§2. Affinoid algebras *

2.1. The algebra of convergent power series. Let k be a non-Archimedean field. For a tuple $r = (r_1, \ldots, r_n)$ of positive numbers, we denote by $k\{r^{-1}T\} = k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ the subalgebra of the algebra of formal power series $k[[T_1, \ldots, T_n]]$ consisting of series of the form $f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu$ with $|a_\nu| r^\nu \to 0$ as $|\nu| = \nu_1 + \ldots + \nu_n \to \infty$. It is a Banach k-algebra with respect to the norm $||f|| = \max_\nu \{|a_\nu| r^\nu\}$. It is easy to see that the norm on $k\{r^{-1}T\}$ is multiplicative. We set $\mathcal{T}_n = k\{T_1, \ldots, T_n\}$. Notice that $||\mathcal{T}_n|| = |k|$. One sets $\mathcal{T}_n^\circ = \{f \in \mathcal{T} \big| ||f|| \le 1\}$ and $\mathcal{T}_n^{\circ\circ} = \{f \in \mathcal{T} \big| ||f|| < 1\}$. Then $\mathcal{T}_n^\circ$ is a ring, $\mathcal{T}_n^{\circ\circ}$ is an ideal in $\mathcal{T}_n^\circ$, and the quotient ring $\widetilde{\mathcal{T}}_n = \mathcal{T}_n^\circ / \mathcal{T}_n^{\circ\circ}$ is canonically isomorphic to the ring of polynomials $\widetilde{k}[T_1, \ldots, T_n]$.

2.1.1. Lemma. *(i) A nonzero element $f \in k\{r^{-1}T\}$ is invertible if and only if $|f(0)| = ||f||$ and $||f - f(0)|| < ||f||$;*

(ii) for every nonzero element $f \in \mathcal{T}_n$ there exists $a \in k$ with $|a| = ||f||$ such that the element $f + a$ is not invertible in $\mathcal{T}_n$.

Proof. (i) is easy. (ii) If $|f(0)| < ||f||$, then, by (i), the element $f + a$ is not invertible for any element $a \in k$ with $|a| = ||f||$. If $|f(0)| = ||f||$, then, also by (i), the element $f - f(0)$ is not invertible. ∎

2.1.2. Corollary. *The Jacobson radical $j(\mathcal{T}_n)$ (i.e., the intersection of maximal ideals) of $\mathcal{T}_n$ is zero).*

Proof. Assume now that such a nonzero element $f \in \mathcal{T}_n$ lies in the Jacobson radical of $\mathcal{T}_n$, and let a be an element from k with $|a| = ||f||$ for which the element $f + a$ is not invertible. Then it is contained in a maximal ideal $\mathbf{m}$ and, by the assumption, $f \in \mathbf{m}$. It follows that $a \in \mathbf{m}$, which is a contradiction. ∎

2.1.3. Corollary. *Every k-algebra homomorphism $\varphi : \mathcal{T}_n \to \mathcal{T}_m$ is a contraction, i.e., $||\varphi(f)|| \le ||f||$ for all $f \in \mathcal{T}_n$.*

Proof. Assume that there is an element $f \in \mathcal{T}_n$ with $||\varphi(f)|| > ||f||$, and let $a \in k$ be an element with $|a| = ||\varphi(f)||$ such that the element $\varphi(f) + a$ is not invertible. Then the element $f + a$

* Notes of a part of Vladimir Berkovich's course

is not invertible. But $||f|| < |a|$, which contradicts Lemma 2.1.1. ∎

2.1.4. Corollary. *If $\varphi : \mathcal{T}_n \to \mathcal{T}_m$ is a k-algebra isomorphism, then $m = n$ and φ is an isometric isomorphism.*

Proof. That φ is an isometric isomorphism follows from Corollary 2.1.3. It follows that φ induces an isomorphism of $\widetilde{k}$-algebras $\widetilde{\varphi} : \widetilde{\mathcal{T}}_n = \widetilde{k}[T_1, \dots, T_n] \widetilde{\to} \widetilde{\mathcal{T}}_m = \widetilde{k}[T_1, \dots, T_m]$ and of fields of rational functions $\widetilde{k}(T_1, \dots, T_n) \widetilde{\to} \widetilde{k}(T_1, \dots, T_m)$. We therefore get $m = n$. ∎

2.1.4. Corollary. *A k-algebra automorphism φ of $\mathcal{T}_n$ is bijective if and only if $\widetilde{\varphi}$ is bijective.*

Proof. The direct implication follows from Corollary 2.1.3. Assume that $\widetilde{\varphi}$ is bijective. Then φ is obviously isometric, and so it remains to show that it is surjective. It suffices to show that there exists $\varepsilon < 1$ such that for every $f \in \mathcal{T}_n$ one can find $g \in \mathcal{T}_n$ with $||f - \varphi(g)|| \le \varepsilon ||f||$. By the assumption, there are elements $h_1, \dots, h_n \in \mathcal{T}_n^\circ$ with $\varepsilon := \max\limits_{1 \le i \le n} ||T_i - \varphi(h_i)|| < 1$. Using the inequality

$$\begin{aligned} ||u_1 \cdot \ldots \cdot u_n - v_1 \cdot \ldots \cdot v_n|| &= \sum_{i=1}^n (u_1 \cdot \ldots \cdot u_i v_{i+1} \cdot \ldots \cdot v_n - u_1 \cdot \ldots \cdot u_{i-1} v_i \cdot \ldots \cdot v_n) \\ &\le (\max_{1 \le i \le n} |u_i - v_i|) \cdot (\max_{1 \le i \le n} \{||u_i||, ||v_i||\})^{n-1} \end{aligned}$$

it is easy to show that, if $f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu$, the above fact is true for $g = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu h^\nu$. ∎

For a non-Archimedean field K over k, we set $E^n(K) = \{x = (x_1, \dots, x_n) \in K^n \big| |x_i| \le 1$ for all $1 \le i \le n\}$. Any element $f \in \mathcal{T}_n$ defines a continuous function $E^n(K) \to K$. Conversely, if a formal power series $f \in k[[T_1, \dots, T_n]]$ is convergent at all points $x \in E^n(K)$, then $f \in \mathcal{T}_n$.

2.1.5. Lemma *(Maximum modulus principle). If the residue field $\widetilde{K}$ is infinite, then for any $f \in \mathcal{T}_n$ there exists a point $x \in E^n(K)$ with $|f(x)| = ||f||$.*

As the proof shows one can find such a point x with the additional property $|x_i| = 1$ for all $1 \le i \le n$.

