{R}, +)$, so in particular symmetric.

Let us fix a dense subset \mathcal{C} of the measure algebra of $(X \times Y, \mu \otimes \nu)$. By [3, Lemma 2.2], to prove (1.5) for every $\mu \otimes \nu$ -measurable subset of $X \times Y$ with positive measure, it suffices to show that there exists $M \geq 1$ with the property that for $\varepsilon > 0$ and $t \geq 0$ (or $t \leq 0$) and $C \in \mathcal{A}$, we have

$$\mu \otimes \nu(C \cap \left(\bigcup_{g \in G} g^{-1}C \cap \{|r_{\mu \otimes \nu}(g, \cdot) - t| < \varepsilon\} \right)) \geq \frac{1}{M} \mu \otimes \nu(C). \quad (1.6)$$

Let \mathcal{A} denote the subset of the measure algebra of $(X \times Y, \mu \otimes \nu)$ consisting of finite unions of product sets of the form $A_i \times B_i$, where $A_i \subset X$ are disjoint *cylinder sets* and $B_i \subset Y$ disjoint Borel sets. Then \mathcal{A} is dense, and to prove (1.6) for every set in \mathcal{A} , it suffices by disjointness of the sets $A_i \times B_i$, to show that there is a constant M such that (1.6) holds for every set of the form $C = A_i \times B_i$.

This is the content of the following proposition which is our main technical result, and whose proof will occupy the rest of the paper.

Proposition 1.7. *Fix $\varepsilon > 0$, a cylinder set $A \subset X$ and a Borel set $B \subset Y$.*

- *If $\lambda(0) \geq 1/2$, then for every $t \geq 0$,*

$$\mu \otimes \nu((A \times B) \cap \left(\bigcup_{g \in G} g^{-1}(A \times B) \cap \{|r_{\mu \otimes \nu}(g, \cdot) - t| < \varepsilon\} \right)) \geq \frac{1}{3} \mu \otimes \nu(A \times B). \quad (1.7)$$

- *If $\lambda(0) < 1/2$, then for every $t \leq 0$,*

$$\mu \otimes \nu((A \times B) \cap \left(\bigcup_{g \in G} g^{-1}(A \times B) \cap \{|r_{\mu \otimes \nu}(g, \cdot) - t| < \varepsilon\} \right)) \geq \frac{1}{3} \mu \otimes \nu(A \times B). \quad (1.8)$$

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2. PROOF OF PROPOSITION 1.7

Let G be a countable amenable group and let $(X, \mu) = (\{0, 1\}^G, \mu)$ and λ be as in Theorem 1.2. We also fix a probability measure-preserving ergodic G -space (Y, ν) .

2.1. Notation

2.1.1. Shapes of cylinder sets

If $K \subset G$ is a finite set and $\sigma : K \rightarrow \{0, 1\}$, we define the *cylinder set* A with $\text{shape}(A) = (K, \sigma)$ by

$$A = \{x \in X : x_g = \sigma(g), \text{ for all } g \in K\}.$$

2.1.2. Radon-Nikodym cocycles

By [9, Lemma 5], there exists a μ -conull subset $X' \subset X$ such that the Radon-Nikodym cocycle of (X, μ) satisfies

$$r_\mu(g, x) = \sum_{h \in G} (\log \mu_h(x_h) - \log \mu_{gh}(x_h)), \quad \text{for all } x \in X'. \quad (2.1)$$

Since ν is G -invariant, we have

$$r_{\mu \otimes \nu}(g, (x, y)) = r_\mu(g, x), \quad \text{for all } (x, y) \in X' \times Y.$$

2.1.3. The homoclinic relation and its cocycles

We define the *homoclinic relation* \mathcal{H}_μ on X' by

$$\mathcal{H} = \{(x, x') \in X' \times X' : x_g = x'_g, \text{ for all but finitely many } g\}.$$

For any family (η_g) of functions on $\{0, 1\}$ we can define the *homoclinic cocycle* (or *Gibbs cocycle*) associated to this family by

$$c_\eta(x, x') = \sum_{g \in G} (\eta_g(x_g) - \eta_g(x'_g)), \quad \text{for } (x, x') \in \mathcal{H}_\mu.$$

