Alterations and resolution of singularities

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Contents

0 Introduction

H. Hironaka, 1964:

In characteristic zero
any variety can be modified into a nonsingular variety.

A. J. de Jong, 1995:

Any variety can be altered into a nonsingular variety.

On July 26, 1995, at the University of California, Santa Cruz, a young Dutch mathematician by the name Aise Johan de Jong made a revolution in the study of the arithmetic, geometry and cohomology theory of varieties in positive or mixed characteristic. The talk he delivered, first in a series of three entitled "Dominating Varieties by Smooth Varieties", had a central theme: a systematic application of fibrations by nodal curves. Among the hundreds of awe struck members of the audience, participants of the American Mathematical Society Summer Research Institute on Algebraic Geometry, many recognized the great potential of Johan de Jong's ideas even for *complex* algebraic varieties, and indeed soon more results along these lines began to form.

0.1 The alteration paradigm

A.J. de Jong's main result was, that for any variety X, there is a nonsingular variety Y and an *alteration,* namely a proper, surjective and generically finite morphism, $Y \to X$ (see Theorem 2.3 for a precise statement). This is in contrast with Hironaka's result, which uses only a *modification,* namely a proper birational morphism.

Here is the basic structure of the proof by De Jong:

- **Projection.** For a given variety X of dimension d we produce a morphism $f: X \to P$ with $\dim P = d-1$, and all fibers of f are curves (we may first have to apply a modification to X).

- **Desingularization of fibers.** After an alteration of the base P, we arrive at a new morphism $f: X \to P$ where all fibers are curves with only ordinary nodes as singularities. The main tool here is the theory of moduli of curves.

- **Desingularization of base.** After a further alteration on the base P, we arrive at a new morphism $f: X \to P$ as above, where P is regular. Here we use induction, i.e. supposing that the theorem is already true for varieties of dimension $d-1$. So here we "desingularize the base".

- **Desingularization of total space.** Given the last two steps, an explicit and easy method of resolution of singularities finishes the job.

0.2 The purpose of this paper

This paper is an outgrowth of our course material prepared for the Working Week on Resolution of Singularities, which was held during September 7-14, 1997 in Obergurgl, Tirol, Austria. As we did in the workshop, we intend to explain Johan de Jong's results in some detail, and give some other results following the same paradigm, as well as a few applications, both arithmetic and in characteristic zero. We hope that the reader will come to share some of the excitement we felt on that beautiful July day in Santa Cruz.

In the rest of this introduction we give an overview of the proof and the material involved. We hope that this introduction will give most readers a general feeling of what the results are about. The body of the paper is divided in two parts. We begin part I by expanding on some of the preliminary material necessary for understanding the proofs by any student of algebraic geometry. Then we go back to the proof of de Jong's main theorem, as well as some generalizations. Proofs of some variants and generalizations of de Jong's theorems are indicated in the form of exercises, with sufficient hints and references, which we hope will enable the reader to appreciate de Jong's work. Part II is an introduction to an ingredient of the proof - the theory of moduli of curves. We aim to indicate the main ideas behind the proofs of the main theorems about existence and properties of moduli spaces, again accompanied with a collection of exercises.

As a result, this account is mostly expository. The only point where some novelty appears is in Section 13, where we show the existence of tautological families of stable curves over the moduli spaces of *stable pointed curves with level structure.* This has been "well known to the experts" for years, and can be collected from the literature. However a complete account under one roof has not been published. For the definition of a "tautological curve" we refer to Section 10.4.

0.3 Historical context

There are many cases in geometry in which one wants to transform a singular variety into a non-singular one: once arrived in such a situation, various technical steps can be performed, not possible on singular varieties.

Since the beginning of the century, partial results in this direction appeared, crowned by Hironaka's theorem on resolution of singularities in characteristic zero, in 1964.

Hironaka's ingenuous proof had many applications, but it was not easy to understand the fine details of his proof. Generalizing that method to varieties in positive characteristic has failed up to now. Indeed, resolution of singularities in positive characteristic has been a topic to which many years of intensive research have been devoted, and up to now the status is not yet clear: for the general question of resolution of singularities in positive characteristic we have neither a fully verified theorem nor a counterexample. In addition, the algorithms involved in Hironaka's theory were difficult to generalize, even in characteristic 0, to some important more complicated situations.

It seemed that a lull in development of this subject had been reached, until a totally new idea came about. In 1995 Johan de Jong approached the problem above, of transforming a variety into a nonsingular one, from a different angle. The idea of the proof is surprisingly easy, and for many applications his result is sufficient. His approach is very geometric, and hence it works in a wide range of situations. The alteration paradigm automatically works in all characteristics, and a suitable version works in mixed characteristic as well. It easily gives rise to some new "semistable reduction" type results which are new even over the complex numbers. Moreover, without much effort it give birth to new, "conceptually easy" proofs of a weaker form of Hironaka's theorem.

0.4 Comparison of approaches

Let us take a moment to make a qualitative comparison of Hironaka's result and de Jong's result.

In the approach taken by Hironaka, singularities of a variety are studied closely, invariants measuring the difficulty of the singularities are defined, and a somewhat explicit algorithm is applied in order to improve the singularities, in the sense that the given invariants get "better". One needs to show that the algorithm terminates (and indeed in characteristic zero it does), resulting in the construction of a regular variety. A big advantage of this process developed by Hironaka (and by many others) is the fact that usually it is very explicit, it is canonical in a certain sense and once it works, the result is in its strongest form, see [70] and [11], as well as [19] in this volume.

In the approach by Johan de Jong, the singularities are, at first, completely ignored. The idea is to first bring the variety to a special form: a fibration by nodal curves. Here one pays a big price: in order to arrive at this special form one needs to use an operation - called alteration - which extends the function field of the variety. However, once we arrive at this form, we can use induction on the dimension for the base space of the fibration, and automatically arrive at a situation where the variety has very mild singularities. Only then, finally, attention is paid to the singularities. But these are so mild that an easy and explicit blowing up finishes the job.

0.5 A sketch of the construction of an alteration giving a regular variety

Here we give a much simplified form of the proof of A.J. de Jong's main Theorem (Theorem 2.3 in this text). We break up the proof in steps. A star attached to a step means that in that phase of the proof a finite extension of the function field might be involved, i.e. the alteration constructed might not be a modification. In steps without a star only modifications are used.

Before starting, a small technical point is necessary. In the course of the proof we use induction on the dimension of the variety X, and it turns out that for the induction to work we need the statement of the theorem to involve a closed subset $Z \subset X$ as well. Our final goal will be to find an alteration $f : Y \to X$ such that $f^{-1}Z$ is a normal crossings divisor.

We start with a field k, a variety X and a closed subset $Z \subset X$, over the field k.

Step 0.

> *We can reduce to the case where k is algebraically closed, the variety X is projective and normal, and the closed subset Z is the support of an effective Cartier divisor.*

We intend to say: if we prove the theorem with this new additional data, then the theorem in the original, more general form follows. Reducing to the algebraically closed field case is standard - in the main body of the paper we avoid it, assuming k is algebraically closed. The main ingredient for projectivity is Chow's Lemma (see [Red Book], pp. 85-89, or [HAG], Exercise II.4.10): *for a variety X over k, there exists a modification $X' \to X$, such that X' is quasi-projective.* To make Z into a divisor we simply blow it up inside X.

Replacement convention. From now on, in each step, we shall replace X by a new variety X' over k which admits a modification or an alteration $X' \to X$, arriving finally at a regular variety and an alteration of the variety produced in Step 0.

Step 1.

> *After modifying X, construct a morphism $f : X \to P$ of projective varieties whose generic fiber is an irreducible, complete, non-singular curve.*

Note: $\dim(P) = \dim(X) - 1$, which suggests using induction later.

Actually we need a little more, but the technical details will be discussed in the main text.

This step follows a classical, geometric idea. Set $\dim(X) = d$, and assume $X \subset \mathbb{P}^N$. Using Bertini's theorem we see that we can find a linear subvariety $L \subset \mathbb{P}^N$ "in general position" with $\dim(L) = N - d$ such that the projection with center L gives a rational map $X \dashrightarrow \mathbb{P}^{d-1}$ where the generic fiber is a regular curve. After modifying X we can make this rational map into a morphism.

The strict transform. We will use an operation which de Jong called the "strict transform". (In [10], 815-12 the terminology "strict alteration" is used). Consider a morphism $X \to S$, and a base change $T \to S$. Assume T to be integral, and let $\eta \in T$ be its generic point. Then define $X' \subset T \times_S X$ as the closure of the generic fiber $(T \times_S X)_\eta$ in $(T \times_S X)$. A more thorough discussion of this operation will follow in Section 3.1.

In our situation $X \to P$, we will often replace P by an alteration, and then simply replace X by its strict transform.

STEP 2*.

After applying alterations to X and to P we can arrive at a morphism $f : X \to P$ as in Step 1, and sections $\sigma_1, \cdots, \sigma_n : P \to X$, such that every geometric component C' of every geometric fiber of f meets at least three of these sections in the smooth locus of f, i.e. in $C' \cap \mathrm{Sm}(f)$.

There is a "multi-section" in the situation of Step 1 having this property. After an alteration on Y and on X this becomes a union of sections.
Stable pointed curves. Here we follow Deligne-Mumford and Knudsen. An algebraic curve is called *nodal* if it is complete, connected and if the singularities of C are not worse than ordinary double points. Its arithmetic genus is given by $g = \dim_k \mathrm{H}^1(C, \mathcal{O}_C)$.

Suppose C is a nodal curve of genus g over a field k, and let $P_1, \cdots, P_n \in C(k)$ with $2g - 2 + n > 0$; we write $\mathcal{P} = \{P_1, \cdots, P_n\}$; this is called a *stable n-pointed curve* if:

- the points are mutually different, $i < j \Longrightarrow P_i \neq P_j$,
- none of these marked points is singular, $P_i \notin \mathrm{Sing}(C)$,
- and $\mathrm{Aut}(C, \mathcal{P})$ is a finite group; under the previous conditions (and k algebraically closed) this amounts to the condition that for every regular rational irreducible component

$$\mathbb{P}_1 \cong C' \subset C, \quad \text{then} \quad \#(C' \cap (\mathcal{P} \cup \mathrm{Sing}(C))) \geq 3.$$

A flat family of curves is called "a family of stable n-pointed curves" if all geometric fibers are stable n-pointed curves in the sense just defined, the markings given by sections.

Historically, stable curves and stable pointed curves were introduced in order to construct, in a natural way, compactifications of moduli spaces (see [17]). Certainly the following names should be mentioned: Zariski, A. Mayer, Deligne, Mumford, Grothendieck, Knudsen, and many more. It came a bit as a surprise when de Jong used these for a desingularization-type problem!

STEP 3^*.

After an alteration on the base P, we can assume that $X \to P$ is a projective family of stable n-pointed curves.

We briefly sketch the heart of the proof of this step - it will be discussed in detail later.
Extending families of curves. We need the following fundamental fact: suppose we are given a variety P, an open dense subset $U \subset P$, and a family of stable curves $C_U \to U$:

$$\begin{array}{ccc} C_U & \subset & ? \\ \downarrow & & \downarrow \\ U & \subset & P. \end{array}$$

Then there is an alteration $a : P_1 \to P$ such that the pullback family $C_{U_1} \to U_1$ over the open set $U_1 = a^{-1}U$ can be extended to a family of stable curves $C_1 \to P_1$:

$$\begin{array}{ccc} C_{U_1} & \subset & C_1 \\ \downarrow & & \downarrow \\ U_1 & \subset & P_1. \end{array}$$

The first result behind this is the existence of a moduli space of stable curves ([39], see also Section 12). Then one needs the fact that a finite cover $M \to \overline{M_{g,n}}$ of the moduli space admits a "tautological family" - namely, a family $C \to M$ such that the associated morphism $M \to \overline{M_{g,n}}$ is the given finite cover. One could consult [16] (the precise statement we need follows from that paper), or use [21], where a tautological family of nodal curves is constructed over a moduli space of stable curves with a level structure.

The sectionss of the family $X \to P$ correspond to those of the stable n-pointed curve $\mathcal{C} \to P$, under the birational transformation thus defined. We want to show this extends to a morphism $\mathcal{C} \to X$.

Flattening of the graph. We take the closure $T \subset X \times_P \mathcal{C}$ of the graph of $\beta_0 : \mathcal{C}_U \to X_U$, and apply the "Flattening Lemma", see 3.2 below. We arrive at new X, T, and $\mathcal{C}$ flat over P. All we have to show (modulo some technicalities) is that no point of a fiber of $\mathcal{C} \to P$ is blown up to a component of a fiber of $X \to P$.

The Three Point Lemma. Using the markings, and studying carefully the geometry we show that indeed β_0 extends to a morphism β. The crucial point here was that every component of every fiber of X over P has at least three nonsingular points marked by the sections σ_i (see 4.18 - 4.20 of [Alteration]).

STEP 4^*.

After an alteration of P, we may assume that P is nonsingular.

We simply apply induction on the dimension of the base: we suppose that the theorem we want to prove is valid for all varieties having dimension less than dim X. Thus after an alteration of the base P we can suppose P is regular and the strict transform of X has all the previous properties.
Following Z. The argument for the previous two steps should be carried through with a proper care given to the divisor Z. At the end, we can guarantee that Z is contained in the union of two types of sets:

- the images of the sections σ_i, and
- the inverse image of a normal crossings divisor $\Delta \subset P$.

Moreover, in the induction hypothesis we can guarantee that the final family of curves $X \to P$ degenerates only over the normal crossings divisor Δ.

STEP 5.

> *The singularities of the resulting family $X \to P$ are so mild that it is very easy to resolve them explicitly.*

Indeed, each singular point can be described in formal coordinates by the equation $xy = t_1^{k_1} \cdots t_r^{k_r}$. It is a fairly straightforward exercise to resolve these singularities.

Part I
The alteration theorem

1 Some preliminaries and generalities on varieties

1.1 Varieties

To fix notation, we use the following definition of a variety:

Definition. By a *variety* defined over k we mean a separated geometrically integral scheme of finite type over k. If $k \subset k_1$ we write X_{k_1} for $X \times_{\operatorname{Spec} k} \operatorname{Spec} k_1$.

In more down to earth terms this means: an *affine* variety defined over k is given as a closed subvariety of an affine space $\mathbb{A}^n_k$ defined by an ideal $I \subset k[T_1, \cdots, T_n] = k[T]$ such that $k_1 \cdot I \subset k_1[T]$ is a prime ideal for every (equivalently, for some) algebraically closed field k_1 containing k. In general, a variety then is defined by gluing a finite number of affine varieties in a separated way. See [Red Book], I.5, Definition 1 (p. 35) and I.6, Definition 2 (p. 52).

Remark. This definition differs slightly from that in [Alteration]. De Jong requires the algebraic scheme to be integral, and we require that the schemes stay integral after extending the field. For example for any finite field extension $k \subset K$, the scheme $\operatorname{Spec}(K)$ is called a k-variety by de Jong, but we only say it is a variety defined over k if $k = K$. For most geometric situations the differences will not be important.

1.2 Operations on varieties

Definition. A morphism of varieties $Y \to X$ is called a *modification* if it is proper and birational.

A modification is the type of "surgery operation" usually associated with resolution of singularities. Johan de Jong introduced the following important variant:

Definition (de Jong). A morphism of varieties $Y \to X$ is called an *alteration* if it is proper, surjective and generically finite. This notion of alteration will also be used for integral schemes.

See [Alteration], 2.20.

Remark. A modification is a birational alteration.

Exercise 1.1. Show that an alteration $\varphi : Y \to X$ can be factored as

$$Y \xrightarrow{\pi} Z \xrightarrow{f} X,$$

where π is a modification, and f is a finite morphism.

Exercise 1.2. Suppose moreover that a finite group G acts on Y by automorphisms such that the field of invariants $K(Y)^G$ contains the function field $K(X)$. Formulate other factorizations of φ.

Remark. Given a variety X and a nonzero coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, the blowing up $Bl_{\mathcal{I}}X = \mathrm{Proj}_X(\oplus_{j\geq 0}\mathcal{I}^j) \to X$ gives naturally a modification, such that the inverse image of $\mathcal{I}$ becomes invertible. If $Z \subset X$ is a subscheme with ideal sheaf $\mathcal{I}$, the blowing up $Bl_Z(X)$ of X with *center* Z is defined to be the blowing up $Bl_{\mathcal{I}}X$.

See [HAG], II.7, p. 163.

1.3 Smooth morphisms and regular varieties

The terminology "smooth" will only be used in a relative situation. Thus a morphism can be smooth. The terminology "regular", or "non-singular", will be used in the absolute sense. Thus a variety can be regular. This means that for every point P in the variety the local ring at P is a regular local ring. If a morphism $X \to \mathrm{Spec}(K)$ is smooth, then X is regular. It is not recommended to use the terminology "a smooth variety", which can be misleading and confusing.

1.4 Resolution, weak and strong

We state what we mean by a resolution of singularities. There are two variants we will use:

Definition. Let X be a variety. A *resolution of singularities in the weak sense* is a modification $Y \to X$ such that Y is nonsingular.

Definition. Let X be a variety. A *resolution of singularities in the strong sense* is a modification $Y \to X$, which is an isomorphism over the nonsingular locus X_{reg}, such that Y is nonsingular.

1.5 Normal crossings

The following type of "nice subschemes" of a variety are quite useful in desingularization problems and applications:

Definition. Let X be a variety. A subscheme $Z \subset X$ is called a *strict normal crossings divisor* if for each point $x \in Z$, there is a regular system of parameters $y_1, \ldots, y_k$ for x in X (in particular the point $x \in X$ is supposed to be a regular point on X), such that Z is given on a Zariski neighborhood of x by the equation $y_1 \cdots y_l = 0$.

Suppose furthermore we have a finite group acting on Z and X equivariantly: $G \subset \mathrm{Aut}(Z \subset X)$. We say that Z is a *G-strict normal crossings divisor* if it has normal crossings, and for any irreducible component $Z' \subset Z$, the orbit $\cup_{g\in G}\, g(Z')$ is normal.

We say that a closed subset $Z \subset X$ is a strict normal crossings divisor, if ths reduced subscheme it supports is a strict normal crossings divisor.

See [Alteration], 7.1.

Strict normal crossings divisors have played an important role in resolution of singularities, and are essential in the proof of de Jong's result.

1.6 Flatness

A crucial idea for studying "families of schemes" is Serre's notion of *flatness* (see [HAG], III.9).

Definition. Let A be a ring and M and A-module. Recall that M is said to be a *flat* A-module if the functor $N \mapsto M \otimes_A N$ is exact.

A morphism of schemes $X \to Y$ is *flat* if at any point $x \in X$, whose image is $y \in Y$, the local ring $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module.

There are many important examples of flat morphisms which we will discuss later. The reader is advised to consult [HAG] or [43] for a more detailed discussion. The general picture should be that in a proper flat morphism, many essential numerical invariants (e.g. dimension, degree...) are "constant" from fiber to fiber, so we should really think about it as a "family".

Here are some instructive examples of morphisms which are *not* flat:

Example 1.3. (See [HAG], III 9.7.1.) Let Y be a curve with a node (say, the locus $xy = 0$ in the affine plane). Let $X \to Y$ be the normalization (in the specific example, the disjoint union of two lines mapping onto the locus $xy = 0$). Then $f : X \to Y$ is not flat. The idea one should have in mind is that since over a general point in Y we have one point in X, and over the node we have two points in X, this is not really a nice family - it jumps in degree.

The same reasoning gives a more general example:

Example 1.4. Let $f : X \to Y$ be a modification. Then f is flat if and only if it is an isomorphism.

In particular, a nontrivial blowup is not flat.

1.7 Stable curves

We give a formal definition of the fundamental notion introduced in the introduction:

Definition. An S scheme $C \to S$ is called a *family of nodal curves* over S if it is of finite presentation, proper and flat, and all geometric fibers are connected reduced curves with at most ordinary double points (locally $xy = 0$) as singularities.

Remark. The terminology *a nodal curve over* S can be used interchangeably with *a family of nodal curves over* S. Indeed, if $C \to S$ comes by way of an extension of a nodal curve C_η over the generic point η_S of S, it may be natural to call it *a nodal curve over* S.

Definition. The *discriminant locus* $\Delta \subset S$ is the closed subset over which $C \to S$ is not smooth.

Definition (Deligne and Mumford). A family of nodal curves

$$f : C \to S,$$

together with sections $s_i : S \to C, \quad i = 1, \ldots, n$ with image schemes $S_i = s_i(S)$, is called a *family of stable* n*-pointed curves of genus* g if

1. The schemes S_i are mutually disjoint.
2. The schemes S_i are disjoint from the non-smooth locus $\mathrm{Sing}(f)$.
3. All the geometric fibers have arithmetic genus g.
4. The sheaf $\omega_{C/S}(\sum S_i)$ is f-ample (namely, it is ample on fall fibers of f).

In case $n = 0$ we simply call these *stable curves* (rather than stable 0-pointed curves).

The definition is made so that a stable pointed curve has a finite automorphism group (relative over S). It agrees with that made (informally) in the introduction. It is discussed in detail in [17].

Remark. In the litterature one sometimes finds the terminology "n-pointed stable curve" instead of "stable n-pointed curve". We try to stick to the latter, since it effectively conveys the idea that the curve *with* the points is stable. The other terminology might give the impression we are dealing with stable curve with some points on them. This would be a different notion in general!

1.8 Minimal models, existence and uniqueness

An important stepping stone for understanding moduli of stable curves is the notion of minimal models of 1-parameter families of curves.

Let K be a field, and C a complete, geometrically irreducible algebraic curve smooth over K; suppose the genus of C is at least 2. Let v be a discrete valuation on K, and $R \subset K$ its valuation ring. Pick a projective model $\mathcal{C}_0$ of C over R. Following Abhyankar (1963) we can resolve singularities in dimension 2, therefore we may assume $\mathcal{C}_0$ is nonsingular. Following Shafarevich (1966) and Lipman (1969) we have the notion of the *minimal model* of C over $S := \mathrm{Spec}(R)$ (see Lichtenbaum, [41], Th. 4.4; also (see [17], page 87). We thus arrive at a family of curves $\mathcal{C} \to \mathrm{Spec}(R)$ which is a regular 2-dimensional scheme, and which is relatively minimal.

Remark. Here we use a special case of resolution of singularities, namely in the case of schemes of dimension 2.

