Math599, HW7

1. Mandatory problems

Please submit these problems by December 14 midnight.

Disclaimer: the problems are new, so some mistakes may happen. If you think you see a question with a mistake please contact me.

Definition 1.1. Recall that we defined the Zariski topology on the affine spaces $\mathbf{A}_k^n$ by taking the closed sets to be (affine) varieties $V = V(I) \subseteq \mathbf{A}_k^n$. The Zariski topology on any variety $V \subseteq \mathbf{A}_k^n$ is the topology induced from the Zariski topology of $\mathbf{A}_k^n$. In particular, closed subsets of V are precisely varieties $U \subseteq \mathbf{A}_k^n$ contained in V, so we will say that such U is a *closed subvariety* of V.

1. (i) Show that sending an ideal I of $k[V]$ to its set of zeros on V provides a bijection between the radical ideals of $k[V]$ and the closed subvarieties of V.

(ii) Show that if U, W are closed subsets of V corresponding to radical ideals I and J of $k[V]$, then $U \cup W$ corresponds to $I \cap J$, which is radical.

(iii) Keep above notation. Show that $U \cap W$ corresponds to the radical of $I + J$ and $k[U \cap W]$ is the reduction of $k[U] \otimes_{k[V]} k[W]$, in particular, $U \cap W = U \times_V W$. Give an example, where $I + J \neq \sqrt{I + J}$ and $k[U] \otimes_{k[V]} k[W]$ is not reduced.

2. Let $f\colon U \to V$ be a morphism of algebraic spaces and $\phi\colon k[V] \to k[U]$ the corresponding homomorphism.

(i) Show that if V' is a closed subvariety of V corresponding to $I \subseteq k[V]$ then $U' = f^{-1}(V')$ is a closed subvariety of U corresponding to the radical of $I^e = Ik[U]$ and $U' = U \times_V V'$.

(ii) Show that if U' is a closed subvariety of U corresponding to J, then $J^c = \phi^{-1}(J)$ is radical and corresponds to the Zariski closure V' of $f(U')$. Give an example when V' is strictly larger than $f(U')$.

Definition 1.2. Given a variety V with a point $x \in V$, consider the function ring $k[V]$ with the corresponding ideal m_x. Then $T^*_{V,x} := m_x/m_x^2$ is a vector space over $k = k[V]/m_x$ called the *cotangent space* to V at x. Its dual $T_{V,x} = (m_x/m_x^2)^*$ is called the *tangent space* at x.

3. (i) Show that for any point of $x \in \mathbf{A}_k^n$, the tangent and cotangent spaces to x are n-dimensional.

(ii) For any morphism $f\colon U \to V$ and points $x \in U$ and $y = f(x) \in V$ construct natural maps $\alpha_x\colon T_{U,x} \to T_{V,y}$ and $\alpha_x^*\colon T^*_{V,y} \to T^*_{U,x}$. (In fact, α_x^* is often called the differential of f at x.)

(iii) Show that if U is a closed subvariety of V then α_x is injective and α_x^* is surjective for any $x \in U$. Deduce that these spaces are always finite-dimensional.

(iv) Let C be the union of n coordinate axis in $\mathbf{A}^n$. Prove that C cannot be embedded into $\mathbf{A}^m$ with $m < n$. (Hint: show that the tangent space to C at the origin is n-dimensional.)

Remark 1.3. (i) The dimension $e_x = \dim_k(m_x/m_x^2)$ is an important invariant of the pair (V, x). In a more advanced theory (that we will probably not reach), one uses it to study smoothness of varieties: one always has that e_x is at least the dimension of V at x and the equality holds iff V is smooth at x. Moreover, e_x is called the embedding dimension because it is the minimal dimension of a smooth variety M so that V can be embedded into M locally at x. (This M does not have to be $\mathbf{A}^n$, though.)

(ii) There is a (rather easy) theorem that any smooth affine curve can be embedded into $\mathbf{A}^3$. We saw that for singular curves the situation is different.

4. Consider the map $f\colon X = \mathbf{A}^2_k \to Y = \mathbf{A}^2_k$ corresponding to the homomorphism $\phi\colon k[y_1, y_2] \to k[x_1, x_2]$ given by $\phi(y_1) = x_1, \phi(y_2) = x_1x_2$. For any point $y \in Y$ find the fiber $f^{-1}(y)$. In particular, show that almost all fibers consist of a single point (in other words, f is generically an isomorphism), there is a line of empty fibers, and one fiber is a line.

Remark 1.4. The map in problem 4 is a simplest example of a so-called blow up (or its affine chart). In this case, one "blows up" Y at the origin. We saw that the dimension of the fibers jumps at the origin. Such a thing can only happen for non-flat morphisms (i.e. morphisms with $k[X]$ non-flat over $k[Y]$). A rather difficult theorem states that the dimension is additive for flat morphisms:

$$\dim_x(X) = \dim_x(f^{-1}(y)) + \dim_y(Y),$$

i.e. the dimension of X at x is the dimension of Y at $y = f(x)$ plus the dimension at x of the fiber over y. In a loose sense, the geometric meaning of flatness is that the fibers of a morphism vary in a "continuous way".

The following exercise corrects what I said in class in the end about non-algebraically closed fields.

5. (i) Assume that k is an arbitrary field with algebraic closure $\overline{k}$. You can assume for simplicity that k is perfect, so $\overline{k}/k$ is separable. Deduce from the weak Nullstellensatz that $\mathrm{Max}(k[t_1, \dots, t_n]) = \overline{k}^n/\mathrm{Gal}_{\overline{k}/k}$, where any $\sigma \in \mathrm{Gal}_{\overline{k}/k}$ acts via $\sigma(a_1, \dots, a_n) = (\sigma a_1, \dots, \sigma a_n)$. (Hint: the idea is to compute the fibers of the map $\mathrm{Max}(\overline{k}[t_1, \dots, t_n]) \to \mathrm{Max}(k[t_1, \dots, t_n])$, which boils down to studying the $\overline{k}$-algebras $(k[t_1, \dots, t_n]/m) \otimes_k \overline{k}$. You can try to solve it now, or wait until the next class where a brief outline will be given.)

(ii) Show that $\overline{k}^n/\mathrm{Gal}_{\overline{k}/k} \neq (\overline{k}/\mathrm{Gal}_{\overline{k}/k})^n$ already for $k = \mathbf{R}$ and $n = 2$.

2. Non-mandatory problems

Non-mandatory problems – do not submit them but I will be glad to discuss them if you wish.

6. Compute $X \times_Y X$ in problem 4. Namely, show that it is a union of two components – one canonically isomorphic to X and another is also a copy of $\mathbf{A}^2_k$, which is canonically isomorphic to the square of the exceptional fiber of f.