Proof. We may assume that $||f|| = 1$. Since the field $\widetilde{K}$ is infinite, the nonzero polynomial takes a nonzero value at some point $\widetilde{x} = (\widetilde{x}_1, \dots, \widetilde{x}_n) \in \widetilde{k}^n$. If x is a point of from $E^n(K)$ whose image in $\widetilde{k}^n$ is $\widetilde{k}$, then $|f(x)| = 1$. ∎

2.2. Weierstrass' theorems. Let $\mathcal{A}$ be a non-Archimedean commutative Banach ring, and let $\mathcal{A}\{r^{-1}T\}$ be the ring of formal power series $f = \sum_{n=0}^\infty a_n T^n$ with $||a_n|| r^n \to 0$. It is a non-Archimedean commutative Banach ring with respect with the norm $||f|| = \max\{||a_n|| r^n\}$. *Assume that the norm on $\mathcal{A}$ is multiplicative*, i.e., $||ab|| = ||a|| \cdot ||b||$ for all $a, b \in \mathcal{A}$. Then the norm on

$\mathcal{A}\{r^{-1}T\}$ is also multiplicative and, in particular, any principal ideal of $\mathcal{A}\{r^{-1}T\}$ (i.e. an ideal of the form $\mathcal{A}\{r^{-1}T\}f$) is closed. The *order* $\mathrm{ord}(f)$ of a nonzero element $f = \sum_{n=0}^{\infty} a_n T^n \in \mathcal{A}\{r^{-1}T\}$ is the maximal n with $||f|| = ||a_n||r^n$. The element f is said to be *distinguished* if $a_n \in \mathcal{A}^*$. It is easy to see that for nonzero elements $f, g \in \mathcal{A}\{r^{-1}T\}$ the following is true:

(1) $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$ and $||f|| \cdot ||g||$;

(2) if $||f|| < ||g||$, or $||f|| = ||g||$ and $\mathrm{ord}(f) < \mathrm{ord}(g)$, then $\mathrm{ord}(f+g) = \mathrm{ord}(g)$ and $||f+g|| = ||g||$.

Let $\mathcal{A}\{r^{-1}T\}_{<n}$ the free Banach $\mathcal{A}$-submodule of polynomials of degree less than n.

2.1.1. Proposition *(Weierstrass division theorem). Let f be a nonzero element of $\mathcal{A}\{r^{-1}T\}$ of order n. Then*

(i) the following homomorphism of Banach $\mathcal{A}$-modules is an isometric monomorphism

$$\mathcal{A}\{r^{-1}T\}f \oplus \mathcal{A}\{r^{-1}T\}_{<n} \to \mathcal{A}\{r^{-1}T\} : (Qf, R) \mapsto g = Qf + R \; ;$$

(ii) the above map is an isomorphism if and only if the element f is distinguished.

Proof. (i) By the property (1), all nonzero summands in the right hand side have pairwise distinct orders, and so the statement follows from the property (2).

(ii) Assume first that the map considered is a bijection. Then $T^n = Qf + R$ for some $Q \in \mathcal{A}\{r^{-1}T\}$ and $R \in \mathcal{A}\{r^{-1}T\}_{<n}$. From the properties (1)-(2) it follows that $\mathrm{ord}(Q) = 0$, i.e., if f is as above and $Q = \sum_{i=0}^{\infty} c_i T^i$, then $||c_0|| > ||c_i||r^i$ for all $i \geq 1$. One has $1 = c_0 a_n + c_1 a_{n-1} + \ldots + c_n a_0$ and $||c_i a_{n-i}|| < ||c_0 a_n||$ for all $1 \leq i \leq n$. It follows that the element $c_0 a_n = 1 - \sum_{i=1}^{n} c_i a_{n-i}$ is invertible in $\mathcal{A}$ and, in particular, $a_n \in \mathcal{A}^*$. Conversely, assume that $a_n \in \mathcal{A}^*$. Then the number $\varepsilon' = \max\limits_{i>n}\{\frac{||a_i||r^i}{||f||}\}$ is strictly less than one. Let ε is a positive number with $\varepsilon' \leq \varepsilon < 1$. We set $\widetilde{f} = \sum_{i=0}^{n} a_i T^i$ and, for a nonzero element $g = \sum_{i=0}^{\infty} c_i T^i \in \mathcal{A}\{r^{-1}T\}$, we set $\widetilde{g} = \sum_{i=0}^{m} c_i T^i$, where m is maximal with the property $||a_m||r^m > \varepsilon||g||$. By the construction, $||f - \widetilde{f}|| \leq \varepsilon||f||$ and $||g - \widetilde{g}|| \leq \varepsilon||g||$. By Euclid's division algorithm, there exist $Q, R \in \mathcal{A}[T]$ with R of degree at most $n-1$ such that $\widetilde{g} = Q\widetilde{f} + R$. By (i), we have $||\widetilde{g}|| = \max(||Q|| \cdot ||f||, ||R||)$ and, in particular, $||Q|| \cdot ||f|| \leq ||g||$. It follows that $||g - (Qf+R)|| = ||g - \widetilde{g} - Q(f - \widetilde{f})|| \leq \max\{||g - \widetilde{g}||, ||Q|| \cdot ||f - \widetilde{f}||) \leq \varepsilon||g||$. Thus, if M denotes the closed Banach $\mathcal{A}$-submodule of $\mathcal{A}\{r^{-1}T\}$ which is the image of the map (i), then, for any element $g \in \mathcal{A}\{r^{-1}T\}$, there exists an element $h \in M$ with $||g - h|| \leq \varepsilon||g||$. It follows that $M = \mathcal{A}\{r^{-1}T\}$. ∎

2.2.2. Proposition *(Weierstrass preparation theorem). Let $f \in \mathcal{A}\{r^{-1}T\}$ be a distinguished element of order n. Then there exists a unique decomposition $f = e \cdot w$, where $w \in \mathcal{A}[T]$ is a monic polynomial of degree and order n, and e is an invertible element of $\mathcal{A}\{r^{-1}T\}$.*

Proof. By Proposition 2.2.1, there exist unique $Q \in \mathcal{A}\{r^{-1}T\}$ and $R \in \mathcal{A}\{r^{-1}T\}_{<n}$ with $T^n = Qf + R$ and $\max\{||Qf||, R\} = ||T^n|| = r^n$, and we define $w = T^n - R$. Then w is a monic polynomial of degree n. Since $||R|| \le r^n$ and $\mathrm{ord}(R) < n$, it follows that w is of order n and $||w|| = r^n$. We have $w = Qf$. It follows that $||Q|| = ||a_n||^{-1}$ and $\mathrm{ord}(Q) = 0$. Since w is monic, the latter implies that the free coefficient of Q is invertible in $\mathcal{A}$. It follows that Q is invertible in $\mathcal{A}\{r^{-1}T\}$ and, setting $e = Q^{-1}$, we get $f = e \cdot w$. Finally, is such a decomposition is given, one has $T^n = e^{-1}f + (T^n - w)$, and the uniquence follows from the correspondent property in Proposition 2.2.1. ∎

2.2.3 Definition. A *Weierstrass polynomial* is a monic polynomial $w \in \mathcal{A}[r^{-1}T] \subset \mathcal{A}\{r^{-1}T\}$ whose order is equal to its degree.

Notice that, given two monic polynomials $w_1, w_2 \in \mathcal{A}[r^{-1}T]$, their product $w_1 \cdot w_2$ is a Weierstrass polynomial if and only if so are both of them.

