Math599, HW9

1. Mandatory problems

Please submit these problems by December 28 midnight.

An algebraic variety (or a scheme) is called *quasi-affine* if it can be embedded as an open subvariety into an affine variety. Here is an example of a quasi-affine but not affine variety.

1. Let k be an algebraically closed field and consider the open subvariety $V = \mathbf{A}^2_k \setminus \{0\}$ of $\mathbf{A}^2_k$ (the affine plane punched at the origin). Prove the the quasi-affine variety V is not affine. (Hint: show that $\mathcal{O}_V(V) = k[x, y]$ and so not all maximal ideals of the ring of regular functions on V correspond to the points of V.)

2. Show that $\mathbf{P}^n_k$ with $n > 0$ is not quasi-affine. (Hint: compute the ring of regular functions on $\mathbf{P}^n_k$ and show that it is "too small".)

3. Let A be a finitely generated k-algebra provided with a grading $A = \oplus_{d\in\mathbf{N}} A_d$ such that $A_0 = k$. Let $X = \mathrm{MaxProj}(A)$ denote the set of maximal homogeneous ideals not equal to $\oplus_{d>0} A_d$, and provide X with the topology whose basis is formed by the sets X_h, where $h \in A_d$ is homogeneous and X_h is the set of ideals not containing h. A function $f\colon X \to k$ is called *regular* if X can be covered by $X_i = X_{h_i}$ with $h_i \in A_{d_i}$ such that for each i we have that $f|_{X_i} = f_i/h_i^{n_i}$, where $n_i \in \mathbf{N}$ and $f_i \in A_{n_i d_i}$. Let $\mathcal{O}_X$ denote the sheaf of regular functions on X.

(i) Prove that $\{X_{h_i}\}_{i\in I}$ cover X if and only if the elements h_i generate the ideal $\oplus_{d>0} A_d$.

(ii) Prove that $(X, \mathcal{O}_X)$ is an algebraic variety and for any homogeneous $h \in A_d$ we have that $(X_h, \mathcal{O}_X|_{X_h})$ is an open affine subvariety associated to the weight-zero component B_0 of the localization $B = A[h^{-1}]$ provided with the natural grading $B = \oplus_{d\in\mathbf{Z}} B_d$.

4. Let M be a noetherian module.

(i) Show that if $u : M \twoheadrightarrow M$ is a surjective homomorphism then u is an isomorphism. (Hint: look at $\mathrm{Ker}(u^n)$.)

(ii) Give an example of a noetherian M with a non-isomorphic embedding $u : M \hookrightarrow M$.

5. Let N be an artinian module.

(i) Show that if $v : N \hookrightarrow N$ is an injective homomorphism then u is an isomorphism.

(ii) Give an example of an artinian N with a non-isomorphic surjective homomorphism $v : N \twoheadrightarrow N$. (Hint: you have to take N which is artinian but not noetherian.)

2. Non-mandatory problems

Non-mandatory problems – do not submit them but I will be glad to discuss them if you wish.

6*. Generalize problem 3 to schemes. Namely, for any graded ring $A = \oplus_{d\in\mathbf{N}} A_d$ provide the set $\mathrm{Proj}(A)$ of prime homogeneous ideals not equal to $\oplus_{d>0} A_d$ with a structure of a scheme covered by affine schemes of the form $\mathrm{Spec}((A[h^{-1}])_0)$ with homogeneous h.

7*. We know that if A is noetherian then any its localization A_p is noetherian. Show that the converse is not true in general. Namely, construct a non-noetherian A such that any localization A_p is noetherian. (Hint: take A to be an infinite product of fields and show that any its localization is a field.)

8*. Show that A is noetherian if and only if any prime ideal of A is finitely generated. (Hint: this is exercise 1 after chapter 7; look at the hint to that exercise.)