1. Basic definitions

1.1. **Categories.**

Definition 1.1.1. A category $\mathcal{C}$ consists of

(0) A class $\mathcal{C}^0$, whose elements are called objects of $\mathcal{C}$.

(1) A class $\mathcal{C}^1 = \coprod_{X,Y \in \mathcal{C}^0} \mathcal{C}^1(X,Y)$, whose elements are called morphisms of $\mathcal{C}$ and are often denoted by arrows $f\colon X \to Y$. One often writes $\mathcal{C}^1(X,Y) = \mathrm{Hom}(X,Y)$ or $\mathcal{C}^1(X,Y) = \mathrm{Mor}(X,Y)$. We will always assume that each $\mathrm{Hom}(X,Y)$ is a set.

(2) The class $\mathcal{C}^2$ of *composition laws*, whose elements are maps
$$\mathcal{C}^2(X,Y,Z)\colon \mathcal{C}^1(X,Y) \times \mathcal{C}^1(Y,Z) \to \mathcal{C}^1(X,Z)$$
for each $X, Y, Z \in \mathcal{C}^0$. Usually the composition is denoted as $g \circ f$ for $f\colon X \to Y$ and $g\colon Y \to Z$.

This data should satisfy the following two conditions: (i) the composition is associative, (ii) each set $\mathcal{C}^1(X,Y)$ contains an element Id_X neutral with respect to the composition.

Remark 1.1.2. (i) A category $\mathcal{C}$ is called *small* if $\mathcal{C}^0$ is a set.

(ii) Sometimes, one does not assume that Hom's are sets. Then the categories in our sense are called *locally small.*

Example 1.1.3. (i) Very useful categories are the category of sets Sets, the category of groups Groups, the category of abelian groups Ab, the category of unital associative rings Rings, the category of commutative rings, categories of non-unital rings, lie algebras, etc. In a sense, these are categories of sets with algebraic structures given by few n-ary operations that satisfy certain compatibility relations (e.g. distributivity, commutativity, etc.). The morphisms are maps that preserve all the structure. We will call such categories *algebraic categories.*

(ii) For any category $\mathcal{C}$ its *opposite* or *dual* category $\mathcal{C}^{\mathrm{opp}}$ is defined by $(\mathcal{C}^{\mathrm{opp}})^0 = \mathcal{C}^0$ and $(\mathcal{C}^{\mathrm{opp}})^1(X,Y) = \mathcal{C}^1(Y,X)$ with the composition being the composition in $\mathcal{C}$ in reversed order (i.e. $f \circ^{\mathrm{opp}} g = g \circ f$). One often uses $\mathcal{C}^{\mathrm{opp}}$ to introduce *dual* construction, definitions, proofs obtained by reversing the arrows.

(iii) One can consider various subcategories by restricting objects and/or morphisms. For example, the category of all finite sets, the category of groups with injective homomorphisms, etc.

(iv) If $\mathcal{C}$ is a category and $X \in \mathcal{C}^0$ then by $\mathcal{C}/X$ one denotes the category of *objects over* X. Its objects are morphisms $f\colon Y \to X$ in $\mathcal{C}$, and one usually says that Y is an object over X and f is its *structure morphism.* A morphism between $f\colon Y \to X$ and $f'\colon Y' \to X$ is a morphism $g\colon Y \to Y'$ that is compatible with the structure morphisms, i.e. $f' \circ g = f$.

(v) There are much more interesting categories in mathematics: topological spaces with continuous maps, Lie groups with continuous homomorphisms, manifolds with differentiable maps, etc.

1.2. **Functors.**

Definition 1.2.1. (i) A *functor* $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ between categories consists of maps $\mathcal{F}^0\colon \mathcal{C}^0 \to \mathcal{D}^0$ and $\mathcal{F}(X,Y)\colon \mathcal{C}^1(X,Y) \to \mathcal{D}^1(\mathcal{F}(X), \mathcal{F}(Y))$, such that $\mathcal{F}$ takes identities to identities and preserves the composition.

(ii) A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{F}\colon \mathcal{C}^{\mathrm{opp}} \to \mathcal{D}$.

Sometimes one says *covariant functor* instead of a functor, as opposed to a contravariant functor that, when viewed as a map $\mathcal{C} \to \mathcal{D}$, reverses the arrows (i.e. maps $\mathcal{C}^1(X,Y)$ to $\mathcal{D}^1(Y,X)$).

Example 1.2.2. (i) Very important functors are obtained by fixing one argument in Hom. Namely, any $X \in \mathcal{C}^0$ defines two functors h_X and h^X to Sets via $\mathcal{F}(T) = \mathrm{Hom}(T,X)$ and $\mathcal{G}(T) = \mathrm{Hom}(T,X)$. The first one is contravariant on $\mathcal{C}$ while the second one is covariant. For example, in the category Vect_k of k-vector spaces the dual space is a contravariant functor $V \mapsto V^* = \mathrm{Hom}(V,k)$.