The case when $\eta_g(a) = \log \frac{\mu_g(a)}{\lambda(a)}$ for $a \in \{0, 1\}$ will be of special importance to us, and to distinguish this case, we omit η as an index. In other words, we write

$$c(x, x') = \sum_{g \in G} \left(\log \frac{\mu_g(x_g)}{\lambda(x_g)} - \log \frac{\mu_g(x'_g)}{\lambda(x'_g)} \right), \quad \text{for } (x, x') \in \mathcal{H}_\mu. \quad (2.2)$$

We denote by $[\mathcal{H}_\mu]$ the collection of partially defined one-to-one measurable maps between measurable subsets of X' whose graphs are subsets of \mathcal{H}_μ . If $\phi \in [\mathcal{H}_\mu]$, we write $\text{Dom}(\phi)$ for its domain of definition, and set

$$\text{supp}(\phi) = \{g \in G : \phi(x)_g \neq x_g, \text{ for some } x \in \text{Dom}(\phi)\}.$$

We note that if $A \subset X$ is a cylinder set with $\text{shape}(A) = (K, \sigma)$ and $K \cap \text{supp}(\phi) = \emptyset$, then

$$\phi^{-1}(A) \cap \text{Dom}(\phi) = \{x \in \text{Dom}(\phi) : \phi(x)_g = \sigma(g), \text{ for all } g \in K\} = A \cap \text{Dom}(\phi), \quad (2.3)$$

and $\mu(A \cap \text{Dom}(\phi)) = \mu(A)\mu(\text{Dom}(\phi))$.

2.2. Three lemmas

In what follows, we shall show how Proposition 1.7 can be reduced to three lemmas, whose proofs are postponed to the next sections.

Our first lemma roughly says that the cocycle c defined in (2.2) attains "most values" quite frequently, and its proof is very much inspired by [7, Proposition 3.1].

Lemma 2.1. Fix $\varepsilon > 0$ and a finite subset $K \subset G$. If either

$$\lambda(0) \geq 1/2 \quad \text{and} \quad t \geq 0$$

or

$$\lambda(0) < 1/2 \quad \text{and} \quad t \leq 0,$$

then there exists $\phi \in [\mathcal{H}_\mu]$ with finite support such that

$$K \cap \text{supp}(\phi) = \emptyset \quad \text{and} \quad \nu(\text{Dom}(\phi)) \geq \frac{1}{3}$$

and

$$|c(x, \phi(x)) - t| < \varepsilon, \quad \text{for all } x \in \text{Dom}(\phi).$$

Our second lemma provides a way to compare the cocycles r_μ and c , and roughly says that if $\phi \in [\mathcal{H}_\mu]$ and $r_\mu(g, \phi(x))$ is small and $c(x, \phi(x))$ is close to t , then $r_\mu(g, x)$ is close to t as well.

Lemma 2.2. For every $\varepsilon > 0$ and $\phi \in [\mathcal{H}_\mu]$ with finite support, there is a finite set $L \subset G$ such that if

$$L \cap g(\text{supp}(\phi)) = \emptyset, \tag{2.4}$$

then

$$|r_\mu(g, x) - r_\mu(g, \phi(x)) - c(x, \phi(x))| < \varepsilon, \quad \text{for all } x \in \text{Dom}(\phi). \tag{2.5}$$

Our third lemma is very general and says that if $G \curvearrowright (X \times Y, \mu \otimes \nu)$ is conservative, then the cocycle $r_{\mu \otimes \nu}(g, \cdot)$ is "frequently" very small for arbitrarily large g .

Lemma 2.3. For every $\varepsilon > 0$ and finite subset $F \subset G$ and $\mu \otimes \nu$ -measurable set $C \subset X \times Y$, we have

$$\mu \otimes \nu \left(C \cap \left(\bigcup_{g \notin F} g^{-1} C \cap \{ |r_{\mu \otimes \nu}(g, \cdot)| < \varepsilon \} \right) \right) = \mu \otimes \nu(C). \tag{2.6}$$

2.3. Proof of Proposition 1.7 assuming Lemmas 2.1 and 2.2 and 2.3

We only present the proof when $\lambda(0) \geq 1/2$; the case $\lambda(0) < 1/2$ is completely analogous. Let us fix $\varepsilon > 0$, a cylinder set $A \subset X$ with $\text{shape}(A) = (K, \sigma)$ and a Borel set $B \subset Y$. We fix $t \geq 0$ and use Lemma 2.1 to find $\phi \in [\mathcal{H}_\mu]$ with

$$K \cap \text{supp}(\phi) = \emptyset \quad \text{and} \quad \mu(\text{Dom}(\phi)) \geq \frac{1}{3}$$

such that $|c(x, \phi(x)) - t| < \varepsilon/3$.

