Algebraic Structures II (80446)

The Hebrew University of Jerusalem   Department of Mathematics

The training exam's duration is three hours. Parts a, b and c sum up to 100 points. In addition, there is also a bonus problem (10 points) given in the end and supposed to be more difficult and less typical. If your total is between 100 and 110, the grade will be 100. (My recommendation: do not touch the bonus until you have made your maximum with the regular problems.)

Part a (27 points). Prove one of the following two theorems.

1. If $n \in K^\times$, $\mu_n \subset K$ and L/K is cyclic of degree n, then there exists $\alpha \in L$ such that $L = K(\alpha)$ and $\alpha^n \in K$.
2. If L is a field and $G \subseteq \mathrm{Aut}(L)$ is a finite subgroup, then L is a finite Galois extension of $K = L^G$ and $G = G_{L/K}$.

Part b (5 points each problem, total 25). Decide whether each one of the following five statements if correct or incorrect. Just answer yes or no, there is no need to justify.

1. If L/K is a finite separable extension, then there are finitely many intermediate fields between K and L.
2. If L, F are finite extensions of a field K, then $[LF : F]$ divides $[L : K]$.
3. If p, q are prime numbers and K is the splitting field of $x^q - p$ over $\mathbf{Q}$, then $[K : \mathbf{Q}] = p(q-1)$.
4. One can construct the right polygon with 51 edges using compass and straightedge.
5. If $x, y \in \mathbf{C}$ are such that $\text{tr.deg.}(\mathbf{Q}(x)/\mathbf{Q}) = \text{tr.deg.}(\mathbf{Q}(y)/\mathbf{Q}) = 1$, then $\text{tr.deg.}(\mathbf{Q}(x,y)/\mathbf{Q}) = 2$.

Part c (16 points each problem, total 48 points). Solve three from the following four problems. Explain your solution; any result proved in class can be used, once you state it clearly.

1. Show that the polynomial $x^5 - 4x + 2$ can not be solved in radicals over $\mathbf{Q}$.
2. Prove that if K is of characteristic $p > 2$ and $K(x)/K$ is an inseparable extension, then the extension $K(x^2)/K$ is inseparable too.
3. Let $\xi = \xi_7$ be the primitive root of unity of order 7. Assume one is given a plane $\mathbf{R}^2$ with marked points $(0,0), (1,0)$. Show that the point $\xi + \xi^6$ cannot be constructed by compass and straightedge, and the point $\xi + \xi^2 + \xi^4$ can be constructed by compass and straightedge, where we use the complex number $x + iy$ to describe a point $(x, y) \in \mathbf{R}^2$.
4. For a prime number $p \neq 3$ let K_p denote the splitting field of $x^9 - 1$ over $\mathbf{F}_p$. Find the list of all possible Galois groups $G_{K_p/\mathbf{F}_p}$, and specify one concrete p for each group in your list.

Part d (10 points). Bonus problem.

1. a) (5 points) How many groups G with 81 elements exist so that the following is true: there exists a field K and an irreducible polynomial $f(x) \in K[x]$ of degree 10 such that G is the Galois group G_f of the splitting filed of f over K.

b) (5 points) How many groups G with 81 elements exist so that the following is true: there exists a field K and an irreducible polynomial $f(x) \in K[x]$ of degree 9 such that G is the Galois group G_f of the splitting filed of f over K.