(ii) Another family of important functors is given by free objects of an appropriate category. For example, $S \mapsto \mathbf{Z}[S]$ is such a functor from Sets to commutative unital rings, while $S \mapsto \oplus_{s\in S}\mathbf{Z}$ is such a functor from Sets to Ab.

1.3. **Natural transformations.**

Definition 1.3.1. (i) A *morphism* or a *natural transformation* $\phi\colon \mathcal{F} \to \mathcal{G}$ between functors $\mathcal{F},\mathcal{G}\colon \mathcal{C} \to \mathcal{D}$ is a family of morphisms $\phi_X : \mathcal{F}(X) \to \mathcal{G}(X)$ for any $X \in \mathcal{C}^0$ such that for any $f\colon X \to Y$ in $\mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

(ii) The family of all natural transformations between $\mathcal{F}$ and $\mathcal{G}$ is denoted $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ or $\mathrm{Trans}(\mathcal{F},\mathcal{G})$ or $\mathrm{Mor}(\mathcal{F},\mathcal{G})$.

Remark 1.3.2. The name natural transformation comes form the following point of view: for each X we give the transformation ϕ_X and the compatibility condition means that the transformation is natural. On one hand, this condition is very important and one should always remember that it should be checked in proofs and constructions. On the other hand, usually it is check reduces to a triviality (often rather messy), so as a rule such a check "is left to the reader". Informally speaking, usually construction of a transformation is natural if it makes no "random choices" (such as in the axiom of choice).

Definition 1.3.3. Given categories $\mathcal{C}$ and $\mathcal{D}$, the functors between them form a category $\mathcal{D}^{\mathcal{C}}$, also denoted as $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, $\mathrm{HOM}(\mathcal{C},\mathcal{D})$, and $\mathrm{Mor}(\mathcal{C},\mathcal{D})$. The morphisms in this category are natural transformations between functors.

Remark 1.3.4. It is problematic to consider all categories (they do not form a class), but one can consider the class Cat of all small categories. It forms so called 2-category (from two-dimensional category), whose objects are categories, and morphisms between two objects form a category and not a set. For this reason, one usually writes $\mathrm{HOM}_{\mathcal{C}}(X,Y)$ to denote the category of morphisms between objects of a 2-category $\mathcal{C}$. The objects of $\mathrm{HOM}_{\mathcal{C}}(X,Y)$ are called morphisms or 1-morphisms, while its morphisms are called 2-morphisms of $\mathcal{C}$. So, 2-morphisms of Cat are natural transformations. We do not describe here the axiomatic of 2-categories (which is slightly more involved, e.g. there are two ways to compose 2-morphisms – so called vertical and horizontal compositions), but the reader may try to guess what it should be using Cat as a model example of a 2-category.

2. Yoneda lemma

2.1. **Functors h_X and h^X.** The following two functors will turn out to be very important in the sequel.

Definition 2.1.1. (i) For any $X \in \mathcal{C}$ set $h_X(T) = \mathrm{Hom}(T, X)$, and for any $f\colon T \to T'$ let $h_X(f)\colon h_X(T') \to h_X(T)$ denotes the map induced by composition with f. This defines a functor $h_X\colon \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$.

(ii) For any $X \in \mathcal{C}$ set $h^X(T) = \mathrm{Hom}(X, T)$, and for any $f\colon T \to T'$ let $h^X(f)\colon h^X(T) \to h^X(T')$ denotes the map induced by composition with f. This defines a functor $h^X\colon \mathcal{C} \to \mathrm{Sets}$.

There are many trivial things to check around this definition. The main point here is to gain some practice, so we suggest that the reader checks the following claims or at least convince himself that everything works fine.

Exercise 2.1.2. (i) Check that, indeed, h_X and h^X are functors.

(ii) Check that any morphism $f\colon X \to X'$ induces the natural transformation $h^-(f)\colon h_X \to h_{X'}$ given by the maps $h^T(f)\colon h_X(T) \to h_{X'}(T)$. Furthermore, check that $h^-\colon \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{opp}}, \mathrm{Sets}) = \mathrm{Sets}^{\mathcal{C}^{\mathrm{opp}}}$ is a functor. Check that in the same way, f induces a natural transformation $h_-(f)\colon h^{X'} \to h^X$ and $h_-\colon \mathcal{C}^{\mathrm{opp}} \to \mathrm{Hom}(\mathcal{C}, \mathrm{Sets}) = \mathrm{Sets}^{\mathcal{C}}$ is a functor.

(iii) Show that $\mathrm{Hom}(\cdot, \cdot)\colon \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathrm{Sets}$ is a bifunctor from $\mathcal{C}$ to sets whose dependence on the first (resp. second) argument is contravariant (resp. covariant).

