Math879, HW9

46. Let $p(x) = \sum_{i=0}^{n} a_i x_i \in \mathbf{R}[x]$ be a polynomial with real coefficients. Show that the number of positive roots of $p[x]$ is bounded by the number of sign changes in the sequence of coefficients: $a_n, a_{n-1}, \dots, a_0$.

47. Show that the three heights of a spherical triangle meet at a point.

48. Show that an infinite dimensional Hilbert space is the union of two disjoint dense convex sets.

49. Assume given $n > 2$ and two disjoint circles Γ and γ, such that the second one is encircled by the first one. Assume that there exists a point P on Γ such that the following n-loop property is satisfied: set $P = P_0$ and define $P_1, P_2, \dots$ inductively so that the interval $[P_i P_{i+1}]$ touches γ so that the orientation of the interval is compatible with the clockwise orientation of γ, then $P_n = P_0$. Prove that in this case any point of Γ satisfies the same n-loop property.

50. (i) Does there exist an uncountable set S of subsets of $\mathbf{N}$ such that for any $A, B \in S$ either $A \subset B$ or $B \subset A$?

(ii) Does there exist an uncountable set S of subsets of $\mathbf{N}$ such that for any $A, B \in S$ the intersection $A \cap B$ is finite?

51. Let (S_i, f_{ij}) be a filtered family of sets. This means that I is a partially ordered set such that for any $i, j \in I$ there exists k such that $k > i, k > j$, for each $i \in I$ we are given a set S_i and for any pair $i > j$ we are given a map $f_{ij} \colon S_i \to S_j$ so that for any triple $i > j > k$ one has $f_{ik} = f_{jk} \circ f_{ij}$. Recall that the limit $S = \lim_{i \in I} S_i$ is the set of tuples $(s_i)_{i \in I}$ such that $s_i \in S_i$ and for any $i > j$ one has that $f_{ij}(s_i) = s_j$. Can it happen that all maps f_{ij} are surjective but the limit S is empty?