2.2.4. Proposition *(Weierstrass finiteness theorem). Let $\mathcal{B}$ be a finite Banach $\mathcal{A}\{r^{-1}T\}$-algebra, and assume that the kernel of the corresponding homomorphism $\mathcal{A}\{r^{-1}T\} \to \mathcal{B}$ contains a Weierstrass polynomial. Then $\mathcal{B}$ is a finite Banach $\mathcal{A}$-algebra.* ∎

We now represent $\mathcal{T}_n$ in the form $\mathcal{T}_{n-1}\{T_n\}$, and call distinguished elements T_n-*distinguished.*

2.2.5. Proposition. *For every nonzero element $f \in \mathcal{T}_n$, there exists a k-automorphism $\sigma : \mathcal{T}_n \xrightarrow{\sim} \mathcal{T}_n$ such that $\sigma(f)$ is T_n-distinguished.*

Proof. We may assume that $||f|| = 1$. Let $f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu$. There are only finitely many tuples ν with $|a_\nu| = 1$. Let μ be the maximal among them with respect to the lexicographical ordering, and let d be an integer strictly greater than all of the coordinates ν_i of such ν's. Define a k-endomorphism σ of $\mathcal{T}_n$ by $\sigma(T_1) = T_1 + T_n^{d^{n-1}}$, $\sigma(T_2) = T_2 + T_n^{d^{n-2}}$, ..., $\sigma(T_{n-1}) = T_{n-1} + T_n^d$, and $\sigma(T_n) = T_n$. It is an automorphism since $\widetilde{\sigma}$ is an automorphism of $\widetilde{k}[T_1, \dots, T_n]$. *We claim that $\sigma(f)$ is T_n-distinguished of order $m = \mu_1 d^{n-1} + \ldots + \mu_{n-1}d + \mu_n$.* Indeed, this follows from the fact that, by our choice of d, if ν is such that $|a_\nu| = 1$ and $\nu \ne \mu$, then the similar sum for ν is strictly smaller than m. ∎

2.3. Rückert's theory. Let A be a commutative ring with unity, and let B be a commutative ring that contains the ring of polynomials $A[T]$.

2.3.1. Definition. The ring B is said to be *Rückert* over A if there is a set of monic polynomials $W \subset A[T]$ with the following properties:

(1) if the product of two monic polynomials lies in W, then so do the factors;

(2) for every $w \in W$, $A[T]/wA[T] \xrightarrow{\sim} B/wB$;

(3) for every nonzero $f \in B$ there is an automorphism σ of B such that $\sigma(f) = e \cdot w$, where $e \in B^*$ and $w \in W$.

For example, $\mathcal{T}_n$ is Rückert over $\mathcal{T}_{n-1}$.

2.3.2. Proposition. *Assume that B is Rückert over A. Then*

(i) if A is Noetherian, then so is B;

(ii) if A is Jacobson, then $\mathrm{rad}(B/\mathbf{b}) = j(B/\mathbf{b})$ *for any* nonzero *ideal* $\mathbf{b} \subset B$*;*

(iii) if A is factorial, then so is B.

It is not true in general that in (ii) B is Jacobson. Example is as follows: $A = k$ is a field and $B = k[[T]]$. One has $\mathrm{nil}(k[[T]]) = 0$ and $j(k[[T]])$ is the unique maximal ideal (generated by T).

Proof. (i) Let $\mathbf{b}$ be a nonzero ideal in B. By the property (3), we may assume it contains a polynomial $w \in W$. Furthermore, by the property (2), $B/wB = A[T]/wA[T]$. Since the latter is Noetherian, by Hilbert's basis theorem, then so is B/wB. But B/wB is a finite A-module. It follows that the ideal $\mathbf{b}$ is finitely generated.

(ii) Since $\mathrm{rad}(B/\mathbf{b})$ is the intersection of all prime ideals, we may assume that $\mathbf{b}$ is a nonzero prime ideal, and so we have to show that $j(B/\mathbf{b}) = 0$. Let $\mathbf{a} = A \cap \mathbf{b}$. Since $\mathbf{b} \neq 0$, we may assume $\mathbf{b}$ contains a polynomial from W, and it follows that $B/\mathbf{b}$ is finite over $A/\mathbf{a}$. For $b \in j(B/\mathbf{b})$, let $b^n + a_1 b^{n-1} + \ldots + a_n = 0$ be an equation over $A/\mathbf{a}$ of minimal degree. Then $a_n = -(b^n + a_1 b^{n-1} + \ldots + a_{n-1} b) \in j(B/\mathbf{b}) \cap A/\mathbf{a} \subset j(A/\mathbf{a}) = 0$. It follows that $b = 0$.

(iii) By the assumption, A is an integral domain, and every nonzero element of A is a finite product of prime elements. (An element of A is prime if it generates a prime ideal.) Any such product decomposition is unique up to invertible elements, i.e., if $p_1 \cdot \ldots \cdot p_m = q_1 \cdot \ldots \cdot q_n$, then $m = n$ and, after a permutation, $q_i = e_i p_i$ with $e_i \in A^*$. Let f be a nonzero element of B. Applying an automorphism and multiplying by an invertible element, we may assume that $f \in W$. Let K be the fraction field of A. Since $K[T]$ is factorial, there is a factorization $f = p_1 \cdot \ldots \cdot p_m$ into monic irreducible polynomials in $K[T]$. Furthermore, since A is factorial, there exist elements $a_1, \ldots, a_n \in A$ such that the polynomials $a_1 p_1, \ldots, a_n p_n$ lie in $A[K]$ and are primitive (i.e., p is primitive if no prime elements of A divide all of the coefficients of p). Then $(\prod_{i=1}^n a_i) f = \prod_{i=1}^n (a_i p_i)$ is a primitive polynomial (by the Gauss lemma). It follows that the product $\prod_{i=1}^n a_i$ is invertible and, therefore, all a_i's are invertible. Thus, the above factorization of f takes place in $A[X]$. *We claim that the elements $p_1, \ldots, p_n$ are primes in B.* By the property (1), these elements belong to W and, by (2), $B/p_iB = A[T]/p_iA[T]$, i.e., it suffices to verify that p_i's are prime elements of $A[T]$. By the Gauss lemma, one has $A[T] \cap p_iK[T] = p_iA[T]$, which implies that the canonical homomorphism $A[T]/p_iA[T] \to K[T]/p_iK[T]$ is injective. ▪

2.3.3. Corollary. *The ring $\mathcal{T}_n$ is Noetherian, factorial and Jacobson.*

Proof. The first two properties follow directly from Proposition 2.3.2(i) and (iii), and the third one follows from (ii) and Corollary 2.1.2 stating that $j(\mathcal{T}_n) = 0$. ▪

A Banach module M over a commutative Banach ring $\mathcal{A}$ is said to be *finite* if there is an admissible epimorphism $\mathcal{A}^n \to M$. It is easy to show that the evident functor from the category of finite Banach $\mathcal{A}$-modules to that of finite $\mathcal{A}$-modules is fully faithful.

2.3.4. Proposition. *(i) If $\mathcal{A} = \mathcal{T}_n$, the above functor is an equivalence of categories;*

(ii) all ideals of $\mathcal{T}_n$ are closed.

Proof. It suffices to verify the following fact. *If M is an submodule of a Banach $\mathcal{T}_n$-module N such that its closure $\overline{M}$ is finitely generated over $\mathcal{T}_n$, then $M = \overline{M}$.* Indeed, this immediately implies (ii) and, by fully faithfulness, it suffices to show that any finite $\mathcal{T}_n$-module M has a structure of a finite Banach $\mathcal{T}_n$-module. For this we take an arbitrary surjective homomorphism $\varphi : \mathcal{T}_n^m \to M$. By the above fact, the kernel $\mathrm{Ker}(\varphi)$ is a closed $\mathcal{T}_n$-submodule of $\mathcal{T}_n^m$. Thus, the homomorphism $\mathcal{T}_n^m/\mathrm{Ker}(\varphi) \xrightarrow{\sim} M$ defines a structure of a finite Banach $\mathcal{T}_n$-module on M.