2 Results

First recall Hironaka's theorem:

Theorem 2.1 (Hironaka). *Let X be a variety over a field k of characteristic 0. Then there exists a sequence of modifications*

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X,$$

where each $X_i \to X_{i-1}$ is a blowing up with nonsingular center, and the center lies over the singular locus $\mathrm{Sing}(X)$. *In particular, $X_n \to X$ is a resolution of singularities in the strong sense.*

See the original [29]. Hironaka's theorem and its refinements will be discussed in [19] in this volume.
Our main goal is to prove the following result, due to A. J. de Jong:

Theorem 2.2. *Let X be a variety over an algebraically closed field. There is a separable alteration $Y \to X$ such that Y is quasi projective and regular.*

Corollary. *Let X/k be a variety. There is a finite extension $k \subset k_1$ and a separable alteration $Y \to X_{k_1}$ such that Y is quasi projective and regular.*

In order for the induction in the proof to work, de Jong's theorem gives more:

Theorem 2.3 (de Jong). *Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset. There is a separable alteration $f : Y \to X$, and an open immersion $j : Y \subset \bar{Y}$, such that $\bar{Y}$ is projective and regular, and the subset $j(f^{-1}Z) \cup (\bar{Y} \smallsetminus Y)$ is the support of a strict normal crossings divisor.*

See [Alteration], 4.1. The proof of this result will be given in Section 4.

De Jong's theorem has a few important variants. First, a theorem of semistable reduction up to alteration over a one dimensional base:

Theorem 2.4 (de Jong). *Let R be a discrete valuation ring, with fraction field K and residue field k. Let $X \to \operatorname{Spec} R$ be an integral scheme of finite type such that X_K is a variety. There exists a finite extension $R \subset R_1$, where R_1 is a discrete valuation ring with residue field k_1, and an alteration $Y \to X_{R_1}$, such that Y is nonsingular, and the special fiber Y_{k_1} is a reduced, strict normal crossings divisor.*

See [Alteration], 6.5. The proof is detailed in Section 5.

This theorem belongs to a class of theorems about "desingularization of morphisms". A "dual" case, which can actually serve as a building block in proving the alteration type theorems, is the case where the base is arbitrary dimensional, and the fibers are curves. A proof can be found in [31].

Theorem 2.5 (de Jong). *Let $\pi : X \to B$ be a proper surjective morphism of integral schemes, with $\dim X = \dim B + 1$. Let $Z \subset X$ be a proper closed subset. There exists an alteration $B_1 \to B$, a modification $X_1 \to \tilde{X}_{B_1}$ of the strict transform $\tilde{X}_{B_1}$ (see Section 3.1), sections $s_i : B_1 \to X_1$, and a proper closed subset $\Sigma \subset B_1$ such that*

1. *$\pi_1 : X_1 \to B_1$ is a family of pointed nodal curves,*
2. *s_i are disjoint sections, landing in the smooth locus of π_1, and*
3. *the inverse image Z_1 of Z in X_1 is contained in the union of $\pi_1^{-1}\Sigma$ (the "vertical part") and $s_i(B_1)$ (the "horizontal part").*

The reader who has solved the exercises in Section 5 will be able to complete the proof of this theorem. From this de Jong deduced the following refinement of Theorem 2.4:

Theorem 2.6 (de Jong). *Let $\pi : X \to B$ be a proper surjective morphism of integral schemes, $\dim X = \dim B + r$. Assume that B admits a proper morphism to an excellent two-dimensional scheme S. Then there are alterations $B_1 \to B$ and $X_1 \to \tilde{X}_{B_1}$, a factorization $X_1 \to X_2 \to \cdots \to X_r \to X_{r+1} = B$, and subschemes $\Sigma_i = \Sigma_i^{hor} \cup \Sigma_i^{ver}$, such that*

1. *X_i are nonsingular and Σ_i are normal crossings divisors, $i = 1, \dots, r+1$;*
2. *$\pi_i : X_i \to X_{i+1}$ are families of nodal curves, smooth away from Σ_{i+1}, and*
3. *Σ_i^{hor} is the union of disjoint sections of π_i, lying in the smooth locus of π_i.*

See [31]. Alternative proofs of different versions of this theorem were provided in [1] and [44].

Next, we consider a finite group action:

Theorem 2.7 (de Jong). *Let X be a variety over an algebraically closed field, $Z \subset X$ a proper closed subset, $G \subset \mathrm{Aut}(Z \subset X)$. There is an alteration $f : Y \to X$, and a finite subgroup $G_1 \subset \mathrm{Aut}\, Y$, satisfying:*

1. *there is a surjection $G_1 \to G$ such that f is G_1 equivariant, and the field extension $K(X)^G \subset K(Y)^{G_1}$ is purely inseparable;*
2. *Y is quasi projective and nonsingular; and*
3. *$f^{-1}Z$ is the support of a G-strict normal crossings divisor.*

See [Alteration], 7.3. The proof is detailed in exercises in Section 5.

Note that, taking $G = \{id\}$, this implies:

Corollary. *Let X be a variety over an algebraically closed field. There is a purely inseparable alteration $Y \to X$ where Y is a quotient of a nonsingular variety by the action of a finite group.*

Remark. For generalizations which combine both Theorem 2.4 and Theorem 2.7, see [31].

In characteristic 0, any purely inseparable alteration is birational, and the quotient singularities can be improved:

Theorem 2.8 (See [2] and [12]). *Let X be a variety over an algebraically closed field of characteristic 0. Then there is a projective resolution of singularities in the weak sense $Y \to X$.*

Remark. This is a rather weak version of Hironaka's theorem. The point is, that new proofs, by Abramovich and de Jong [2], and by Bogomolov and Pantev [12], were given based on de Jong's ideas. The proof by Bogomolov and Pantev is extremely simple, drawing only on toric geometry. Its proof is detailed in Section 7.

Question 2.9. *Can we improve the methods and obtain a weak resolution of singularities in all characteristics? Or, at least weak resolution up to purely inseparable alterations?*

The proof by Abramovich and de Jong, detailed in Section 8, lends itself to generalizations in the flavor of de Jong's semistable reduction theorem, such as the following two results:

Theorem 2.10 (Abramovich - Karu). *Let $X \to B$ be a dominant morphism of complex projective varieties. There exists a commutative diagram*

$$\begin{array}{ccccc} U_{X'} & \subset & X' & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ U_{B'} & \subset & B' & \to & B \end{array}$$

such that

1. *$X' \to X$ and $B' \to B$ are modifications,*
2. *X' and B' are nonsingular,*
3. *$U_{X'} \subset X'$ and $U_{B'} \subset B'$ are toroidal embeddings, and the morphism $X' \to B'$ is a toroidal morphism (see definition in 6).*

Theorem 2.11 (Abramovich - Karu). *Let $X \to B$ be a dominant morphism of complex projective varieties. There exists a commutative diagram*

$$\begin{array}{ccccc} U_X & \subset & X_1 & \to & X \\ \downarrow & & \downarrow \pi_1 & & \downarrow \pi \\ U_B & \subset & B_1 & \to & B \end{array}$$

where $B_1 \to B$ is an alteration, $X_1 \to \tilde{X}_{B_1}$ is a modification *of the strict transform, $U_X \subset X_1$ and $U_B \subset B_1$ are toroidal, the morphism $\pi_1 : X_1 \to B_1$ is toroidal with $\pi_1^{-1} U_B = U_X$, the variety B_1 is nonsingular and*

1. *the morphism π_1 is equidimensional and*
2. *all fibers of π_1 are reduced.*

See [3] for details. A refinement is given in [33], and an application in [34].

3 Some tools

In this section we gather some basic tools which we are going to use. Some of these tools seem to be of vital importance in algebraic geometry, and it is instructive to see them functioning in the context of de Jong's theorem. We have included some indications of proofs for the interested reader. For the proof of the alteration theorem only the following will be necessary: Section 3.1, Lemmas 3.1, 3.2 and 3.4, and Theorem 3.6.

3.1 The strict transform

See [Alteration], 2.18.

As mentioned in the introduction, we need an operation called the "strict transform". Let us recall the definition.

Definition. Consider a morphism $X \to S$, and a base change $T \to S$. Assume T to be integral, and let $\eta \in T$ be its generic point. Then define the *strict transform* $\tilde{X}_T \subset T \times_S X$ as the Zariski closure of the generic fiber $\eta \times_S X$:

$$\begin{array}{ccccccc} \tilde{X}_T & \stackrel{\text{def}}{=} & \overline{\eta \times_S X}^{Zar} & \subset & T \times_S X & \longrightarrow & X \\ & & & \searrow & \downarrow & & \downarrow \\ & & & & T & \longrightarrow & S. \end{array}$$

Note that if the image of η is not in the image of $X \to S$ (i.e. if $T \times_S X \to T$ is not dominant), then the strict transform in the sense explained here is empty.

Remark. The notion given here is different from the usual notion of the "strict transform" of a subvariety under a modification (compare with [HAG], II.7, the definition after 7.15). For example consider a blowing up $T \to S$ of a surface S in a point $P \in S$, and let $C \subset S$ be a curve in S passing through P. The "strict alteration" (or "strict transform" in the terminology above) of C under $T \to S$ is empty; the "strict transform" of C under $T \to S$ in the classical sense, as explained in [HAG], II.7, is a curve in T.

Some people have suggested the use of terminology "essential pullback of X along $T \to S$", which may have some merits. After all, $\tilde{X}_T$ contains only the "part" of $T \times_S X$ which dominates T, which is in some sense its essential part.

3.2 Chow's lemma

An algebraic curve and a regular algebraic surface are quasi-projective. However in higher dimension an "abstract variety" need not be quasi-projective. A beautiful example by Hironaka (of a variety of dimension three) is described in [HAG], Appendix B, Example (3.4.1). However in certain situations (such as the alteration method described below) we like to work with projective varieties.

Lemma 3.1. *Given a variety X, there is a modification $Y \to X$ such that Y is quasi-projective.*

See [Red Book] , I.10, p. 85, or [HAG], Exc. II.4.10 p. 107. ○

3.3 The flattening lemma

In some situations we want to replace a morphism by a flat morphism. One can show this is possible after a *modification* of the base. The general situation is studied in [60]. We only need this in an easier, special situation, as follows:

Lemma 3.2 (The Flattening Lemma). *Let X and Z be varieties over a perfect field K (more generally, integral schemes of finite presentation) and $X \to Z$ a projective, dominant morphism. There exists a modification $f : Y \to Z$ such that the strict transform $f' : \tilde{X}_Y \to Y$ is flat.*

The main ingredient in the proof is the existence and projectivity of the Hilbert scheme. Hilbert schemes were introduced and constructed by Grothendieck in [24], Exp. 221 (see [47] for simplified proofs, [18] for discussion). We will come back to them in Section 10. Their purpose is to parametrize all subschemes of a fixed projective space $\mathbb{P}^N$. Of course, the set of all subschemes of a projective space is rather large, so we cut it down into bounded pieces by fixing the Hilbert polynomial $P_W(T) = \chi(W, \mathcal{O}_W(T))$ for a subscheme $W \subset \mathbb{P}^N$. Grothendieck's result may be summarized as follows:

Theorem 3.3. *There is a projective scheme $\mathcal{H}_{\mathbb{P}^N,P(T)}$ over $\operatorname{Spec}\mathbb{Z}$ and a closed subscheme $\mathcal{X}_{\mathbb{P}^N,P(T)} \subset \mathbb{P}^N \times \mathcal{H}_{\mathbb{P}^N,P(T)}$ which is flat over $\mathcal{H}_{\mathbb{P}^N,P(T)}$, such that $\mathcal{H}_{\mathbb{P}^N,P(T)}$ parametrizes subschemes of $\mathbb{P}^N$ with Hilbert polynomial $P(T)$, and where $\mathcal{X}_{\mathbb{P}^N,P(T)} \to \mathcal{H}_{\mathbb{P}^N,P(T)}$ is a universal family, in the following sense:*

Given a scheme T, let $X \subset \mathbb{P}^N \times T$ be a closed subscheme which is flat over T and such that the Hilbert polynomial of the fibers is $P(T)$. Then there exists a unique morphism $h : T \to \mathcal{H}_{\mathbb{P}^N,P(T)}$, such that

$$X = T \underset{\mathcal{H}_{\mathbb{P}^N,P(T)}}{\times} \mathcal{X}_{\mathbb{P}^N,P(T)}$$

Back to the proof of Lemma 3.2. Since $X \to Z$ is projective, we can choose an embedding $X \subset \mathbb{P}^N \times T$ for some N. Note that the generic fiber of f is reduced. By generic flatness, there exists a dense, open subset $i : U \hookrightarrow Z$ such that

$$f_U : X_U := X|_U \longrightarrow U$$

is flat. Let P be Hilbert polynomial of the fibers of f_U (all fibers in a flat family over an irreducible base have the same Hilbert polynomial), and let $\mathcal{X} \to \mathcal{H}$ be the universal family over the Hilbert scheme associated to this polynomial. We have a cartesian commutative diagram:

$$\begin{array}{ccc} X_U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & \mathcal{H}. \end{array}$$

Note that $\mathcal{X} \to \mathcal{H}$ is a flat morphism. We have a morphism $i \times g : U \to Z \times \mathcal{H}$. Define

$$Z' := \overline{i \times g(U)}^{Zar} \quad \subset \quad Z \times \mathcal{H},$$

and let $X' \to Z'$ be the pull back:

$$X' = Z' \underset{\mathcal{H}}{\times} \mathcal{X}.$$

Note that the base change of a flat morphism is flat, hence $X' \to Z'$ is flat. It follows from [HAG] III.9.8 that X' is the strict transform under $Z' \to Z$ of $X \to Z$. ♻3.2

Remark. We cited [HAG] III.9.8, which is in fact an important building block in the construction of Hilbert schemes.

Remark. We can delete the word "dominant" in the flattening lemma, and still prove the conclusion, but we do not gain much: if $X \to Z$ is not dominant, the identity on Z gives a strict transform (in the sense explained above) of X such that $X' = \emptyset$, and flatness trivially follows.

Remark. In the proof above we note a general method, which will also be used in the question of extending curves below: suppose we study a certain property (e.g. flatness of a map). Suppose there is a "universal family" having this property (e.g., the Hilbert scheme). Suppose also that in a given family the property holds over a dense open subset U in the base. Then, after a modification, or an alteration of the base, depending on the situation, we can achieve that property by mapping U to the base of the universal family, taking the closure of the graph, and pulling back the universal family.

We encounter a similar situation, in the context of extending stable curves, in Section 3.6 below.

3.4 Deforming a node

An important fact underlying the role of stable curves, which is implicitly invoked in several places in this paper, is that a node $uv = 0$ can only deform in a certain way. To be precise:

Lemma 3.4. *Let R be a complete local ring with maximal ideal m and algebraically closed residue field. Let $S = \operatorname{Spec} R$ and denote the special point by s. Let $X \to S$ be the completion of a nodal curve at a closed point x on the fiber X_s over s, so $X_s = \operatorname{Spec}(R/m[\bar{u}, \bar{v}]/(\bar{u}\bar{v}))^\wedge$. Then there is an element $f \in m$, and liftings u of $\bar{u}$ and v of $\bar{v}$, such that $X \simeq \operatorname{Spec}(R[u, v]/(uv - f))^\wedge$.*

One can prove this using the deformation theory of a node: the versal deformation space (see [6]) of the completion X_s of a nodal curve has dimension $\dim \mathrm{Ext}^1(\Omega^1_{X_s}, \mathcal{O}_{X_s}) = 1$, and it is easy to see that the equation $uv = t$ is versal. An elementary proof by lifting the equation is sketched in [Alteration], Section 2.23.

3.5 Serre's lemma

A critical result in the theory of moduli of curves is, that a 1-parameter family of curves admits stable reduction after a base change (see Theorem 11.2). A crucial point in the proof is the relationship between the automorphisms of a curve and the automorphisms of its jacobian, as in the following lemma.

Lemma 3.5. *Let C be a stable curve defined over an algebraically closed field k, let $m \in \mathbb{Z}_{\geq 3}$, not divisible by the characteristic of k, and let $\varphi \in \mathrm{Aut}(C)$ such that the induced map on locally free sheaves of order m*

$$\varphi_* : \mathbf{Pic}^0_C[m] \longrightarrow \mathbf{Pic}^0_C[m]$$

is the identity map. Then $\varphi = 1_C$, the identity morphism on C.

Proof (see [65], or [16], 3.5.1). Let $\tilde{C} \to C$ be the normalization of C (namely the disjoint union of the normalizations of all irreducible components). Denote $J := \mathbf{Pic}^0_C$ and $X := \mathbf{Pic}^0_{\tilde{C}}$. Consider the "Chevalley decomposition" (as in [14]):

$$0 \to T \longrightarrow J \longrightarrow X \to 0,$$

i.e. $T \subset J$ is the maximal connected linear subgroup in J, the quotient is an abelian variety, and $J/T \cong \mathbf{Pic}^0_{\tilde{C}}$. Note that $T \cong (\mathbb{G}_m)^s$ is a split torus. Define $f := \varphi_* - 1_J \in \mathrm{End}(J)$. Using

$$\mathrm{Hom}(T, X) = 0$$

we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & T & \longrightarrow & J & \longrightarrow & X & \to & 0 \\ & & g\downarrow & & f\downarrow & & h\downarrow & & \\ 0 & \to & T & \longrightarrow & J & \longrightarrow & X & \to & 0. \end{array}$$

By the original lemma of Serre we deduce that $h = 0$; let us sketch the argument. The automorphism φ is of finite order (because C is stable), hence the induced $\psi \in \mathrm{Aut}(\tilde{C})$ is of finite order, hence $\psi_* = 1 + h$ is of finite order. Note that the ring $\mathrm{End}(X)$ is torsion free, and since ψ is of finite order the subring $\mathbb{Z}[\psi_*] \subset \mathrm{End}(X)$ is cyclotomic. By assumption the element $\psi_* \otimes_{\mathbb{Z}} \mathbb{Z}/m = 1$ in $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m$. Since ψ_* is a root of unity, and $m \geq 3$, this implies $\psi_* = 1_X$, hence $h = 0$.

Moreover an analogous reasoning implies that $g = 0$: use that $\mathrm{End}(T) = \mathrm{Mat}(s, \mathbb{Z})$ is torsion-free. From $h = 0$ and $g = 0$ we deduce that $f : J \to J$ factors as

$$J \to X \xrightarrow{f'} T \to J.$$

Using

$$\mathrm{Hom}(X, T) = 0$$

we conclude $f' = 0$, hence $f = 0$. Hence $\varphi_* = 1_J$, and this implies that $\varphi = 1_C$. ◌3.5

Remark. We have used the fact that for $p \geq 3$, even modulo p, the root of unity ζ_p is not equal to 1: indeed, the ring $\mathcal{O}_{\mathbb{Q}(\zeta_p)}/p$ is artinian, with generator ζ_p.

3.6 Extending stable curves

Suppose we are given a stable curve $\mathcal{C}_U \to U$ over an open set $U \subset S$ of a base scheme S. Can it be extended to a stable curve $\mathcal{C} \to S$? In general the answer is negative. This question is discussed in [32] and [44], where we find criteria which ensure that in certain cases this is possible. The general situation has the following answer: an extension to a stable curves is possible after an *alteration* on the base. Note the difference from the Flattening Lemma, which has to do with extending families of flat subschemes of a fixed scheme.

Theorem 3.6 (Stable Extension Theorem). *Let S be a locally noetherian integral scheme, let $U \subset S$ be a dense open subset, and let $\mathcal{C} \to U$ with sections $s_i^U : U \to \mathcal{C}$ be a stable pointed curve. There exists an alteration $\varphi : T \to S$, and a stable pointed curve $\mathcal{D} \to T$ with sections $\tau_i : T \to \mathcal{D}$, such that, if we write $\varphi^{-1}(U) =: U' \subset T$, we have an isomorphism*

$$\mathcal{D}|_{U'} \quad \xrightarrow{\phi} \quad U' \quad \times_U \quad \mathcal{C},$$

such that $\phi^* s_i^U = \tau_i$.

Remark. A proof for unpointed curves can be found in [16], Lemma 1.6. We present here a somewhat different proof. For simplicity of notation the proof is stated in the case of unpointed curves.

The first step is to extend *isomorphisms* of stable curves. The first lemma is the following:

Lemma 3.7. *Suppose T is the spectrum of a discrete valuation ring, and $\mathcal{D} \to T$ and $\mathcal{D}' \to T$ are stable (pointed) curves, such that the generic fibers are isomorphic: $\mathcal{D}_\eta \cong \mathcal{D}'_\eta$. Then this extends a unique isomorphism: $\mathcal{D} \cong_T \mathcal{D}'$.*

For the proof see [17], Lemma 1.12. The main point is that the *minimal models* of $\mathcal{D}$ and $\mathcal{D}'$ coincide, and $\mathcal{D}$ or $\mathcal{D}'$ are obtained from the minimal model in a unique way by blowing down (-2)-curves. This lemma implies the following (see [17], 1.11):

Lemma 3.8. *Suppose T is a scheme, and $\mathcal{D} \to T$ and $\mathcal{D}' \to T$ are stable (pointed) curves. Then* $\mathrm{Isom}_T(\mathcal{D}, \mathcal{D}') \to T$ *is finite and unramified.*

Indeed, the previous lemma implies that $\mathrm{Isom}_T(\mathcal{D}, \mathcal{D}') \to T$ is proper. Since stable curves have a finite automorphism groups, the morphism is finite. And since stable curves have no nonzero vector fields, the morphism is unramified.