Remark 2.1.3. Note that the functors h_X and h^X are obtained by fixing an argument in the bifunctor Hom. The transformations between them come from the functoriality of Hom in the fixed argument. In particular, h_X is a contravariant functor, but its transformations are covariant, while the situation with h^X is opposite.

2.2. **Representable functors.**

Definition 2.2.1. A functor $\mathcal{F}\colon \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$ (resp. $\mathcal{F}\colon \mathcal{C} \to \mathrm{Sets}$) is called *representable* (resp. *corepresentable*) if there is a (natural) isomorphism of functors $\psi : \mathcal{F} \widetilde{\to} h_X$ (resp. $\psi\colon \mathcal{F} \widetilde{\to} h^X$). In such case one says that X *represents* the functor $\mathcal{F}$ (where ψ is a part of this universal picture).

Remark 2.2.2. (i) Often one says representable instead of corepresentable since there is no place for mistake (the sources in the above definition are different).

(ii) This definition formalizes in various situations the notion of a universal object. Here, X is the universal object for $\mathcal{F}$ and the isomorphism ψ. Without fixing ψ one can only determine the isomorphism class of X, but the isomorphism is not unique.

The following result shows that once $\mathcal{F}$ is fixed, the universal object is unique up to a unique isomorphism. It is a part of Yoneda lemma that will be proved in §2.3. So, the reader may practice his skills and prove it now, or skip to §2.3.

Exercise 2.2.3. (i) Assume that $\psi : \mathcal{F} \widetilde{\to} h_X$ and let $x \in \mathcal{F}(X)$ be the object corresponding to Id_X, i.e. $x = \psi(X)^{-1}(\mathrm{Id}_X)$. Let $Y \in \mathcal{C}^0$ and $y \in \mathcal{F}(Y)$, and let $f\colon Y \to X$ be the morphism corresponding to y, i.e. $f = \psi(Y)(y)$. Show that y is induced from x via f, i.e. $y = \mathcal{F}(f)(x)$.

(ii) Show that any other isomorphism $\mathcal{F}\widetilde{\to}h_{X'}$ is induced from ψ via uniquely defined isomorphism $X\widetilde{\to}X'$. In particular, X is essentially unique.

(iii) Deduce that the functors h_- and h^- from Exercise 2.1.2 take non-isomorphic objects to non-isomorphic functors.

2.3. **Yoneda lemma and applications.** If S is a set then the incidence relation induces the diagonal embedding $S \hookrightarrow \{0,1\}^S$. Somewhat similarly, any category $\mathcal{C}$ admits a functor $h_- : \mathcal{C} \hookrightarrow \mathrm{Sets}^{\mathcal{C}^{\mathrm{opp}}}$, and we will now prove that it is an embedding in the sense of categories, i.e. a fully faithful functor. Actually, we will prove even a stronger statement.

Theorem 2.3.1 (Yoneda Lemma). *(i) If $\mathcal{F}\colon \mathcal{C}^{\mathrm{opp}} \to \mathrm{Sets}$ is a functor then for any $X \in \mathcal{C}^0$ there exists a bijection $\alpha_X\colon \mathrm{Mor}(h_X,\mathcal{F})\widetilde{\to}\mathcal{F}(X)$ functorial in X, i.e. for any $f\colon X \to X'$ one has that $\alpha_X = h^-(f)\circ\alpha_{X'}$.*

(ii) If $\mathcal{F}\colon \mathcal{C} \to \mathrm{Sets}$ is a functor then for any $X \in \mathcal{C}^0$ there exists a bijection $\alpha_X\colon \mathrm{Mor}(h^X,\mathcal{F})\widetilde{\to}\mathcal{F}(X)$ functorial in X.

Proof. We will check (ii), and the proof of (i) is obtained by switching to $\mathcal{C}^{\mathrm{opp}}$. Also, all stages of the construction will be easily seen to be natural in X, so we will skip these checks. To any morphism $\phi\colon h^X \to \mathcal{F}$ we associate the element $\alpha(\phi) = \phi(X)(\mathrm{Id}_X)$ of $\mathcal{F}(X)$. This gives a map $\alpha\colon \mathrm{Mor}(h^X,\mathcal{F}) \to \mathcal{F}(X)$. Conversely, given an element $u \in \mathcal{F}(X)$, to any $f\colon X \to Y$ in $h^X(Y)$ we can associate $\mathcal{F}(f)(u) \in \mathcal{F}(Y)$. Functoriality of this construction in Y follows from the fact that $\mathcal{F}$ preserves compositions, hence we obtain a map $\beta\colon \mathcal{F}(X) \to \mathrm{Mor}(h^X,\mathcal{F})$. Clearly, $\alpha\circ\beta$ is the identity on $\mathcal{F}(X)$, hence it remains to show that α is injective. In other words, we want to show that any $\phi\colon h^X \to \mathcal{F}$ is uniquely determined by the element $\phi(X)(\mathrm{Id}_X)$ of $\mathcal{F}(X)$.