Consider a surjective homomorphism of $\mathcal{T}_n$-modules $\varphi : \mathcal{T}_n^m \to \overline{M} : e_i \mapsto x_i$. By the Banach theorem, it is an open map. It follows that, for any $1 \le i \le m$, one has $x_i \in M + \sum_{j=1}^m \mathcal{T}_n^{\circ\circ} x_j$, i.e., $y_i = x_i - \sum_{i=1}^n f_{ij}x_j \in M$ for some $f_{ij} \in \mathcal{T}_n^{\circ\circ}$. If X and Y are the vector columns of x_i and y_i, respectively, and F is the matrix $(f_{ij})_{1\le i,j\le m}$, we get $Y = (1-F)X$. The matrix $1-F$ is invertible and, therefore, $x_i \in M$ for all $1 \le i \le m$, i.e., $\overline{M} = M$. ▪

2.4. Noether normalization. A *chart* of the algebra $\mathcal{T}_n$ is a system $(f_1, \dots, f_n)$ of elements in $\mathcal{T}_n^\circ$ such that the homomorphism $k\{S_1, \dots, S_n\} \to \mathcal{T}_n : S_i \mapsto f_i$ is an isomorphism.

2.4.1. Proposition. *Let $\mathcal{A}$ be a nonzero strictly k-affinoid algebra. Then for any bounded finite homomorphism $\varphi : \mathcal{T}_n \to \mathcal{A}$ there exist a chart $(S_1, \dots, S_n)$ of $\mathcal{T}_n$ and an integer $d \ge 0$ such that the induced homomorphism $\mathcal{T}_d = k\{S_1, \dots, S_d\} \to \mathcal{A}$ is finite and injective.*

Proof. The statement is trivially true for $n = 0$. Assume that $n \ge 1$. If $\mathrm{Ker}(\varphi) = 0$, there is nothing to prove. Otherwise, we can find a chart $(S_1, \dots, S_n)$ and a Weierstrass polynomial $w \in \mathcal{T}_{n-1}[S_n]$ which lies in $\mathrm{Ker}(\varphi)$. By the Weierstrass finiteness theorem, the induced homomorphisms $\mathcal{T}_{n-1} \to \mathcal{T}_n/w\mathcal{T}_n \to \mathcal{A}$ are finite. Continuing this process, we get the statement. ▪

2.4.2. Corollary. *For every nonzero strictly k-affinoid algebra $\mathcal{A}$ there exists a finite bounded monomorphism $\mathcal{T}_n \hookrightarrow \mathcal{A}$.* ∎

2.4.3. Corollary. *Let* $\mathbf{a}$ *be an ideal of a strictly k-affinoid algebra $\mathcal{A}$ such that its radical* $\mathrm{rad}(\mathbf{a})$ *is a maximal ideal. Then $\mathcal{A}/\mathbf{a}$ is of finite dimension over k.*

Proof. By Corollary 2.4.2, there is a bounded finite monomorphism $\varphi : \mathcal{T}_n \to \mathcal{A}/\mathbf{a}$. Since $\mathcal{T}_n$ is reduced (i.e., it has no nilpotent elements), the induced homomorphism $\mathcal{T}_n \to \mathcal{A}/\mathrm{rad}(\mathbf{a})$ is finite and injective. But the latter quotient is a field. It follows that $\mathcal{T}_n$ is a field, i.e., $n = 0$. ∎

2.4.4. Proposition. *Any homomorphism between strictly k-affinoid algebras $\varphi : \mathcal{A} \to \mathcal{B}$ is bounded.*

Proof. The statement is trivial if the valuation on k is trivial. Thus, assume it is not the case. To show that φ is bounded, we use the Banach Closed Graph Theorem. It suffices to verify that the graph of φ in $\mathcal{A} \times \mathcal{B}$ is closed in the product topology. Assume that there is a sequence of elements $\{f_i\}_{i \geq 1}$ in $\mathcal{A}$ that tends to zero, but $\varphi(f_i) \to g \neq 0$ as $i \to \infty$. Given a maximal ideal $\mathbf{m} \subset \mathcal{B}$ and an integer $n \geq 1$, consider the induced injective homomorphism $\mathcal{A}/\varphi^{-1}(\mathbf{m}^n) \to B/\mathbf{m}^n$. Both spaces are of finite dimension over k and, therefore, the homomorphism between them is bounded. It follows that the images of the elements $\varphi(f_i)$ tend to zero in $B/\mathbf{m}^n$, i.e., $g \in B/\mathbf{m}^n$ for all $\mathbf{m}$ and n. Thus, it remain to show that $\bigcap_{\mathbf{m},n} \mathbf{m}^n = 0$. Indeed, if $f \in \bigcap_n \mathbf{m}^n$ then, by the Krull intersection theorem, there exists an element $g_{\mathbf{m}} \in \mathbf{m}$ with $f = fg$, i.e., $f(1 - g_{\mathbf{m}}) = 0$, and so the ideal generated by the elements $1 - g_m$ coincides with A (it is not contained in any maximal ideal); it follows that $f = 0$. ∎

2.5. Complete tensor products. Given Banach modules M, N and P over a non-Archimedean commutative Banach ring $\mathcal{A}$, an $\mathcal{A}$-bilinear homomorphism $\varphi : M \times N \to P$ is said to be a *bounded* if there is a constant $C > 0$ such that $||\varphi(f, g)|| \leq C||f|| \cdot ||g||$ for all $(f, g) \in M \times N$. A *complete tensor product* of M and N over $\mathcal{A}$ is a Banach $\mathcal{A}$-module $M \widehat{\otimes}_{\mathcal{A}} N$ provided with bounded A-bilinear homomorphism $M \times N \to M \widehat{\otimes}_{\mathcal{A}} N$ such that, for any bounded A-bilinear homomorphism $M \times N \to P$, there is a unique bounded homomorphism $M \widehat{\otimes}_{\mathcal{A}} N \to P$ which is compatible with all of the above homomorphism. The complete tensor product exists and is unique up to a unique isomorphism. It is constructed as follows. The usual tensor product $M \otimes_A N$ is provided with a non-Archimedean seminorm

$$||x|| = \inf \max_{1 \leq i \leq n} \{||f_i|| \cdot ||g_i||\} ,$$

where the infimum is taken over all representations of $x \in M \otimes_A N$ in the form $\sum_{i=1}^n f_i \otimes g_i$. The complete tensor product is the completion of $M \otimes_A N$ with respect to that seminorm. For example, for $n \geq 1$ the above seminorm on $\mathcal{A}^n \otimes_{\mathcal{A}} N$ is in fact a norm, the tensor product is complete with respect to it, and the canonical bijection $\mathcal{A}^n \otimes_{\mathcal{A}} N \to N^n$ is an isometric isomorphism. in particular, $\mathcal{A}^n \otimes_{\mathcal{A}} N \widetilde{\to} \mathcal{A}^n \widehat{\otimes}_{\mathcal{A}} N$.