As a consequence we get the following general result about extending isomorphisms:

Lemma 3.9. *Suppose T is an integral* normal *scheme, and $\mathcal{D} \to T$ and $\mathcal{D}' \to T$ are stable (pointed) curves, such that the generic fibers are isomorphic: $\mathcal{D}_\eta \cong \mathcal{D}'_\eta$. Then this induces an isomorphism: $\mathcal{D} \cong_T \mathcal{D}'$.*

Proof. The given isomorphism over the generic point η gives a lifting $\eta \to \mathrm{Isom}_T(\mathcal{D}, \mathcal{D}')$. The closure of its image in $\mathrm{Isom}_T(\mathcal{D}, \mathcal{D}')$ maps finitely and birationally to T. By Zariski's Main theorem it is isomorphic to T, and therefore gives a section of $\mathrm{Isom}_T(\mathcal{D}, \mathcal{D}') \to T$. ↺

Exercise 3.10. We show that the condition "normal" in the previous lemma is needed. To this end, choose a regular curve T_0, and a smooth curve $\mathcal{D}_0 \to T_0$. Choose it in such a way that the geometric generic fiber has only the identity as automorphism, and such that there exist closed points $x, y \in T_0$ and two *different* isomorphisms

$$\alpha, \beta : (\mathcal{D}_0)_x \xrightarrow{\sim} (\mathcal{D}_0)_y.$$

Let $T_0 \to T$ be the nodal curve obtained by identifying x and y as a nodal point $P \in T$ (and the curves isomorphic outside these points). Construct $\mathcal{D}_\alpha \to T$ by "identifying $(\mathcal{D}_0)_x$ and $(\mathcal{D}_0)_y$ via α". Analogously $\mathcal{D}_\beta \to T$. Show that

$$\mathcal{D}_\alpha \not\cong_T \mathcal{D}_\beta, \quad \text{and} \quad (\mathcal{D}_\alpha)_{\eta_T} = (\mathcal{D}_0)_{\eta_{T'}} = (\mathcal{D}_\beta)_{\eta_T}.$$

It is instructive to describe $\mathrm{Isom}_T(\mathcal{D}_\alpha, \mathcal{D}_\beta)$.

Remark. The phenomenon described in the exercise is characteristic of situations where one has a *coarse* moduli space rather than a *fine* one. See Section 10.4 for details.

The following is an analogous lemma about isomorphisms of the geometric generic fibers:

Lemma 3.11. *Suppose T is an integral scheme, $\mathcal{D} \to T$ and $\mathcal{D}' \to T$ stable (pointed) curves, such that the* geometric *generic fibers are isomorphic:*

$$\mathcal{D}_{\overline{\eta}} \quad \cong \quad \mathcal{D}'_{\overline{\eta}}.$$

Then there exists a finite surjective morphism $T' \to T$ and an isomorphism

$$\mathcal{D} \times_T T' \quad \cong_{T'} \quad \mathcal{D}' \times_T T'.$$

Remark. It is easy to give examples where the isomorphism requested does not exist even over the generic point of T.

Proof. As in [17], 1.10 we consider $\mathrm{Isom}_T(\mathcal{D},\mathcal{D}')$. The condition in the lemma assures that this is not empty, it is finite and dominant over T, and the lemma follows. ◌

Proof of Theorem 3.6: Here we use the fact that there exists a "tautological family" of curves over the compactified moduli space of curves with a level structure. For stable curves without points, this is given in [21]. For the case of stable pointed curves, use Theorem 13.2. Another proof, more in the line of [17], is sketched in Section 13.4

Let us suppose that there exists $m \in \mathbb{Z}_{\geq 3}$ such that $S \to \mathrm{Spec}(\mathbb{Z}[1/m])$. Hence the family $\mathcal{C} \to U$ defines a moduli morphism

$$f : U \to \overline{M_g}[1/m] := \overline{M_g} \underset{\mathrm{Spec}\,\mathbb{Z}}{\times} \mathrm{Spec}\,\mathbb{Z}[1/m].$$

We write $M := \overline{M_g^{(m)}}$ (after having fixed g and m) for the moduli scheme of stable curves of genus g with level-m structure (see [21] and Section 13.2). We have a curve $\mathcal{Z} \to M$ such that the associated moduli morphism to $\overline{M_g}[1/m]$ is the natural morphism $\pi : M \to \overline{M_g}[1/m]$ (we say that $\mathcal{Z} \to M$ is a *tautological family*; see Section 10.4). Let $U'' := U \times_{\overline{M_g}} M$, let $U' \subset U''$ be a reduced, irreducible component of U'' dominant over U, and let $\mathcal{C}'$ be the pull back $\mathcal{C}' = \mathcal{C} \times_U U'$. Let $\mathcal{Z}' \to U'$ be the pull back of the tautological family, $\mathcal{Z}' = \mathcal{Z} \times_M U'$. The very fact that $\overline{M_g}$ os a coarse moduli scheme (see Section 10.4, condition 1) guarantees that over the *geometric* generic point we have $\mathcal{C}'_{\overline{\eta}} \simeq \mathcal{Z}'_{\overline{\eta}}$. By the previous lemma we can replace U' by a finite cover (call it again U') for which there is a U' isomorphism $\mathcal{C}' \simeq \mathcal{Z}'$. Let S' be the normalization of S in the function field of U'. We define $V \subset S' \times M$ to be the image of U' by the two morphisms into S' and M, and let $T = \overline{V}^{Zar} \subset S' \times M$.

By construction there is a stable curve over T, obtained by pulling back $\mathcal{Z} \to M$, which moreover by construction extends the pull back of $\mathcal{C}' \to V$. This proves the theorem in case $S \to S_m := \mathrm{Spec}(\mathbb{Z}[1/m])$.

In case $S \to \mathrm{Spec}(\mathbb{Z})$ is surjective, one does the construction for two different values of m, and then one pastes the result using Lemma 3.11. ◌3.6

3.7 Contraction and stabilization

In [39], II, Section 3, pp. 173-179, we find a description of the following two constructions.

1. Consider a stable $(n+1)$-pointed curve $(\mathcal{X},\mathcal{P}) \to S$ with $2g-2+n > 0$. Deleting one section gives a nodal n-pointed curve (with an extra section), which need not be a stable n-pointed curve. However, if necessary one can contract "non-stable components" of fibers (regular rational curves containing not enough singularities and marked points). After this blowing down one

obtains a stable n-pointed curve $(X', \mathcal{Q}) \to S$, and an S-morphism $\mathcal{X} \to \mathcal{X}'$ mapping the first n sections of $\mathcal{P}$ to $\mathcal{Q}$. This process, which arrives at a unique solution to this problem, is called *"contraction"*.

2. Consider a stable n-pointed curve $(\mathcal{Y}, \mathcal{Q}) \to S$ plus an extra section $\sigma : S \to \mathcal{Y}$ not in $\mathcal{Q}$. This extra section may meet sections in $\mathcal{Q}$, or meet the nodes of $\mathcal{Y} \to S$. One can blow up $\mathcal{Y}$ in such a way that the strict transforms (in the old sense) of elements of $\mathcal{Q}$ and of the extra section give a stable $(n+1)$-pointed curve $(\mathcal{X}, \mathcal{P}) \to S$, and an S-morphism $\mathcal{X} \to \mathcal{Y}$ mapping the first n sections of $\mathcal{P}$ to $\mathcal{Q}$. This process, which arrives at a unique solution to this problem, is called *"stabilization"*.

4 Proof of de Jong's main theorem

One striking feature of the proofs of de Jong's theorem and its derivatives is, that all the ingredients, with the exception of one subtle, but still natural, result (the Three Point Lemma), were known and understood nearly two decades before. The way they are put together is quite ingenious.

4.1 Preparatory steps and observations

The proof of de Jong's theorem starts with a series of simple reduction steps.
The situation. We want to prove de Jong's Theorem 2.3. Thus we are given a variety X defined over an algebraically closed field k, and a Zariski - closed subset $Z \subset X$. We perform some elementary reductions:
Replacing X by an alteration. In order to prove the theorem for a variety X and a closed subset Z, it is enough to prove it for an alteration X' of X while replacing Z by its inverse image Z' in X'. Thus in several stages of the proof, once we find an alteration $X' \to X$ which we like better than X, we simply replace the pair (X, Z) by (X', Z').
Making Z into a divisor. By blowing up Z in X, and using the observation above, we may assume that Z is the support of an effective Cartier divisor. We will slightly abuse terminology, and say that "Z is a divisor" when we mean that Z is a closed subset supporting an effective Cartier divisor.
Enlarging Z. Suppose $Z_i \subset X$ are divisors and $Z_1 \subset Z_2$, then to prove the theorem for (X, Z_1) it suffices to prove it for (X, Z_2). Indeed, if $f : Y \to X$ is an alteration such that Y is nonsingular and $f^{-1}(Z_2)$ is a strict normal crossings divisor, then $f^{-1}(Z_1)$ is a Cartier divisor contained in $f^{-1}(Z_2)$, and it is clear from the definition that it is a strict normal crossings divisor as well. Thus we may always enlarge the divisor Z.
Making X quasi-projective. Using Chow's Lemma 3.1, we may assume X is quasi-projective. Indeed, by Chow's lemma there is a modification $X' \to X$ such that X' is quasi-projective. We may replace X by X'.

Enlarging X. Suppose $X \subset X_1$ is an open embedding of varieties, $Z_1 \subset X_1$ a divisor which containing $X_1 \smallsetminus X$, and $Z = X \cap Z_1$. Then evidently to prove the theorem for (X, Z) it suffices to prove it for (X_1, Z_1).
Making X projective. Since X is quasi-projective, there is an open embedding $X \subset \overline{X}$ where $\overline{X}$ is projective. Denote $Z_1 = \overline{Z} \cup (\overline{X} \smallsetminus X)$. We may replace $\overline{X}$ by the blowup of Z_1, thus we may assume that Z_1 is the support of a Cartier divisor. By the previous observation it is enough to prove the result for $(\overline{X}, Z_1)$.
We may assume X is normal. Indeed, we can simply replace X by its normalization.

To summarize, one may assume that the variety X is projective and normal, and the subset Z is the support of an effective Cartier divisor. Moreover, one may always replace Z by a larger divisor.

4.2 Producing a projection

The next step is to produce a projection with some nice properties. We first start with some general facts about projections in projective spaces.

Let $Y \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field (in fact, separably closed would suffice). For any closed point $p \in \mathbb{P}^N \smallsetminus Y$ we have a projection $pr_p : Y \to \mathbb{P}^{N-1}$.

Lemma 4.1. *Suppose* $\dim Y < N-1$. *Then there is a nonempty open set* $U \subset \mathbb{P}^N$, *such that if* $p \in U$ *then* pr_p *sends* Y *birationally to its image.*

Proof. Let $q \in Y$ be a regular point. Define the cone $C_{Y,q}$ over Y with vertex q to be the Zariski closure of the union of all secant lines lines containing q and another q', for all $q' \in Y$. It is easy to see that $C_{Y,q}$ has dimension $\leq \dim Y + 1 < N$. Note that $C_{Y,q}$ contains (as "limit points") the projective tangent space $\mathbb{T}_{Y,q}$ at q. Therefore if $p \in \mathbb{P}^N \smallsetminus C_{Y,q}$ then the line through p and q meets Y transversally, at q only. This property holds as well for the line through p and q', for any $q' \in Y$ in a neighborhood of q. Hence the lemma. ♁

Lemma 4.2. *Suppose* $\dim Y = N-1$. *Then there is a nonempty open set* $U \subset \mathbb{P}^N$, *such that if* $p \in U$ *then* pr_p *maps* Y *generically étale to* $\mathbb{P}^{N-1}$.

Proof. Same as before, using $\mathbb{T}_{Y,q}$ instead of $C_{Y,q}$. ♁
We go back to our X and Z.

Lemma 4.3. *There exists a modification* $\phi : X' \to X$ *and a morphism* $f : X' \to \mathbb{P}^{d-1}$ *such that*

1. *There exists a finite set of nonsingular closed points* $S \subset X_{ns}$ *disjoint from* Z, *such that* X' *is the blowup of* X *at the points of* S.

2. f *is equidimensional of relative dimension 1*

3. *The smooth locus of f is dense in all fibers*

4. *Let $Z' = \phi^{-1}Z$. Then $f|_{Z'}$ is finite and generically étale*

5. *At least one fiber of f is smooth.*

Proof. First project $\pi : X \to \mathbb{P}^d$ using the previous lemmas $N - d - 1$ times.

Let $B \subset \mathbb{P}^d$ be the locus over which π is not étale.

If we choose a general $p \in \mathbb{P}^d$, then $pr_p : \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale - simply use the lemma above for all irreducible components of $\pi(Z)$.

We choose such a p away from B. By the local description of blowing up, we can identify the variety

$$X' = \{(x, \ell) \in X \times \mathbb{P}^{d-1} | \pi(x) \in \ell\}$$

with the blowing up of X at the points in $\pi^{-1}(p)$.

We define $f : X' \to \mathbb{P}^{d-1}$ to be the second projection.

We can identify the fibers: the fiber over a point ℓ is the scheme theoretic inverse image $\pi^{-1}(L)$ where L is the line corresponding to ℓ.

It follows immediately that f is equidimensional: all fibers have dimension at most 1, and are defined by $d - 1$ equations (the equations of L).

Since no line through p is contained in B, every fiber has a dense smooth locus.

The last assertion follows by Bertini's theorem, since the fibers are obtained by intersecting X with linear subspaces.

Lemma 4.4. *The morphism f has connected fibers.*

Proof. Since the smooth locus is dense in every fiber, the Stein factorization is étale. Since projective space has no nontrivial finite étale covers, the Stein factorization is trivial.

Remark. 1. The last assertion is not really necessary: if f did not have connected fibers, we could replace $f : X' \to \mathbb{P}^{d-1}$ by its Stein factorization.

2. The projection above is the only point where it is crucial that X should be normal, to guarantee that the generic fiber is smooth. From here on we will allow ourselves to make reductions after which X might not be normal.

To summarize, one may assume that we have a morphism of varieties $X \to P$, for some variety P, which makes X into a generically smooth family of curves, satisfying some nice properties, in particular $Z \to P$ is finite and generically étale.

4.3 Enlarging the divisor Z

In order to "rigidify" the situation, it will be useful to enlarge Z so it meets every fiber "sufficiently". This is done as follows:

Lemma 4.5. *Let $X \to P$ be as above. There exists a divisor $H \subset X$ such that*

1. *$f|_H : H \to P$ is finite and generically étale, and*
2. *for any irreducible component C of a geometric fiber of f, we have*

$$\#sm(X/P) \cap C \cap H \geq 3.$$

Here we count the points without *multiplicities.*

Proof. Let $n \geq 1$ be an integer. Given a very ample line bundle $\mathcal{L}$ on X, consider the embedding

$$i : X \hookrightarrow \mathbb{P} = \mathbb{P}(\Gamma(X, \mathcal{L}^{\otimes n}))$$

associated to $\mathcal{L}^{\otimes n}$.

Claim. Given any irreducible curve $C \subset X$, the image $i(C) \subset \mathbb{P}$ is not contained in any linear subspace of dimension $n-1$.

Proof of claim. Since $\mathcal{L}$ is very ample, the image of $\Gamma(X, \mathcal{L}) \to \Gamma(C, \mathcal{L}_{|C})$ contains a rank-2 subspace $V \subset \Gamma(C, \mathcal{L}_{|C})$ such that the corresponding linear series (of dimension 1) has no base points. The map $\mathrm{Sym}^m V \to \Gamma(C, \mathcal{L}_{|C}^{\otimes n})$ has rank $\geq n+1$, therefore $\Gamma(X, \mathcal{L}^{\otimes n}) \to \Gamma(C, \mathcal{L}_{|C}^{\otimes n})$ has rank $\geq n+1$, which is what we claimed. ↺(Claim)

The divisors of sections of $\mathcal{L}^{\otimes n}$ are parametrized by the dual projective space $\mathbb{P}^\vee$. We consider the collection of "bad" divisors and show that there are "good" ones left. So consider

$$T = \{(H, y) \in \mathbb{P}^\vee \times P | \dim f^{-1}y \cap H = 1\} \subset \mathbb{P}^\vee \times P.$$

It is clear that T is a Zariski closed subset. We can describe the fibers of $pr_2 : T \to P$ using irreducible components of the fibers:

$$pr_2^{-1}(y) = \bigcup_{C \subset f^{-1}y} \{H | i(C) \subset H\}.$$

But by the fact that $i(C)$ is not contained in any linear subspace of dimension $n-1$, we have

$$\mathrm{codim}(pr_2^{-1}(y), \mathbb{P}^\vee) \geq n.$$

Therefore $\dim T \leq \dim P + \dim \mathbb{P}^\vee - n$.

Thus if n is large enough, $pr_1(T) \subset \mathbb{P}^\vee$ is of large codimension (at least $n - \dim P$). In particular $pr_1(T) \neq \mathbb{P}^\vee$.

We fix such large n. Thus there are plenty of H which map finitely to P. For a fixed closed point $y \in P(k)$ consider the set

$$U(y) = \left\{ H \in \mathbb{P}^\vee(k) \;\middle|\; \begin{array}{l} H \notin pr_1(T) \\ H \cap f^{-1}y \subset sm(X/P) \\ H \cap f^{-1}y \text{ is reduced} \end{array} \right\}.$$

This is clearly a nonempty open set of $\mathbb{P}^\vee$. Moreover, if $H \in U(y)$ then $H \in U(y')$ for all y' in a neighborhood of y.

If moreover $n \geq 3$, then we have that $\#H \cap f^{-1}y \geq 3$. so we are done for all points in a neighborhood V of y.

We deal with points in $P \setminus V$ in the same way. Using Noetherian induction we are done. ♁(Lemma)

Summarizing, one may assume that Z meets every irreducible component of every geometric fiber in at least three smooth points.

4.4 The idea of simplifying the fibers

De Jong's idea is to simplify the fibers of the morphism $X \to P$. Then by induction on dimension one can simplify the base P, and finally put these simplifications together.

The method of simplifying the fibers uses the deepest ingredient in the program: the theory of moduli of curves (see Section 10 for discussion).

Here is the general plan. First, as we will see below, it is easy to make an alteration of P, and replace X and Z by their pullbacks, such that Z becomes the union of sections of $X \to P$.

We can think of the generic fiber of $X \to P$ as a smooth curve with a number of points marked on it. Say the genus of this curve is g, and the number of points is n. By the Stable Extension Theorem 3.6, the generic fiber can be extended, *after an alteration* $P_1 \to P$, to a family of stable curves $X_1 \to P_1$:

$$\begin{array}{ccc} X_1 & \dashrightarrow & X \\ \downarrow & & \downarrow \\ P_1 & \to & P \end{array}$$

The new morphism $f_1 : X_1 \to P_1$ is much nicer than f, since at least the fibers are as nice as one can expect: they are nodal curves. Moreover, Z was made much nicer: it is replaced by n sections which are mutually disjoint, and pass through the smooth locus of f_1.

If we can resolve P_1 (say using induction on dimension), then it is easy to resolve X_1 as well.

There is a problem though: if we want to repeat this inductively, we cannot allow a rational map $X_1 \dashrightarrow X$ which is not a morphism - since we cannot pull back nicely along rational maps. So we want to find a way to make sure that $X_1 \dashrightarrow X$ is actually a regular map.

Remark. If one is satisfied with proving a weaker result, namely that every variety admits a "rational alteration" by a nonsingular variety, then there is an alternative way to avoid the issue. This is carried out by S. Mochizuki in [44].

Remark. Another way to circumvent the issue of extending β to a morphism, is to ensure that it extends automatically, by using a moduli space into which a morphism is built in: the space of stable maps. This was carried out, in characteristic

0, in [1], Lemma 4.2. Unfortunately the details of constructing moduli spaces of stable maps have not yet been written out in positive or mixed characteristics, although this would not be difficult: the results of [9] imply that the moduli of stable maps forms a proper Artin stack, and the results of [37] imply that this stack admits a proper algebraic space as a coarse moduli space. One should even be able to modify the argument of [40], Proposition 4.5 and show that this space is projective, but this is not essential for the argument.

Let us go into details.

4.5 Straightening out Z

Lemma 4.6. *There exists a normal variety P_1 and a separable finite morphism $P_1 \to P$ satisfying the following property:*

Let $X_1 = \tilde{X}_{P_1}$ be the strict transform (see Section 3.1), and let Z_1 be the inverse image of Z in X_1. Then there is an integer $n \geq 3$, and n distinct sections $s_i : P_1 \to X_1$, $i = 1, \ldots, n$ such that

$$Z_1 = \bigcup_{i=1}^{n} s_i(P_1).$$

Proof. This can be proven by induction on the degree n of $Z \to P$ as follows: Let Z_1 be an irreducible component of Z and let $P' := Z_1^\nu$ be its normalization. We have a generically étale morphism $P' \to P$. Denote $X' = \tilde{X}_{P'}$ and let Z' be the inverse image of Z. The morphism $P' \to Z$ gives rise to a section $s_{k+1} : P' \to Z'$, and therefore we can write $Z' = s_{k+1}(P') \cup Z''$. We have $\deg(Z'' \to P') = \deg(Z \to P) - 1$, and therefore the inductive assumption holds for Z''. ♻

Thus one can assume Z is the union of sections of $X \to P$.

4.6 Producing a family of stable pointed curves

Let $X \to P$, $\quad s_i : P \to X$ be the new family. Let $U \subset P$ be an open set satisfying the following assumptions:

1. $X_U \to U$ is smooth;
2. the sections $s_i|_U : U \to X_U$ are disjoint.

Such an open set clearly exists.

Since $n \geq 3$ this gives the morphism $X_U \to U$ the structure of a family of stable n-pointed curves.

And here comes the point where moduli theory is used: by Theorem 3.6, there exists an alteration $P_1 \to P$, a family of stable pointed curves $\mathcal{C} \to P_1$, with sections $\tau_i : P_1 \to \mathcal{C}$, such that over the open set $U_1 = P_1 \times_P U \subset P_1$ we have an isomorphism $\beta : \mathcal{C}_{U_1} \to U_1 \times_P X$, satisfying $\beta^* s_i = \tau_i$.

4.7 The three point lemma

As usual, we replace P by P_1 and X by its strict transform. Thus we may assume that we have a diagram as follows:

$$\begin{array}{ccc} C & \overset{\beta}{\dashrightarrow} & X \\ & \searrow & \downarrow \\ & & P \end{array}$$

The crucial point, for which we needed to "enlarge Z" in a previous step, is the following:

Lemma 4.7 (Three Point Lemma). *Suppose Z meets the smooth locus of every irreducible component of every fiber in at least three points. Then, at least after a modification of P, the rational map $\beta : C \dashrightarrow X$ extends to a morphism.*

The proof of this lemma, which is detailed in the next few paragraphs, is probably the most subtle point in this chapter.

4.8 Flattening the graph

Let $T \subset X \times_P C$ be the closure of the graph of the rational map β. We have two projection maps $pr_1 : T \to C$ and $pr_2 : T \to X$.