Set $A' = A \cap \text{Dom}(\phi)$ and note that since $K \cap \text{supp}(\phi) = \emptyset$, we have

$$\phi^{-1}(A) \cap \text{Dom}(\phi) = A \cap \text{Dom}(\phi),$$

by (2.3), whence $\phi(A') \subset A$, and $\mu(A') \geq \frac{1}{3}\mu(A)$.

By Lemma 2.2, we can find a finite set $L \supset K$ such that whenever $g \in G$ and $\phi \in [\mathcal{H}_\mu]$ satisfy

$$L \cap g(\text{supp}(\phi)) = \emptyset,$$

then

$$|r_\mu(g, x) - r_\mu(g, \phi(x)) - c(x, \phi(x))| < \frac{\varepsilon}{3}, \quad \text{for all } x \in \text{Dom}(\phi). \tag{2.7}$$

We now apply Lemma 2.3 to $C = \phi(A') \times B$ and the finite set

$$F = \{g \in G : L \cap g(\text{supp}(\phi)) \neq \emptyset\} \subset G$$

to conclude that

$$E = (\phi(A') \times B) \cap \left(\bigcup_{g \notin F} g^{-1}(\phi(A') \times B) \cap \{ |r_{\mu \otimes \nu}(g, \cdot)| < \varepsilon/3 \} \right)$$

satisfies $\mu \otimes \nu(E) = \mu(\phi(A'))\nu(B)$. In particular, $(\phi \times \text{id})^{-1}(E) = A' \times B$ modulo $\mu \otimes \nu$ -null sets, and thus

$$\mu \otimes \nu((\phi \times \text{id})^{-1}(E)) = \mu(A')\nu(B) \geq \frac{1}{3}\mu \otimes \nu(A \times B). \quad (2.8)$$

For every $z = (x, y) \in (\phi \times \text{id})^{-1}(E)$, there exists $g_z \notin F$ such that

$$|r_{\mu}(g_z, \phi(x))| < \frac{\varepsilon}{3} \quad \text{and} \quad g_z(\phi(x)) \in A \quad \text{and} \quad g_z(y) \in B.$$

Set $\phi_z = g_z \circ \phi \circ g_z^{-1}$ and note that

$$g_z(x) \in \phi_z^{-1}(A) \quad \text{and} \quad \text{supp}(\phi_z) = g_z(\text{supp}(\phi)) \quad \text{and} \quad \text{Dom}(\phi_z) = g_z(\text{Dom}(\phi)).$$

Since $K \cap g_z(\text{supp}(\phi)) = \emptyset$, we see that

$$\phi_z^{-1}(A) \cap \text{Dom}(\phi_z) = A \cap \text{Dom}(\phi_z),$$

by (2.3), and since $x \in A'$, and thus $g_z(x) \in \text{Dom}(\phi_z)$, we conclude that $g_z(x) \in A$.

By (2.7),

$$\begin{aligned} |r_{\mu}(g_z, x) - t| &\leq |r_{\mu}(g_z, x) - r_{\mu}(g_z, \phi(x)) - c(x, \phi(x))| + \\ &\quad + |r_{\mu}(g_z, \phi(x))| + |c(x, \phi(x)) - t| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

so we can conclude that

$$z = (x, y) \in (A \times B) \cap g_z^{-1}(A \times B) \cap \{ |r_{\mu \otimes \nu}(g_z, \cdot) - t| < \varepsilon \},$$

and thus,

$$(\phi \times \text{id})^{-1}(E) \subset (A \times B) \cap \left(\bigcup_{g \notin F} g^{-1}(A \times B) \cap \{ |r_{\mu \otimes \nu}(g, \cdot) - t| < \varepsilon \} \right). \quad (2.9)$$

By combining (2.9) and (2.8), we get

$$\mu \otimes \nu((A \times B) \cap \left(\bigcup_{g \notin F} g^{-1}(A \times B) \cap \{ |r_{\mu \otimes \nu}(g, \cdot) - t| < \varepsilon \} \right)) \geq \frac{1}{3}\mu \otimes \nu(A \times B),$$

which finishes the proof.

