Compliments in Linear Algebra 80146

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- Theorem: Every vector-space has a *basis* (not necessarily finitely-generated)

  Proof: Via Zorn's Lemma (or transfinite induction)

  Note: Zorn's lemma may help prove that every basis has the same cardinatility (its' dimension)

  $\mathbb{R}^{\mathbb{N}} = \{(a_0, a_1, ...)|a_i \in \mathbb{R}\} \supset \oplus_{\mathbb{N}}\mathbb{R} = \{(a_0, a_1, ...)|a_i \in \mathbb{R}(\exists n \in \mathbb{N}, a_i = 0, i > n\}$ (is countable)

  Candidate for a basis:

  $$\begin{cases} e_1 = (1, 0, 0, ...) \\ e_2 = (0, 1, 0, ...) \\ e_3 = ... \\ \Sigma_{i=1}^{n} a_i e_i = (a_1, ..., a_n, 0, 0, ...) \end{cases}$$

  Examples:

  Polynomials over $\mathbb{F}$:

  $\mathbb{F}[x] = \{a_n x^n + ... + a_0 | a_i \in \mathbb{F}\}$ (formal sum) $\equiv \{(a_n, ..., a_0)|a_i \in \mathbb{F}\}$, thus $\mathbb{F}[x] = \oplus_{\mathbb{N}}\mathbb{F}$

  Formal series: (divergent)

  $\mathbb{F}[[x]] = \{a_0 + a_1 x + ... a_n x^n + ...|a_i \in \mathbb{F}\}$ is *isomorphic* to $\mathbb{R}^{\mathbb{N}}$.

  $\Sigma_{i=0}^{\infty}(a_i x^i) \cdot \Sigma_{j=0}^{\infty}(b_j x^j) =_{def} \Sigma_{n=0}^{\infty}(\Sigma_{i=0}^{n} a_i b_{n-i}) x^n$

- Exercise (*): $\mathbb{F}^{\mathbb{N}}$ has no *countable* basis over $\mathbb{F}$.

  Proposition: $\mathbb{R}$ over $\mathbb{Q}$ has no *countable* basis.

  $\mathbb{Q} = \{\pm \frac{m}{n}\}$

  $\mathbb{R}$ is not *countable*, suppose $\mathbb{R} = \{x_0, x_1, x_2, x_3, ...\}$

  Clue: Prove: if V over $\mathbb{Q}$ has a countable basis, then V itself is a *countable* set.

  Note: Zorn's Lemma is the only way to prove since $\mathbb{R}$ has a basis over $\mathbb{Q}$.

  (side note: $dim_{\mathbb{C}}\mathbb{C} = 1$, $dim_{\mathbb{C}}\mathbb{C} = 2$)

  Basis of $\mathbb{R}$ over $\mathbb{Q}$ $(1, \sqrt{2}, \sqrt{3}, e, \pi, ...)$

If $\mathbb{F}$ is $finite$ ($countable$) then $\mathbb{F}[[x]]$ is not $countable$,

However, every vector space $\mathbb{V}$ over $\mathbb{F}$ with countable basis is still countable.

- Sum and product of vector spaces:

  Definition: Suppose I is an indices set , and $\{V_i|i \in I\}$ is a set of $vector-spaces$ indexed by I.

  Then $\prod_{i\in I} V_i = \{(v_i)|i \in I, v_i \in V_i\}$.

  Examples:

  (1) $I = \{1, 2\}$

  $V_1 \times V_2 = \{(v_1, v_2)\}$

  (2) $I = \mathbb{N}$

  $V_1 \times V_2 \times ... \times V_n \times ... = \{(v_1v_2v_3...)$

  Notes:

  Just like with sets:

  $\prod_{\mathbb{N}} \mathbb{F} = \mathbb{F}^{\mathbb{N}}$

  Note: Union of sets is usualy not a vector-space. Instead, we define an external sum of vector-spaces.

- Definition: Suppose I is a set of indices, $\{V_i|i \in I\}$ is a set of $vector-spaces$.

  Define: $\oplus_{i\in I}V_i = \{(v_i)|i \in I$, there is a finite subset in $I' \subset I$ such that if $i \in I \setminus I'$ then $v_i = 0\}$

  Note:

  If $\oplus V_i \subseteq \prod V_i$ is a $subspace$, then there is equality $\iff$ only a $finite$ amount V_i is $nonzero$.

  Example:

  $\mathbb{F}[x] = \oplus_{\mathbb{N}}\mathbb{F} \subset \mathbb{F}[[x]] = \prod_{\mathbb{N}} \mathbb{F}$.

  Exercise (proof next week):

  Finding a basis B for vector-space V is equivalent to obtaining an $isomorphism$.

  $V \cong \oplus_{\mathbb{N}}\mathbb{F}$ (at least when $dim\mathbb{V} < \infty$)