Math879, HW10

52. Let γ be a closed curve on the surface of a unit cube which is smooth outside of finitely many points and passes through all six facets. What is the minimal possible length of C?

53. Let $p = (p_1, p_2, \dots, .p_n)\colon \mathbf{C}^n \to \mathbf{C}^n$ be a polynomial map.
(a) Show that if p is injective then it is onto.
(b) Is it true that if p is onto then it is injective?
(c) Is the claim of (a) true for polynomial maps $\mathbf{R}^n \to \mathbf{R}^n$?

54. Show that any compact metric space can be embedded isometrically in $L^\infty[0,1]$, the Banach space of essentially bounded integrable functions on the interval $[0,1]$ with the supremum norm.

55. Four circularly ordered players P_i, $i \in \mathbf{Z}/4\mathbf{Z}$ use four dices A_1, A_2, A_3, A_4, and player i uses only dice A_i. Show that one can assign numbers to the sides of the dices A_i so that for each of the four pairs P_i, P_{i+1} the winning probability of P_i against P_{i+1} is larger than $1/2$. (Naturally, the game is that they independently throw the dices, and the larger result wins.)

56.
(a) Can one find polynomials $P, Q, R \in \mathbf{R}[x, y, z]$ so that

$$P(x,y,z)(2x-3y+z-1)^3 + Q(x,y,z)(-x+y-z)^4 + R(x,y,z)(x-2y-1)^5 = 1?$$

(b) Can one find polynomials $P, Q, R \in \mathbf{R}[x, y, z]$ so that

$$P(x,y,z)(2x-3y+z-1)^3 + Q(x,y,z)(-x+y-z-1)^4 + R(x,y,z)(x-2y-1)^5 = 1?$$

57. Let $K = \mathbf{Q}^{\mathrm{ab}}$ be the maximal cyclotomic extension of $\mathbf{Q}$ (i.e. K is obtained by adjoining all roots of 1 to $\mathbf{Q}$), and let $G = G_{K/\mathbf{Q}}$ be its Galois group over $\mathbf{Q}$. Is it true that for any subgroup $H \subset G$ of index 2 the fixed field K^H is a quadratic extension of $\mathbf{Q}$?