Claim. *There exists a modification P' of P such that the strict transform of X, and the closure of the graph of $C \dashrightarrow X$ are both flat. Thus we might as well assume $X \to P$ and $T \to P$ are flat.*

Proof. By the Flattening Lemma 3.2 there exists a modification $P' \to P$ such that $\tilde{X}_{P'}$ and $\tilde{T}_{P'}$ are both flat. Evidently the closure of the graph of the rational map $C \times_P P' \to \tilde{X}_{P'}$ is contained in $\tilde{T}_{P'}$, and since $\tilde{T}_{P'}$ is flat they coincide by [HAG] III.9.8. ↺(Claim)

Let p be a geometric point on P, and denote by X_p, T_p, C_p the fibers over p. There exists a finite set $W \subset X_p$ such that $T_p \to X_p$ is finite away from W. Indeed, the flatness implies that $\dim T_p = \dim X_p = 1$.

Thus, for any $x \in X_p \smallsetminus W$, there is an open neighborhood $x \in V \subset X$ such that $pr_2^{-1} V \to V$ is finite and birational.

In case $x \in \mathrm{Sm}(X_p) \smallsetminus W$, we may choose $V \subset \mathrm{Sm}(X \to P)$. Using the assumption that P is normal, it follows that V is normal. In this case, by Zariski's main theorem, $pr_2^{-1} V \to V$ is an isomorphism.

Note that the assumption that $x \in \mathrm{Sm}(X_p)$ excludes only finitely many points, since our projection $X \to P$ is smooth at the generic point of each component of the geometric fiber X_p. Therefore we conclude that the following lemma holds:

Lemma 4.8. *If $X' \subset X_p$ is an irreducible component, then there is a* unique *irreducible component T' of T_p mapping finitely onto X' via $pr_2 : T \to X$. Moreover, $T' \to X'$ is birational.*

Repeating the argument for $pr_1 : T \to C$, we also have:

Lemma 4.9. *If $C'' \subset C_p$ is an irreducible component, then there is a unique irreducible component T'' of T_p mapping finitely onto C'' via $pr_1 : T \to C$. Moreover, $T'' \to C''$ is birational.*

Let $x \in \mathrm{Sm}(X_p)$ be a closed point. Considering the Stein factorization $T \to \tilde{X} \to X$, we have that the fiber $pr_2^{-1}(x)$ is connected. Indeed, since X is normal at x, we have that $\tilde{X} \to X$ is an isomorphism at x.

4.9 Using the three point assumption

Let $X' \subset X_p$ be an irreducible component, and $T' \subset T_p$ the unique component mapping finitely (and birationally) onto it, as in Lemma 4.8 above. We will prove that $pr_1 : T' \to C$ is non-constant. Assume by contradiction that $pr_1(T') = \{c\}$ is a point.

We will use the three point assumption. Let $s_i : P \to X, i = 1, \ldots, 3$ be three of the given sections such that $s_i(p) = x_i$ are three distinct points on $\mathrm{Sm}(X')$. Let $T_i = pr_2^{-1} x_i$. Let $\tau_i(p) = c_i \in C_p$.

Note that the point $t_i = (c_i, x_i) \in C \times X$ is in T.

Assume $c \notin \{c_i, i = 1, \ldots, 3\}$. Then each of $T_i, i = 1, \ldots, 3$ contains an irreducible component T_i' whose image in C is again a curve passing through c. These image components are *distinct*. Indeed, T_i are disjoint subschemes of T_p, whose images in C connect c with c_i, and therefore each has an irreducible component whose image contains c. These components are distinct, and by Lemma 4.9 their images are distinct.

This contradicts the fact that C_p is nodal. Thus c is among the c_i.

Assume, without loss of generality, $c = c_1$. Repeating the argument of the previous paragraph we conclude that there are two distinct components of C_p passing through c. This contradicts the fact that C_p has a marked point at $c = c_1$.

Thus we conclude that $pr_1 : T \to C$ is finite and birational.

By Serre's criterion C is normal: it is clearly regular in codimension 1, and condition S_2 follows since $C \to P$ has reduced one-dimensional fibers and P is normal.

We conclude that $T \to C$ is an isomorphism, hence β extends as a morphism!

♁4.7

4.10 Induction

We arrived at the following situation:

$$\begin{array}{ccc} C & \stackrel{\beta}{\to} & X \\ & \searrow & \downarrow \\ & & P \end{array}$$

We may replace X by C, and Z by its inverse image in C. Note that Z is no longer finite over P: it has a"finite part", the union of the sections $\tau_i : P \to C = X$, but there is a "vertical" part Z_{vert}, which is the union of irreducible components of *singular* fibers of $X \to P$.

Let $\Sigma \subset P$ be the closed subset over which $f : X \to P$ is not smooth. By the induction assumption there is a projective alteration $P_1 \to P$ such that P_1 is nonsingular and the inverse image of Σ is a strict normal crossings divisor. We may replace X by its pullback to P_1, and replace P by P_1. It is convenient to replace Z by its union with $f^{-1}(\Sigma)$.

We arrived at a situation where both P, and the morphism $f : X \to P$, are simplified. The resulting variety has very simple singularities, and its desingularization results from the following exercises.

4.11 Exercises on blowing up of nodal families

The exercises below, which aim at completing the proof, are adapted from De Jong's complete exposition in [Alteration]. We have not reproduced his proofs here. The reader may consult [Alteration], pages 63-64 (Section 3.4) and 75-76 (Sections 4.23-4.28). We find it hard to improve upon that text, but we hope the reader will enjoy unraveling the details by following the exercises below.

Let $f : X \to S$ be a flat morphism of varieties over an algebraically closed field k, with $n = \dim X = \dim S + 1$. Let $D \subset S$ be a reduced divisor. We make the following assumptions.

N1 The base S is nonsingular.

N2 The divisor D has strict normal crossings.

N3 The morphism f is smooth over $S \setminus D$.

N4 The morphism $f : X \to S$ is a nodal curve.

Let $x \in X$ be a closed point and let $s = f(x) \in S$. By assumption we may choose a regular system of parameters $t_1, ..., t_{n-1}$ at s such that D coincides on a neighborhood with the zero locus of $t_1 \cdots t_r$ for some $r \leq n-1$. It can be seen that if x is a singular point of X, then the completed local ring of X at x can be described as

$$(*) \quad k[[u, v]]/(uv - t_1^{n_1} \cdot \cdots \cdot t_r^{n_r})$$

.

Step 1: Assume $\operatorname{codim}_X \operatorname{Sing}(X) = 2$.

1. Show that there is an irreducible component $D_1 \subset D$ and $\Sigma_1 \subset \operatorname{Sing}(X)$ such that $f(\Sigma_1) = D_1$.

2. Fix a point $x \in \Sigma_1$, and use formal coordinates as in $(*)$, such that $D_1 = V(t_1)$. Show that the power n_1 of t_1 is > 1.

3. Show that the ideal of Σ_1 in the formal completion is (u, v, t_1).

4. Conclude that $\Sigma_1 \to D_1$ is étale, in particular Σ_1 is nonsingular.

5. Let $X_1 = Bl_{\Sigma_1} X$. Show that $X_1 \to S$ satisfies conditions N1-N4, there is at most one component of $\operatorname{Sing}(X_1)$ over Σ_1, with the exponent n_1 replaced by $n_1 - 2$

6. Conclude by induction that there is a blowup $X' \to X$ centered above $\operatorname{Sing}(X)$, such that X' satisfies N1-N4, and $\operatorname{codim}_X \operatorname{Sing}(X) > 2$.

7. Show that each component of $\operatorname{Sing}(X')$ is defined by $u = v = t_i = t_j$ in equation $(*)$, in particular it is nonsingular.

Step 2: Assume $\operatorname{codim}_X \operatorname{Sing}(X) > 2$. Define $Z = f^{-1}D$. Unfortunately here we need to abandon the structure $X \to B$ of a family of nodal curves. Instead we look at X itself. The situation is as follows:

T1 whenever x is a nonsingular point of X, Z has normal crossings at x.

T2 whenever $x \in \operatorname{Sing}(X)$, we have formal description

$$(**) \quad k[[u, v]]/(uv - t_1 \cdots t_s), \quad 2 \leq s \leq r \leq n - 1$$

and $Z = V(t_1 \cdots t_r)$.

T3 All components of $\operatorname{Sing}(X)$ are nonsingular.

1. Let $E \subset \operatorname{Sing}(X)$ be an irreducible component. Show that the blowup $Bl_E X$ satisfies T1-T3, and its singular locus has one fewer irreducible component.

2. Conclude by induction that there is a resolution of singularities $X' \to X$.

This concludes the proof of Theorem 2.3 ◯

5 Modifications of the proof for Theorems 2.4 and 2.7

5.1 Exercises on removing the conditions on the projection

An important step in the proof of de Jong's theorem was, that given the projection $X \to P$, one could construct an alteration $P_1 \to P$ and a diagram

$$\begin{array}{ccc} C & \stackrel{\beta}{\to} & X \\ \downarrow & & \downarrow \\ P_1 & \to & P \end{array}$$

where $C \to P_1$ was a family of nodal curves. In order for the proof to go through, we made several assumptions on the projection $X \to P$. Here we will show that even if these conditions fail, we can still reduce to the case where they do hold.

Exercise 5.1. Using an alteration, show that the condition that $Z \to P$ be finite in Lemma 4.3 (4) is unnecessary for the rest of the proof.

Exercise 5.2. Show that, if one is willing to accept inseparable alterations in the theorem, the condition that $Z \to P$ be generically étale in Lemma 4.3 (4) is unnecessary for the rest of the proof.

Exercise 5.3. $\ast$ By reviewing the arguments, show that the condition that every component of every fiber of $X \to P$ be generically smooth is unnecessary.
Here a modification of the three point lemma is be necessary! In [31], de Jong uses a trick of "raising the genus of the curves" with finite covers. Another way goes as follows: in the proof of the Three Point Lemma 4.7, after flattening X and T, one works with fibers of *the normalizations* X^ν and T^ν. This way one avoids the need for $\mathrm{Sm}(X_p)$ to be dense. One notes that the sections s_i lift to X^ν, and at least three meet every component of every fiber, since Z is the support of a Cartier divisor! The details are left to the reader.

Exercise 5.4. Using the flattening lemma and the previous exercise, show that the condition that $X \to P$ be equidimensional is unnecessary.

Exercise 5.5. Show that, if one is willing to accept inseparable alterations in the theorem, the condition that the generic fiber of $X \to P$ be smooth is unnecessary.

5.2 Exercises on Theorem 2.4

Let us address Theorem 2.4 on semistable reduction up to alteration. Suppose $S = \operatorname{Spec} R$ where R is a discrete valuation ring, $X \to S$ is a morphism as in the theorem, and Z a proper closed subset.

Exercise 5.6. Show that we may assume X projective over S, that Z is the support of a Cartier divisor.

Exercise 5.7. Show that we may assume that the generic fiber is a normal variety, and that X is a normal scheme. (You may need an inseparable base change!)

Exercise 5.8. Let d be the dimension of X_η. Produce a projection $X \to \mathbb{P}_S^{d-1}$ with connected fibers.

Exercise 5.9. Use the semistable reduction argument, with the Three Point Lemma, and the results of Section 5.1 to replace X by a nice family of curves $X \to P \to S$.

Exercise 5.10. Use induction on the dimension to conclude the proof of the theorem.

Exercise 5.11. Can you think of other situations where a similar theorem can be proven, where S is not necessarily the spectrum of a discrete valuation ring? (This is interesting even in characteristic 0!)

5.3 Exercises on Theorem 2.7

We address the equivariant version of the theorem. Suppose X is a variety, Z a proper closed subset, and a finite group G acts on X stabilizing Z. We wish to prove Theorem 2.7.

Exercise 5.12. Produce an equivariant version of Chow's lemma, so that we may assume X is projective.

Exercise 5.13. Show that, to prove the theorem, it suffices to consider the case where Z is a divisor.

Exercise 5.14. Show that we may replace Z by a bigger *equivariant* divisor; in particular we may assume Z contains the fixed point loci of elements in G.

Exercise 5.15. Using a projection of X/G, show that we may assume we have an equivariant projection $X \to P$ making X into a nice family of curves.

Exercise 5.16. Consider the case $X = \mathbb{A}_k^2$ where $\operatorname{char} k = p$, and $G = \mathbb{Z}/p\mathbb{Z}$ acting via $(x, y) \mapsto (x, x + y)$. Show that the fixed point set maps inseparably to the image. In particular, the map $Z \to P$ in the previous exercise might be inseparable!

Exercise 5.17. Making an alteration "Galois": Given a variety W, a finite group action $H \subset \operatorname{Aut} W$ and an alteration $V_0 \to W$, show that there exists an alteration $V \to V_0$, and a finite group H' with a surjection $H' \to H$, and a lifting of the H action $H' \subset \operatorname{Aut}(V \to W)$ such that the extension of fixed fields $K(W)^H \subset K(V)^{H'}$ is purely inseparable.

Exercise 5.18. $\ast$ Use the uniqueness in the stable reduction theorem to show that there is an alteration $P' \to P$, a family of stable pointed curves $C \to P'$ and a finite group G' with a quotient $G' \to G$ and a diagram

$$\begin{array}{ccc} C & \to & X \\ \downarrow & & \downarrow \\ P' & \to & P \end{array}$$

on which G' acts equivariantly, such that C is birational to $\widetilde{X_{P'}}$, and the extensions $K(X)^G \subset K(C)^{G'}$ and $K(P)^G \subset K(P')^{G'}$ are purely inseparable.

Exercise 5.19. Use induction on the dimension and a suitable modification of the elementary blowups argument to conclude the theorem.

6 Toroidal geometry

Toroidal geometry is a generalization of the more well known geometry of toric varieties. In this section we will show that various aspects of toric varieties generalize with few difficulties to the toroidal case. The reader is assumed to be familiar with the basic facts about toric varieties, as given in [15] in this volume.

6.1 Basic definitions

For simplicity we work over an algebraically closed field. We recall the notion of a toric variety (a more thorough discussion is available in [15]):

Definition. A variety X together with an open dense embedding $T \subset X$ is called a *toric variety* if X is normal, T is a torus (geometrically isomorphic to $\mathbb{G}_m^k$), and the action of T on itself by translations extends to an action on X.

To get an intuitive idea about the singularities of a toric variety, it is worth noting that a normal, affine variety, defined by equations between monomials (such as $z^2 = xy$) is toric, and every toric variety is locally of this type.

For many purposes toric varieties are too restrictive. A more general notion was introduced by Mumford in [38]:

Definition. A variety X together with an open embedding $U \subset X$ is called a *toroidal embedding* if any point $x \in X$ has an étale neighborhood X' such that X' is isomorphic to an étale neighborhood of a point on a toric variety, and the isomorphism carries the open subset $U' = X' \times_X U \subset X'$ to the torus of the toric variety.

Thus a toroidal embedding looks locally like a toric variety, and the big open set U is a device which ties together these "local pictures". In a sense, this notion is suitable for studying varieties whose singularities are like those of toric varieties.

In this section we recall facts about toric varieties and briefly indicate how one can obtain analogous facts about toroidal embeddings. The details are available in [38].

Remark. A more sheaf - theoretic approach was introduced by K. Kato, see [35], [36].

Definition. A toroidal embedding is said to be *strict* (or a *toroidal embedding without self intersections*) if every irreducible component of $X \smallsetminus U$ is normal.

For instance, if X is a nonsingular variety, $D \subset X$ is a strict normal crossings divisor, and $U = X \smallsetminus D$, then $U \subset X$ is a strict toroidal embedding.

We will only work with strict toroidal embeddings.

Definition. If $G \subset \mathrm{Aut}(U \subset X)$ is a finite group, we say that G *acts toroidally* if for any point $x \in X$, the stabilizer G_x of x can be identified with a subgroup of the torus in an appropriate étale neighborhood of x.

Definition. A morphism between toric varieties is called a *toric morphism*, if it is surjective and torus-equivariant. A morphism of toroidal embeddings $(U_X \subset X) \to (U_Y \subset Y)$ is called a *toroidal morphism* if locally on X it looks like a toric morphism.

6.2 The cone

First recall some notation (see [15]):

$M = \mathrm{Hom}(T, \mathbb{G}_m)$ - this is the group of algebraic characters of T;

$M_{\mathbb{R}} = M \otimes \mathbb{R}$

$N = \mathrm{Hom}(M, \mathbb{Z}) = \mathrm{Hom}(\mathbb{G}_m, T)$ - this is the group of 1-parameter subgroups on T;

$N_{\mathbb{R}} = N \otimes \mathbb{R}$

It is common to call the functions defined by elements of M the *monomials*. One uses the notation x^m for the monomial associated with the element $m \in M$.

Recall the basic correspondence between

$$\{\text{affine toric varieties } T \subset X\}$$
$$\text{and}$$
$$\{\text{strictly convex rational polyhedral cones } \sigma \subset N_{\mathbb{R}}\}$$

which can be defined in one direction via

$$X = V_\sigma := \mathrm{Spec}\, k[\sigma^\vee \cap M],$$

and in the other direction by

$$\sigma \;=\; \begin{array}{l}\text{the cone spanned by the 1-parameter subgroups } \phi : \mathbb{G}_m \to T \\ \text{such that “the limit } \lim_{z\to 0}\phi(z) \text{ exists in } X\text{”, that is, } \phi \text{ extends} \\ \text{to a morphism } \mathbb{A}^1 \to X.\end{array}$$

There is another, less well known characterization of σ, which is less dependent on the torus action, and is therefore useful for toroidal embeddings:

Any monomial $m \in M$ defines a Cartier divisor $\mathrm{Div}(x^m)$ supported on $X \smallsetminus T$. If $\sigma^\vee$ contains a line through the origin, then for any m on this line the divisor is easily seen to be trivial (both m and $-m$ give regular functions).

We use the following notation:

$\sigma^\perp = \{m \in M_\mathbb{R} | \langle m, \sigma \rangle = 0\}$

$M^\sigma =$ Cartier divisors supported on $X \smallsetminus T$.

One can easily see that $M^\sigma = M/\sigma^\perp \cap M$.

$N_\sigma = span(\sigma)$. Clearly $N_\sigma = \mathrm{Hom}(M^\sigma, \mathbb{Z})$.

Let $M^\sigma_+ \subset M^\sigma$: the *effective* Cartier divisors.

We have that $M^\sigma_+ = \sigma^\vee \cap M/\sigma^\perp \cap M$.

It is not hard to see that $\sigma = (M^\sigma_+)^\vee_\mathbb{R}$, the dual cone of the cone spanned by M^σ_+.

In short: σ is the dual cone to the cone of effective Cartier divisors supported on $X \smallsetminus T$.

6.3 The toroidal picture

We wish to mimic the construction of the cone in the toroidal case. We follow [38], Chapter II.

Let $U \subset X$ be a strict toroidal embedding. $X \smallsetminus U = \cup D_i$, where D_i normal.

We decompose $\cap_{i \in I} D_i = \cup X_\alpha$; the locally closed subsets X_α are called *strata*.

Each stratum has its star: $\mathrm{Star}(X_\alpha) = \cup_{X_\alpha \subset \overline{X_\beta}} X_\beta$.

Note: X_α is the unique closed stratum in $\mathrm{Star}(X_\alpha)$. In a sense it is analogous to the unique closed orbit in an affine toric variety.

Define:

$M^\alpha =$ group of Cartier divisors supported on $\mathrm{Star}(X_\alpha) \smallsetminus U$;

$M^\alpha_+ =$ subset of effective Cartier divisors;

$N_\alpha = \mathrm{Hom}(M_\alpha, \mathbb{Z})$;

$\sigma_\alpha = (M^\alpha_+)^\vee_\mathbb{R}$.

Thus, to each stratum we associated a strictly convex rational polyhedral cone.

Remark. The cone σ_α has a description analogous to the toric one using 1-parameter subgroups, in terms of valuations. Let $RS(X)$ be the discrete valuations on X. Let v be a valuation centered in $\mathrm{Star}(X_\alpha)$. Let f_j be rational function defining generators of M_α on a small affine open. Then $v(f_j)$ is a vector in σ_α, and in fact σ_α can be described as a set of equivalence classes of discrete valuations centered in $\mathrm{Star}(X_\alpha)$, the equivalence being defined by equality of the valuations of these functions f_j.

6.4 Birational affine morphisms

Recall: if $\tau \subset \sigma$ are two strictly convex rational polyhedral cones, then $\tau^\vee \supset \sigma^\vee$ gives rise to a morphism $V_\tau \to V_\sigma$, which is birational and affine.

Note that $V_\tau \to V_\sigma$ can be described in the following invariant manner:

$$V_\tau = \mathrm{Spec}_{V_\sigma} \sum_{E \in M_+^\tau} \mathcal{O}_{V_\sigma}(-E),$$

where the sum is taken inside the field of rational functions of V_σ.

This clearly works over $\mathrm{Star}(X_\alpha)$ in the toroidal case as well.

6.5 Principal affine opens

If $m \in \sigma^\vee$ then $\tau = \{n \in \sigma | < n, m >= 0\}$ is a face of σ. We have $\tau^\vee = \sigma^\vee + \mathbb{R}\cdot m$, and therefore V_τ is the principal open set on V_σ obtained by inverting the monomial x^m.

Again, this can be described divisorially in terms of $\mathrm{Div}(m)$. Thus the same is true for $\mathrm{Star}(X_\alpha)$: given a face τ of σ_α, we get an open set

$$\mathrm{Star}(X_\beta) \subset \mathrm{Star}(X_\alpha)$$

such that $\tau = \sigma_\beta$.

The most important face of a cone is the vertex. It corresponds to the open set $T \subset V$. In the toroidal case you get U.

6.6 Fans and polyhedral complexes

Recall: if σ_1 and σ_2 intersect along a common face τ, then V_{σ_1} and V_{σ_2} can be glued together along the common open set V_τ, forming a new toric variety.

In general, whenever you have a *fan* Σ in N, namely a collection of cones σ_i intersecting along faces, you can glue together the V_{σ_i} and get a toric variety V_Σ.

It is not hard to see that *every* toric variety is obtained in this way in a unique manner. The point is that every toric variety is covered by affine open toric varieties.

In the toroidal case, X is covered by the open sets $\{\mathrm{Star}(X_\alpha)\}_\alpha$.