If $\mathcal{B}$ is a Banach $\mathcal{A}$-algebra and N is a Banach $\mathcal{A}$-module, then the complete tensor product $\mathcal{B}\widehat{\otimes}_{\mathcal{A}} N$ carries the structure of a Banach $\mathcal{B}$-module. If $\mathcal{B}$ and $\mathcal{C}$ are Banach $\mathcal{A}$-algebras, then so is $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}$. For example, there is a canonical isometric isomorphism

$$\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \widetilde{\to} \mathcal{B}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \ .$$

2.5.1. Lemma. *If $\varphi : M \to M'$ and $\psi : N \to N'$ are admissible epimorphisms of Banach $\mathcal{A}$-modules, then the induced homomorphisms $M \otimes_{\mathcal{A}} N \to M' \otimes_{\mathcal{A}} N'$ and $M\widehat{\otimes}_{\mathcal{A}} N \to M'\widehat{\otimes}_{\mathcal{A}} N'$ are admissible epimorphisms.*

Proof. The assumption means that there is a constant $C > 0$ such that, for any elements $m' \in M'$ and $n' \in N'$, there exist $m \in \varphi^{-1}(m')$ and $n \in \psi^{-1}(n)$ with $||m|| \leq C||m'||$ and $||n|| \leq C||n'||$. Given $x' \in M' \otimes_{\mathcal{A}} N'$ and $\varepsilon > 0$, take a representation as a finite sum $x' = \sum m'_i \otimes n'_i$ with $\max\{||m'_i|| \cdot ||n'_i||\} \leq ||x'|| + \varepsilon$. Furthermore, take elements $m_i \in \varphi^{-1}(m'_i)$ and $n_i \in \varphi^{-1}(n'_i)$ with $||m_i|| \leq C||m'_i||$ and $||n_i|| \leq C||n'_i||$. Then for the element $x = \sum m_i \otimes n_i$ we have

$$||x|| \leq \max\{||m_i|| \cdot ||n_i||\} \leq C^2 \max\{||m'_i|| \cdot ||n'_i||\} \leq C^2(||x'|| + \varepsilon) \ .$$

This implies that the tensor product seminorm on $M \otimes_{\mathcal{A}} N$ is equivalent to the quotient seminorm induced from $M \otimes_{\mathcal{A}} N$. ∎

2.5.2. Corollary. *Let k be a non-Archimedean field, and let $\mathcal{A}$ be a (strictly) k-affinoid algebra Then*

(i) if $\mathcal{B}$ and $\mathcal{C}$ are (strictly) $\mathcal{A}$-affinoid algebras, then so is $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}$;

(ii) if k' is a bigger non-Archimedean field, then $\mathcal{A}\widehat{\otimes}_k k'$ is a (strictly) k'-affinoid algebra. ∎

2.5.3. Lemma. *Let $\mathcal{A}$ be a strictly k-affinoid algebra, M and N finite Banach $\mathcal{A}$-modules, and $\mathcal{B}$ an $\mathcal{A}$-affinoid algebra. Then*

(i) $M \otimes_{\mathcal{A}} N \widetilde{\to} M\widehat{\otimes}_{\mathcal{A}} N$;

(ii) $M \otimes_{\mathcal{A}} \mathcal{B} \widetilde{\to} M\widehat{\otimes}_{\mathcal{A}}\mathcal{B}$.

Proof. Let $\mathcal{A}^m \xrightarrow{\varphi} \mathcal{A}^n \xrightarrow{\psi} M \to 0$ be an exact sequence. Then both φ and ψ are admissible homomorphisms. Tensoring by N over $\mathcal{A}$, we get a commutative diagram of bounded homomorphisms

$$\begin{array}{ccccccc}
\mathcal{A}^m \otimes_{\mathcal{A}} N & \xrightarrow{\varphi\otimes\mathrm{id}} & \mathcal{A}^n \otimes_{\mathcal{A}} N & \xrightarrow{\psi\otimes\mathrm{id}} & M \otimes_{\mathcal{A}} N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{A}^m\widehat{\otimes}_{\mathcal{A}} N & \xrightarrow{\varphi\widehat{\otimes}\mathrm{id}} & \mathcal{A}^n\widehat{\otimes}_{\mathcal{A}} N & \xrightarrow{\psi\widehat{\otimes}\mathrm{id}} & M\widehat{\otimes}_{\mathcal{A}} N & \longrightarrow & 0
\end{array}$$

The first row is an exact sequence. The first and second vertical arrows are isomorphisms, and the homomorphism $\varphi \otimes \mathrm{id}$ is admissible. Together with Lemma 2.5.1, this implies that the seminorm on $M \otimes_{\mathcal{A}} N$ is in fact a norm, and $\psi \otimes \mathrm{id}$ is an admissible epimorphism. In particular, (i) is true. (ii) is verified in the same way. ∎

2.5.4. Corollary. *Let $\mathcal{A}$ be a strictly k-affinoid algebra. Then*

(i) the canonical functor from the category of finite Banach $\mathcal{A}$-algebras to that of finite $\mathcal{A}$-algebras is an equivalence of categories;

(ii) every finite Banach $\mathcal{A}$-algebra is strictly k-affinoid.

Proof. Let $\mathcal{B}$ be a finite $\mathcal{A}$-algebra. We know that it can be provided with a (unique) structure of a finite Banach $\mathcal{A}$-module. By Lemma 2.5.3, there is an isomorphism of Banach $\mathcal{A}$-modules $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} \mathcal{B} \widehat{\otimes}_{\mathcal{A}} \mathcal{B}$ and, in particular, the algebra $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ is a finite Banach $\mathcal{A}$-module. Furthermore, the multiplication homomorphism $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ is bounded, i.e., there exists a constant $C > 0$ such that $||fg|| \le C||f|| \cdot ||g||$ for all $f, g \in \mathcal{B}$. We can therefore replace the norm on $\mathcal{B}$ by an equivalent one so that $||fg|| \le ||f|| \cdot ||g||$. To prove (ii), take an admissible epimorphism of Banach $\mathcal{A}$-modules $\mathcal{A}^n \to \mathcal{B} : e_i \mapsto f_i$, and let r_i be a number from $|k^*|$ with $r_i \ge ||f_i||$. Then there is an admissible $\mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{B}$ that takes T_i to f_i. ∎

2.5.5. Proposition. *The k-affinoid algebra $\mathcal{T}_{n,r} = k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is strictly k-affinoid if and only if $r_1, \dots, r_n \in \sqrt{|k^*|}$.*

Proof. Assume first that $\mathcal{T}_{n,r}$ is strictly affinoid. then there is a finite bounded homomorphism $\varphi : \mathcal{T}_m \to \mathcal{T}_{n,r}$. It is admissible, by Corollary 2.5.4(i). since the norms on both Banach algebras are multiplicative, it follows that φ is an isometry. Given $1 \le l \le n$, let $T_l^d + f_1 T_l^{d-1} + \ldots + f_d = 0$ be an equation of T_l over $\mathcal{T}_m$. Then there exist $1 \le i \ne j \le d$ with $||f_i|| r_l^{d-i} = ||f_j|| r_l^{d-j}$. It follows that $r_l \in \sqrt{|k^*|}$.

