Complements in Linear Algebra 80146

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- Examples for groups:

  (1) Group of permutations ($S_n = Aut(\{1,...,n\})$), not commutative ($n = 3$)

  (2) Let $(\mathbb{F}, +, \circ)$, then this is a group,

  also: $(\mathbb{F} \setminus \{0\}, \circ, 1)$ is a group

  (3) Let G_1, G_2 be groups, then $G_1 \times G_2$ is also a group when $\circ$ be defined as:

  $(g_1, g_2) \circ (h_1 \circ h_2) = (g_1 h_1, g_2 h_2)$

  $1_{G_1 \times G_2} = (1_{G_1}, 1_{G_2})$

  For example, $G^2 = G \times G$, and in general, for every set S , there exists a structure of a group on

  $Func(S, G) = G^S = \{g_i | i \in S, g_i \in G\}$

- If $f, h \in G^S \Rightarrow (f \cdot h)$ is defined by $(f \cdot h)(s) = f(s) \cdot h(s)$

  Cyclics groups (created by a single element)

  example: $1 \in \mathbb{Z}$ is a creator since $\{-(1 + ... + 1), 1 + 1 + ... + 1\}$ (every element in $\mathbb{Z}$)

  $\forall 0 < n \in \mathbb{N}$, consider $\mathbb{Z}_n = \{0, 1, ...n - 1\}$

  Exercise: The method described above can be used to describe all *cyclic* groups.

  hint: if $0 \neq 1 + 1 + ... + 1 \Rightarrow G = \mathbb{Z}$

  If $0 = 1 + 1 + ... + 1$ (n times) $\Rightarrow G = \mathbb{Z}_n$.

  Example: The perfect polygon of n sides has $2n$ symmetries .

  n rotations and n mirrorings

  We denote this group D_n.

  $2n - - > D_n \supset \mathbb{Z}_n < - - n$

  The group of n rotations, $0 \leq k \leq n - 1$, $\frac{360 \cdot k}{n}$

- Definition: A *homomorphism* between *groups* G, H is a map $f : G \to H$ which satisfies the group structure.

  $f(1_G) = 1_H$

  $f(g_1 \cdot g_2)_G = f(g_1)_H \cdot f(g_2)$

  (0) $0 \hookrightarrow G$ (*addition* notation)

  $G \to 0$

  (1) $\mathbb{Z} \to \mathbb{Z}_n$,$x \mapsto x(mod n)$

  $x + y \mapsto x + y(mod n) < --$depends on $x(mod n)$, $y(mod n)$

  (-1) $(0...n-1) \hookrightarrow \mathbb{Z}$

  $\mathbb{Z}_n \hookrightarrow \mathbb{Z}$

  Not a *homomorphism* if $n > 1$ since $n \in \mathbb{Z}$, e.g. $= n - 1 + 1$

  $0 \in \mathbb{Z}_n$.

  Example: $D_n \supset \mathbb{Z}_n$ (explains orientation e.g. in physics: orientation of right hand rule, x, y, z axis orientations)

  $\varphi : D_n \to \mathbb{Z}_2 = \{\pm 1\}$

  $$\varphi(g) = \begin{cases} 1 & g - satisfies - orientation \\ -1 & otherwise \end{cases}$$

  Note: The orientation structure for $n = 3$ from last lecture, $D_3 = S_3 \to \mathbb{Z}_2$

  At the end of the course we define $S_n \to \mathbb{Z}_2\ \forall n$.

  Exercise: build the *homomorphism* as described above.

- Definition: A *subgroup* $H \subset G$ is a *subset* such that:

  1) $1_G \in H$

  2) $\forall x, y \in H(x_G \cdot y \in H)$

  3) $\forall x \in H(x^{-1} \in H)$ (and x^{-1} exists!)

  Proposition: H is a *group*

  Example: $\mathbb{Z}_n \subset D_n$.

- Definition: If $\varphi : G \to H$ is a *homomorphism*, then $ker\varphi$ (the *kernel* of φ) is:

  $ker\varphi = \{g \in G | \varphi(g) = 1\} = \varphi^{-1}(1_H)$

  Proposition: $ker\varphi$ is a *subgroup*. (immediate test)

  Examples:

  1)$\varphi : \mathbb{Z} \to \mathbb{Z}_n$, $ker\varphi = n\mathbb{Z} = \{n \cdot x | x \in \mathbb{Z}\}$

  2) $\psi : D_n \to \mathbb{Z}_2$, $ker\psi = \mathbb{Z}_n$