In general

$$\mathrm{Star}(X_\alpha) \cap \mathrm{Star}(X_\beta) = \cup\, \mathrm{Star}(X_{\gamma_i}),$$

so σ_{γ_i} are possibly several faces of both σ_α and σ_β.

Still these can be glued together, as a rational conical polyhedral complex. The main difference from the toric case, is that it is abstractly defined, and in general it is not linearly contained in some vector space $N_\mathbb{R}$.

6.7 Modifications and subdivisions

Let Σ be a fan, and $\Sigma' \to \Sigma$ a (complete) subdivision. This corresponds to a toric modification $V_{\Sigma'} \to V_\Sigma$.

Since the construction is local (the Spec construction, as in Section 6.4, and gluing) it works word for word in the toroidal case. There is a small issue in checking that the resulting modification is still a strict toroidal embedding; this is discussed in detail in [38].

In [38] (see also [15]) it is shown that a modification is *projective* if and only if the subdivision is induced by a *support function* - one associates to a support function an ideal, whose blowup gives the modification. This works in the toroidal case as well.

6.8 Nonsingularity

Recall: an affine toric variety V_σ is nonsingular if and only if σ is simplicial, generated by a basis of N_σ (namely, part of a basis of N). Such a cone is called nonsingular.

In general: a toric variety V_Σ is nonsingular if and only if every cone $\sigma \in \Sigma$ is nonsingular.

This is a local fact, so it is true in the toroidal case as well.

6.9 Desingularization

Recall that it is easy to resolve singularities of a toric variety: one finds a simplicial subdivision such that every cone is nonsingular.

Obviously, the same works in toroidal case! We obtained:

Theorem 6.1. *For any toroidal embedding $U \subset X$ there is a projective toroidal modification $U \subset X' \to X$ such that X' is nonsingular.*

See [38], Theorem 11*, page 94.

6.10 Exercises on toric varieties and toroidal embeddings

1. Show that $\mathbb{G}_m^n \subset \mathbb{A}^n$ is a toric variety. Describe its cone.

2. Show that $\mathbb{G}_m^n \subset \mathbb{P}^n$ is a toric variety. Describe its fan.

3. Let $X \subset \mathbb{A}^n$ be a normal variety defined by monic monomial equations of type
$$\prod x_j^{n_j} = \prod x_j^{m_j}.$$
Show that X is toric. (Identify the torus!)

4. Do the same if the monomial equations are not necessarily with coefficients $= 1$.

5. Describe the cone associated to the affine toric variety defined by
$$xy = t_1^{k_1} \cdots t_r^{k_r}.$$

6. Look at the affine 3-fold $xy = zw$. Let $X' \to X$ be the blowup of X at the ideal (x, z). Describe this blowup, show that it is toric, and describe the cone subdivision associated to it.
7. Let $X = \mathbb{A}^2$, $D = \{xy(x+y-1) = 0\}$, $U = X \smallsetminus D$. Show that $U \subset X$ is a toroidal embedding. Describe its conical polyhedral complex. (Compare with the fan of $\mathbb{P}^2$!)
8. Do the same for $D = \{y(x^2+y^2-1) = 0\}$. Show that the resulting complex can not be linearly embedded in a vector space.
9. Consider the surface $X = \{z^2 = xy\}$, $U = \{z \neq 0\}$. Show that $U \subset X$ is toric and describe its cone.
10. Consider the surface X above, let $D_1 = \{x = 0\}$, $D_2 = \{y = x(x-1)^2 \text{ and } z = x(x-1)\}$. Let $U = X \smallsetminus (D_1 \cup D_2)$. Show that $U \subset X$ is toroidal. Describe its conical polyhedral complex. Make sure to describe the integral structure!

6.11 Abhyankar's lemma in toroidal terms

Abhyankar's lemma about fundamental groups (see [25], [26]) describes the local tame fundamental group of a variety around a normal crossings divisor. Let $X = \operatorname{Spec} k[[t_1, \ldots, t_n]]$ and let $D = V(t_1 \cdots t_n)$. Let $Y \to X$ be a finite alteration which is tamely branched along D, and étale away from D. For m prime to char k, denote $X_m = \operatorname{Spec} k[[t_1^{1/m}, \ldots, t_n^{1/m}]]$. Abhyankar's lemma says that the normalization of $Y \times_X X_m$ is étale over X_m.

In the following exercises we interpret this in toroidal terms.

Exercise 6.2. Let $U \subset X$ be a *nonsingular* strict toroidal embedding. Let $f : Y \to X$ be a finite cover, which is tame, and étale over U. Then $f^{-1}U \subset Y$ is a strict toroidal embedding.

Exercise 6.3. Suppose further that $Y \to X$ is Galois, with Galois group G. Show that G acts toroidally on Y.

7 Weak resolution of singularities I

Given the existence of toroidal resolution, the proof of weak resolution of singularities in characteristic 0 by Bogomolov and Pantev is arguably the simplest available. It does not even require surface resolution.

We will go through this proof. The steps of proof here include some simplifications on the arguments in [12], which came up in discussions with T. Pantev. These and additional simplifications were discovered independently by K. Paranjape [53], and we have used his exposition in some of the following exercises. The version given in [53] has the advantage that it does not even require moduli spaces.

7.1 Projection

Let X be a variety over an algebraically closed field of characteristic 0, and $Z \subset X$ a proper closed subset. Let $n = \dim X$, and again assume we know the weak resolution theorem for varieties of dimension $n-1$.

First a few reduction steps:

1. Show, as in 4.1 that we may assume X projective and normal, and Z the support of a Cartier divisor.

2. Show that there is a finite projection $X \to \mathbb{P}^n$.

3. Let $P \to \mathbb{P}^n$ be the blowup at a closed point. Show that

$$P \simeq \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)).$$

   Denote by E the exceptional divisor of $P \to \mathbb{P}^n$.

4. By blowing up a general point on $\mathbb{P}^n$, and blowing up X at the points above, show that we may assume we have a finite morphism $f : X \to P$, which is étale along E, such that the image of Z is disjoint from E, and maps finitely to $\mathbb{P}^{n-1}$.

5. By the Nagata - Zariski purity theorem, note that the branch locus of $X \to P$ is a divisor B, disjoint from E, mapping finitely to $\mathbb{P}^{n-1}$.

We replace Z by $Z \cup f^{-1}B$.

7.2 Vector bundles

The next steps are aimed at replacing P by another $\mathbb{P}^1$-bundle $Q \to \mathbb{P}^{n-1}$, such that the branch locus in Q of $X \to Q$ becomes simpler. Let Y be any variety, F a rank-2 vector bundle on Y, $P = \mathbb{P}_Y(E)$. Let $E \subset P$ be a divisor which is a section of $\pi : P \to Y$ and let $D \subset P$ be another effective divisor disjoint from E. Denote by $\mathcal{O}_P(1)$ the tautological bundle, and by d the relative degree of D over Y.

1. Consider the exact sequence

$$0 \to \mathcal{I}_D(d) \to \mathcal{O}_P(d) \to \mathcal{O}_P(d)|_D \to 0.$$

   Use this to show that there is an invertible sheaf $\mathcal{L}_D$ on Y such that $\mathcal{I}_D(d) \simeq \pi^*\mathcal{L}_D$.

2. If $D_1, D_2 \subset P$ are any two *disjoint* divisors finite of degree d over Y, show that there is an embedding of vector bundles $\mathcal{L}_{D_1} \oplus \mathcal{L}_{D_2} \subset sym^d F$ inducing a *surjection* $\pi^*(\mathcal{L}_{D_1} \oplus \mathcal{L}_{D_2}) \to \mathcal{O}_P(d)$.

3. Assume the characteristic is 0. Consider the case $D_1 = dE, D_2 = D$. show that the resulting morphism $P \to \mathbb{P}_Y(\mathcal{L}_{D_1} \oplus \mathcal{L}_{D_2}) = P'$ maps E to a section E' and D to a disjoint section; and its branch locus is of the form $(d-1)E+D'$ where D' has degree $d-1$ over Y and is disjoint from E'.

4. Continue by induction to show that there is a $\mathbb{P}^1$ bundle $Q \to Y$ and a morphism $g : P \to Q$ over Y, such that the image of D and the branch locus of g form a union of sections of $\pi_Q : Q \to Y$.

7.3 Conclusion of the proof

Back to our theorem, where $Y = \mathbb{P}^{n-1}$. Composing with the morphism $f : X \to P$, we obtain that the image $g(f(Z)) \subset Q$ is the image D_1 of a section $s_1 : \mathbb{P}^{n-1} \to Q$ of $Q \to \mathbb{P}^{n-1}$ and the branch locus of $g \circ f$ is the union of images D_i sections $s_i : \mathbb{P}^{n-1} \to Q$ as well. Denote $\Delta = \pi_Q(\cup_{i \neq j} D_i \cap D_j)$.

The following steps use moduli theory; however it has been shown (in the preprint version of [12], and in Paranjape's exposition [53]) that the use of moduli theory can be circumvented within a few pages of work.

1. * Use the stable reduction argument to show that there is a *modification* $Y' \to Y$, and a modification $Q' \to Q \times_Y Y'$ such that $Q' \to Y$ is a family of nodal curves of genus 0, and the sections lift to *disjoint* sections $s_i' : Y' \to Q'$. We replace Y by Y', Δ by its inverse image, etc.

   Hints. The point is that the generic fiber of $Q \to Y$ is a projective line with a number (say k) of points marked by the sections we obtained. This gives a rational map $Y \dashrightarrow \overline{M_{0,k}}$, which can be replaced by a morphism after a modification $Y' \to Y$.

   Since the moduli schemes in genus 0 are *fine* moduli schemes, there is a family of pointed rational curves $Q' \to Y'$. one would like to use the Three Point Lemma to get a morphism $Q' \to Q$. However, the argument above only guarantees that every fiber of Q has two marked points, and not necessarily three. This is easy to correct by adding sections on the $\mathbb{P}^1$-bundle $Q \to Y$ before applying the moduli argument.

   Another approach is to use Knudsen's stabilization method directly. The details of this can be found in [12].

2. Use induction on the dimension to replace Y by a nonsingular variety such that Δ becomes a strict divisor of normal crossings.

3. Use either toroidal geometry, or Section 4.11, to replace Q by a nonsingular variety, such that the inverse image of D is a strict normal crossings divisor. (Note that at this point $Q \to Y$ is a family of nodal pointed curves, degenerating over the divisor of normal crossings D.)

4. Let $\tilde{X}$ be the normalization of Q in the function field of X. Use Abhyankar's lemma (Section 6.11) to show that $\tilde{X}$ has a toroidal structure, such that the inverse image $\tilde{Z}$ of Z is a toroidal divisor.

5. Conclude that there is a weak resolution of singularities $r : X' \to X$ such that $r^{-1}Z$ is a strict divisor of normal crossings.

8 Weak resolution of singularities II

The weak resolution argument according to Abramovich - de Jong starts very much like de Jong's theorem: a projection $X \to P$ is produced, and a Galois alteration $P_1 \to P$ over which one has stable reduction $X_1 \to P_1$, equivariant under the Galois group G, is produced. Induction on the dimension for P allows one to assume that X_1 and P_1 are toroidal, and the Galois action on P_1 is toroidal as well. The only point left is to make the group action on X_1 toroidal, so that the quotient should be toroidal, and therefore admit toroidal resolution.

Let us go through the steps. Let X be a variety over an algebraically closed field of characteristic 0, and let $Z \subset X$ be a Zariski-closed subset. We want to find a nonsingular, quasi-projective variety X' and a modification $f : X' \to X$ such that $f^{-1}Z$ is a divisor with simple normal crossings.

8.1 Reduction steps

Exercise 8.1. Show that it is enough to prove the result when X is projective and normal, and Z a Cartier divisor.

Exercise 8.2. Reduce to the case where there is a projection $X \to P$, such that the generic fiber is a smooth curve.

Exercise 8.3. *Using the trick of enlarging Z and stable reduction, show that there is a diagram as follows:

$$\begin{array}{ccc} X_1 & \to & X \\ \downarrow & & \downarrow \\ P_1 & \to & P \end{array}$$

such that $P_1 \to P$ is an alteration, $X_1 \to \tilde{X}_{P_1}$ is birational, and $X_1 \to P_1$ has section $s_i : P_1 \to X_1$ making it a family of stable pointed curves, and the image of these sections in X contains Z.

Exercise 8.4. Show that you can make $P_1 \to P$ a Galois alteration. Call the Galois group G. Show, using the uniqueness of stable reduction3.7, that the action of G on P_1 lifts to an action on X_1, which permutes the sections s_i.

You can replace X by X_1/G and P by P_1/G

Exercise 8.5. Use induction on the dimension to reduce to the case where:

1. P is nonsingular, with a normal crossings divisor Δ;
2. The branch locus of $P_1 \to P$ is contained in δ;
3. The locus where $X_1 \to P$ is not smooth is contained in δ.

Exercise 8.6. Show that in this case $P_1 \to P$ is a toroidal morphism, G acts toroidally on X, and $X_1 \to P_1$ is a toroidal morphism as well.

The only point left is to make the action of G on X_1 toroidal - if it were, then X would be toroidal and we could easily resolve its singularities.

Looking locally, the question boils down to the following situation:

Let $T_0 \subset X_0$ be an affine torus embedding, $X_0 = \text{Spec } R$. Let $G \subset T_0$ be a finite subgroup of T_0, let $p_0 \in X_0$ be a fixed point of the action of G, and let ψ_u be a character of G. Consider the torus embedding of $T = T_0 \times \text{Spec } k[u, u^{-1}]$ into $X = X_0 \times \text{Spec } k[u]$, and let G act on u via the character ψ_u. Write $p = (p_0, 0) \in X$ and write $D = (X_0 \smallsetminus T_0) \times \text{Spec } k[u]$. We wish to find a canonical blowup $X_1 \to X$, such that if $U \subset X_1$ is the inverse image of T_0, then it is a toroidal embedding, and the group G acts toroidally.

8.2 The ideal

Let $M \subset R[u]$ be the set of monomials. For each $t \in M$ let χ_t be the associated character of T, and let let $\psi_t : G \to k^*$ be the restriction of χ to G. Define $M_u = \{t | \psi_t = \psi_u\}$, the set of monomials on which G acts as it acts on u. Define $I_G = \langle M_u \rangle$, the ideal generated by M_u.

Exercise 8.7 (canonicity). * Show that if X_0', T_0', G', p_0' and ψ_u' is a second set of such data, and if we have an isomorphism of formal completions

$$\varphi : \hat{X}_p \tilde{\to} \hat{X'}_{p'},$$

which induces isomorphisms $G \cong G'$ and $\hat{D}_p \cong \hat{D'}_{p'}$, then φ pulls back I_G to the ideal $I_{G'}$.

Exercise 8.8 (gluing property). * If q_0 is any point of X_0 and if $G_q \subset G$ is the stabilizer of $q = (q_0, 0)$ in G, then the stalk of I_G at q is the same as the stalk of I_{G_q} at q.

Exercise 8.9. Show that I_G is generated by u and a finite number of monomials $t_1, \ldots, t_m$ in $M_u \cap R$.

Exercise 8.10. Let $X' = B_{I_G}(X)$ be the blowup. Let X_u' be the chart with coordinates $u, t_j/u$. Show that the action of G on X_u' is toroidal.

Exercise 8.11. Let X'_i be the chart on X' with coordinates $t_i, v = u/t_i, s_j = t_j/t_i$. Show that G acts trivially on v, and that $X'_i = \text{Spec } R'_i[v]$ where R'_i is generated over R by s_j.

Exercise 8.12. Let X_1 be the normalization of X'. Show that if $U \subset X_1$ is the inverse image of T_0, then it is a toroidal embedding, and the group G acts toroidally.

9 Intersection multiplicities

Intersection theory has a long history, and certainly we are not going to say much about it here. One aspect is, that it is not so easy to have a good definition for intersection multiplicities.
Remark, exercise: let $C \subset \mathbb{P}^2_k$ be a plane algebraic curve, $P \in C$ a closed point at which C is regular, and $D = \mathcal{Z}(F) \subset \mathbb{P}^2$ a plane curve given by a homogeneous polynomial F; suppose F is not identical zero on a neighborhood of P in C (i.e. no component of D contains the component of C containing P). Show that the following two definitions of *the intersection multiplicity* $i(C, D; P)$ *of* C *and* D *at* P are equivalent:

- the dimension of the k-vector space

$$\mathcal{O}_{C,P} \quad \otimes_{\mathcal{O}_{\mathbb{P}^2,P}} \quad \mathcal{O}_{D,P},$$

- the value of the valuation $v = v_{C,P}$ defined by the discrete valuation ring $\mathcal{O}_{C,P}$ computed on the function on C given by F,

see [HAG], Exercise (5.4) on page 36, and Remark (7.8.1) on page 54.

Consider two varieties $V, W \subset \mathbb{P}^n$ which have an isolated point of intersection at $P \in V \cap W$. One could try to define the intersection of V and W at P as the length of

$$\mathcal{O}_{V,P} \quad \otimes_{\mathcal{O}_{\mathbb{P}^n,P}} \quad \mathcal{O}_{W,P}.$$

Analogous situations of intersections of arbitrary schemes in some regular ambient scheme can be considered.

Exercise 9.1. (See Gröbner [23], 144.10/11, also see [66], [62], [10], see [HAG], I.7): **a)** Let $C \subset \mathbb{P}^3$ be the space curve with parameterization

$$(x_1 \ : \ x_2 \ : \ x_3 \ : \ x_4) = (t^4 \ : \ t^3{\cdot}s \ : \ t{\cdot}s^3 \ : \ s^4)$$

(we work over some field K). Show that the prime ideal given by this curve equals

$$j := (T_1^2T_3 - T_2^3 \ , \ T_1T_4 - T_2T_3 \ , \ T_1T_3^2 - T_2^2T_4 \ , \ T_2T_4^2 - T_3^3) \subset K[T_1, T_2, T_3, T_4].$$

b) Consider C as a curve embedded in $\mathbb{P}^4$: choose the hyper plane $\mathbb{P}^3 \cong \mathcal{Z}(T_0) = H$, and we get $C \subset H \subset \mathbb{P}^4$. Let $P := (x_0 = 1:0:0:0:0) \in \mathbb{P}^4$. Define $V \subset \mathbb{P}^4$ as the cone with vertex P over the curve $C \subset \mathbb{P}^4$, i.e. V is defined by the ideal

$$J := K[T_0, T_1, T_2, T_3, T_4] \cdot j, \quad V = \mathcal{Z}(J).$$

Note that the dimension of V equals two, that the degree of $V \subset \mathbb{P}^4$ equals four.
c) Let W be the 2-plane given by

$$I := (T_1, T_4) \subset K[T_0, T_1, T_2, T_3, T_4], \quad W := \mathcal{Z}(I).$$

Note that $P \in W$. Remark that (set-theoretically):

$$W \cap V = \{P\}$$

(use the geometric situation, or give an algebraic computation).
We like to have a Bézout type of theorem for the situation $W \cap V \subset \mathbb{P}^4$, however:
d) Define

$$M := \mathcal{O}_{W,P}, \quad A := \mathcal{O}_{\mathbb{P}^4,P}, \quad N := \mathcal{O}_{V,P},$$

and compute

$$\dim_K (M \otimes_A N)$$

(surprise: this is not equal to four).
e) Compute

$$\dim_K \left(\mathrm{Tor}_i^A(M,N)\right), \quad \forall i$$

(either using, or reproving $\chi_A(M,N) = 4$, for notation see below).

Hence we see that just the length of the appropriate tensor product does not define necessarily the correct concept. Serre proposed in 1957/58 to define the intersection multiplicity as the alternating sum of the lengths of the Tor_i (note that $\mathrm{Tor}_0 = \otimes$), i.e. by

$$\chi_A(M,N) := \sum_{i \geq 0} (-1)^i \ \mathrm{length}_A \mathrm{Tor}_i^A(M,N)$$

(we follow notation of [66], also see [10], 6.1, see [62]), here A is a regular local ring, and M and N are A-modules such that $M \otimes_A N$ has finite length. In equal characteristic this is the right geometric concept (i.e. satisfies Bézout's theorem, coincides with previously defined intersection multiplicities etc.).

The following theorem was conjectured by Serre, proved by Gabber (using de Jong's alteration result), and written up by Berthelot (in [10], 6.1):

Theorem 9.2. *Let the characteristic of A be equal to zero. Suppose $p \in \mathfrak{m}^2$, hence its residue field $A/\mathfrak{m}$ has characteristic $p > 0$. Then:*

$$\chi_A(M,N) \geq 0.$$

Part II
Moduli of curves

10 Introduction to moduli of curves

It is an important feature of algebraic geometry, that the set of all objects (e.g. smooth projective curves) of the same a fixed geometric nature (e.g. genus) often has the structure of an algebraic variety itself. Such a space is a "moduli space", which gives a good algebraic meaning to the problem of "classification". It is fair to say that this "self referential" nature of algebraic geometry is one of the main reasons for the depth of the subject - it is impossible to overestimate its importance.

The first instances of this phenomenon to be discovered were those of *embedded* variety: the projective space as a parameter space for lines in a vector space; Grassmannians parametrizing vector subspaces of arbitrary dimension; the projective space (of dimension $(d^2+3d)/2$) parametrizing all plane curves of degree d, and so on. The case of abstract varieties, such as smooth curves, had to await for some technical advances, although already Riemann knew that algebraic curves of genus g "vary in $3g-3$ parameters"; see [61], page 124:

> "Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter μ-werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $(2p+1)$-fach zusammenhängender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen."

Historically moduli spaces of curves, or of curves with points on them, were constructed with more or less ad hoc methods. Moduli spaces for 3 or 4 points on rational curves have been known for ages, using the so called "cross ratio" (see exercise 10.9 below). For genus 1, the modular function j was used (see exercise 10.12). The case of genus 2 was already quite difficult to achieve by algebraic methods [30]. For years, moduli spaces for higher genus were only known to exist using Teichmüller theory.

One problem which took years to solve was, that no good understanding of "what moduli spaces really are" was available. Then Grothendieck introduced the notion of "representable functor", describing the best possible meaning for moduli spaces. This had a great success with the development of Hilbert schemes. For a while one hoped that nature would be as ideal as expected (see Grothendieck hopeful Conjecture 8.1 in [24] 212-18, and its retraction in the Additif of [24] 221-28). But it was soon seen that in general moduli functors are not representable, or as we say now, some moduli functors do not give rise to "fine moduli spaces" due to existence of automorphisms. Finally, Mumford pinned down the compromise notion of a "coarse moduli scheme", which enables us to have a good insight in various aspects of moduli theory. This is what we shall try to describe here. It should be said that since then, other good approaches were developed, by way of "enlarging the category of schemes" to include some "moduli objects", called stacks. For details see [17], Section 4. We will not pursue this direction here.