3. PROOFS OF LEMMAS 2.1 AND 2.2

Throughout this section, we retain the notation introduced in Section 2. We recall our standing assumptions on (μ_g) and λ from Theorem 1.2 and from (1.4), namely that there exists $\delta > 0$ such that

$$\delta \leq \mu_g(0) \leq 1 - \delta, \quad \text{for all } g \in G, \quad (3.1)$$

and

$$\lim_{g \rightarrow \infty} \mu_g(0) = \lambda(0) \quad \text{and} \quad \sum_{g \in G^+} (\mu_g(0) - \lambda(0))^2 = \infty, \quad (3.2)$$

where

$$G^+ = \{g \in G : \mu_g(0) > \lambda(0)\}.$$

If $\eta_g : \{0, 1\} \rightarrow \mathbb{R}$ is a sequence of functions, indexed by $g \in G$, we define the associated homoclinic (or Gibbs) cocycle $c_\eta : \mathcal{H}_\mu \rightarrow \mathbb{R}$ by

$$c_\eta(x, x') = \sum_{g \in G} (\eta_g(x_g) - \eta_g(x'_g)), \quad \text{for } (x, x') \in \mathcal{H}_\mu.$$

3.1. Proof of Lemma 2.1

Lemma 2.1 is an immediate consequence of the following two lemmas, whose proofs are presented below.

Lemma 3.1. *Let $\eta_g : \{0, 1\} \rightarrow \mathbb{R}$ and suppose that $\lim_{g \rightarrow \infty} \|\eta_g\|_\infty = 0$, and*

$$\sum_{g \in G^+} (\eta_g(0) - \eta_g(1))^2 = \infty. \quad (3.3)$$

Fix $\varepsilon > 0$ and a finite subset $K \subset G$.

- If

$$\sum_{g \in G^+} (\eta_g(0) - \eta_g(1))(\mu_g(0) - \mu_g(1)) = \infty, \quad (3.4)$$

then, for every $t \geq 0$, there exists $\phi_+ \in [\mathcal{H}_\mu]$ with

$$K \cap \text{supp}(\phi_+) = \emptyset \quad \text{and} \quad \nu(\text{Dom}(\phi_+)) \geq \frac{1}{3}$$

such that

$$|c_\eta(x, \phi_+(x)) - t| < \varepsilon, \quad \text{for all } x \in \text{Dom}(\phi_+).$$

- If

$$\sum_{g \in G^+} (\eta_g(0) - \eta_g(1))(\mu_g(0) - \mu_g(1)) = -\infty, \quad (3.5)$$

then, for every $t \leq 0$, there exists $\phi_- \in [\mathcal{H}_\mu]$ with

$$K \cap \text{supp}(\phi_-) = \emptyset \quad \text{and} \quad \nu(\text{Dom}(\phi_-)) \geq \frac{1}{3}$$

such that

$$|c_\eta(x, \phi_-(x)) - t| < \varepsilon, \quad \text{for all } x \in \text{Dom}(\phi_-).$$

Lemma 3.2. *Let $\eta_g = \log \frac{\mu_g}{\lambda}$. Then $\lim_{g \rightarrow \infty} \|\eta_g\|_\infty = 0$, and the sequence (η_g) satisfies (3.3). Furthermore,*

- if $\lambda(0) \geq 1/2$, then (η_g) satisfies (3.4).
- if $\lambda(0) < 1/2$, then (η_g) satisfies (3.5).

3.2. Proof of Lemma 3.1

For $g \in G$, we define $\tau_g : X \rightarrow X$ by

$$\tau_g(x)_{g'} = \begin{cases} x_{g'} & \text{if } g \neq g' \\ 1 - x_{g'} & \text{if } g = g' \end{cases}.$$

We first write c_η on the form

$$c_\eta(x, x') = \sum_{g \in G} (F_g(x) - F_g(x')),$$

where $F_g : X \rightarrow \mathbb{R}$ are defined as

$$F_g(x) = \eta_g(x_g) - \frac{1}{2}(\eta_g(0) + \eta_g(1)), \quad \text{for } x \in X.$$