It now suffices for any morphism $f\colon X \to Y$ to give a formula for $\phi(f) \in \mathcal{F}(Y)$ in terms of $u := \phi(X)(\mathrm{Id}_X)$. We have the following commutative square

$$\begin{tikzcd} \mathrm{Hom}(X,X) \arrow[r, "\phi(X)"] \arrow[d, "h^X(f)"] & \mathcal{F}(X) \arrow[d, "\mathcal{F}(f)"] \\ \mathrm{Hom}(X,Y) \arrow[r, "\phi(Y)"] & \mathcal{F}(Y) \end{tikzcd}$$

Since f is the image of Id_X under $h^X(f)$, it suffices to find the image of Id_X in $\mathcal{F}(Y)$. For this we can go the other way around: its image in $\mathcal{F}(X)$ is u, so its image in $\mathcal{F}(Y)$ is $\mathcal{F}(f)(u)$. $\square$

Corollary 2.3.2. *The functors $h_-\colon \mathcal{C} \hookrightarrow \mathrm{Sets}^{\mathcal{C}^{\mathrm{opp}}}$ and $h^-\colon \mathcal{C}^{\mathrm{opp}} \hookrightarrow \mathrm{Sets}^{\mathcal{C}}$ are fully faithful.*

Proof. By Yoneda Lemma we have the isomorphisms $\alpha_{X,Y}\colon h_Y(X)\widetilde{\to}\mathrm{Hom}(h_X,h_Y)$. So, we should prove that they coincide with the h_--functoriality maps

$$h_-(X,Y)\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(h_X,h_Y)$$

that, by the very definition, associate to $f\colon X \to Y$ the transformation ϕ_f such that $\phi_f(T)\colon h_X(T) \to h_Y(T)$ is the composition with f. From the construction of α's, we know that $\alpha_X(f)$ is the unique transformation $\phi\colon h_X \to h_Y$ that takes $\mathrm{Id}_X \in h_X(X)$ to $f \in h_Y(X)$. By uniqueness, $\phi = \phi_f$ and we obtain that $h_-(X,Y) = \alpha_{X,Y}$, as claimed. The second claim is proved in the same fashion. $\square$

3. Limits: naive approach

3.1. Diagrams.

Definition 3.1.1. (i) A *diagram* in $\mathcal{C}$ is a functor $D: \mathcal{I} \to \mathcal{C}$ from a small category I.

(ii) A diagram I is called *discrete* (resp. *empty*) if so is I, that is, I^1 (resp. I^0) is empty.

To simplify notation, we will often view D as the family $\{D_i\}_{i\in I}$ of objects of $\mathcal{C}$ indexed by some indexing set I. Note that the same objects can repeat few times. Unless the diagram is discrete, there are also non-trivial morphisms in the picture that will sometimes be denoted as $\{\alpha_{ijk}\}_{k\in I^1(i,j)}$ for any $i, j \in I^0$.

Definition 3.1.2. (i) We say that a diagram $\{D_i\}_{i\in I}$ is a *direct system* (in $\mathcal{C}$) if (1) I is a poset, (2) if $i \le j$ in I then the set $I(i,j)$ contains a single element $\alpha_{ij}: D_i \to D_j$, and the set $I(i,j)$ is empty otherwise. We say that the direct system is *filtered* if so is the poset I, i.e. for any $i, j \in I$ there exists $k \in I$ with $i, j \le k$.

(ii) We say that a diagram $\{D_i\}_{i\in I}$ is an *inverse system* (in $\mathcal{C}$) if (1) I is a poset, (2) if $i \ge j$ in I then the set $I(i,j)$ contains a single element $\alpha_{ij}: D_i \to D_j$, and the set $I^1(i,j)$ is empty otherwise. We say that the inverse system is *filtered* if so is the poset I, i.e. for any $i, j \in I$ there exists $k \in I$ with $i, j \le k$.

Example 3.1.3. (i) All subsets (resp. finite subsets) of a set S form a filtered direct system with respect to inclusions.

(ii) All finite quotients of a group G form an inverse system with respect to quotients. This system is filtered when the homomorphism from G to its profinite completion is injective, but it does not have to be filtered in general.

3.2. Limits and colimits.

Definition 3.2.1. (i) If X is an object of $\mathcal{C}$ then a *morphism* $f: X \to D$ to a diagram D is a family of morphisms $f_i: X \to D_i$ compatible with respect to the morphisms of D, i.e. for any $\alpha_{ijk}: D_i \to D_j$ one has that $\alpha_{ijk} \circ f_i = f_j$.