In this section we gather some basic definitions on functors of moduli for curves. In the next sections we discuss existence theorems for moduli spaces of curves, and for complete moduli spaces with extra structure carrying a "tautological family". Section 14 is devoted to some further questions, examples and facts, not needed for the methods of alterations, but in order to give a more complete picture of this topic.

10.1 The functor of points and representability

To any scheme M one naturally associates a contravariant functor

$$\mathcal{F}_M : \{\text{Schemes}\} \to \{\text{Sets}\}$$

via

$$X \mapsto \text{Mor}(X, M).$$

This is known as *the functor of points of* M, see [Red Book], II §6.

We say that a contravariant functor $\mathcal{F} : \{\text{Schemes}\} \to \{\text{Sets}\}$ is *representable* by a scheme M, if it is isomorphic to $\mathcal{F}_M$, i.e. there is a functorial isomorphism

$$\xi : \mathcal{F}(-) \xrightarrow{\sim} \text{Mor}(-, M).$$

Remark. Strictly speaking, it is the *pair* (M, ξ), consisting of the object M *and* the isomorphism ξ, which represent $\mathcal{F}$. But it has become customary to say "M represents $\mathcal{F}$", suppressing ξ.

Already in the early Bourbaki literature one finds this notion in the disguise of a "universal property". The question of representability of functors can also be posed in categories other than the category of schemes.

Exercise 10.1.
1. Fix an integer N, and let V be a vector space of dimension $N+1$ over $\mathbb{C}$. Consider the functor $\mathcal{F}_{\mathbb{C}}$ that associates to a scheme T over $\mathbb{C}$, the set $\{\mathcal{L} \subset T \times V\}$ of all line sub-bundles of the trivial vector bundle $T \times V$. Show that $\mathcal{F}_{\mathbb{C}}$ is represented by $\mathbb{P}^N_{\mathbb{C}}$.
2. Let $\mathcal{F}$ be the functor that associates to any scheme T (over $\mathbb{Z}$) the set $\{\mathcal{L} \subset \mathcal{O}_T^{N+1}\}$ of all locally free subsheaves of rank 1 of the trivial sheaf $\mathcal{O}_T^{N+1}$ having locally free quotient. Show that $\mathcal{F}$ is represented by the projective scheme $\mathbb{P}^N_{\mathbb{Z}}$.
3. In general, show that the Grassmannian scheme $\mathbf{Grass}(n, r)$ represents the functor of locally free subsheaves of rank r of the trivial free sheaf $\mathcal{O}_T^n$ of rank n having locally free quotients.

Exercise 10.2. Fix integers N and d, and let $\mathcal{G}$ be the functor that associates to a scheme T the set $\{\mathcal{X} \subset \mathbb{P}^N_T\}$ of all flat families of hypersurfaces of degree d in projective N-space over T. Show that $\mathcal{G}$ is represented (over $\mathbb{Z}$) by a projective space $\mathbb{P}^{M-1}$, where $M = \binom{N+d}{d}$ is the dimension of the space of homogeneous polynomials of degree d in $N+1$ variables.

Exercise 10.3. Show that the Hilbert scheme $\mathcal{H}_{\mathbb{P}^N, P(T)}$ represents the "Hilbert" functor, that associates to a scheme T, the set of all subschemes $X \subset \mathbb{P}^N_T$, which are flat over T and such that the geometric fibers have Hilbert polynomial $P(T)$.

10.2 Moduli functors and fine moduli schemes

Suppose a contravariant functor $\mathcal{F}$ has the nature of a *moduli functor*, namely, it assigns to a scheme S the set $\{C \to S\}/\cong$ of isomorphism classes of certain families of objects over S. As a guiding example, let us fix an integer g, with $g \in \mathbb{Z}_{\geq 0}$, and define the moduli functor for smooth curves:

$$\mathcal{M}_g(S) = \{\text{isom. classes of families of curves of genus } g \text{ over } S\}.$$

A morphism $T \to S$ defines (by pulling back families) a map of sets in the opposite direction: $\mathcal{M}_g(T) \leftarrow \mathcal{M}_g(S)$, and we have obtained a contravariant functor.

Assume the functor $\mathcal{F}$ were represented by a scheme M. Then we would call M a *fine moduli scheme* for this functor $\mathcal{F}$, and the object

$$\mathcal{C} \to M \quad \text{corresponding to the identity} \quad id \in= \mathrm{Mor}(M, M)$$

would be called a *universal family.*

Remark. Note that in the exercise above on the Hilbert scheme, we can view it as a fine moduli scheme, if we agree that "families up to isomorphism" means "up to isomorphisms as subfamilies of the fixed $\mathbb{P}^N_T$, namely up to equality.

It is a fact of life that *for every $g \geq 0$ the functor $\mathcal{M}_g$ is not representable.* We will explain later why this is true in general, but for the moment let us consider the easiest case:

Exercise 10.4. Let us say that C is a "curve of genus 0", if it is an algebraic curve defined over a field K, and over some extension of $K \subset L$ it is isomorphic with $\mathbb{P}^1_L$. In other words: C is geometrically irreducible, reduced, it is complete and of genus equal to zero.

1. Let K be a field. Show there exist an extension $K \subset K'$, and two curves of genus 0 over K' which are not isomorphic.

2. For every algebraically closed field k, the set $\mathcal{M}_0(k)$ consists of one element, $\mathcal{M}_0(k) = \{\mathbb{P}^1_k\}$.

3. Show that the moduli functor $\mathcal{M}_0$ is not representable.

10.3 Historical interlude

The first case of a highly nontrivial algebraic construction of a moduli space of curves in all charactersitics, appeared in Igusa's work [30]. This is a construction of a "moduli scheme for non-singular curves of genus two in all characteristics", which would now be denoted by $M_2 \to \mathrm{Spec}(\mathbb{Z})$. This happened almost concurrently with Grothendieck's study of representability of functors. But notice that, when Samuel discussed these beautiful results by Igusa in Séminaire Bourbaki (see [63]), his very first comment was:

> *"Signalons ausitôt que le travail d'* IGUSA *ne résoud pas pour les courbes de genre 2, le "problème des modules" tel qu'il a été posé par* GROTHENDIECK *à diverses reprises dans ce Séminaire."*

It really seemed that Nature was working against algebraic geometers, refusing to provide us with these fine moduli schemes...

The truth is, Nature does provide us with a replacement. Indeed, not much later, Mumford (see [GIT], 5.2) discovered how to follow nature's dictations and come to a good working definition, requiring that the scheme gives geometrically what you want, and does it in the best possible way.

10.4 Coarse moduli schemes

Here is the definition:

Definition. A scheme M and a morphism of functors

$$\varphi : F \to \mathrm{Mor}_S(-, M)$$

is called a *coarse moduli scheme* for F if:

1. for every algebraically closed field k the map

   $$\varphi(k) : F(\mathrm{Spec}(k)) \to \mathrm{Mor}_S(\mathrm{Spec}(k), M) = M(k)$$

   is bijective, and

2. for any scheme N and any morphism $\psi : F(-) \to \mathrm{Mor}_S(-, N)$ there is a unique $\chi : M \to N$ factoring ψ.

By definition, a coarse moduli scheme does not carry a universal family, *unless* it is a fine moduli scheme. A replacement, called a tautological family, is defined as follows:

Definition. Let $\mathcal{F}$ be a moduli functor. Suppose T is a scheme, and let $f : T \to M$ be a morphism. A family $\mathcal{C} \to T$ giving an element of $\mathcal{F}(T)$, is called a *tautological family* if it defines f, namely $\psi(\mathcal{C} \to T) = f$. In particular this implies that for every geometric point $t \in T$ the fiber $\mathcal{C}_t$ is an object whose isomorphism class defines the image under f, i.e.: $[\mathcal{C}_t] = f(t)$.

Remark. There exist cases (and we shall give examples), where a moduli functor is not representable, where there is no (unique) universal family, but where a tautological family does exist. In such cases the use of the word "tautological", and the distinction between "universal" and "tautological" is necessary, and it pins down the differences.

The terminology "tautological" will also be used in cases such as pointed curves, curves with a level structure, and so on.

Here is the first triumphant success of the notion of coarse moduli scheme:

Theorem 10.5 (Mumford). *Suppose $g \geq 2$. The functor $\mathcal{M}_g$ of smooth curves of genus g admits a quasi-projective coarse moduli scheme.*

See [GIT], Th. 5.11 and Section 7.4, or [17], Coroll. 7.14. We will denote the coarse moduli scheme of $\mathcal{M}_g$ by $M_g \to \operatorname{Spec} \mathbb{Z}$.
We note some properties of M_g:

- As we mentioned before, for every $g \geq 2$ the functor $\mathcal{M}_g$ is not representable: there does not exist a universal family of curves over M_g which can give an isomorphism between $\mathcal{M}_g$ and M_g.
- For every $g \geq 2$ and for any field K, the variety $(M_g)_K = M_g \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} K$ is not complete. A fortiori, the morphism $M_g \to \operatorname{Spec}(\mathbb{Z})$ is not proper.
- At least for the sake of de Jong's theorem, we need a moduli space of curves with points on them.

The first problem is solved by introducing a finite covering $M \to M_g$ admitting a *tautological family*, namely a family realizing the morphism $M \to M_g$ as its moduli morphism. The nicest way of doing this is by introducing a new moduli functor, of smooth curves "enriched" with a finite amount of "extra structure", which does admit a fine moduli scheme. See Section 13.

In order to "compactify" these spaces, the notion of *stable curves* was invented. Historically, the influential paper [17] by Deligne and Mumford seems to be one of the first printed versions in which the concept of stability, especially in the case of algebraic curves is explained and used. In [46] we see that already in 1964 Mumford was trying to find the appropriate notions assuring good compactifications. In [GIT], page 228, Mumford attributes the notion of a stable curve to unpublished joint work with Alan Mayer.

As it turns out, the third problem was solved almost concurrently with the second.

First, the moduli space of smooth pointed curves:

Theorem 10.6. *Let $g \in \mathbb{Z}_{\geq 0}$, and $n \in \mathbb{Z}_{\geq 0}$ such that $2g-2+n > 0$. Consider the functor $\mathcal{M}_{g,n}$ of isomorphism classes of families of stable smooth n-pointed curves of genus g. This functor admits a quasi-projective coarse moduli scheme.*

We will denote this moduli space by $M_{g,n} \to \operatorname{Spec}(\mathbb{Z})$.

Remark. 1. Note that this includes the previous theorem.

2. For $g = 0$ and $n = 3$ this space is proper over $\mathrm{Spec}(\mathbb{Z})$. However in all other cases in the theorem $M_{g,n}$ is not proper. In many cases it will not represent the functor (see Section 14 for a further discussion), in other words, in general this is not a fine moduli scheme.

3. It is important to note that these spaces exist over $\mathrm{Spec}(\mathbb{Z})$, which is useful for arithmetical applications.

4. The litterature poses difficulties in choosing notations. In [GIT] the subscript n denotes a level structure, but in [39] it indicates the number of marked points. We have chosen to indicate the markings as lower index, using n, and the level structure as upper index, using (m).

Finally the moduli space of stable pointed curves:

Theorem 10.7 (Knudsen and Mumford). *Let $g \in \mathbb{Z}_{\geq 0}$, and $n \in \mathbb{Z}_{\geq 0}$ such that $2g - 2 + n > 0$. Consider the functor $\overline{\mathcal{M}_{g,n}}$ of isomorphism classes of families of stable n-pointed curves of genus g. This functor admits a* projective *coarse moduli scheme.*

See [39], part II, Theorem 2.7 and part III, Theorem 6.1, or [22], Theorem 2.0.2. We will denote this moduli scheme by $\overline{M_{g,n}} \to \mathrm{Spec}(\mathbb{Z})$.
The following exercise should give you an idea why the moduli space $\overline{\mathcal{M}_{0,n}}$ is complete. This is discussed in further detail in the next section.

Exercise 10.8. Let K be a field, and let $R \subset K$ be a discrete valuation ring having K as field of fractions. Consider the projective line $\mathbb{P}^1$ over K and suppose $n \geq 3$, let $P_1, \cdots, P_n \in \mathbb{P}^1(K)$ are distinct points. Write $P = \{P_1, \cdots, P_n\}$. Construct a stable n-pointed curve $(\mathcal{C}, \mathcal{P}) \to \mathrm{Spec}(R)$ extending $(\mathbb{P}^1, P)$. (You will need to blow up closed points over the special fiber where the Zariski closures of P_i meet. Then you may need to blow down some components! See [39].)

Exercise 10.9. 1. Let K be a field. Given three distinct finite points P_1, P_2 and P_3 on $\mathbb{P}^1_k$ consider the cross ratio

$$\lambda(P_1, P_2, P_3, z) = \frac{(z - P_1)(P_2 - P_3)}{(z - P_3)(P_2 - P_1)}.$$

Show that, as a function of z, the cross ratio is an automorphism of $\mathbb{P}^1$ carrying P_1, P_2, P_3 to $0, 1$, and ∞, respectively. Show that this automorphism is the unique one with this property. Check that this definition can be extended to the case where one of the points is ∞.

2. Using the cross ratio λ defined above, describe $M_{0,3}$.

3. Show that $M_{0,3}$ is a fine moduli scheme by exhibiting a universal family over it!

4. Show that $M_{0,3} = \overline{M_{0,3}}$.
5. Use the cross ratio to give an explicit description of $M_{0,4}$. Show that it is a fine moduli scheme by explicitly constructing a universal family.
6. Use the above (possibly together with the previous exercise) to describe $M_{0,4} \subset \overline{M_{0,4}}$.
7. Show that $\overline{M_{0,4}}$ is a fine moduli scheme, and give explicit descriptions of the universal family.
8. Show that the universal family over $\overline{M_{0,4}}$ is canonically isomorphic to $\overline{M_{0,5}}$.

Exercise 10.10. Give an alternative description of $\overline{M_{0,4}}$ as follows: consider the projective space $\mathbb{P}$ of dimension 5 parametrizing conics in $\mathbb{P}^2$. Choose four points in general position in $\mathbb{P}^2$ (for instance $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ will do). Let $M \subset \mathbb{P}$ be the subscheme parametrizing conics passing through these four points. Show that $M = \overline{M_{0,4}}$ and the universal family of conics is a universal family for $\overline{M_{0,4}}$.

Exercise 10.11.

1. Show that $M_{0,n}$ exists and is a fine moduli scheme (you may exhibit it as an open subscheme of $(\mathbb{P}^1)^{n-3}$).
2. Show that, assuming $\overline{M_{0,n}}$ is a fine moduli scheme, then there is a canonical morphism $\overline{M_{0,n+1}} \to \overline{M_{0,n}}$ which exhibits $\overline{M_{0,n+1}}$ as the universal family over $\overline{M_{0,n}}$.
3. * Show that for every $n \geq 3$, the scheme $\overline{M_{0,n}}$ is a fine moduli scheme. (You may want to use Knudsen's stabilization technique.)

Remark. For every $n \geq 3$, let (C, P) be a stable n-pointed rational curve. Then $\mathrm{Aut}(C, P) = \{id\}$. You do not need to know this in the previous exercise, but it "explains" why the result should be true.

Exercise 10.12. Let k be a field of characteristic $\neq 2$ and let (E, O) be an elliptic curve, namely a projective, smooth and connected curve E of genus 1 with a k-rational point O on it.

1. Considering the linear series of $\mathcal{O}_E(2O)$, show that E can be exhibited as a branch covering of $\mathbb{P}^1$ of degree 2.
2. Show that the branch divisor B on $\mathbb{P}^1$ is reduced and has degree 4.
3. If k is algebraically closed, show that E is determined up to isomorphism by the divisor B.
4. Conclude that $M_{1,1}$ is isomorphic to the quotient of $M_{0,4}$ by the action of the symmetric group S_4, permuting the four points.

5. Assume further that char $k \neq 3$, so that every elliptic curve can be written in affine coordinates as $y^2 = x^3 + ax + b$. Show that

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

is an invariant characterizing the $\bar{k}$-isomorphism class of E, exhibiting $M_{1,1} = \mathbb{A}^1$.

11 Stable reduction and completeness of moduli spaces

11.1 General theory

In order to understand the reason why $\overline{M_{g,n}}$ is projective, let us recall the following:

Theorem 11.1 (The valuative criterion for properness). *A morphism*

$$f : X \to Y$$

of finite type is proper, if and only if the following holds:

Let R be a discrete valuation ring, and let $S := \mathrm{Spec}(R)$ be the corresponding "germ of a non-singular curve", with generic point η. Let $\varphi : S \to Y$ and let $\psi_\eta : \eta \to X$ be a lifting:

$$\begin{array}{ccc} \eta & \stackrel{\psi_\eta}{\to} & X \\ \downarrow & & \downarrow \\ S & \stackrel{\varphi}{\to} & Y. \end{array}$$

Then there is an extension $\psi : S \to X$, lifting φ:

$$\begin{array}{ccc} \eta & \stackrel{\psi_\eta}{\to} & X \\ \downarrow & \stackrel{\psi}{\nearrow} & \downarrow \\ S & \stackrel{\varphi}{\to} & Y. \end{array}$$

See [HAG], II, Theorem (4.7) for a precise formulation.
Let us translate this to our moduli scheme. Keeping in mind the relationship between the functor $\overline{\mathcal{M}_{g,n}}$ and the space $\overline{M_{g,n}}$, one might hope that every family of stable pointed curves over η as in the theorem above might extend to R. This is not the case, as we shall see later. However, a weaker result, sometimes called "the *weak* valuative criterion for properness", does hold for the functor $\overline{\mathcal{M}_{g,n}}$, and it does imply the valuative criterion for $\overline{M_{g,n}}$. The first case to consider is when the generic fiber is *smooth* and $n = 0$. This is the content of the following result, the Stable Reduction Theorem for a one parameter family of curves:

Theorem 11.2. *Let $S = \operatorname{Spec}(R)$ be the spectrum of a discrete valuation ring R, $\eta \in S$ the generic point, corresponding with the field of fractions K of R. Let $C_\eta \to \eta$ be a smooth stable curve of genus $g > 1$. There exists a finite extension of discrete valuation rings $R \hookrightarrow R_1$, with $S_1 = \operatorname{Spec} R_1$ and generic point η_1, and an extension*

$$\begin{array}{ccc} C_{\eta_1} & \hookrightarrow & C_1 \\ \downarrow & & \downarrow \\ \{\eta_1\} & \hookrightarrow & S_1, \end{array}$$

such that $C_1 \to S_1$ is a family of stable curves.

Proofs of this theorem, using different methods, may be found in various references. One proof which works in pure characteristic 0 is relatively simple. As the reader will notice, none of the general proofs is easy or elementary.

Most proofs of this theorem use resolution of singularities of 2-dimensional schemes (Abhyankar).

Exercise 11.3. Suppose R is of pure characteristic 0. Let $s \in \operatorname{Spec} R$ be the closed point.

1. Show that there exists an extension $\pi : C \to S$ such that π is proper and flat, C is nonsingular, and $C_s \subset C$ is a normal crossings divisor.
2. Let $x \in C_s$ be a singular point. After passing to the algebraic closure of the field of constant, let $\bar{x} \in \hat{C}_{\bar{k}}$ be the completion. Show that one can find local parameters u, v at $\bar{x}$ and t at $\bar{s} \in \hat{S}_{\bar{k}}$, and positive integers k_x, l_x, such that $t = u^{k_x} v^{l_x}$.
3. Let $S_1 \to S$ be a finite cover obtained by extracting the n-th root of a uniformizer, where n is divisible by all the non-zero k_x, l_x given above. Let C_1' be the normalization of $C \times_S S_1$. Show that the special fiber is reduced and nodal.
4. Show that the minimal model C_1 of $C_1' \to S_1$ is stable.

We list some approaches for positive and mixed characteristic:
Artin-Winters. This proof can be found in [7]. A precise and nice description and analysis of the proof is given by Raynaud, see [59].

In this proof one attaches an numerical invariant to a given genus, and one proves that by choosing a prime number q larger than this invariant, and not equal to the residue characteristic, and by extending the field of definition of a curve of that genus such that all q-torsion point on the jacobian are rational over the extension, then one acquires stable reduction. The proof consists of a careful numerical analysis of the possible intersection matrices of components of degenerating curves. The proof does not rely on a lot of theory, but is quite subtle.
Grothendieck, Deligne-Mumford. This proof can be found in [17], Theorem (2.4) and Corollary (2.7).

In this proof one shows that a curve has stable reduction if and only if its jacobian has stable reduction. Then one shows following Grothendieck that eigenvalues of algebraic ℓ-adic monodromy are roots of unity (see [67], Appendix). Moreover, again following Grothendieck one shows that these eigenvalues are all equal to one iff the abelian variety in question has stable reduction. The advantage of this proof is that it has a more conceptual basis. The big disadvantage is that it relies on the theory of Néron models, whose foundations are quite difficult.

Hilbert schemes and GIT - Gieseker. See [22], Chapter 2, Proposition(0.0.2). He says on the first page of the introduction: "...we use results of Chapter 1 to give an indirect proof that the n-canonical embedding of a stable curve is stable if $n \geq 10$, and to construct the projective moduli space for stable curves. As corollaries, we obtain proofs of the stable reduction theorem for curves, and of the irreducibility for smooth curves." The proof uses Geometric Invariant Theory to prove directly that $\overline{M_g}$ exists and is projective, and then one can easily derive the theorem. This proof does not use resolution of singularities for surfaces in any explicit manner.

Remark. This theorem is an instance of the *semistable reduction problem.* In [10], 1.3, the definition of semistable reduction, over a one-dimensional base, and *arbitrary* fiber dimension, is recalled. As we have seen above, it is true that if the relative dimension is one, stable reduction, hence semistable reduction, exists over a one-dimensional base. For higher relative dimension an analogous result holds in pure characteristic zero - see [38]. The general case is an important open problem, which seems difficult.