We note that all (F_g) are independent of each other,

$$\lim_{g \rightarrow \infty} \|F_g\|_\infty = 0 \quad \text{and} \quad F_g - F_g \circ \tau_g = 2F_g, \quad (3.6)$$

and

$$\int_X F_g \, d\mu = \frac{1}{2} (\eta_g(0) - \eta_g(1)) (\mu_g(0) - \mu_g(1)). \quad (3.7)$$

and

$$\int_X F_g^2 \, d\mu = \frac{1}{4} (\eta_g(0) - \eta_g(1))^2. \quad (3.8)$$

Let us fix $\varepsilon > 0$ and choose a finite subset $M \supset K$ such that $\|F_g\|_\infty < \varepsilon/2$ for all $g \notin M$. We enumerate $G^+ \setminus M = \{g_1, g_2, \dots\}$ and set

$$G_n = \{g_1, \dots, g_n\} \quad \text{and} \quad S_n = \sum_{g \in G_n} F_g \quad \text{and} \quad A_n = \sum_{g \in G_n} \int_X F_g \, d\mu.$$

Since $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$, it is not hard to see that there is a constant C_δ such that

$$\frac{1}{C_\delta} \sum_{g \in G_n} (\eta_g(0) - \eta_g(1))^2 \leq \int_X |S_n - A_n|^2 \, d\mu \leq C_\delta \sum_{g \in G_n} (\eta_g(0) - \eta_g(1))^2,$$

for all n , and thus the variance $B_n^2 := \int_X |S_n - A_n|^2 \, d\mu$ tends to infinity as $n \rightarrow \infty$ by (3.3). Hence, by Lyapunov's CLT (see e.g. [4, Theorem 7.1.2]),

$$\lim_n \mu \left(\left\{ x \in X : \frac{S_n(x) - A_n}{B_n} \geq r \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_r^\infty e^{-u^2/2} \, du,$$

for all $r \in \mathbb{R}$. In particular, there exist $n_+, n_- \geq 1$ such that

$$\mu(\{x \in X : S_n(x) > A_n\}) = \mu\left(\left\{x \in X : \frac{S_n(x) - A_n}{B_n} > 0\right\}\right) \geq \frac{1}{3}, \quad \text{for all } n \geq n_+$$

and

$$\mu(\{x \in X : S_n(x) < A_n\}) = \mu\left(\left\{x \in X : \frac{S_n(x) - A_n}{B_n} < 0\right\}\right) \geq \frac{1}{3}, \quad \text{for all } n \geq n_-.$$

If (3.4) holds, then $A_n \rightarrow \infty$, so for any $t \geq 0$, the inequality $A_n \geq t$ holds eventually, and thus we can find $N_+ = N_+(t) \geq n_+$ such that

$$\mu(\{x \in X : S_{N_+}(x) > t\}) \geq \frac{1}{3}.$$

Let us from now on fix $t \geq 0$ and an integer $N_+ = N_+(t)$ as above, and set

$$E_t^+ = \{x \in X : S_{N_+}(x) > t\}.$$

We define $T_+ : E_t^+ \rightarrow \{1, \dots, N_+\}$ by

$$T_+(x) = \min \left\{ n \geq 1 : S_n(x) > \frac{t}{2} \right\},$$

and $\phi_+ : E_t^+ \rightarrow X$ by

$$\phi_+(x)_g = \begin{cases} \tau_g(x) & \text{if } g \in G_{T_+(x)} \\ x & \text{if } g \notin G_{T_+(x)}. \end{cases}$$

Clearly,

$$\text{Dom}(\phi_+) = E_t^+ \quad \text{and} \quad \text{supp}(\phi_+) \subset G_{N_+} \quad \text{and} \quad (x, \phi_+(x)) \in \mathcal{H}_\mu, \quad \text{for all } x \in \text{Dom}(\phi_+).$$

We note that by (3.6),

$$c_\eta(x, \phi_+(x)) = \sum_{g \in G_{T_+(x)}} (F_g(x) - F_g(\tau_g(x))) = 2S_{T_+(x)}(x) > r,$$

and since $\|F_g\|_\infty < \varepsilon/2$ for all $g \notin M$, we also have

$$2S_{T_+(x)}(x) = 2S_{T_+(x)-1}(x) + 2F_{g_{T_+(x)}}(x) < r + \varepsilon,$$

and thus $|c_\eta(x, \phi_+(x)) - r| < \varepsilon$.