(ii) Similarly, a morphism $D \to X$ is a family of morphisms $g_i: D_i \to X$ such that $g_i \circ \alpha_{ijk} = g_j$ for any $\alpha_{ijk}: D_i \to D_j$.

Remark 3.2.2. (i) One can compose morphisms to/from diagrams with morphisms of objects. For example, if $\phi: T \to X$ is in $\mathcal{C}^1$ and $f: X \to D$ is as above then $f' = f \circ \phi$ is defined as the set of morphisms $f'_i = f_i \circ \phi$.

(ii) We will later define also morphisms between diagrams with the same categories of indexes.

Definition 3.2.3. Let $h_D(T)$ (resp. $h^D(T)$) be the set of all morphisms $T \to D$ (resp. $D \to T$). Then by the above remark we actually obtain a functor $h_D: \mathcal{C}^{\overline{p}} \to$ Sets (resp. $h^D: \mathcal{C} \to$ Sets).

Definition 3.2.4. (i) A limit

$$X = \lim D = \lim_{\mathcal{C}} D$$

of a diagram D is a universal morphism $u: X \to D$, i.e. any other morphism $T \to D$ is the composition of u with a uniquely defined morphism $T \to X$. In other words, X is the object that represents h_D.

(ii) A colimit

$$X = \operatorname{colim} D = \operatorname{colim}_{\mathcal{C}} D$$

of a diagram D is a universal morphism $u\colon D \to Z$, i.e. any other morphism $D \to T$ is the composition of u with a uniquely defined morphism $Z \to T$. In other words, Z is the object that represents h^D.

We will include the category into the notation only when this might be informative for the reader. Note that the limit is defined uniquely up to a unique isomorphism. (Often one says up to a canonical isomorphism.) To simplify notation, we may thus write $X = \lim D$ – it is a standard abuse of language to write the sign of equality when the object is defined up to a unique isomorphism. We will also write $X = \lim_{i\in I} D_i$ or $Z = \operatorname{colim}_{i\in I} D_i$, although the latter notation ignores the morphisms of D.

Remark 3.2.5. Any diagram D induces an opposite (or dual) diagram D^{opp} in the opposite category $\mathcal{C}^{\mathrm{opp}}$: it has the same objects and reverse arrows. It follows from the very definitions that lim and colim are dual notions in the sense that $\lim D = \operatorname{colim} D^{\mathrm{opp}}$.

3.3. **Certain classes of limits.** Let us now discuss various classes of limits/colimits that have a special name. We start with limits.

(0) The limit of the empty diagram is called the *final* object of $\mathcal{C}$. It is the object $X = \mathbf{1}_{\mathcal{C}}$ such that any set $\operatorname{Hom}(T, X)$ consists of an element. For example, $\mathbf{1}_{\mathrm{Sets}}$ is a one element set, $\mathbf{1}_{\mathrm{Groups}} = \mathbf{1}$, $\mathbf{1}_{\mathrm{Ab}} = \mathbf{0}$, and the final object of various categories of rings (commutative, associative, unital, etc.) is the zero ring.

(1) If D is discrete then $\lim D$ is called the *product* of elements of D and is denoted $\prod_{i\in I} D_i$. In particular, the final object is the empty product (that explains why $0! = 1$). Products in the categories algebraic categories, such as Sets, Rings, Groups, and Ab, you the usual products you are familiar with.

(2) If D consists of two morphisms $f\colon Y \to X$ and $g\colon Z \to X$ then $\lim D$ is called *fibered product* of Y and Z over X or *pullback* and is denoted $Y \times_X Z$. In Sets it is the subset of $Y \times Z$ given by $f(y) = g(z)$, and the same is true for algebraic categories.

(3) If D consists of two morphisms $f, g\colon X \to Y$ then $\lim D$ is called the *equalizer* of f and g and is denoted $\operatorname{Eq}(X \rightrightarrows Y)$ or (sometimes) $\operatorname{Ker}(X \rightrightarrows Y)$. For sets it consists of elements $x \in X$ with $f(x) = g(x)$, and the same is true for algebraic categories.

(4) Finally, if D is a (resp. filtered) inverse system then $\lim D$ is called the (resp. filtered) *inverse limit* and is denoted $\operatorname{proj\,lim}_{i\in I} D_i$.

Exercise 3.3.1. (i) Show that $\prod_{i\in I \coprod J} D_I = (\prod_{i\in I} D_i) \times (\prod_{j\in J} D_j)$ and use this to give another explanation to the fact that the empty product of sets is a one element set.

(ii) Check that the product in Top is the Tychonoff product (which is not the box product when the product is infinite).

(iii) Let $\mathcal{C} = $ Sets. Check that the fiber of $Y \times_X Z$ over $x \in X$ is the product of the fibers of Y and Z over x (hence the name "fibered product").

