Math599, HW3

1. Mandatory problems

Please submit these problems by November 16 midnight.

1. (i) For any set X consider the functors $T_X : \text{Sets} \to \text{Sets}$ and $h^X : \text{Sets} \to \text{Sets}$ given by $h^X(Z) = \text{Hom}_{\text{Sets}}(X, Z)$ and $T_X(Z) = X \times Z$. Show that h^X is continuous and T_X is cocontinuous.

(ii) Show that if $|X| \neq 1$, then h^X does not possess a right adjoint and T_X does not possess a left adjoint.

2. Describe all adjoint pairs $\mathcal{F}, \mathcal{G}$ of functors from the category of sets to itself.

3. Let I be a set and $\mathcal{C}$ be a complete and cocomplete category.

(i) Find left adjoint to the functor T_I defined by $T_I(X) = X^I = \prod_{i \in I} X$ for any $X \in \mathcal{C}^0$.

(ii) Specify I and $\mathcal{C}$ so that T_I is adjoint to itself (both from the left and from the right) but is not isomorphic to the identity (in particular, $|I| = 1$ does not count as an answer).

4. Let $R_1, \dots, R_n$ be rings and $R = \prod_{i=1}^n R_i$. Show that any R module M decomposes as $M = \oplus_{i=1}^n M_i$ where each M_i is an R_i-module and R acts on M_i through R_i. (Hint: probably, the shortest argument is to use the projectors $M \mapsto e_i M$, where $e_i \in R$'s are the idempotents that correspond to the decomposition of R.)

5. Show that (unlike the category of rings) both monomorphisms and epimorphisms in the category of R-modules are compatible with the forgetful functor to the category of sets:

(i) A homomorphism of R-modules $M \to N$ is a categorical monomorphism if and only if it is injective.

(ii) A homomorphism of R-modules $M \to N$ is a categorical epimorphism if and only if it is surjective.

6. (i) Let $f : L \to M$ and $g : L \to N$ be two homomorphisms of R-modules show that $M \sqcup_L N$ is isomorphic to the module $M \oplus_L N$, which, by definition, is the cokernel of $(f, -g) : L \to M \oplus N$.

(ii) Let $f : M \to L$ and $g : N \to L$ be two homomorphisms of R-modules show that $M \times_L N$ is isomorphic to the kernel of $(f, 0) - (0, g) : M \oplus N \to L$.

2. Non-mandatory problems

Non-mandatory problems – do not submit them but I will be glad to discuss them if you wish.

7*. Prove Yoneda's lemma or check the argument in the notes (or even in the wikipedia).

8*. Prove Freyd's theorem on existence of adjoint functors or read the proof in the literature (e.g. MacLane's book on categories for working mathematician).