Math879, HW1

1. By a box B in $\mathbf{R}^n$ we mean a product of n closed intervals $[a_i, b_i]$, that is, B is the set of points $x = (x_1, \dots, x_n)$ such that $a_i \leq x_i \leq b_i$ for any i with $1 \leq i \leq n$. The numbers $b_i - a_i$ will be called the edge lengthes of B. Assume that a box B is presented as a union of boxes $B_1, \dots, B_l$ so that each intersection $B_j \cap B_k$ has empty interior. Prove that if each B_j has an integral edge length, then B also has an integral edge length. Your proof should not use calculus (no integrals and derivatives, please).

2. Let $r > 0$ be an integer with primary decomposition $r = 2^l \prod_{i=1}^a p_i^{n_i} \prod_{j=1}^b q_j^{m_j}$. Let N_r be the total number of natural solutions $(x, y) \in \mathbf{N}^2$ of the equation $x^2+y^2 = r$. Prove that $N_r > 0$ if and only if each exponent m_j is even, and in the latter case $N_r = \prod_{i=1}^a (n_i + 1)$.

3. For a natural $r > 0$ and $i \in \{1, 3\}$ let $D_{i,r}$ denote the number of natural divisors of r whose remainder modulo 4 is i. Prove that $N_r = D_{1,r} - D_{3,r}$, where N_r is as in problem 2. (To solve this problem you can assume that the assertion of problem 2 is known.)

4. Assume that one fixed in $\mathbf{R}^2$ a closed interval $I = [P, Q]$ and a line l parallel to I (and not containing I). Show how to find the center of I using the straightedge only.

5. Assume that one fixed a circumference C in $\mathbf{R}^2$. Prove that using straightedge only one cannot find the center of the circle encompassed by C.

6. Give an example of a finitely generated group with no proper subgroups of finite index.