# Compliments in Linear Algebra 80146

by Amit Novick

First compiled: 12/12/2016

- (Continuing from previous lesson's Set game challenge)

  $\{0,1,2\}^4 = \mathbb{F}_3^4 \simeq deck$ (81 cards)

  $v \in \mathbb{F}_3^4$, $v = (a,b,c,d)$

  $\forall Set\ (v_1, v_2, v_3)$ satisfies $\Sigma v_i = 0$, we have to check that $(\Sigma a_i = 0, \Sigma b_i = 0, ...)$ in each coordinate such that

  there is a Set :

  case (a): $a_1 = a_2 = a_3$, and then $\Sigma_i a_i = 3a_i = 0$

  case (b): $\{a_1, a_2, a_3\} = \{0,1,2\}$ $0+1+2 = 0 \in \mathbb{F}_3$.

  In addition $\Sigma_{v\in\mathbb{F}_3^4} v = 0$, $S_{x,y,z} = \{(0,x,y,z),(1,x,y,z),(2,x,y,z)\}$ (is a Set) for $x,y,z \in \mathbb{F}_3$.

  $\sqcup_{x,y,z} S_{x,y,z} = \mathbb{F}_3^4 \Rightarrow \Sigma_{v\in\mathbb{F}_3^4} v = \Sigma_{x,y,z}(\Sigma_{v\in S_{x,y,z}} v)$ with the innermost equal 0.

  Exercise: A *set* $S \subset \mathbb{F}_3^4$ is a Set $\iff$ $S$ is a *line* (maybe not passing through 0)

  $dimL = 1$, $L \subset V$ when the *line* is a $translation$ (*) of a *subspace* $L$ of $dimension$ 1.

  Essentially, that is $v_0 + L$ such that $v_0 \in \mathbb{F}_3^4$, $L \subset \mathbb{F}_3^4$ is a *subspace* of $dimension$ 1.

  (*) : Each point of the line is moved by the same amount in a given direction

  Question: Is $L$ unique? is $v_0$unique?

  Theorem: $B$ is a $base$ $\iff$ $B$ is $maximal\ linearly-independent\ set$ $\iff$ $B$ is a $minimal\ generating\ set$

  Note: A set $B$ may be $infinite$ !

  (a) In the case where $V$ is a $finitely-generated\ vector-space$ : we start with some $finite\ generating\ set\ B$, and

  remove members from this set until it is $minimal$.

  This method does not work if $B$ is $infinite$.

(b) In the general case: it is preferable to expand $linearly-independent$ $sets$ in order to build $B$.

- Definition: A $partial\ order$ on a $set$ $X$ is a $subspace$ $P \subset X^2$ such that $(x,y) \in P$ may

  be written $x \leq y$ such that:

  1) Reflexive: $\forall x \in X(x \leq x)$

  2) Transitive: $\forall x,y,z \in X(x \leq y, y \leq z) \Rightarrow x \leq z$

  3) Anti-Symmetric: if $x \leq y, y \leq x$ then $x = y$

  Example: $P = \{(x,x)|x \in X\}$, not every $x, y$ are $comparable$.

- Definition: A $total\ order$ is a $partial\ order$ such that every pair $(x,y)$ is $comparable$:

  $x \leq y$ or $y \leq x$.

  Example: $(\mathbb{R}, \leq)$ is a $total\ order$

  $X$ be some $set$ and $2^X$ be defined as $2^X = \{All - subsetes - in - X\}$ (note: a.k.a $Power - Set(X)$)

  Relation of $containment$ $(\subset)$ is $partial\ order$.

  $Y \subset X$ is a $subset$, then $c \in X$ is an $upper\ bound$ on $Y$ if $\forall y \in Y(y \leq c)$

- Zorn's Lemma:

  If $x \neq \emptyset$ is $set$ with $partial\ order$ $\leq$such that $\forall Y \subset X$ the $order$ $\leq$on $Y$ is $full\ order$ ,

  then there is an $upper\ bound$ $C_Y \in X$ , thus $X$ has at least one $maximal$ $member$.

- Axiom of choice:

  If $\{X_i\}_{i\in I}$ is a $nonempty\ set$ of $nonempty\ sets$ , then $\emptyset \neq \sqcap_{i\in I} X_i$, $\{(x_i)_{i\in I}|x_i \in X_i\}$

- Theorem: $\forall V(V$ is a $vector-space$ over some $field$ $\mathbb{F})$ there is a $base$.

  Proof: $X = \{linearly-independent\ sets\ B_i \subset V\}$.

  In order to utilise $Zorn$ we check that $\forall Y \subset X(Y$ has a $total-order)$ there is an $upper-bound$ $C_Y$

  such that $C_Y = \bigcup_{B_i \in Y} B_i$.

  Proposition: $C_Y$ is a $linearly-independent\ set$.

  Suppose there is a $linear{-}combination$ $\Sigma_{i=1}^n a_i V_i (v_i \in C_Y, a_i \in \mathbb{F})$

  Then $\forall v_i$ there is a $set$ $B_{j(i)} \in Y$ such that $v_i \in B_{j(i)}$.

  We have $n \in \mathbb{N}$ $linearly-independent\ sets$:

  $B_{j(1)}...B_{j(n)}$ is a $finite\ set$ with $total-order$ $\Rightarrow$there is a $maximal$ $B_l$ such that

$\forall i(v_i \in B_{j(i)} \subset B_l)$ and since it is a $linearly - independent\ set \Rightarrow a_1 = ... = a_n = 0$ , means that

$C_Y$ is $linearly - independent$ (every $linear - combination$ of 0 is trivial)

$C_Y \in X$ is an $upper\ bound$ of $Y \Rightarrow X$ has $maximal\ member$.