Remark 3.3.2. In all examples, the limits in the categories Sets, Rings, Groups, and Ab agreed. In other words, the forgetful functors Groups $\to$ Sets, Rings $\to$ Ab, etc., commute with limits. We will later see that this was not a coincidence.

3.4. **Certain classes of colimits.** Similarly to the above description, analogous classes of colimits also have special names.

(0) The colimit of the empty diagram is called the *initial* object of $\mathcal{C}$. It is the object $X = \mathbf{0}_\mathcal{C}$ such that any set $\mathrm{Hom}(X, Y)$ consists of an element. For example, $\mathbf{0}_{\mathrm{Sets}} = \emptyset$, $\mathbf{0}_{\mathrm{Groups}} = \mathbf{1}$, $\mathbf{0}_{\mathrm{Ab}} = \mathbf{0}$, the initial object of categories of non-unital rings is 0, while in their unital analogs the initial object is $\mathbf{Z}$, since the homomorphisms should preserve 1.

(1) If D is discrete then $\operatorname{colim} D$ is called the *coproduct* of elements of D and is denoted $\coprod_{i\in I} D_i$. In particular, the initial object is the empty coproduct. Morally, coproducts in algebraic categories may be rather complicated and they are not compatible with forgetful functors. For example, coproduct in Sets is the disjoint union, coproduct in Ab is the direct sum (so it coincides with the direct product for finite diagrams, but differ from it in general), coproduct of groups is a free or amalgamated product, coproduct of finitely many commutative unital rings is their tensor product over $\mathbf{Z}$. Morally, in all these cases $\coprod_i D_i$ is obtained by taking the disjoint union of generators of D_i and imposing the disjoint union of relations they satisfy in D_i's.

(2) If D consists of two morphisms $f\colon X \to Y$ and $g\colon X \to Z$ then $\operatorname{colim} D$ is called *pushout* of Y and Z with respect to X and is denoted $Y \coprod_X Z$.

(3) If D consists of two morphisms $f, g\colon X \to Y$ then $\operatorname{colim} D$ is called the *co-equalizer* of f and g and is denoted $\mathrm{Coeq}(X \rightrightarrows Y)$ or (sometimes) $\mathrm{Coker}(X \rightrightarrows Y)$. In an algebraic category, it is always the quotient of Y by the minimal equivalence relation (normal subgroup, two-sided ideal, etc.) generated by the relations $f(x) \sim g(x)$ for any $x \in X$ (resp. $xy^{-1} = 1$, resp. $x - y = 0$, etc.).

(4) Finally, if D is a (resp. filtered) direct system then $\operatorname{colim} D$ is called the (resp. filtered) *direct limit* and is denoted $\operatorname{inj\,lim}_{i\in I} D_i$.

Exercise 3.4.1. (i) Show that any set is the filtered direct limit of its finite subsets, any group is the direct filtered limit of its finitely generated subgroups, etc.

(ii) We saw that almost all colimits in algebraic categories do not commute with the forgetful functors, but there is one important exception: show that filtered direct limits in algebraic categories commute with filtered direct colimits.

3.5. **Limits and colimits of sets.** It turns out that despite the variety of different limits, there is a general way to construct them in terms of products and equalizers (and similarly for the dual theory). It will be convenient to deal with sets first.

Proposition 3.5.1. *Let D be a diagram of sets, then*

(i) $\lim D$ *exists and can be described as the subset of* $\prod_{i\in I^0} D_i$ *that consists of elements* $(d_i)_{i\in I^0}$ *such that* $f(d_i) = d_j$ *for any* $f\colon D_i \to D_j$ *in* D.

(ii) $\operatorname{colim} D$ *exists and can be described as the quotient of* $\coprod_{i\in I^0} D_i$ *by the equivalence relation generated by the relations* $d_i \sim d_j$ *whenever* $f(d_i) = d_j$ *for some* $f\colon D_i \to D_j$ *in* D.

Proof. Direct check. □

Corollary 3.5.2. *Keep the above notation, then*

(i) For any $\alpha_{ijk}\colon D_i \to D_j$ *in* D *let* $p_f\colon \prod_{l\in I} D_l \to D_j$ *be the projection and let* $q_f\colon \prod_{l\in I} D_l \to D_j$ *be the projection onto* D_i *composed with* f*. Then the maps* $p, q\colon \prod_{i\in I} D_i \rightrightarrows \prod_{\alpha_{ijk}\in D^1} D_j$ *arise, and* $\lim D$ *is the equalizer of* p *and* q*.*

(ii) For any $\alpha_{ijk}\colon D_i \to D_j$ *in* D *let* $p_f\colon D_i \to \coprod_{l\in I} D_l$ *be the embedding and let* $q_f\colon D_i \to \prod_{l\in I} D_l$ *be the composition of* f *with the embedding of* D_j *into the coproduct. Then the maps* $p, q\colon \coprod_{\alpha_{ijk}\in D^1} D_j \rightrightarrows \coprod_{i\in I} D_i$ *arise, and* $\operatorname{colim} D$ *is the coequalizer of* p *and* q*.*