Conversely, assume that $r_1, \dots, r_n \in \sqrt{|k^*|}$. Then there are elements $a_1, \dots, a_n \in k^*$ and integers $m_1, \dots, m_n \ge 1$ with $|a_i^{-1}| = r_i^{m_i}$ for all $1 \le i \le n$. Define a homomorphism $\varphi : \mathcal{T}_n = k\{S_1, \dots, S_n\} \to \mathcal{T}_{n,r}$ by setting $\varphi(S_i) = a_i T_i^{m_i}$. It is easy to verify that $\mathcal{T}_{n,r}$ is generated by the monomials $T_1^{l_1} \cdot \ldots \cdot T_n^{l_n}$ with $0 \le i \le m_i$, i.e., $\mathcal{T}_{n,r}$ is a finite $\mathcal{T}_n$-module. Corollary 2.5.4(ii) implies that $\mathcal{T}_{n,r}$ is strictly k-affinoid. ∎

Let $\mathcal{A}$ be a strictly k-affinoid algebra. From Corollary 2.4.3 it follows that there is a canonical injective map $\mathrm{Max}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$.

2.5.6. Proposition *(i) If the valuation on k is trivial, the topology of $\mathcal{M}(\mathcal{A})$ induces the discrete topology on* $\mathrm{Max}(\mathcal{A})$*;*

(ii) if the valuation on k is nontrivial, then the above map induces a homeomorphism of $\mathrm{Max}(\mathcal{A})$ *with a dense subset of* $\mathcal{M}(\mathcal{A})$.

Proof. (i) Let x_0 be a point of $\mathcal{M}(\mathcal{A})$, and let $f_1, \dots, f_n$ be generators of the corresponding maximal ideal $\mathbf{m}_{x_0} \subset \mathcal{A}$. Then the only point from $\mathrm{Max}(\mathcal{A})$ which lies in the open subset $\{x \in \mathcal{M}(\mathcal{A}) \big| |f_i(x)| < \frac{1}{2}$ for all $1 \le i \le n\}$ is the point x_0.

(ii) Let $\mathcal{U}$ be an open neighborhood of a point $x_0 \in \mathcal{M}(\mathcal{A})$. We may assume that $\mathcal{U} = \{x \in \mathcal{M}(\mathcal{A}) \big| |f_i(x)| < a_i,\ |g_j(x)| > b_j,\ 1 \le i \le m,\ 1 \le j \le n\}$. Choose $p_i, q_j \in \sqrt{|k^*|}$ with $|f(x_0)| < p_i < a_i$ and $|g_j(x_0)| > q_j > b_j$. By Proposition 2.5.5, the algebra

$$\mathcal{B} = \mathcal{A}\{p_1^{-1}T_1, \dots, r_m^{-1}T_m, q_1^{-1}S_1, \dots, q_n^{-1}S_m\}/(T_i - f_i, g_jS_j - 1)$$

is strictly k-affinoid and nonzero. Hence, $\mathrm{Max}(\mathcal{B}) \ne \emptyset$. Since the image of $\mathrm{Max}(\mathcal{B})$ in $\mathcal{M}(\mathcal{A})$ lies in $\mathcal{U}$ and contains the point x_0, $\mathrm{Max}(\mathcal{A})$ is dense in $\mathcal{M}(\mathcal{A})$. Furthermore, let $x_0 \in \mathrm{Max}(\mathcal{A})$. A fundamental system of open neighborhoods of the point x_0 in $\mathrm{Max}(\mathcal{A})$ is formed by sets of the form $U = \{x \in \mathrm{Max}(\mathcal{A}) \big| |f_i(x)| < a_i, 1 \le i \le n\}$ for $f_1, \dots, f_n \in \mathbf{m}_{x_0}$. The set $\mathcal{U} = \{x \in \mathcal{M}(\mathcal{A}) \big| |f_i(x)| < a_i, 1 \le i \le n\}$ is open in $\mathcal{M}(\mathcal{A})$, and $\mathcal{U} \cap \mathrm{Max}(\mathcal{A}) = U$. ∎

2.6. Properties of the spectral norm. Let $\mathcal{A}$ be a strictly k-affinoid algebra. Proposition 2.5.6 implies that, for any element $f \in \mathcal{A}$, one has

$$\rho(f) = \sup_{x \in \mathrm{Max}(\mathcal{A})} |f(x)| \ .$$

2.6.1. Proposition *(Maximum Modulus Principle). Let $\mathcal{A}$ be a strictly k-affinoid algebra. Then for any element $f \in \mathcal{A}$ there exists a point $x \in \mathrm{Max}(\mathcal{A})$ with $\rho(f) = |f(x)|$.*

Let $P(T) = T^n + a_1T^n + \dots + a_n$ be a monic polynomial in $k[T]$. The quotient $K = k[T]/(P)$ is a finite k-algebra which, therefore, has the structure of a strictly k-affinoid algebra with $\mathcal{M}(\mathcal{K}) = \mathrm{Max}(\mathcal{A})$ a finite set. Of course, in this case the Maximum Modulus Principle holds.

2.6.2. Lemma. *In the above situation, let f be the image of T in K. Then*

$$\rho(f) = \max_{1 \le i \le n} |a_i|^{\frac{1}{i}} \ .$$

Proof. If we replace k by a bigger non-Archimedean field and K by the corresponding tensor product, both sides do not change. We may therefore assume that the filed k is algebraically closed. Let $\sigma(P)$ denote the righthand side. If k is of characteristic $p > 0$ and $P(T) = Q(T)^{p^m}$, then $\sigma(P) = \sigma(Q)$, and $\rho(f)$ does not change if we replace K by $K[T]/(Q)$. Thus, we may assume that all roots of P are pairwise different, i.e., $P(T) = \prod_{i=1}^{d}(T - \alpha_i)$ with $\alpha_i \neq \alpha_j$ for $i \neq j$. The required fact is then equivalent to the equality

$$\max_{1 \le i \le d} |\alpha_i| = \max_{1 \le i \le n} |a_i|^{\frac{1}{i}} .$$

Since a_i is the value of the symmetric function of degree i at $\alpha_1, \dots, \alpha_n$, the left hand side is greater or equal to the right hand side. On the other hand, if $|\alpha_1| = \dots = |\alpha_m| > |\alpha_{m+1}|, \dots, |\alpha_n|$, then the norm of the value of m-th symmetric function is equal to $|\alpha_1|^m$, i.e., the left hand side is less or equal to the right hand side. ∎

Proof of Proposition 2.6.1. The statement is trivial if the valuation on k is trivial. Assume therefore that the valuation on k is nontrivial, and $f \neq 0$.

Case 1: $\mathcal{A} = \mathcal{T}_n$. We may assume that $\rho(f) = ||f|| = 1$. Since the residue field $\widetilde{k^{\mathrm{a}}}$ of the algebraic closure k^{a} of k has infinitely many elements, there exists a point $x = (x_1, \dots, x_n) \in k^{\mathrm{a}}$ with $|x_i| \le 1$ and $\widetilde{f}(\widetilde{x}_1, \dots, \widetilde{x}_n) \neq 0$, which is equivalent to $|f(x) = 1$. The image of the point x in $\mathcal{M}(\mathcal{T}_n)$ lies in $\mathrm{Max}(\mathcal{T}_n)$, i.e., the required fact is true.

