Math990, HW1

The first problem mainly repeats facts explained in class.

1. Let K/k be an extension of fields.

(i) If K/k is finite and separable then $\Omega_{K/k} = 0$. (Hint: realize K as $k[x]/(f(x))$ with a separable f and use the second fundamental sequence.)

(ii) If $\mathrm{char}(k) = 0$ and K/k is finitely generated with transcendence basis $x_1, \dots, x_n$ then $\Omega_{K/k}$ is a vector space with basis $dx_1, \dots, dx_n$. (Hint: you know what $\Omega_{k[x_1,\dots,x_n]/k}$ is, hence by localization you can compute $\Omega_{k(x_1,\dots,x_n)/k}$. Then use the first fundamental sequence for $k \hookrightarrow k(x_1, \dots, x_n) \hookrightarrow K$.)

(iii) If $K = k(\alpha)$ is a purely inseparable extension of degree p then $\Omega_{K/k}$ is a free module of rank one. (Hint: $K = k[x]/(x^p - a)$.)

(iv) If $\mathrm{char}(k) = p > 0$ and K is perfect then $\Omega_{K/k} = 0$. (Hint: if $x \in K^p$ then $dx = 0$.)

Remark 0.0.1. (i) The last problem indicates that differentials behave rather fancy in positive characteristic. For example, if $K = k(x)$ is purely transcendental or $K = k(\alpha)$ is purely inseparable of degree p then $\Omega_{K/k}$ is a free module of rank 1 generated by dx or $d\alpha$, but $\Omega_{K^{1/p^\infty}/k} = 0$. So the map $\phi\colon \Omega_{K/k} \otimes_K K^{1/p^\infty} \to \Omega_{K^{1/p^\infty}/k}$ from the first fundamental sequence is not injective.

(ii) There is a way to extend the first fundamental sequence to the left (with the output called cotangent complex), and the first few terms can be described rather explicitly. In particular, both the first and the second fundamental sequences are obtained by truncating the cotangent complex. Also, this extended sequence explains "what is responsible" for non-injectivity of ϕ above.

2. (i) Show that Kahler differentials are compatible with filtered colimits: if B is a filtered colimit of A-algebras B_i then $\mathrm{colim}_i \Omega_{B_i/A} \otimes_{B_i} B \widetilde{\to} \Omega_{B/A}$.

(ii) Deduce that if K/k is of characteristic zero and $\{x_i\}_{i \in I}$ is its transcendence basis (maybe infinite) then $\Omega_{K/k}$ is a vector space with basis $\{dx_i\}_{i \in I}$.

3. Solve problem II.8.2(a) about modules of differentials for fibred products.

Definition 0.0.2. A field extension K/k is called *separable* if $K \otimes_k \overline{k}$ is reduced.

4*. Let K/k be a finitely generated extension of transcendence degree n and let d be the dimension of the K-vector space of Kahler differentials $\Omega^1_{K/k}$. Prove that $d \geq n$ and the following conditions are equivalent:

(i) K/k is separable,

(ii) K has a separable transcendence basis $x_1, \dots, x_n$ over k (i.e. the finite extension $K/k(x_1, \dots, x_n)$ is separable),

(iii) $d = n$.