Once Theorem 11.2 is known, it is easy to generalize it. The pointed case can be easily proven using Knudsen's stabilization technique:

Exercise 11.4. Let R be a discrete valuation ring, with field of fractions K, suppose $(C, p_1, \ldots, p_n)$ is a *smooth, stable n-pointed curve* of genus $g > 1$ defined over K. There exists a finite extension $R \subset R_1$ of valuation rings, with K_1 the field of fractions of R_1, such that $C_1 = C \otimes K_1$ extends to a stable n-pointed curve $\mathcal{C}_1 \to \mathrm{Spec}(R_1)$.

The case of genus zero follows from Exercise 10.9. We will discuss the case of genus 1 in Section 11.2 below.

We can also consider the case when the generic fiber is not necessarily smooth:

Exercise 11.5. Let R be a discrete valuation ring, with field of fractions K, suppose C is a stable curve defined over K. There exists a finite extension $R \subset R_1$ of valuation rings, with K_1 the field of fractions of R_1, such that $C_1 = C \otimes K_1$ extends to a stable curve $\mathcal{C}_1 \to \mathrm{Spec}(R_1)$. [Below we formulate a generalization to stable pointed curves of this.]

We give a full generalization of (11.2):

Exercise 11.6. Let S be the spectrum of a discrete valuation ring, $\eta \in S$ the generic point. Let $(C, P) \to \{\eta\}$ be a stable n-pointed curve of genus g, i.e., C is a complete, nodal curve defined over a field K, and $P := \{P_1, \cdots, P_n\}$ are distinct closed points $P_j \in C(K)$, with such that (C, P) is stable n-pointed over K.

Then there exists a finite extension of discrete valuation rings $S \hookrightarrow S_1$, with generic point η_1, and an extension

$$\begin{array}{ccc} (C_{\eta_1}, P) & \hookrightarrow & (\mathcal{C}, \mathcal{P}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K_1) & \hookrightarrow & S_1 \end{array}$$

such that $(\mathcal{C}, \mathcal{P}) \to S_1$ is a family of stable n-pointed curves.

This is the "weak valuative criterion for properness" of the functor $\overline{\mathcal{M}_{g,n}}$.

Remark. Here is a hint about a technical detail which can be used in solving the previous exercises, *"The normalization of a stable n-pointed curve"*: Suppose given a stable n-pointed curve (C, P) over a field K, with $P = \{P_1, \cdots, P_n\}$. There exists a finite extension $K \subset L$, a finite disjoint union (D, Q) of stable pointed curves, and a morphism ("the normalization") $\varphi : (D, Q) \to (C, P)_L = (C, P) \otimes_K L$ such that: $D = \coprod D^{(t)}$, let the singular points of C_L be: $R_j \in C(L)$, with $1 \leq j \leq d$, moreover $Q = \{Q_1, \cdots, Q_n\} \cup \{S_j, T_j \mid 1 \leq j \leq d\}$, for every irreducible component of C_L there is a unique component of D mapping birationally onto it, the morphism φ is an isomorphism outside $\mathrm{Sing}(C_L)$, the markings Q_i corresponds with the markings P_i of the pointed curve (C, P), and the markings $\{S_j, T_j\}$ are precisely the points mapping to R_j.

You need to show this choice can be made, and show it is unique in case $K = k$ is an algebraically closed field.

Corollary. *Let $g \in \mathbb{Z}_{\geq 0}$, and $n \in \mathbb{Z}_{\geq 0}$ such that $2g - 2 + n > 0$. The coarse moduli scheme $\pi : \overline{M_{g,n}} \to \mathrm{Spec}(\mathbb{Z})$ is proper over $\mathrm{Spec}(\mathbb{Z})$.*

Proof. We use the valuative criterion for properness setting $X = \overline{M_{g,n}}$ and $Y = \mathrm{Spec}(\mathbb{Z})$. Suppose R is a discrete valuation ring, with field of fractions K, and suppose given

$$\begin{array}{ccc} \mathrm{Spec}\, K & \stackrel{\psi_K}{\to} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \stackrel{\varphi}{\to} & Y. \end{array}$$

By the definition of a coarse moduli scheme, there is a finite extension $K \subset K'$ such that the point $\psi_K(\mathrm{Spec}(K)) \in X$ corresponds to a stable pointed curve (C, P) over K'. By the stable reduction theorem there is a finite extension $K' \subset K_1$ such that $(C, P) \times_{\mathrm{Spec}\, K'} \mathrm{Spec}\, K_1$ extends to a stable pointed curve; this defines a morphism $\tau : \mathrm{Spec}(R_1) \to X$, "extending" φ and ψ_K. It factors over $\mathrm{Spec}(R)$, because $R = K \cap R_1$. This shows that the condition for the valuative criterion holds in our situation, hence that $\pi : \overline{M_{g,n}} \to \mathrm{Spec}(\mathbb{Z})$ is proper. ◯

11.2 Stable reduction for elliptic curves.

In these exercises we illustrate the concept of stable reduction by studying the case of elliptic curves. The concepts, ideas and examples below can be found in Silverman's book [69]. In this case examples are easy to give because in many cases we can choose plane models (Weierstrass equations). These exercises can be used at motivation for more abstract methods which apply for higher genus. You can do the exercises by explicit methods and calculations.

For details on elliptic curves, Weierstrass equations, the j-function, and related issues, see [69], Chapters III and VII.

A non-singular one-pointed curve of genus one is called an elliptic curve. In other words: an elliptic curve is an algebraic curve E defined over a field, absolutely irreducible, non-singular, of genus one, with a marked point $P \in E(K)$. Morphisms are supposed to respect the marked point.

The following exercise is an easy exercise using the theorem of Riemann-Roch.

Exercise 11.7. Show the following are equivalent:

1. (E, P) is an elliptic curve over K.
2. $E \subset \mathbb{P}^2_K$ is a plane, nonsingular cubic curve, with a marked point $P \in E(K)$.
3. (E, P) is an abelian variety of dimension one over K.

Definition. Let $R = R_v$ be a discrete valuation ring, with $K = \text{fract}(R)$ its field of fractions, and $k = R_v/m_v$ the residue class field.

1. An elliptic curve E defined over K is said to have *good reduction* (at the given valuation) if there exists a smooth proper morphism $\mathcal{E} \to \text{Spec}(R)$ with generic fiber isomorphic to $E \to \text{Spec}(K)$. If E does not have good reduction, we say that it has *bad* reduction.
2. We say E has *stable reduction* at v if either it has good reduction, or there exists a *nodal* $\mathcal{E} \to \text{Spec}(R)$ with generic fiber isomorphic to $E \to \text{Spec}(K)$.

Definition. We say that E has *potentially good reduction*, if there exists a finite extension $K \subset L$, where B is the integral closure of R in L, and w a valuation over v, such that $E \otimes L$ has good reduction at w.

We define *potentially stable reduction* analogously.

Here are some exercises to warm up:

Exercise 11.8. Suppose $R = k[T]$, with $\text{char}(k) \neq 2, \neq 3$, and let E over $K = k(T)$ be given by the equation $Y^2 = X^3 + T^6$. Show that E has good reduction at the valuation given by $v(T) = 1$.

Exercise 11.9. Suppose $R = k[S]$, with $\text{char}(k) \neq 2, \neq 3$, and let E over $K = k(S)$ be given by the equation $Y^2 = X^3 + S$. Show that E has bad reduction at the valuation v given by $v(S) = 1$.

Exercise 11.10. Suppose $R = k[S]$, with $\mathrm{char}(k) \neq 2, \neq 3$, and let E over $K = k(S)$ be given by the equation $Y^2 = X^3 + S$. Show that E has potentially good reduction at the valuation v given by $v(S) = 1$.

Suppose that E is given over K by a Weierstrass equation with coefficients in R (see [69], III). Such an equation defines an *affine* plane curve $\mathcal{E} \subset \mathbb{A}^2_B$ over $\mathrm{Spec}(R) = B$, and it is easy to see that the curve $E_0 := \mathcal{E} \otimes_R k$ is irreducible and has at most one singular point. The curve E is obtained by adding the point at infinity to E_0. Suppose the Weierstrass equation is *minimal* at v. If this singular point is a cusp, we say that this reduction is of *additive type*, if it is a node we say that this reduction is of *multiplicative type*, or we say in this case the reduction is stable.

Exercise 11.11. Show that the notion of "good reduction" as defined earlier is equivalent by saying there is Weierstrass equation defining good reduction. Show that a reduction of multiplicative type is a stable reduction.

A reduction given by a minimal Weierstrass equation of additive type is bad reduction which is non-stable; non-stable bad reduction is sometimes called cuspidal reduction.

Exercise 11.12. Suppose $R = k[T]$, with $\mathrm{char}\, k \neq 2$, and let E over $K = k(T)$ be given by the equation $Y^2 = X \cdot (X-1) \cdot (X-T)$. Show that any model of this curve given by a Weierstrass equation has stable reduction at the valuation given by $v(T) = 1$. Show that this curve does not have potentially good reduction.

Exercise 11.13. Let R be a discrete valuation ring, with residue characteristic $\neq 2$, and fraction field K. Let E be an elliptic curve over K.

1. Show that after a suitable extension of R, the curve E admits a minimal Weierstrass equation of the form
$$y^2 = x(x-1)(x-\lambda)$$
for some $\lambda \in R$.
2. Conclude that this curve has potentially stable reduction.

Exercise 11.14. $\ast$ Let R_v be a DVR, with residue characteristic $\neq 3$. Suppose E is an elliptic curve over K given by a Weierstrass equation $E = V(F) \subset \mathbb{P}^2_K$ such that all flex points of E have coordinates in K.

1. Show that this curve admits a plane equation (not a Weierstrass equation!) over K of the form
$$\lambda(X^3 + Y^3 + Z^3) = 3\mu XYZ,$$
for some $(\lambda : \mu) \in \mathbb{P}^1_{R_v}$.

2. Show that $E \otimes_K L$ has stable reduction at v.
3. Show that E has potentially stable reduction.

Conclusion. Every elliptic curve over a field K with a discrete valuation has potentially stable reduction at that valuation.

This is a special case of 11.6, the stable reduction theorem for stable curves of arbitrary genus.

Exercise 11.15. $*$

1. Let R_v be a DVR, and E an elliptic curve of the fraction field K. Show that E has potentially good reduction at R_v if and only if $j(E) \in R_v$.
2. Can you formulate (and prove?) the same result for curves of arbitrary genus?
3. If E is an elliptic curve over a field K, and $\mathrm{End}_K(E) \neq \mathbb{Z}$, then E has potentially good reduction at every place of K.

11.3 Remarks about monodromy

Let C be a non-singular curve over a field K, and let v be a discrete valuation of K. Consider properties of good reduction, bad reduction at v, and so on. We have quoted that C has stable reduction at v iff $J := \mathrm{Jac}(C)$ has stable reduction, see [17], Proposition (2.3).

However note that it may happen that J has good reduction, and C has bad reduction; this is the case if the special fiber C_0 of the minimal model of C at v has a generalized jacobian $J_0 = \mathrm{Jac}(C_0)$ which is an abelian variety. Such a curve C_0 is called a curve of "compact type", or a "nice curve" (and sometimes called a "good curve", but we do not like that terminology, because a curve reducing to a "good curve" may not have good reduction...). In this case the special fiber C_0 is a tree of non-singular curves, i.e. every irreducible component is non-singular, and in the dual graph of C_0 there are no cycles. The easiest example is: a join of two non-singular curves, each of genus at least one, meeting transversally at one singular point. For example a curve of genus two degenerating to a transversal crossing of two curves of genus one is the easiest example. Here is another example: take $\mathbb{P}^1$ with three marked points, and attach three elliptic tails via normal crossings at the markings, arriving at a nice curve of genus three

Monodromy (action of the local fundamental group of the base on cohomology) decides about the reduction of an abelian variety being bad or good, see [67], Theorem 1 on page 493. In the analytic context one can take the local fundamental group of a punctured disc acting on cohomology; in all cases one considers the inertia-Galois group of v acting on ℓ-adic cohomology, where ℓ is a prime number not equal to the residue characteristic of v.

Note that algebraic monodromy has *eigenvalues which are roots of unity.* This was proved by Landman, Steenbrink, Brieskorn in various settings, and we find a proof by Grothendieck in the appendix of [67]. For a sketch of that proof, see [49], for further references, see [52].

Algebraic monodromy is trivial iff $X = J$ has good reduction, iff C has compact type reduction (which may be either good reduction or bad but “nice” as explained above).

The algebraic monodromy is unipotent (all eigenvalues are equal to one) if and only if X has stable reduction, if and only if C has stable reduction.

But, how can we distinguish for curves the difference between good reduction and bad compact type reduction? As we have seen, this is not possible via algebraic monodromy on cohomology. But, in a beautiful paper, [8] we find a method which for curves unravels these subtle differences for curves: the local fundamental group of the base acts via outer automorphisms on the fundamental group of the generic fiber (again, here one can work in the analytic-topological context, or in the ℓ-adic algebraic context). *This action is trivial iff C has good reduction.*

12 Construction of moduli spaces

Early constructions of the moduli spaces of smooth curves M_g included a complex - analytic constructions via Teichmüller theory and via the construction of moduli of abelian variety using locally symmetric spaces. These constructions are not algebraic in nature and therefore cannot be generalized to positive or mixed characteristics.

A first algebraic approach, which is still commonly used today, was given by Mumford using his Geometric Invariant Theory [GIT]. We will sketch one version of this approach, due to Gieseker, which automatically gives also the moduli spaces of stable curves $\overline{M_g}$. There is another approach, due to Artin and Kollár [40], which circumvents the use of Geometric Invariant Theory. Nowadays both approaches work over $\mathbb{Z}$.

How does one start? It is evident that if we want to parametrize *all* stable curves of a certain genus, we had better have *some* family of curves in which all these curves appear. We know of two general approaches for that. One method uses parameter spaces for curves embedded in projective space, such as Hilbert schemes (or Chow varieties). We will follow this approach. The other approach, due to Artin [6], uses versal deformation spaces. It works in greater generality but involves a number of technicality which we would rather avoid here.

It is easy to see that for any stable curve C of genus $g > 1$, and any $\nu \geq 3$, the ν-canonical series $H^0(C, \omega_C^\nu)$ gives an embedding of C as a curve of degree $d := \nu(2g-2)$ in a projective space of dimension $N := \nu(2g-2) - g$. Thus the Hilbert scheme $\mathcal{H}_{\mathbb{P}^N, P(T)}$ (over $\mathbb{Z}$!) parametrizing subschemes of $\mathbb{P}^N$ with Hilbert polynomial $P(T) := dT + 1 - g$ carries a universal family $\mathcal{C}_{\mathbb{P}^N, P(T)} \to \mathcal{H}_{\mathbb{P}^N, P(T)}$ in which each stable curve of genus g appears at least once.

There are two problems with this family:

1. Each curve appears more than once in the family. Indeed, the embedding of the curve C in $\mathbb{P}^N$ involves two choices: a choice of a line bundle of degree d, and a choice of a basis for the linear series. And of course the curves could also be embedded in a projective subspace using a subseries.

2. There are many curves in $\mathbb{P}^N$ with Hilbert polynomial $P(T)$ which are far from stable.

Since a nodal curve can only deform into nodal curves, it is easy to see that there is an open subset $\mathcal{H}_{st} \subset \mathcal{H}_{\mathbb{P}^N,P(T)}$ which parametrizes *stable* curves, embedded by a *complete* linear system in $\mathbb{P}^N$. Denote the restriction of the universal family to $\mathcal{H}_{st}$ by $\pi : \mathcal{C}_{st} \to \mathcal{H}_{st}$. Considering the locus in $\mathcal{H}_{st}$ where $R^1\pi_*(\mathcal{O}(1) \otimes \omega^{\nu}_{\mathcal{C}_{st}\to\mathcal{H}_{st}})$ jumps in dimension, we immediately see that there is a *closed* subscheme $\mathcal{H}_g \subset \mathcal{H}_{st}$ parametrizing stable curves embedded by a complete ν-canonical series. The restriction of the universal family will be denoted $\mathcal{C}_g \to \mathcal{H}_g$.

There is a natural action of the projective linear group $PGL(N+1)$ on $\mathcal{H}_g$ via changing coordinates on $\mathbb{P}^N$. It is easy to see that the "ambiguity" for choosing the embedding of a curve C in the latter universal family is fully accounted for by the action of this group. In other words, stable curves correspond in a one-to one manner with $PGL(N+1)$ orbits in $\mathcal{H}_g$. Thus, at least set theoretically, $\overline{M_g} = \mathcal{H}_g/PGL(N+1)$.

12.1 Geometric Invariant Theory and Gieseker's approach

We arrived at the following questions:

1. Does the quotient $\mathcal{H}_g/PGL(N+1)$ exist as a scheme?

2. Can we show that it is projective?

3. Does it satisfy the requirements of a coarse moduli scheme?

Geometric Invariant Theory is a method which allows one to approach the first two questions simultaneously. The third question then becomes an easy gluing exercise.

The general situation is as follows: Let $X \subset \mathbb{P}^n$ be a quasi-projective scheme and suppose G is an algebraic group acting on $\mathbb{P}^n$ and stabilizing X. One wants to know whether or not a quotient X/G exists as a scheme and whether or not it is projective.

A natural approach is to look for a space of invariant sections of some line bundle. Thus assume that the action of G on $\mathbb{P}^n$ lifts to $\mathcal{O}_{\mathbb{P}^n}(l)$. Then it also lifts to any power $\mathcal{O}_{\mathbb{P}^n}(l\cdot m)$, and we can look at the ring of invariants $R := \oplus(\mathcal{O}_{\mathbb{P}^n}(l\cdot m)^G$. We have a natural rational map $q : \mathbb{P}^n \dashrightarrow \operatorname{Proj} R$. We would like to know whether or not this map is well defined along X, and what the image is like.

First, an easy observation. For any point $x \in X$, the map q is well defined at x if and only if there exists a nonconstant invariant $f \in R$ such that $f(x) \neq 0$.

We want to check whether q is a quotient map in a neighborhood of x. To go any further, we need to assume that the group G is *reductive*. Assuming that G is reductive, then the question whether map q is a quotient map at a neighborhood of x can be translated to a question about the closure $\overline{Gx}$ of the orbit of x: one needs to check that for any point $y \in \overline{Gx} \setminus Gx$ there is an invariant $f \in R$ which vanishes at y but not at x. A point x is called *GIT-stable* if it satisfies this condition.

Mumford's numerical criterion for stability (see [GIT]) gives a way to check GIT-stability in some situations.

Let us consider our situation. The scheme $\mathcal{H}_g$ is quasi projective - from its construction one sees that it naturally sits inside a Grassmannian, which has a Plücker embedding in some $\mathbb{P}^n$. It is easy to see that the action of $PGL(N+1)$ extends to $\mathbb{P}^n$, and lifts to some line bundle $\mathcal{O}_{\mathbb{P}^n}(k)$. Applying this criterion systematically, Gieseker verified in [22] that

1. If a point $x \in \mathcal{H}_{\mathbb{P}^N, P(T)}$ corresponds to a scheme which is not a stable curve, or to a curve which is not embedded by a complete linear series, then *every* nonconstant invariant vanishes at x.

2. If a point $x \in \mathcal{H}_{\mathbb{P}^N, P(T)}$ corresponds to a stable curve embedded by the complete ν-canonical linear series, then x is GIT-stable.

Using the two statements, and the fact that G is reductive, it is not difficult to realize that

1. the map $\mathcal{H}_g \to \operatorname{Proj} R$ is a quotient map, and

2. the image of $\mathcal{H}_g$ is projective.

This proves the existence and projectivity of $\overline{M_g}$.

12.2 Existence of $\overline{M_{g,n}}$

There is no known analogue of Gieseker's result for stable pointed curves. It is not difficult to construct a Hilbert-type scheme for stable pointed curves, with a reductive group action, and such that the quotient is set-theoretically $\overline{M_{g,n}}$. But in order to tell that the quotient is isomorphic to $\overline{M_{g,n}}$ as a scheme, we first need to construct $\overline{M_{g,n}}$ in some other way.

But there is a very useful trick, which reduces the construction of $\overline{M_{g,n}}$ to the existence of $\overline{M_g}$ for some larger value of g. We give the reduction over a field, but it works similarly over $\mathbb{Z}$.

Fix n irreducible stable curves C_i of genus $g_i > g$, all nonisomorphic to each other, and fix a rational point $x_i \in C_i$. For any stable n-pointed curve $(C, p_1, \dots, p_n)$ of genus g, we can construct a stable curve C' of genus $g' = g + \sum g_i$ as follows: $C' = (\cup C_i) \cup C$, where we glue together C and C_i by identifying p_i with x_i.

Clearly this construction gives a set theoretic embedding $\overline{M_{g,n}} \to \overline{M_{g'}}$. The image set is easily seen to be a scheme, and by working the construction in a family it is easy to see that it is a coarse moduli scheme.

13 Existence of tautological families

For almost any application of moduli spaces of curves including the alteration theorem, it is necessary to know that there exists a family $C \to M$ over a scheme M such that the associated morphism to the moduli space is finite and surjective. Such a family is called a *tautological family*; see Section 10.4. Various authors have devised general methods of showing this, but for the moduli spaces of curves there is a “very nice” way to find such a cover, using level structures. The case of the moduli space $\overline{M_g}$ of stable (unpointed) curves is discussed in detail in [21]. In this section we describe how this can be generalized for sable *pointed* curves as well. We rely throughout on the treatment in [21]. In Section 13.4 we outline another way to construct tautological families, which works in greater generality.

13.1 Hilbert schemes and level structures

Fix:

- an integer $g \in \mathbb{Z}_{\geq 0}$ (the genus),
- an integer $n \in \mathbb{Z}_{\geq 0}$ (the number of marked points),
- such that $2g - 2 + n > 0$,
- and an integer $m \in \mathbb{Z}_{\geq 1}$ (the level).
- Fix an integer $\nu \in \mathbb{Z}_{\geq 5}$, which will be used to study ν-canonical embeddings of curves into a projective space.

Remark. If $n = 0$ or $m = 1$ these data will be omitted from the notation, e.g. $M_{g,0} = M_g$. If $g = 0$, the level structure is irrelevant, $M_{0,n}^{(m)} = M_{0,n}$.

Let C be a curve whose jacobian is an abelian variety. By a level-m structure on C we mean a symplectic level structure as explained in [21]. If a level-m structure is considered we assume that all schemes, varieties are over a base on which m is invertible, i.e. are schemes over $\mathrm{Spec}(\mathbb{Z}[1/m])$.