Case 2: $\mathcal{A}$ is an integral domain. By the Noether Normalization Lemma, there exists a finite monomorphism $\varphi : \mathcal{T}_n \hookrightarrow \mathcal{A}$. Let $P(T) = T^d + g_1 T^d + \dots + g_d$ be the minimal polynomial of f over the fraction field of $\mathcal{T}_n$. Since $\mathcal{T}_n$ is integrally closed, it follows that $g_i \in \mathcal{T}_n$ for all $1 \le i \le d$. We may assume that $\mathcal{A} = \mathcal{T}_n[f]/(P)$. One has

$$\sup_{y \in \mathrm{Max}(\mathcal{A})} |f(y)| = \sup_{x \in \mathrm{Max}(\mathcal{T}_n)} \max_{y \mapsto x} |f(y)| = \sup_{x \in \mathrm{Max}(\mathcal{T}_n)} \max_{1 \le i \le d} |g_i(x)|^{\frac{1}{i}} .$$

By case 1, there exists a point $x \in \mathrm{Max}(\mathcal{T}_n)$ with $\rho(g_i) = |g(x)|$ for all $1 \le i \le d$. Then there exists a point $y \in \mathrm{Max}(\mathcal{A})$ over x where the supremum on the right hand side is achieved.

Case 3: $\mathcal{A}$ is arbitrary. Let $\wp_1, \dots, \wp_n$ be the minimal prime ideals of $\mathcal{A}$, and f_i is the image of f in the quotient ring $\mathcal{A}_i = \mathcal{A}/\wp_i$. Then $\rho(f) = \max\limits_{1 \le i \le n} \rho(f_i)$, and the required fact follows from Case 2. ∎

2.6.3. Corollary. *$\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ for any element f of a strictly k-affinoid algebra $\mathcal{A}$.* ∎

The following statement is proved using the reasoning from the proof of Proposition 2.6.1.

2.6.4. Proposition. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a finite homomorphism of strictly k-affinoid algebras. Then for every element $g \in \mathcal{B}$ there exists a monic polynomial $P(T) = T^n + f_1 T^{n-1} + \dots + f_n \in \mathcal{A}[T]$ such that $P(g) = 0$ and*

$$\rho(g) = \max_{1 \le i \le n} \rho(f_i)^{\frac{1}{i}} .$$

Proof. Case 1: $\mathcal{B}$ is an integral domain. By the Noether Normalization Lemma, we can find a homomorphism $\mathcal{T}_d \to \mathcal{A}$ whose composition with φ is a finite monomorphism. we can therefore assume that $\mathcal{A} = \mathcal{T}_d$, and the required fact was obtained in the proof of Proposition 2.6.1.

Case 2: $\mathcal{B}$ is arbitrary. Let $\wp_1, \ldots, \wp_m$ be the minimal prime ideals of $\mathcal{B}$, and let g_i denotes the image of g in $\mathcal{B}_i = \mathcal{B}/\wp_i$. By the Case 1, there are monic polynomials $P_i(T) \in \mathcal{A}[T]$ with $P_i(g_i) = 0$ and $\rho(g_i) = \sigma(P_i)$. We set $Q(T) = \prod_{i=1}^m P_i(T)$. Then the element $Q(g)$ lies in the intersection of all minimal prime ideals of $\mathcal{B}$, i.e., there is $e \geq 1$ such that for $P(T) = Q(T)^e$ one has $P(g) = 0$. We get

$$\sigma(P) \leq \max_{1 \leq i \leq m} \sigma(P_i) = \max_{1 \leq i \leq m} \rho(g_i) = \rho(f) \ .$$

The converse inequality is trivial. ∎

2.6.5. Proposition. *The following properties of an element f of a strictly k-affinoid algebra $\mathcal{A}$ are equivalent:*

(a) f is power bounded, i.e., there is $C > 0$ such that $||f^n|| \leq C$ for all $n \geq 1$;

(b) $\rho(f) \leq 1$, i.e., $f \in \mathcal{A}^\circ$.

Proof. The implication (a)$\Longrightarrow$(b) is trivial. Assume that $\rho(f) \leq 1$. Take a finite homomorphism $\varphi : \mathcal{T}_d \to \mathcal{A}$. By Proposition 2.6.4, there exists a monic polynomial $P(T) = T^m + g_1 T^{m-1} + \ldots + g_m \in \mathcal{T}_n[T]$ with $P(f) = 0$ and $\rho(f) = \max\limits_{1 \leq i \leq d} \rho(g_i)^{\frac{1}{i}}$. It follows that $\rho(g_i) \leq 1$ and, therefore, $f^n \in \sum_{i=0}^{m-1} \varphi(\mathcal{T}_d^\circ) f^i$ for all $n \geq m$. Since $\varphi(\mathcal{T}_d^\circ)$ is bounded in $\mathcal{A}$, it follows that the above sum is bounded, and we are done. ∎

2.7. Properties of k-affinoid algebras. Given positive numbers $r_1, \ldots, r_2$, let $K_{r_1,\ldots,r_n}$ denote the space of all formal series $f = \sum_{\nu \in \mathbf{Z}^n} a_\nu T^\nu$ with $a_\nu \in k$ and $|a_\nu| r^\nu \to 0$ as $|\nu| = |\nu_1| + \ldots + |\nu_n| \to \infty$ provided with the multiplicative norm $||f|| = \max_{\nu \mathbf{Z}^n}\{|a_\nu| r^\nu\}$. It is a k-affinoid algebra since there is an admissible epimorphism

$$k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n, rS_1, \ldots r_n S_n\} \to K_{r_1,\ldots,r_n} : T_i \mapsto T_i, \ S_i \mapsto T_i^{-1} \ .$$

Notice that there is an isometric isomorphism $K_{r_1} \widehat{\otimes} \ldots \widehat{\otimes} K_{r_n} \xrightarrow{\sim} K_{r_1,\ldots,r_n}$. *Assume now that the images of $r_1, \ldots, r_n$ in the $\mathbf{Q}$-vector space $\mathbf{R}_+^*/\sqrt{|k^*|}$ are linearly independent.* Then $K_{r_1,\ldots,r_n}$ is a field, i.e., a non-Archimedean field over k.

2.7.1. Lemma. *Let X be a k-Banach space. Then*

(i) the canonical map $X \to X \widehat{\otimes} K_{r_1,\ldots,r_n}$ is an isometric embedding;

(ii) a sequence of bounded homomorphisms of k-Banach spaces $X \to Y \to Z$ is exact and admissible if and only if the corresponding sequence of $K_{r_1,\dots,r_n}$-banach spaces $X\widehat{\otimes}K_{r_1,\dots,r_n} \to Y\widehat{\otimes}K_{r_1,\dots,r_n} \to Z\widehat{\otimes}K_{r_1,\dots,r_n}$ is exact and admissible. ∎

2.7.2. Corollary. *Given a k-affinoid algebra $\mathcal{A}$, there exist $r_1, \dots, r_n > 0$, whose images in the $\mathbf{Q}$-vector space $\mathbf{R}^*_+/\sqrt{|k^*|}$ are linearly independent, such that $\mathcal{A}\widehat{\otimes}K_{r_1,\dots,r_n}$ is a strictly $K_{r_1,\dots,r_n}$-affinoid algebra.* ∎

2.7.3. Proposition. *Let $\mathcal{A}$ be a k-affinoid algebra, and $f \in \mathcal{A}$. Then*

(i) $\mathcal{A}$ is Noetherian, and all of its ideals are closed;

(ii) $\rho(f) = 0$ if and only if f is nilpotent;

(iii) if f is not nilpotent, there exists a constant $C > 0$ with $||f^n|| \le C\rho(f)^n$ for all $n \ge 1$.

