Math599, HW4

1. Mandatory problems

Please submit these problems by November 23 midnight.

1. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ and $0 \to N_1 \to N_2 \to N_3 \to 0$ be two short exact sequences of R-modules with a family of compatible homomorphisms $f_i : M_i \to N_i$ for $1 \le i \le 3$. The snake lemma formulated in class produces an exact sequence

$$0 \to \mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \stackrel{\partial}{\to} \mathrm{Coker}(f_1) \to \mathrm{Coker}(f_2) \to \mathrm{Coker}(f_3) \to 0$$

We have constructed in class the connecting homomorphism ∂. Prove by a direct computation (or a diagram chase) that, indeed, the sequence is exact at the terms $\mathrm{Ker}(f_3)$ and $\mathrm{Coker}(f_1)$.

Recall that a module P is projective if the functor h^P is exact. Equivalently, P is projective if and only if for any surjective homomorphism $M \twoheadrightarrow N$ the induced map $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ is surjective. The latter property means that any homomorphism $P \to N$ lifts to a homomorphism $P \to M$. Similarly, recall that a module Q is injective if h_Q is exact. Equivalently, if $L \hookrightarrow M$ is an embedding of modules then the map $\mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(L, Q)$ is surjective, that is, any homomorphism $L \to Q$ extends to a homomorphism $M \to Q$.

2. (i) Show that if P is a projective module and $f : M \twoheadrightarrow P$ is a surjective homomorphism, then P splits off as a direct summand of M. In other words, $M \widetilde{\to} P \oplus Q$ so that $f = (\mathrm{Id}_P, 0)$.

(ii) Deduce that P is projective if and only if it is a direct summand of a free module, i.e. there exists a module Q such that $P \oplus Q$ is a free R-module.

(iii) Show that a finitely generated $\mathbf{Z}$-module is projective if and only if it is torsion-free, but $\mathbf{Q}$ is not a projective $\mathbf{Z}$-module.

3. (i) Show that if an injective module I is a submodule in a module M then I splits off as a direct summand, i.e. $M = I \oplus Q$.

(ii) Assume that R is a PID (i.e. a principal ideal domain). Show that a module I is injective iff it is *divisible*, i.e. for any $m \in I$ and $0 \ne r \in R$ there exists $x \in I$ with $rx = m$. (Hint: extend homomorphisms to I from a submodule L of M using Zorn's lemma.)

4. (From Exercise 11 in Atyah-Macdonald.) Let $\phi : R^l \to R^n$ be a homomorphism of free R-modules of finite rank. Show that if ϕ is an isomorphism then $l = n$, and if ϕ is surjective then $l \ge n$. (Hint: pick up a maximal ideal m and study $\phi \otimes_R R/m$.)

Recall that an R-module M is *finitely presented* if it admits a presentation $M = \mathrm{Coker}(R^l \to R^n)$ with finite l, n. Equivalently, there exists an exact sequence $R^l \to R^n \to M \to 0$.

5. Let M be a *finitely presented* R-module. Show that any epimorphism $R^k \to M$ with $k \in \mathbf{N}$ has a finitely generated kernel, and hence can be extended to such a presentation $R^m \to R^k \to M \to 0$.

6. (i) Show that if abelian groups A and B have no torsion then $A \otimes_{\mathbf{Z}} B$ has no torsion. (Hint: use that tensor products are compatible with direct colimits to reduce to the case of finitely generated subgroups of A and B.)

(ii) Let k be a field and $A = k[x, y]$. Consider the ideal $m = (x, y)$. Show that $m \otimes_A m$ contains a non-trivial m-torsion, that is, there exists an element $t \in m \otimes_A m$ such that $t \neq 0$ but $mt = 0$. (Hint: it is easy to see that $t = x \otimes y - y \otimes x$ is killed by m, but difficult questions about tensor products are usually about non-vanishing (or injectivity). To show that $t \neq 0$, you will, probably, have to use an exact sequence of the form $A^n \to A^l \to m \to 0$ to compute this.)

2. Non-mandatory problems

Non-mandatory problems – do not submit them but I will be glad to discuss them if you wish.

7. Extend exercise 2(iii) to a PID R.

8*. Find an example of a ring R and an embedding $R^n \hookrightarrow R^m$ with $n > m$.