Recall that there is a Hilbert scheme H_P parametrizing curves $C \subset \mathbb{P}^N$, where $N = \nu\cdot(2g-2+n) - g$, with Hilbert polynomial $P(t) = \nu\cdot(2g-2+n)\cdot t - g + 1$. We want to find a scheme parametrizing *pointed* curves - this is done in a standard way as follows. Observe that there is a closed subscheme $H_{P,n} \subset H_P \times (\mathbb{P}^N)^n$ parametrising pairs $(C, (p_1, \ldots, p_n))$ where $p_i \in C$. There is an open subscheme $H_{st} \subset H_{P,n}$ where the curves are nodal, the points are distinct and regular points on the curves, and the pairs $(C, (p_1, \ldots, p_n))$ are stable. Last, there is a closed subscheme $H_{g,n} \subset H_{st}$ where the embedding line bundle of $C \subset \mathbb{P}^N$ is isomorphic to $(\omega_C(p_1 + \ldots + p_n))^\nu$.

Over $H_{g,n}$ there is a universal family $C_{g,n} \to H_{g,n}$ with sections $s_i : H_{g,n} \to C_{g,n}$ of stable pointed curves, embedded in $\mathbb{P}^N$ by the chosen line bundle. The linear group $\mathrm{PGL} = PGL(N)$ acts on $C_{g,n} \to H_{g,n}$ equivariantly, and $\overline{M_{g,n}} = H_{g,n}/\mathrm{PGL}$ is the quotient.

Note that there is an open subset $H^0_{st} \subset H_{g,n}$ parametrizing *smooth* stable pointed curves.

13.2 Moduli with level structure

Theorem 13.1. *For $m \geq 3$, and $2g-2+n > 0$, there exists a* fine *moduli scheme $M^{(m)}_{g,n}$ for smooth stable n-pointed curves with level-m structure. In particular there exists a universal curve with level structure over $M^{(m)}_{g,n}$. This moduli scheme is smooth over* $\mathrm{Spec}(\mathbb{Z}[1/m])$.

Note in particular that $M^{(m)}_{g,n}$ is a normal scheme, and that $M^{(m)}_{g,n} \to M_{g,n}$ is a Galois cover with Galois group $\mathrm{Sp}(2g, \mathbb{Z}/m)$.

We use the notation $S_m := \mathrm{Spec}(\mathbb{Z}[1/m])$.

Definition. Let $g \in \mathbb{Z}_{\geq 1}$. Fix $n \in \mathbb{Z}_{\geq 0}$, with $2g-2+n > 0$. For any $m \in \mathbb{Z}_{\geq 3}$, the scheme

$$\overline{M^{(m)}_{g,n}} \longrightarrow S_m$$

is defined as the normalization of $\overline{M_{g,n}}[1/m] = \overline{M_{g,n}} \times_{\mathbb{Z}} S_m$ in $M^{(m)}_{g,n}$.

For simplicity of notation in this section, we write $M = \overline{M^{(m)}_{g,n}}$ and $M^0 = M^{(m)}_{g,n} \subset M$.

Theorem 13.2. *Fix g, n, and m as above. Suppose $m \geq 3$. There exist a stable n-pointed curve $(\mathcal{C}, \mathcal{P}) \to M$, and a level-$m$-structure α on $\mathcal{C}^0 := \mathcal{C}|_{M^0}$ such that*

$$(\mathcal{C}, \mathcal{P}) \to M \quad \textit{is tautological for} \quad M \to \overline{M_{g,n}},$$

and such that

$$(\mathcal{C}^0, \mathcal{P}^0, \alpha) \to M^0$$

represents the functor $\mathcal{M}^{(m)}_{g,n}$.

We give an argument for 13.1 and 13.2 following the line of [21]. This is a kind of "boot-strap" argument, which uses the idea that *once one quotient space exists, many others follow*. We also sketch another argument which reduces the problem to the case of [21].

There is a relative jacobian scheme $J(C^0_{g,n}) \to H^0_{g,n}$. This is an abelian scheme, so we can look at its group-subscheme of m-torsion points. Taking a symplectic rigidification of this group scheme we arrive at $H^{(m),0}_{g,n}$ - the Hilbert scheme of smooth stable n-pointed curves with symplectic level-m structure - embedded in projective space as above.

The action of PGL on $H^0_{g,n}$ clearly lifts to $H^{(m),0}_{g,n}$. This immediately implies that

$$M^{(m)}_{g,n} \ = \ H_{g,n} \ / \ \mathrm{PGL}$$

exists, since it is finite over $M_{g,n}$. By Serre's lemma this action has no fixed points, and it also lifts to $C^{(m),0}_{g,n} = C_{g,n} \times_{H_{g,n}} H^{(m),0}_{g,n}$. This means that the quotient $\mathrm{PGL}\backslash C^{(m),0}_{g,n} \to M^{(m)}_{g,n}$ is a universal family of smooth stable pointed curves with level structure.

This proves Theorem 13.1.

The normalization of $H_{g,n}$ in $H^{(m),0}_{g,n}$ will be denoted by $H^{(m)}_{g,n}$. The argument of [21], (2.6) works word for word, and shows that PGL still acts without fixed points on $H^{(m)}_{g,n}$. This gives the existence of the quotient

$$\overline{M^{(m)}_{g,n}} \ = \ H^{(m)}_{g,n} \ / \ \mathrm{PGL}.$$

Again the universal family over $H^{(m)}_{g,n}$ descends to a family over $\overline{M^{(m)}_{g,n}}$, this extends the universal family over $M^{(m)}_{g,n}$, and clearly it is tautological. This proves Theorem 13.2. ♁

13.3 Proof by reduction to the unpointed case

Starting from $\overline{M^{(m)}_g}$ and its tautological family we can construct $\overline{M^{(m)}_{g,n}}$ and its tautological family by induction on the number of points n in the manner described below.

Denote by $D \to \overline{M^{(m)}_{g,n}}$ the tautological family. It is easy to see that in fact $D = \overline{M^{(m)}_{g,n+1}}$. So $D \times_{\overline{M^{(m)}_{g,n}}} D \to \overline{M^{(m)}_{g,n+1}}$ is a family of stable n-pointed curves with level structure, but with an additional section given by the diagonal. Using the stabilization process as described in [39] (see Section 3.7 above) one blows this scheme up, to obtain the tautological family over $\overline{M^{(m)}_{g,n+1}}$ as desired.

Remark. The moduli space $M^{(m)}_{g,n}$ is smooth over S_m for $m \geq 3$; this follows from Serre's lemma and deformation theory. However, the moduli space $\overline{M^{(m)}_{g,n}}$ is singular if $g > 2$; Serre's lemma holds also in this situation, but the space is not the coarse (or fine) moduli space of a moduli functor whose deformation spaces coincide with the deformations of stable curves. For more explanation, see [45] or [21].

The argument above works for $g > 1$ when $\overline{M^{(m)}_g}$ exists. For rational curves these theorems are relatively easy, and known, since the moduli spaces are fine moduli spaces in genus 0. For elliptic curves these theorems are known by the theory of modular curves.

13.4 Artin's approach via slicing

A general approach for constructing tautological families over finite covers of coarse moduli space was developed by Artin (see description in [40]). Here we present a version of this approach adapted to stable pointed curves.

Step 1: slicing Consider the locally closed subset of the Hilbert scheme $H_{g,n}$ discussed above. It carries a universal family of stable pointed curves $C_{g,n} \to H_{g,n}$ suitably embedded in a projective space. This family induces a natural morphism $H_{g,n} \to \overline{M_{g,n}}$. The fibers coincide with the G-orbits associated to the embedded curves, where $G = \mathrm{PGL}$.

Fix a point $x \in H_{g,n}$. By repeatedly taking hyperplane sections, we can find a locally closed subscheme $V_x \subset H_{g,n}$ such that

1. $Gx \cap V_x \neq \emptyset$;
2. If $x' \in H_{g,n}$ and $Gx' \cap V_x \neq \emptyset$, then there exists a neighborhood $x' \in U$ such that for any $y \in U$ we have that $Gy \cap V \neq \emptyset$; and
3. for any $y \in H_{g,n}$ we have that $V_x \cap Gy$ consists of finitely many closed points.

These V_x are "multi-sections" of the map $H_{g,n} \to \overline{M_{g,n}}$ in a neighborhood of Gx. The essential point is that all orbits in $H_{g,n}$ are of the same dimension.

Using the Noetherian property, we can choose finitely many of these, say $V_1, \ldots, V_l$, such that every orbit meets at least one of them.

Step 2: normalization. Let K be the join of the function fields of V_i over $\overline{M_{g,n}}$. Let V be the normalization of $\overline{M_{g,n}}$ in the Galois closure of K. The scheme V admits many rational maps to the V_i. It is not hard to see that for every point $v \in V$ at least one of these maps is well defined at v! Pulling back the families on V_i, we see that V is covered by open sets, each of which carries a family of stable pointed curves, compatible with the given morphism $V \to \overline{M_{g,n}}$.

Step 3: Gluing. Now we can use Lemma 3.11 inductively. We obtain a finite surjective $M \to V$ over which the families glue together to a family $C \to M$ such that the associated moduli morphism is the composition $M \to V \to \overline{M_{g,n}}$. Since V is finite over $\overline{M_{g,n}}$, this forms a tautological family.

Remark. It is not hard to construct a tautological as above *without using the existence of* $\overline{M_{g,n}}$*!* One can use this to construct the moduli space "from scratch" as a proper algebraic space, which is roughly speaking a quotient of a scheme by a finite equivalence relation. Kollár in [40] has shown how to use this to prove, without GIT, that $\overline{M_{g,n}}$ is projective.

14 Moduli, automorphisms, and families

This section will not be needed in the proofs above. The central theme here is the relationship between automorphisms, coarseness of moduli, and the existence of families. The main principle which will emerge is:

$$\text{a moduli space } M \text{ is a fine moduli space}$$
$$\Updownarrow$$
$$\text{objects parametrized by } M \text{ have no nontrivial automorphisms}$$
$$\Updownarrow$$
$$M \text{ carries a unique tautological family.}$$

We also touch on the issue of singularities of moduli spaces.

For rational curves, and $n \geq 3$, the moduli schemes $M_{0,n}$ and $\overline{M_{0,n}}$ exist, these are smooth over $\mathrm{Spec}(\mathbb{Z})$, these are fine moduli schemes, i.e. they carry a universal family.

However, the moduli space $M_{1,1}$ and the moduli spaces M_g for $g > 1$ are not fine for the related moduli functor.

Exercise 14.1 (Deuring). Let K be a field, let $x \in K$. Then there exists an elliptic curve E defined over k with $j(E) = x$. [Suppose $\mathrm{char}(K) \neq 2, \neq 3$, suppose E is is given over K by the equation $Y^2 = X^3 + AX + B$, with $4A^3 + 27B^2 \neq 0$. Then define

$$j(E) := 1728{\cdot}4{\cdot}A^3/(4A^3 + 27B^2).$$

For the definition of the j-invariant, see [69].]

This can partly be made more precise as follows:

Exercise 14.2. Consider $M_{0,1} \cong \mathbb{A}^1_{\mathbb{Z}}$, and remove the sections $j = 0$ and $j = 1728$:

$$U := \mathbb{A}^1_{\mathbb{Z}} \smallsetminus \{0, 1728\}_{\mathbb{Z}}.$$

There exists a tautological curve

$$\mathcal{E} \to U.$$

1. This cannot be extended over any of the deleted points.
2. This family is not at all unique.

Exercise 14.3. Consider $U := \mathbb{C} \smallsetminus \{0, 1728\}$. Show: up to isomorphisms there exist exactly 4 tautological curves (stable, one pointed smooth curves of genus 1 with j invariant different from 0 and 1728) over this moduli space. Show that for the ground field $K = \mathbb{Q}$ there are *infinitely many* tautological curves over the moduli space $\mathbb{A}^1_{\mathbb{Q}} \smallsetminus \{0, 1728\}$. Characterize them all.

We have seen the difference between a universal curve and a tautological curve: the moduli problem for elliptic curves with geometrically no non-trivial automorphisms admits a coarse moduli scheme; over that scheme there is a tautological curve, but the scheme is not a fine moduli scheme (not every family is a pull-back from one chosen tautological curve). Here is another example:

Definition. A curve $\mathcal{C} \to S$ is called a *hyperelliptic curve* if it is smooth, of relative genus g with $g \geq 2$, and if there exists an involution $\iota \in \mathrm{Aut}(\mathcal{C}/S)$ such that the quotient $\mathcal{C}/ < \iota > \to S$ is a smooth family of rational curves.

Remark. Elliptic curves and rational curves are not called "hyperelliptic", but sometimes the terminology "quasi-hyperelliptic" is used for curves having an involution with rational quotient.

Theorem 14.4. *Consider the moduli space* Hip_g *of hyperelliptic curves of genus* $g \geq 2$ *(even over* $\mathbb{C}$*). If* g *is even there* does not exist *a curve defined over the function field* $\mathbb{C}(\mathrm{Hip}_g)$ *having as moduli point the generic point of* Hip_g.

(See Shimura [68], Theorem 3.)

In different terminology: For no open dense subset $U \subset H_g$ does there exist a tautological curve when g is *even*.

There does exist a open dense subset $U \subset H_g$ and a tautological curve $\mathcal{C}_U$ when g is *odd*.

Corollary. *No dense open subset in* M_2 *or in* $M_2 \otimes K$ *carries a tautological curve.*

Exercise 14.5. Choose $g \in \mathbb{Z}_{>2}$, and consider nonsingular curves of genus g.

1. Show that there exists such a curve which has no nontrivial automorphisms.
2. (variant:) Show that a general curve of genus > 2 has no nontrivial automorphisms.

Remark. There is a morphism $M_{g,n+1} \to M_{g,n}$ ("forgetting the last marking"). Sometimes this is called the "universal curve over $M_{g,n}$", but we think in general this terminology is not justified in all cases possible.

Theorem 14.6. *Let* $U \subset M_g$ *with* $g \geq 3$ *fixed, be the set of points corresponding with curves which have geometrically no non-trivial automorphisms. This set is dense and open. Let* $\mathcal{M}_U$ *be the corresponding moduli functor. This functor is representable.*

In other terminology: there does exist a (unique) universal curve $\mathcal{C}_U \to U$ for the moduli problem of curves of genus $g \geq 3$ with geometrically no non-trivial automorphisms.

In particular: Let K be a field, $g \in \mathbb{Z}_{\geq 3}$, and η be the generic point of $M_g \otimes K$. There exists an algebraic curve defined over $K(\eta)$ having η as moduli point. However the universal family as indicated above over $U \subset M_g$ does not extend to any smooth family of curves over M_g.

Exercise 14.7. Formulate and prove a generalization of previous theorems to the case of stable pointed curves.

Exercise 14.8. Let $n > 2g + 2$ and let $(C, P_1, \cdots, P_n)$ be any stable n-pointed curve of genus g. Suppose that C is *regular* (and hence irreducible). Show that

$$\mathrm{Aut}((C, P_1, \cdots, P_n)) = \{1\}$$

(if you want, assume that $\mathrm{char}(k) = 0$).

Exercise 14.9. Let $g \in \mathbb{Z}_{\geq 1}$ and $2 - 2g < n \leq 2g + 2$ and $0 \leq n$. Show that $M_{g,n}$ is a coarse, but not a fine moduli space.

Exercise 14.10. Choose $g \in \mathbb{Z}_{\geq 0}$, and let $n > 2g + 2$. Show that $M_{g,n}$ is a fine moduli space. Show that the universal curve over $M_{g,n}$ is not smooth if $n \geq 2$.

Exercise 14.11. Consider all stable n-pointed curves of genus g. Suppose that

$$2g - 2 + n \geq 3.$$

1. Show that there exists such a curve which has no nontrivial automorphisms.
2. (variant:) Show that a general curve as above has no nontrivial automorphisms.

Exercise 14.12. Choose some g (e.g. $g = 3$), choose a very large integer n (e.g. $n = 1997$), and construct a stable n-pointed curve of genus g which has a non-trivial group of automorphisms.

Variant: Let $2g - 2 + n \geq 2$; show that there exist stable n-pointed curves of genus $g \geq 3$ in codimension two in the moduli space with non-trivial groups of automorphisms.

Exercise 14.13. Let $g \in \mathbb{Z}_{\geq 1}$, and $n > 2 - 2g$ and $n \geq 0$. Show that $\overline{M_{g,n}}$ is not a fine moduli space.

Choose $2g - 2 + n > 0$, choose $m \geq 1$ and let M be one of the following spaces: $M_{g,n}^{(m)}$, or $\overline{M_{g,n}}$ (all these spaces are defined by a moduli functor). Let $x \in M(k)$ be a geometric point, and let $X_0 := (C, P, \alpha)$ be the corresponding object over k (if C is non-smooth there is no level structure, the genus of C is g, we have $P = \emptyset$ if $n = 0$, we have $\alpha = id$ if $m = 1$). Let $D = \mathrm{Def}(X_0)$ be the universal deformation space; i.e. consider $\Lambda = k$ if $\mathrm{char}(k) = 0$, and $\Lambda = W_\infty(k)$ in case of positive characteristic, consider all local artin Λ-algebras, and consider the object prorepresenting all deformations of X_0 over such algebras (see [64]). This universal deformation object exists, and it is formally smooth over Λ on $3g - 3 + n$ variables; in case $n = 0$ this can be found in [17], page 81, the case of pointed curves follows along the same lines; in case $m > 1$, we have required that m is invertible in k, finite, flat group schemes of m-power order on such bases are étale, and deformations of level structures are unique by EGA IV^4, 18.1. Let $G := \mathrm{Aut}(X_0)$. Note that G *is a finite group* (because we work with stable curves). Note that G acts in a natural way on $D = \mathrm{Def}(X_0)$ by “transport of structure”.

Theorem 14.14. *In the cases described, the formal completion of M at x is canonically isomorphic with the quotient*

$$\mathrm{Def}(X_0) \;/\; G \;\xrightarrow{\sim}\; M_x^{\wedge}.$$

This is well-known, e.g. see [27], §1.

Exercise 14.15. (Rauch, Popp): Let $g \in \mathbb{Z}_{\geq 4}$, and let $A \subset M_g$ be an irreducible component of the set of all points corresponding with curves with non-trivial automorphisms. Show that the codimension of $A \subset M_g$ is ≥ 2. (In positive characteristic this is also correct, but you might need some extra insight to prove also those cases.)

Remark. Stable *rational* pointed curves have no non-trivial automorphisms. For *elliptic* curves there are curves with more than 2 automorphisms in codimension one. For curves *of genus two* we find a description of all curves with "many automorphisms" in [30]. Note that hyperelliptic curves of genus three are in codimension one.

Exercise 14.16. Show that non-hyperelliptic curves of genus three with non-trivial automorphisms are in codimension at least two.

Exercise 14.17. (Rauch [58], Popp [57]): Let $g \in \mathbb{Z}_{\geq 4}$, and let $[C] = x \in M_g$ be a geometric point. Show that x is a singular point on M_g iff $\mathrm{Aut} \neq \{id\}$. [You might like to use: [5], Coroll. 3.6 on page 95: A quasi-finite local homomorphism of regular local rings having the same dimension is flat. Also you might like to use purity of branch locus: a ramified *flat* covering is ramified in codimension one.]

Remark. For singularities of M_2 see [30]. Show that for genus three non-hyperelliptic points are singular iff there are non-trivial automorphisms, e.g. see [50]. For singularities of moduli schemes of abelian varieties, see [51].

Remark. As we have seen in [21], the moduli schemes $\overline{M_g^{(m)}}$ have singularities for all $g \geq 3$ and $m \geq 3$ (these spaces cannot be handled with the methods just discussed, these spaces are not given by "an obvious" moduli functor !). As Looijenga, see [42], in characteristic zero, and Pikaart and De Jong, see [54] showed, there exist a finite map $M \to M_g$ with M regular (using non-abelian level structures) (it is even true that M is smooth over $\mathbb{Q}$, or smooth over $\mathbb{Z}[1/r]$ for some natural number $r > 1$).

Summary about

$$M_{g,n}^{(m)} \hookrightarrow \overline{M_{g,n}^{(m)}} \longrightarrow \mathrm{Spec}(\mathbb{Z}[1/m]) =: S_m$$

for

$$g \in \mathbb{Z}_{\geq 0}, \quad n \in \mathbb{Z}_{\geq 0}, \quad m \in \mathbb{Z}_{\geq 1}, \quad \text{with} \quad 2g-2+n > 0,$$

$M_{g,n}^{(m)}$ and $\overline{M_{g,n}}$ exist as coarse moduli schemes, we have constructed $\overline{M_{g,n}^{(m)}}$. We have seen:

- For $g \geq 2$ the coarse moduli scheme $M_g \to S = \mathrm{Spec}(\mathbb{Z})$ exists. These are not fine moduli spaces. They do not carry a tautological family. For every g this is singular.
- For $g \geq 2$ the coarse moduli scheme $\overline{M_g} \to S = \mathrm{Spec}(\mathbb{Z})$ exists. These are not fine moduli spaces. They do not carry a tautological family. They are singular.
- A dense open set in $M_{1,1}$ carries a tautological family, and it is not universal.
- No dense open set in M_2 carries a tautological family.
- For $g \geq 3$ a dense open set in $M_{g,n}$ carries a universal family.
- For $n \geq 3$ the moduli spaces $M_{0,n} \subset \overline{M_{0,n}}$ exist, they are fine moduli spaces, they are smooth over $S = \mathrm{Spec}(\mathbb{Z})$.
- For $2g-2+n > 0$, and $m \geq 0$ the moduli spaces $M_{g,n} \to S$, and $\overline{M_{g,n}} \to S$ and $M_g^{(m)} \to S_m$ exist, they coarsely represent a moduli functor. For $n > 2g+2$ the moduli space $M_{g,n}$ is fine, and smooth over $\mathrm{Spec}(\mathbb{Z})$ (but the universal family is not smooth for $n > 1$). For $m \geq 3$ the space $M_g^{(m)}$ is fine and smooth over S_m.
- For $2g-2+n > 0$, and $m \geq 0$ there is a moduli space, and a tautological family, with properties as in 13.2. For $g \geq 3$ the morphism $\overline{M_{g,n}^{(m)}} \to S_m$ is not smooth.

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