Compliments in Linear Algebra 80146

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Examples of Dual Spaces

- Let V be some $vector - space$.

  Suppose $B = (e_1, ..., e_n)$ is some *basis* of V (not necessarily the standard basis)

  As such, $\forall v \in V$ there is a *unique* representation $v = \sum_{i=0}^{n} x^i e_i$.

  Then, B defines an *isomorphism* $[]_B : V \to \mathbb{F}_{col}^n$:

  $\forall v \in V$: $[v]_B = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} \in \mathbb{F}_{col}^n$.

  Essentially, $f^i(v) = x^i$, and as such, $f^i : V \to \mathbb{F}$ is a $linear\ transformation$,

  specifically, $f^i \in V^*$, i.e. f^i is a $functional$.

  Indeed, every *basis* B of V defines $functionals$ $f^1, ..., f^n \in V^*$.

- Lemma: Suppose $B = (e_1, ..., e_n) \subset V$ is a *basis*.

  If $\forall i : 1 \leq i \leq n$, $f^i : V \to \mathbb{F}$ is a $linear\ transformation$ such that $f^i(v) = x^i$,

  then $(f^1, ..., f^n) \subset V^*$ is a *basis*.

  Proof: Let $u \in V^*$. We want to show that there is a *unique* representation for u as a *linear combination* of $(f^1, ..., f^n)$.

  - proposition: $u = \sum_{i=1}^{n} u(e_i) f^i$
    proof: Let $v \in V$. Since B is a *basis* of V, then there is a *unique* representation $v = \sum_{i=1}^{n} x^i e_i$.
    Note that $u(v) = \sum_{i=1}^{n} x^i u(e_i)$
    In fact, $v = \sum_{i=1}^{n} u(e_i) f^i = \sum_{i=1}^{n} x^i u(e_i)$ ($f^i(v) = x^i$ per f's definition)
    Thus, $Span(f^1, ..., f^n) = V^*$.

- proposition: $(f^1, ..., f^n)$ is *linearly independent*.
  proof: (negative proof) Suppose $\exists a_1, a_2, ..., a_n \in \mathbb{F}$
  such that $\sum_{i=1}^n a_i f^i = 0$, and $\exists a_{i_0} \neq 0$.
  Then, $0 = 0(e_{i_0}) = \sum_{i=1}^n a_i f^i(e_{i_0}) = a_{i_0} \neq 0$, in contradiction.

With the last proposition, we completed our proof. ■

- Notes: (i) The *basis* $(f^1, ..., f^n) \subset V^*$ is called *dual* in relation to the *basis* $(e_1, ..., e_n) \subset V$, and satisfies:

  $$f_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

  In fact, every *functional* in $f^1, ..., f^n$ is dependent upon the choice of the respective *vector* in the *basis* $(e_1, ..., e_n)$.

  (ii) Exercise: Suppose $(e_1, ..., e_n) \subset V$ is a *basis*,
  and suppose $(f^1, ..., f^n) \subset V*$ is its *dual basis*,
  Let $v \in V$, and let $u \in V^*$, suppose $v = \sum_{i=1}^n e^i e_j$, and $u = \sum_{i=1}^n a_i f^i$.

  Then $u(v) = \sum_{i=1}^n a_i b_i = (a_1, ..., a_n) \cdot \begin{bmatrix} b^1 \\ . \\ . \\ . \\ b^n \end{bmatrix}$

  (iii) Corollary: $dimV = dimV* = dimV**$

Reminder on *Linear Transformations* and *Matrices*

- Theorem:
  Let V, U each be $vector-spaces$ on some $field$ $\mathbb{F}$.
  Suppose $B = (u_1, ..., u_n) \subset U$ is some *basis*,
  And suppose $C = (v_1, ..., v_m) \subset V$ is some set of vectors.
  Define $T : U \to V$ such that $T_{u_i} = \sum_{j=1}^m a_i^j v_j$

  If $\forall u \in U$, $[T_u]_C = \begin{bmatrix} y^1 \\ . \\ . \\ . \\ y^m \end{bmatrix}$, $[u]_B = \begin{bmatrix} x^1 \\ . \\ . \\ . \\ x^n \end{bmatrix}$

  Then, $y^j = \sum_{i=1}^n a_i^j x^i$

  Examples:

  Suppose $\begin{cases} y^1 = a_1^1 x^1 + a_2^1 x^2 + a_3^1 x^3 \in U_3 \\ y^2 = a_1^2 x^1 + a_2^2 x^2 + a_3^2 x^3 \in U_3 \end{cases}$

Define $T : U_3 \to V_2$ such that:

$T(u_1) = a_1^1 v_1 + a_1^2 v_2$

$T(u_2) = a_2^1 v_1 + a_2^2 v_2$

$T(u_3) = a_3^1 v_1 + a_3^2 v_2$

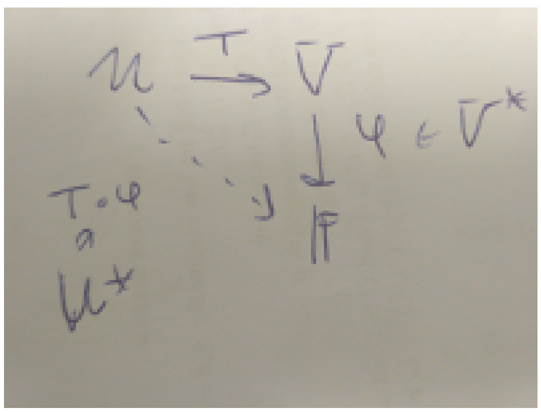
$\to A = \begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix}$, and $A^t = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ a_3^1 & a_3^2 \end{bmatrix}$.

- Definition: Suppose $T : U \to V$ is a *linear transformation*,

  The *transformation* $T^* : V^* \to U^*$such that $\forall \varphi \in V^*$:$T^*(\varphi)(u) = \varphi(T(u))$,

  is a *linear transformation*, and may also be called T's *dual linear transformation*.

  An illustration:

- Theorem: Let U, V be two $vector - spaces$ each over some $field$ $\mathbb{F}$.

  Suppoes $B \subset U$ is a *basis*,

  and suppose $C \subset V$ is also a *basis*.

  Suppose $T : U \to V$ is a *linear transformation*, such that $A = [T]_C^B$.

  If $B^*, C^*$are the *dual bases* of *bases* B, C respectively,

  Then, $[T^*]_{B^*}^{C^*} = A^t$

  and, suppose $dimU = n$, $dimV = m$ then:

  $T : U \to V$ in matrix form is in $Mat(m, n)$

  $T* : V^* \to U^*$in matrix form is in $Mat(n, m)$.