It remains to show that $\phi_+ \in [\mathcal{H}_\mu]$, that is to say, ϕ_+ is one-to-one on E_t^+ . Pick two different points $x, x' \in E_t^+$. If $T_+(x) = T_+(x')$, then clearly $\phi_+(x)$ and $\phi_+(x')$ are distinct, so we may without loss of generality assume that $T_+(x) < T_+(x')$, whence

$$S_{T_+(x)}(x) > S_{T_+(x)}(x'), \quad \text{and thus} \quad \sum_{g \in G_{T_+(x)}} F_g(x) > \sum_{g \in G_{T_+(x)}} F_g(x').$$

Since $G_{T_+(x)} \subset G_{T_+(x')}$, we have

$$F_g(x) = -F_g(\phi_+(x)) \quad \text{and} \quad F_g(x') = -F_g(\phi_+(x')), \quad \text{for all } g \in G_{T_+(x)},$$

and thus

$$-\sum_{g \in G_{T_+(x)}} F_g(\phi_+(x)) > -\sum_{g \in G_{T_+(x)}} F_g(\phi_+(x')),$$

which shows that $\phi_+(x) \neq \phi_+(x')$.

If (3.5) holds, then $A_n \rightarrow -\infty$, so for any $t \leq 0$, the inequality $A_n \leq t$ holds eventually, and thus we can find $N_- = N_-(t) \geq n_-$ such that

$$\mu(\{x \in X : S_{N_-}(x) < t\}) \geq \frac{1}{3}.$$

Let us from now on fix $t \leq 0$ and an integer $N_- = N_-(t)$ as above, and set

$$E_t^- = \{x \in X : S_{N_-}(x) < t\}.$$

We define $T_- : E_t^- \rightarrow \{1, \dots, N_+\}$ by

$$T_-(x) = \min \{n \geq 1 : S_n(x) < \frac{t}{2}\},$$

and $\phi_- : E_t^- \rightarrow X$ by

$$\phi_-(x)_g = \begin{cases} \tau_g(x) & \text{if } g \in G_{T_-(x)} \\ x & \text{if } g \notin G_{T_-(x)}. \end{cases}$$

Clearly,

$$\text{Dom}(\phi_-) = E_t^- \quad \text{and} \quad \text{supp}(\phi_-) \subset G_{N_-} \quad \text{and} \quad (x, \phi_-(x)) \in \mathcal{H}_\mu, \quad \text{for all } x \in \text{Dom}(\phi_-).$$

The rest of the argument is now completely analogous to previous case.

3.3. Proof of Lemma 3.2

Set $\eta_g = \log \frac{\mu_g}{\lambda}$. We recall the standard inequalities:

$$\frac{t}{1+t} \leq \log(1+t) \leq t, \quad \text{for all } t > -1.$$

In particular, applied to $t = \frac{\mu_g(\cdot)}{\lambda(\cdot)} - 1$, these yield

$$\frac{\mu_g - \lambda}{\mu_g} \leq \eta_g \leq \frac{\mu_g - \lambda}{\lambda}, \quad \text{on } \{0, 1\},$$

and thus, by (3.1),

$$\eta_g(0) - \eta_g(1) \geq \left(\frac{1}{\mu_g(0)} + \frac{1}{\lambda(1)} \right) (\mu_g(0) - \lambda(0)) \geq \left(\frac{1}{1-\delta} + \frac{1}{\lambda(1)} \right) (\mu_g(0) - \lambda(0)),$$

for all $g \in G$. In particular, $\eta_g(0) > \eta_g(1)$ for all $g \in G^+$, and by the second condition in (3.2), we see that (3.3) holds.

Set

$$I = \sum_{g \in G^+} (\eta_g(0) - \eta_g(1)) (\mu_g(0) - \mu_g(1)) = 2 \sum_{g \in G^+} (\eta_g(0) - \eta_g(1)) (\mu_g(0) - \frac{1}{2}).$$

If $\lambda(0) \geq 1/2$, then

$$\mu_g(0) - \frac{1}{2} = \mu_g(0) - \lambda(0) + \lambda(0) - \frac{1}{2} \geq \mu_g(0) - \lambda(0),$$

and thus, by the second condition in (3.2),

$$I \geq 2 \left(\frac{1}{1-\delta} + \frac{1}{\lambda(1)} \right) \sum_{g \in G^+} (\mu_g(0) - \lambda(0))^2 = \infty.$$

If $\lambda(0) < 1/2$, then there exists $\varepsilon > 0$ such that the set

$$F = \{g \in G^+ : \mu_g(0) - \frac{1}{2} \geq -\varepsilon\}$$

is finite. The second condition in (3.2), combined with the fact that $\mu_g(0) - \lambda(0) < 1$ for all $g \in G^+$, shows that we have $\sum_{g \in G^+ \setminus F} (\mu_g(0) - \lambda(0)) = \infty$, whence

$$\sum_{g \in G^+ \setminus F} (\eta_g(0) - \eta_g(1)) (\mu_g(0) - \frac{1}{2}) < -\varepsilon \left(\frac{1}{1-\delta} + \frac{1}{\lambda(1)} \right) \sum_{g \in G^+ \setminus F} (\mu_g(0) - \lambda(0)) = -\infty.$$

We conclude that $I = -\infty$.