Proof. Using the explicit formula for the equalizer (either from §3.3 (3) or from the above proposition), one can easily describe the equalizer of p and q on the level of elements, and then it follows from Proposition 3.5.1 that this is precisely the limit of D. This establishes (i) and (ii) is done similarly. □

This corollary will allow us to reinterpret the notion of limit/colimit in an arbitrary category. Assume that D be a diagram in a category $\mathcal{C}$, $T \in \mathcal{C}^0$ and consider the diagram $h^T(D)$ of sets: its objects are $h^T(D_i) = h_{D_i}(T)$ and its morphisms are $h^T(\alpha_{ijk})$. The diagram $h_T(D)$ is defined similarly.

Corollary 3.5.3. *Keep the above notation, then*

(i) $h_D(T)\widetilde{\to} \lim_{\mathrm{Sets}} h^T(D)$*, in particular,* $\lim_{\mathcal{C}} D$*, if exists, is the object that represents the functor* $\lim_{\mathrm{Sets}} h^-(D)$*.*

(ii) $h_D(T)\widetilde{\to}\operatorname{colim}_{\mathrm{Sets}} h^T(D)$*, in particular,* $\operatorname{colim}_{\mathcal{C}} D$*, if exists, is the object that corepresents the functor* $\lim_{\mathrm{Sets}} h_-(D)$*.*

Proof. As usually, it is enough to establish (i). To give a morphism $f\colon T \to D$ is the same as to give morphisms $f_i\colon T \to D_i$ compatible with the diagram, but the latter means that we pick elements $f_i \in h^T(D_i)$ that are compatible with the morphisms α_{ijk}, i.e. are taken one to another by the maps $h^T(\alpha_{ijk})$. Thus, the latter gadget is a family (f_i) compatible with the maps of the diagram $h^T(D)$. By Proposition 3.3(i), we can thus identify it with an element of $\lim_{\mathrm{Sets}} h^T(D)$. □

Remark 3.5.4. The terminology suggests that limits are slightly "more basic" or "simpler" objects than colimits (though we can always pass from one to another using duality). Note that both limits and colimits are expressed in terms of *limits* of sets. This gives an additional justification for the terminology.

3.6. **Complete and cocomplete categories.**

Definition 3.6.1. A category $\mathcal{C}$ is called *complete* (resp. *cocomplete*) if all limits (resp. colimits) exist in $\mathcal{C}$.

Proposition 3.6.2. *Let* $\mathcal{C}$ *be a category with a diagram* D*, then*

(i) For any $\alpha_{ijk}\colon D_i \to D_j$ *in* D *let* $p_f\colon \prod_{l\in I} D_l \to D_j$ *be the projection and let* $q_f\colon \prod_{l\in I} D_l \to D_j$ *be the projection onto* D_i *composed with* f*. Then the maps* $p, q\colon \prod_{i\in I} D_i \rightrightarrows \prod_{\alpha_{ijk}\in D^1} D_j$ *arise, and* $\lim D$ *is the equalizer of* p *and* q*.*

(ii) For any $\alpha_{ijk}\colon D_i \to D_j$ *in* D *let* $p_f\colon D_i \to \coprod_{l\in I} D_l$ *be the embedding and let* $q_f\colon D_i \to \prod_{l\in I} D_l$ *be the composition of* f *with the embedding of* D_j *into the coproduct. Then the maps* $p, q\colon \coprod_{\alpha_{ijk}\in D^1} D_j \rightrightarrows \coprod_{i\in I} D_i$ *arise, and* $\operatorname{colim} D$ *is the coequalizer of* p *and* q*.*

Proof. As usually, it is enough to prove only one part, but let us check (ii) this time. We should check that the functor h^D is isomorphic to h^E, where E is the diagram $p, q\colon X \rightrightarrows Y$ and X, Y are the coproducts from (ii). By Corollary 3.5.3, $h^D(T)\widetilde{\to}\lim_{\mathrm{Sets}} h_T(D)$. Similarly,

$$h^E(T)\widetilde{\to}\lim h_T(E) = \mathrm{Eq}\left(h_T(Y) \rightrightarrows h_T(X)\right)\widetilde{\to}\mathrm{Eq}\left(\prod_{i\in I} h_T(D_i) \rightrightarrows \prod_{\alpha_{ijk}} h_T(D_j)\right)$$

where limits, products and equalizers are taken in Sets. We already proved in Corollary 3.5.2 the isomorphism of (i) for sets, hence the right hand side above is isomorphic to $\lim h_T(D)$ and we obtain that $\psi_T\colon h^D(T)\widetilde{\to}h^E(T)$. It remains to check that the isomorphisms ψ_T are natural, i.e. are compatible with the morphisms $T \to T'$, and we leave it to the reader. □

Exercise 3.6.3. Complete the above proof. (Hint: there is a direct diagram chasing proof, but a smarter approach is to replace (as we already did) $h^D(T)$ and $h^E(T)$ with $h_T(D)$ and $h_T(E)$ and use functoriality of h_-.)

