Math598, HW3

1. Mandatory problems

Please submit these problems not later than the seminar on November 10. Full score will be given for 4 problems. You may submit all 5 problems, then the score will be given for the best four ones.

1. (Problem 28 in text.) Let $V = k^{\mathbf{N}}$ be a vector space of countable dimension over a field k.

(i) Show that the elements of $R = \mathrm{End}_k(V)$ can be represented by infinite $\mathbf{N}\times\mathbf{N}$ matrices such that each row contains only finitely many non-zero elements.

(ii) Let I be the set of endomorphisms ϕ whose image $\phi(V)$ is finite dimensional. Show that I is a two-sided ideal, in particular, R is not simple.

(iii) Show that R/I is not semisimple. (Hint: show that its length is infinite.)

2. (Problem 28 in text, continuation.) Show that R/I is a simple ring.

The following exercise is not essentially new. It just gives a slightly different view on what we proved in Wedderburn's theorem. You do not have to prove this from scratch, but may use anything formulated and used in class.

3. (Problem 18 in text.) Let R be a semisimple ring and let $[M_1], \dots, [M_n]$ be the set of all isomorphism classes of simple R-modules. Set $D_i = \mathrm{End}_R(M_i)$, then the action of R on M_i defines a homomorphism $\phi_i \colon R \to \mathrm{End}_{D_i}(M_i)$. Prove that $M_i \widetilde{\to} D_i^{n_i}$ with $n_i < \infty$ and the homomorphism $\phi = (\phi_1, \dots, \phi_n) : R \to \prod_{i=1}^n \mathrm{End}_{D_i}(M_i)$ is an isomorphism.

4. (Problem 21 in text.) Prove that if R is semisimple and $R = \oplus_{i=1}^n M_i$ is its isotypic decomposition then any partial sum $\oplus_{j\in J} M_j$ with $J \subseteq \{1, \dots, n\}$ is a two-sided ideal and, conversely, any two-sided ideal of R is of this form.

In the following problem we will prove that the condition $(|G|, \mathrm{char}(k)) = 1$ in Maschke's theorem is necessary.

5. (Based on problem 27(c) in text.) Assume that G is a finite group and k is a field, and consider the group ring $k[G] = \oplus_{g\in G} k e_g$. By augmentation ideal we mean the left ideal I_G generated by all elements of the form $e_1 - e_g$ for $g \in G$.

(i) Prove that I_G is a two-sided ideal.

(ii) Prove that $k[G]/I_G \widetilde{\to} k$, where the action of G on k is trivial. The projection $k[G] \to k$ is called the augmentation map.

(iii) Prove that if $\mathrm{char}(k)$ divides the order of G then the (left) submodule I_G of $k[G]$ does not splits off, and deduce that $k[G]$ is not semisimple in this case. (Hint: show that the augmentation map admits no sections in this case.)

2. Non-mandatory problems

Non-mandatory problems – do not submit them but I will be glad to discuss them if you wish.

6. Solve problem 18 after chapter 0. (It proves explicitly that $\mathbf{Q}[S_3] \widetilde{\to} M_2(\mathbf{Q}) \oplus \mathbf{Q}^2$.)

7. Solve problems 42-45,47 after chapter 0. (These are basic properties of Artin and Noether modules, and most of them are very simple.)

8*. Solve Problems 48-50 after chapter 0.