Math879, HW8

40. Choose 4 points on a sphere. Each point is chosen with uniform distribution on the sphere and the choices are independent. What is the probability that the center of the sphere is contained in their convex hull?

41. Let $f(x) \in \mathbf{C}[x]$. Show that the roots of its derivative f' lie in the convex hull of the roots of f.

42. Let $\Gamma(2) = \{A \in PSL_2(\mathbf{Z}) | \ A \equiv I \ (mod\ 2)\}$. Show that $\Gamma(2)$ is a free group.

43. Assume that $k > 1$ is a natural number such that the numbers $n^2 + n + k$ are prime for any n such that $0 \leq n \leq \sqrt{k/3}$. Prove that in this case the numbers $n^2 + n + k$ are prime for any n such that $0 \leq n \leq k - 2$.

44. Assume that there is given an n-gon with edges $P_1P_2, \ldots, P_{n-1}P_n, P_nP_1$ such that the vertices $P_1, \ldots, P_n$ lie on a single circle γ. Assume that you are only given γ and n points $Q_1, \ldots, Q_n$ such that $Q_i \in P_iP_{i+1}$, where we set $P_{n+1} = P_1$ for convenience. How can you reconstruct the polygon from this data?

45. Two people are playing the following game: They are given a finite graph whose vertices are colored white. The first chooses a vertex and colors it green. At each of the next step the person whose turn it is now has to choose a vertex which is adjacent to vertex chosen at the previous step and which is not yet colored green. If there is no such vertex the player looses. Show that if the graph has an odd number of vertices then the first player has a winning strategy.