3.4. Proof of Lemma 2.2

We recall the definitions of the cocycles r_μ and c from Section 2. In particular, by (2.1), we know that for all $x \in X'$,

$$r_\mu(g, x) = \sum_{h \in G} (\log \mu_h(x_h) - \log \mu_{gh}(x_h)), \quad \text{for all } x \in X',$$

and for all $(x, x') \in \mathcal{H}_\mu$,

$$c(x, x') = \sum_{g \in G} \left(\log \frac{\mu_g(x_g)}{\lambda(x_g)} - \log \frac{\mu_g(x'_g)}{\lambda(x'_g)} \right).$$

Fix $\varepsilon > 0$ and $\phi \in [\mathcal{H}_\nu]$ with finite support and set

$$L = \left\{ g \in G : \max_{a \in \{0,1\}} \left| \log \frac{\mu_g(a)}{\lambda(a)} \right| \geq \frac{\varepsilon}{2|\text{supp}(\phi)|} \right\}.$$

Since $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$ and $\mu_g(0) \rightarrow \lambda(0)$ as $g \rightarrow \infty$, the set L is finite. We note that for every $g \in G$ and $x \in \text{Dom}(\phi)$,

$$\begin{aligned} r_\mu(g, x) - r_\mu(g, \phi(x)) &= \sum_{h \in \text{supp}(\phi)} \left(\log \frac{\mu_h(x_h)}{\lambda(x_h)} - \log \frac{\mu_h(\phi(x)_h)}{\lambda(\phi(x)_h)} \right) + \\ &+ \sum_{h \in \text{supp}(\phi)} \log \frac{\lambda(x_h) \cdot \mu_{gh}(\phi(x)_h)}{\mu_{gh}(x_h) \cdot \lambda(\phi(x)_h)} \\ &= c(x, \phi(x)) + \sum_{h \in \text{supp}(\phi)} \log \frac{\lambda(x_h) \cdot \mu_{gh}(\phi(x)_h)}{\mu_{gh}(x_h) \cdot \lambda(\phi(x)_h)}. \end{aligned}$$

If $L \cap g(\text{supp}(\phi)) = \emptyset$, then the absolute value of the last sum is bounded by ε , which finishes the proof.

4. PROOF OF LEMMA 2.3

We retain the notation from Section 2. It is not hard to see that $G \curvearrowright (X, \mu)$ essentially free. Since we have assumed that it is also conservative, so is the diagonal action $G \curvearrowright (X \times Y, \mu \otimes \nu)$. Hence Lemma 2.3 follows from the following general result, which is surely known to experts, but we include it for completeness.

Lemma 4.1. *If $G \curvearrowright (Z, \xi)$ is conservative and essentially free, then, for every $\varepsilon > 0$, Borel set $C \subset Z$ and finite subset $F \subset G$,*

$$\xi\left(C \cap \left(\bigcup_{g \notin F} g^{-1}C \cap \{|r_\xi(g, \cdot)| < \varepsilon\} \right)\right) = \xi(C). \quad (4.1)$$

Proof. Since $G \curvearrowright (Z, \xi)$ is conservative, it follows from [13, Theorem 4.2 and Theorem 5.5] that its Maharam extension $G \curvearrowright (\tilde{Z}, \tilde{\xi})$ is conservative as well. Since $G \curvearrowright (Z, \xi)$ is essentially free, so is $G \curvearrowright (\tilde{Z}, \tilde{\xi})$. Hence by Halmos recurrence theorem [1, Proposition 1.6.2], for every measurable set $\tilde{C} \subset \tilde{Z}$, we have $\tilde{D} = \tilde{C}$ modulo $\tilde{\xi}$ -null sets, where

$$\tilde{D} = \{(z, t) \in \tilde{C} : \sum_{g \in G} \chi_{\tilde{C}}(gz, t + r_\xi(g, z)) = \infty\}. \quad (4.2)$$