Proof. (i) It suffices to show that if the ring $\mathcal{A}\widehat{\otimes}K_r$ with $r \notin \sqrt{|k^*|}$ is Noetherian and all of its ideals are closed, then the ring $\mathcal{A}$ possesses the same properties. Let $\mathbf{a}$ be an ideal of $\mathcal{A}$. Then the ideal $\mathbf{a}(\mathcal{A}\widehat{\otimes}K_r)$ is generated by elements $f_1, \dots, f_n \in \mathbf{a}$. Any $f \in \mathbf{a}$ can be written in the form $f = \sum_{i=1}^n f_i g_i$, where $g_i = \sum_{j=-\infty}^{+\infty} g_{ij} T^i$ with $g_{ij} \in \mathcal{A}$. We have $f = \sum_{i=1}^n f_i g_{i,0}$, i.e., $f_1, \dots, f_n$ generate $\mathbf{a}$ and $\mathbf{a} = \mathcal{A} \cap \mathbf{a}(\mathcal{A}\widehat{\otimes}K_r)$.

(ii) Consider first the case when $\mathcal{A}$ is strictly k-affinoid, and assume that $\rho(f) = 0$. By Corollary 2.4.3, there is an injective embedding $\mathrm{Max}(\mathcal{A}) \hookrightarrow \mathcal{M}(\mathcal{A})$, it follows that $f \in \mathbf{m}$ for all maximal ideals $\mathbf{m} \subset \mathcal{A}$. Since $\mathcal{A}$ is a Jacobson ring (Corollary 2.3.3), it follows that f is a nilpotent element. In the general case, it suffices to show that if the statement is true for the ring $\mathcal{A}\widehat{\otimes}K_r$ with $r \notin \sqrt{|k^*|}$, then it is also true for $\mathcal{A}$. Since the canonical homomorphism $\mathcal{A} \to \mathcal{A}\widehat{\otimes}K_r$ is isomoetric, it follows that the spectral norm of an element $f \in \mathcal{A}$ does not change if it is considered as an element of $\mathcal{A}\widehat{\otimes}K_r$. This immediately implies the requited fact.

(iii) It suffices to assume that $\mathcal{A}$ is strictly k-affinoid. By Corollary 2.6.3, $\rho(f)^m = |a|$ for some $m \ge 1$ and $a \in k^*$. Then $\rho(g) = 1$ for $g = \frac{f^m}{a}$. Proposition 2.6.5 implies that there exists $C > 0$ such that $||g^n|| \le C\rho(g)^n$ for all $n \ge 1$, and the required fact follows. ∎

2.7.4. Corollary. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bounded homomorphism between k-affinoid algebras. Let $f_1, \dots, f_n \in \mathcal{B}$, and let $r_1, \dots, r_n$ be positive numbers with $r_i \ge \rho(f_i)$. then there is a unique bounded homomorphism $\Phi : \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{B}$ extending φ and sending T_i to f_i.* ∎

2.7.5. Corollary. *A k-affinoid algebra $\mathcal{A}$ is strictly k-affinoid if and only if $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ for all $f \in \mathcal{A}$.*

Proof. The direct implication follows from Corollary 2.6.3. Assume that $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ for all $f \in \mathcal{A}$. (We may assume that the valuation on k is nontrivial.) Consider an admissible epimorphism $\varphi : k\{r^{-1}T\} \to \mathcal{A} : T_i \mapsto f_i$, $1 \le i \le n$. Suppose that $r_1 \notin \sqrt{|k^*|}$. Then there is a number $s_1 \in \sqrt{|k^*|}$ with $\rho(f_1) \le s_1 < r_1$. Then the induced homomorphism $k\{s_1^{-1}T_1, r_2^{-1}T_2, \ldots, r_n^{-1}T_n\} \to \mathcal{A}$ is also an admissible epimorphism. Continuing this process, we get the required fact. ∎

For a k-affinoid algebra $\mathcal{A}$, let $\mathrm{Mod}_b^h(\mathcal{A})$ denote the category of finite Banach $\mathcal{A}$-modules, and let $\mathrm{Mod}^h(\mathcal{A})$ denote the category of finite $\mathcal{A}$-modules. An *affinoid $\mathcal{A}$-algebra* is an $\mathcal{A}\widehat{\otimes}K)$-algebra for some non-Archimedean field K over k.

2.7.4. Proposition. *Let $\mathcal{A}$ be a k-affinoid algebra, and $M, N \in \mathrm{Mod}_b^h(\mathcal{A})$. Then*

(i) the canonical functor $\mathrm{Mod}_b^h(\mathcal{A}) \to \mathrm{Mod}^h(\mathcal{A})$ is an equivalence of categories;

(ii) any $\mathcal{A}$-linear map $M \to N$ is admissible;

(iii) $M \otimes_{\mathcal{A}} N \xrightarrow{\sim} M\widehat{\otimes}_{\mathcal{A}}N \in \mathrm{Mod}_b^h(\mathcal{A})$;

(iv) for any affinoid $\mathcal{A}$-algebra $\mathcal{B}$, one has $M \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\sim} M\widehat{\otimes}_{\mathcal{A}}\mathcal{B} \in \mathrm{Mod}_b^h(\mathcal{B})$.

Proof. (i) It suffices to verify that any finite $\mathcal{A}$-module M can be provided with the structure of a finite $\mathcal{A}$-module. For this we consider an arbitrary $\mathcal{A}$-linear epimorphism $\varphi : \mathcal{A}^n \to M$. Since $\mathcal{A}^n$ has a k-affinoid structure, the kernel of φ is closed. Therefore, one can endow M with the residue norm, with respect to which M is complete.

(ii) follows from Lemma 2.7.1(ii) and the fact that the statement is true in the strictly k-affinoid case.

(iii) and (iv) are now proved in the same way as Lemma 2.5.3. ∎

2.7.5. Corollary. *The category of finite Banach $\mathcal{A}$-algebras is equivalent to the category of finite $\mathcal{A}$-algebras.* ∎

2.7.6. Proposition. *Let $\mathcal{B}$ be a finite Banach algebra over a k-affinoid algebra $\mathcal{A}$, and assume the canonical homomorphism $\mathcal{A} \to \mathcal{B}$ is injective. Then the map $\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ is surjective and has finite fibers.*

Proof. That the map has finite fibers is easy: for $x \in \mathcal{M}(\mathcal{A})$, the fiber at x is the spectrum of $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{H}(x)$ which is a finite extension of $\mathcal{H}(x)$. Furthermore, if $r \notin \sqrt{|k^*|}$, then the canonical map $\mathcal{M}(\mathcal{A}\widehat{\otimes}K_r) \to \mathcal{M}(\mathcal{A})$ is surjective. this reduces the situation to the case when $\mathcal{A}$ and $\mathcal{B}$ are strictly k-affinoid and the valuation on k is nontrivial. In this case, the map $\mathrm{Max}(\mathcal{B}) \to \mathrm{Max}(\mathcal{A})$ is surjective. Since $\mathrm{Max}(\mathcal{A})$ is dense in $\mathcal{M}(\mathcal{A})$, the surjectivity follows. ∎