Math599, HW12

To get full credit it suffices to solve 5 problems out of 6.

1. (i) Solve problem 22 after chapter 1.

(ii) Deduce that any Artin ring A possesses a decomposition $A = \prod_{i=1}^{n} A_i$ with A_i local Artin rings.

In problems 2 and 3, (A, m) is a local Artin ring. For simplicity, we also assume that A is a finite-dimensional k-algebra, where k is an algebraically closed field. In particular, $A/m = k$. You can use the following facts shown at the recitation: $D(M) = \mathrm{Hom}_k(M, k)$ is an anti-equivalence of the category $\mathcal{C}$ of finitely generated A-modules on itself, which switches projective and injective modules. In particular, $\omega_A := D(A)$ is injective.

2. *Socle* of a finitely generated A-module M is the maximal submodule $N \subset M$ annihilated by m. In other words, it is the maximal submodule of the form k^n.

(i) Show that socle of M is naturally dual to $D(M)/mD(M)$. Deduce that the socle of ω_A is simple (i.e. isomorphic to k).

(ii) Show that ω_A is the injective hull of the A-module k (that is, $k \to \omega_A$ is the universal map from k to an injective A-module).

3. Show that the following conditions are equivalent; an Artin ring satisfying them is called *Gorenstein.*

(i) A is self-dual, i.e. A is isomorphic to ω_A.

(ii) The socle of A is simple.

(iii) A is injective as an A-module.

In the following exercise we will construct a non-catenary ring. You can use the fact (which will be proved soon) that $\mathbf{A}_k^n = \mathrm{Spec}(k[x_1, \dots, x_n])$ is equidimensional of dimension n for any $n \in \mathbf{N}$ and a field k.

4. (i) Find a field k with an isomorphism $\phi\colon k \widetilde{\to} k(x)$. (Hint: you can take $k = k_0(x_0, x_1, x_2, \dots)$ for any field k_0.)

(ii) Set $A' = k[x, y, z]$, $p' = (x, y, z)$ and $q' = (y-1, z-1)$, and find a localization $A = S^{-1}A'$ such that the ideals $p = p'A$ and $q = q'A$ are maximal. Show that $\dim(A) = 3$ and A is not equidimensional. (Hint: take any S disjoint from p' and q' but such that for any $\alpha \in k$ there exists $f \in S$ with $f(x-\alpha, y-1, z-1) = 0$.)

(iii) Glue the points p and q in $X = \mathrm{Spec}(A)$ due to ϕ. In other words, let B be the subring of A consisting of elements f such that $\phi(f(p)) = f(q)$, where $f(p)$ denotes the image of f in $k(p) \widetilde{\to} k$ and $f(q)$ denotes the image of f in $k(q) \widetilde{\to} k(x)$. Show that A is finite over B and the map $X \to Y = \mathrm{Spec}(B)$ glues p and q and does not affect anything else.

(iv) Deduce that Y and B are not catenary.

5. Let $f : A \to B$ be a homomorphism and $p \in \mathrm{Spec}(A)$. By *rank* of $\mathrm{Spec}(f)$ (or f) over p we mean the dimension of the fiber $B(p) = B \otimes_A k(p)$, viewed as a vector space over $k(p)$. (Usually, one only uses this notion when f is finite.)

(i) Show that if f is finite then its rank over any point is finite.

(ii) Show that if f is finite and flat and B is a domain then the rank is constant (i.e. independent of p). (Hint: use that B is locally free.)

(iii) Let A be a non-normal domain with fraction field K such that the integral closure $\widetilde{A} = \mathrm{Nor}_K(A)$ is finite over A. Show that $\widetilde{A}$ is not flat over A. (Hint: use that the rank of $A \to \widetilde{A}$ over the generic point is one.)

Normalization of an integrally closed domain A in a larger field is often flat over A (e.g. in the one-dimensional case). However, this does not have to be the case in general and here is an example:

6. Show that $k[x,y] = \mathrm{Nor}_{k(x,y)}(k[x^2, xy, y^2])$ and $k[x,y]$ is not flat over $k[x^2, xy, y^2]$.

Remark 0.0.1. Note, by the way, that $\mathrm{Spec}(k[x^2, xy, y^2])$ is a cone (or orbifold) that can be obtained from $\mathrm{Spec}(k[x,y])$ by dividing by the involution which takes x and y to $-x$ and $-y$. This is a very basic and important example and it is worth to have it in mind when studying various problems in algebraic geometry. Note also that there is a general result that any finite covering of a non-regular scheme by a regular one is not flat (so our example is a particular case of that claim).