Math599, HW2

1. Mandatory problems

Please submit these problems by Thursday, November 9 midnight.

Definition 1.1. (i) A morphism $f\colon X \to Y$ in a category $\mathcal{C}$ is a *monomorphism* or *monic* if for any pair of distinct morphisms $g, h\colon T \to X$ one has that $f \circ g \neq f \circ h$ (in other words, the maps $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$ induced by f are injective).

(ii) A morphism $f\colon X \to Y$ is an *epimorphism* or *epic* if for any pair of distinct morphisms $g, h\colon Y \to T$ one has that $g \circ f \neq h \circ f$ (in other words, the maps $\mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T)$ induced by f are injective).

1. Assume that morphisms $f\colon X \to Y$ and $g\colon Y \to X$ satisfy $g \circ f = \mathrm{Id}_X$. Prove that f is monic and g is epic.

2. Show that $f\colon Y \to X$ is monic if and only if the natural (diagonal) morphism $Y \to Y \times_X Y$ is an isomorphism.

3. Prove that in the category of sets, the categorical definitions of monomorphisms and epimorphisms agree with the usual ones. Namely, consider a map of sets $f\colon X \to Y$ and prove that
(i) f is epic if and only if it is surjective (as a map of sets).
(ii) f is monic if and only if it is injective (as a map of sets).

4. Let $f\colon R \to S$ be a homomorphism of rings.
(i) Prove that $f\colon R \to S$ is monic in the category of rings if and only if it is monic in the category of sets.
(ii) Prove that if f is epic t in the category of sets then it is epic in the category of rings, and give an example when it is epic in the category of rings but not in the category of sets.

5. For any ring R show that $R[[x]] \widetilde{\to} \lim\{R[x]/(x^n)\}$ where the transition morphisms are the projections $R[x]/(x^m) \twoheadrightarrow R[x]/(x^n)$ for $m \geq n$.

6. For any pair of natural numbers $m|n$ consider the embedding $f_{m,n} : \mathbf{Z}/m\mathbf{Z} \hookrightarrow \mathbf{Z}/n\mathbf{Z}$ and the surjection $h_{n,m} : \mathbf{Z}/n\mathbf{Z} \twoheadrightarrow \mathbf{Z}/m\mathbf{Z}$.
(i) Construct an isomorphism of abelian groups $\mathrm{colim}\{\mathbf{Z}/m\mathbf{Z}, f_{m,n}\} \widetilde{\to} \mathbf{Q}/\mathbf{Z}$.
(ii) Set $\widehat{\mathbf{Z}} = \lim\{\mathbf{Z}/n\mathbf{Z}, h_{m,n}\}$ and $\widehat{\mathbf{Z}}_p = \lim\{\mathbf{Z}/p^n\mathbf{Z}, h_{p^m,p^n}\}$. Construct a natural isomorphism of rings $\widehat{\mathbf{Z}} \widetilde{\to} \prod_p \mathbf{Z}_p$. (Hint: you may wish to use Chinese remainder theorem.)

7. Let k be a perfect field. Recall that an algebraic closure $\overline{k}$ of k is a separable algebraic extension which does not admit non-trivial separable algebraic extensions (it is unique up to a non-unique isomorphism).
(i) Let $k \subseteq k_i \subseteq \overline{k}$ be the family of all finite Galois subextensions of $\overline{k}$. Show that k_i form a direct filtered family and the filtered direct limit (i.e. colimit) $\mathrm{colim}\{k_i\}$ is canonically isomorphic to $\overline{k}$.
(ii) Show that the Galois group $\mathrm{Gal}(\overline{k}/k) := \mathrm{Aut}_k(\overline{k})$ is canonically isomorphic to the filtered inverse limit $\lim\{\mathrm{Gal}(k_i/k)\}$. (Hint: an element of $\mathrm{Gal}(\overline{k}/k)$ is nothing else but a compatible family of elements of $\mathrm{Gal}(k_i/k)$.)

(iii) In particular, construct a natural isomorphism $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) \to \widehat{\mathbf{Z}}$. (Hint: use the restriction homomorphisms $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) \to \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$ and identify the targets with $\mathbf{Z}/n\mathbf{Z}$ by sending the Frobenius to 1.)

2. Non-mandatory problems

Non-mandatory problems – do not submit them but I will be glad to discuss them if you wish.

8*. In exercise 5 provide all finite Galois groups with the discrete topology and define $\mathrm{Gal}(\overline{k}/k)$ as the limit $\lim\{\mathrm{Gal}(k_i/k)\}$ in the category of topological groups.

(i) Show that open subgroups have finite index in $\mathrm{Gal}(\overline{k}/k)$ and form a fundamental family of neighborhoods of e. Such a topological group is called pro-finite.

(ii) Prove the infinite Galois correspondence: the maps $l/k \mapsto \mathrm{Gal}(\overline{k}/l)$ and $G \subseteq \mathrm{Gal}(\overline{k}/l) \mapsto \overline{k}^G$ establish a bijection between separable algebraic extensions l/k and closed subgroups $G \subseteq \mathrm{Gal}(\overline{k}/l)$.

(iii) Generalize (absolutely straightforwardly) this to the case of a general Galois extension k'/k, i.e. separable algebraic extension k'/k such that all embeddings $k' \to \overline{k}$ have the same image (in other words, k'/k is normal).

9*. Give an example of a filtered inverse family of sets $\{X_\alpha, f_{\alpha,\beta}\}$ such that all transition maps $f_{\alpha,\beta}$ are surjective but the limit is empty. (Hint: the family should be large, e.g. uncountable.)