Let $\delta > 0$ and fix a Borel set $C \subset Z$. If we define $\tilde{C} = C \times (-\delta, \delta)$ and let \tilde{D} denote the corresponding set defined in (4.2), then Fubini's Theorem tells us that there is a ξ -conull subset $C' \subset C$ such that

$$\tilde{D}_z := \{t \in \mathbb{R} : (z, t) \in \tilde{D}\} = (-\delta, \delta), \quad \text{modulo } \xi\text{-null sets,}$$

for all $z \in C'$. In particular, for all $z \in C'$, we have

$$\int_{\tilde{D}_z} \sum_{g \in G} \chi_{\tilde{C}}(gz, t + r_\xi(g, z)) e^t dt = \sum_{g \in G} \chi_C(gz) \rho_\delta(r_\xi(g, z)) = \infty,$$

where

$$\rho_\delta(s) = \int_{-\delta}^{\delta} \chi_{(-\delta, \delta)}(s + t) e^t dt, \quad \text{for } s \in \mathbb{R}.$$

Let us now fix $\varepsilon > 0$ and choose $\delta > 0$ so small so that $\rho_\delta \leq 2\chi_{(-\varepsilon, \varepsilon)}$. Then, with the notations above,

$$\sum_{g \in G} \chi_C(gz) \chi_{(-\varepsilon, \varepsilon)}(r_\xi(g, z)) = \infty, \quad \text{for all } z \in C',$$

and thus, for every finite subset $F \subset G$,

$$\sum_{g \notin F} \chi_C(gz) \chi_{(-\varepsilon, \varepsilon)}(r_\xi(g, z)) = \infty, \quad \text{for all } z \in C',$$

We conclude that for every finite set $F \subset G$,

$$C \cap \left(\bigcup_{g \notin F} g^{-1}C \cap \{|r_\xi(g, \cdot)| < \varepsilon\} \right) \supset \{z \in C : \sum_{g \notin F} \chi_C(gz) \chi_{(-\varepsilon, \varepsilon)}(r_\xi(g, z)) = \infty\} \supset C'.$$

Since $\xi(C') = \xi(C)$, we are done. \square

APPENDIX A. ON THE ROLE OF AMENABILITY

Proposition A.1. *Let G be a finitely generated non-amenable group and suppose that $G \curvearrowright (\{0, 1\}^G, \mu)$ is a non-singular Bernoulli action with the property that there exist $\delta > 0$ such that $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$ and a probability measure λ on $\{0, 1\}$ such that*

$$\lim_{g \rightarrow \infty} \mu_g(0) = \lambda(0).$$

Then μ is equivalent to the G -invariant probability measure $\prod_g \lambda$. In particular, it is not of type III.

Remark A.2. The proof below can be adapted to also deal with infinitely generated groups, but providing references in this generality would become harder.

Since $G \curvearrowright (\{0, 1\}^G, \mu)$ is non-singular and $\delta \leq \mu_g(0) \leq 1 - \delta$ for all $g \in G$, Kakutani's criterion 1.1 shows that

$$\sum_{h \in G} (\mu_{gh}(0) - \mu_h(0))^2 < \infty, \quad \text{for all } g \in G.$$

Let S be a finite and symmetric generating set of G . Since G is non-amenable, we can, by [2, Corollary 3], write

$$\mu_g(0) = u(g) + v(g), \quad \text{for all } g \in G,$$

where $v \in \ell^2(G)$ and

$$\frac{1}{|S|} \sum_{s \in S} u(gh) = u(g), \quad \text{for all } g \in G. \tag{A.1}$$

Since $v \in \ell^2(G)$ and G is infinite, we must have $\lim_g v(g) = 0$, whence $\lim_g u(g) = \lambda(0)$. There are many ways to see why u has to be a constant (and thus equal to $\lambda(0)$). For instance, let (z_n) be the simple random walk on G with steps in S . Then, $z_n \rightarrow \infty$ almost surely (since G is non-amenable), and thus $u(gz_n) \rightarrow \lambda(0)$ for all $g \in G$. On the other hand, by (A.1) and the dominated convergence theorem,

$$u(g) = \lim_n \mathbb{E}[u(gz_n)] = \lambda(0), \quad \text{for all } g \in G.$$

We conclude that $\mu_g(0) - \lambda(0) \in \ell^2(G)$, and thus μ and $\prod_g \lambda$ are equivalent by [9, Corollary 1].

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