Corollary 3.6.4. *Given a category $\mathcal{C}$ the following conditions are equivalent:*
(i) $\mathcal{C}$ is complete (resp. cocomplete),
(ii) products and equalizers (resp. coproducts and coequalizers) exist in $\mathcal{C}$,
(iii) products and fibered products (resp. coproducts and pushouts) exist in $\mathcal{C}$.

Proof. Clearly, (i) implies (ii) and (iii), and (ii) implies (i) by Proposition 3.6.2. The remaining implication (iii) $\Longrightarrow$ (ii) follows form the following lemma. □

Lemma 3.6.5. *Let D be a diagram of the form $X \rightrightarrows Y$, then*
(i) $\mathrm{Eq}(D)\widetilde{\to}X \times_{(X\times X)} (X \times_Y X)$ *whenever the righthand side is defined.*
(ii) $\mathrm{Coeq}(D)\widetilde{\to}Y \coprod_{(Y\coprod Y)} (Y \coprod_X Y)$ *whenever the righthand side is defined.*

Proof. For sets (i) is checked by use of Proposition 3.5.1(i). Both (i) and (ii) for a general category $\mathcal{C}$ are reduced to this case by applying h^- or h_-, respectively. □

Remark 3.6.6. It follows rather easily from Proposition 3.6.2 that various algebraic categories, such as Sets, Groups,Ab, Rings, etc. are complete. In addition, we observed in §3.4(3) how to construct coproducts and coequalizers in algebraic categories, hence they are also cocomplete.

4. Adjoint functors

4.1. **A criterion of continuity.** Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ be two functors in opposite directions. As we discussed, an important example of such a pair is when the functors are essentially inverse. One might expect that the case when $\mathcal{F}$ is only a right inverse of $\mathcal{G}$ should also appear in various situation, but rather surprisingly it turns out that a far more important case is when the functors are adjoint in the sense that we define below.

Definition 4.1.1. Keep the above notation. One says that $\mathcal{F}$ is *right adjoint* to $\mathcal{G}$, or $\mathcal{G}$ is *left adjoint* to $\mathcal{F}$ is there exists a family of isomorphisms

$$\phi_{X,Y}\colon \mathrm{Hom}_{\mathcal{D}}(X, \mathcal{F}(Y))\widetilde{\to}\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(X), Y)$$

that are natural both in $X \in \mathcal{D}^0$ and $Y \in \mathcal{C}^0$. In this case one also says that $\mathcal{F}$ and $\mathcal{G}$ are *adjoint* functors or form an *adjoint pair.*

Remark 4.1.2. In the notation $\mathcal{F}$ stands on the right, hence the name "right adjoint". Nevertheless, it is often very confusing to remember which functor is left or right adjoint. A good heuristics is that in simple cases right adjoint is usually a stupid functor that forgets information, while left adjoint is a creative functor that constructs new things.

Example 4.1.3. The free object on a set S in an algebraic category $\mathcal{C}$ is by definition an object $\mathcal{G}(S)$, where $\mathcal{G}$ is the left adjoint to the forgetful functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{Sets}$.

Importance of adjoint functors is illustrated, in particular, by the following feature.

Definition 4.1.4. A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is *continuous* (resp. *cocontinuous*) if it preserves all limits (resp. colimits), i.e. if a diagram I in $\mathcal{C}$ possesses a limit then $\mathcal{F}(\lim_{\mathcal{C}} I) \widetilde{\to} \lim_{\mathcal{D}} \mathcal{F}(I)$ (and similarly, for colimits).

Proposition 4.1.5. *If $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is a right (resp. left) adjoint of a functor $\mathcal{G}$ then $\mathcal{F}$ is continuous (resp. cocontinuous).*

Proof. Enough to deal with the case when $\mathcal{F}$ is right adjoint. Then for any limit $X = \lim I$ in $\mathcal{C}$ and any $T \in \mathcal{D}^0$ we have that

$$\mathrm{Hom}_{\mathcal{D}}(T, \mathcal{F}(X)) \widetilde{\to} \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(T), X) \widetilde{\to} \lim \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(T), I) \widetilde{\to} \lim \mathrm{Hom}_{\mathcal{D}}(T, \mathcal{F}(I))$$

hence $\mathcal{F}(X)$ is the limit of $\mathcal{F}(I)$, as claimed. $\square$