1. Minors and their applications

1.1. **Row and column decompositions.**

Definition 1.1.1. If $A \in M_{n\times n}(F)$ is a square matrix of size n then its *minor* at the place (i,j) is the determinant of the $(n-1)\times(n-1)$ submatrix obtained by omitting i-th row and j-th column. It is customary to denote this minor as A_{ij}.

A typical application of minors is to compute $\det(A)$ via row or column decompositions established in the following theorem.

Theorem 1.1.2. *Assume that $A \in M_{n\times n}(F)$ is a square matrix of size n. Then,*

(a) Decomposition along i-th row: for any $1 \le i \le n$ the equality $|A| = \sum_{k=1}^n (-1)^{i+k} a_{ik} A_{ik}$ holds.

(b) Decomposition along j-th column: for any $1 \le j \le n$ the equality $|A| = \sum_{k=1}^n (-1)^{j+k} a_{kj} A_{kj}$ holds.

Proof. We will prove (a), but let us first show how this implies (b). Set $B = A^t$ and recall that $|B| = |A|$. Furthermore, the submatrices of B are the transposes of the corresponding submatrices of A, hence we have equalities of all minors $B_{ij} = A_{ji}$. In particular,

$$\sum_{k=1}^n (-1)^{j+k} a_{kj} A_{kj} = \sum_{k=1}^n (-1)^{j+k} b_{jk} B_{jk}.$$

If the claim of (a) holds for the j-th row of B then the righthand side equals to $|B|$, and using that $|B| = |A|$ we obtain (b) for the j-th column of A.

Now, let us prove (a). Note that the i-th row of A equals to $\sum_{k=1}^n a_{ik}\varepsilon_k$. Let B_k denote the matrix obtained from A by replacing its i-th row with ε_k, then $|A| = \sum_{k=1}^n a_{ik}|B_k|$ by linearity of det with respect to the i-th row. It now suffices to prove that $|B_k| = (-1)^{k+i} A_{ik}$. Set $C = B_k$ to simplify notation. Then $C_{ik} = A_{ik}$ because the corresponding submatrices are equal. We prove in the lemma below that $|C| = (-1)^{i+k} C_{ik}$, so $|B_k| = (-1)^{i+k} C_{ik} = (-1)^{i+k} A_{ik}$, as claimed. □

Lemma 1.1.3. *Assume that $C \in M_{n\times n}(F)$ is a square matrix of size n such that the i-th row of C equals to $\varepsilon_k = (0,\dots,0,1,0,\dots,0)$. Then $|C| = (-1)^{i+k} C_{ik}$.*

Proof. First we consider the case of $i = k = n$. Since $c_{nj} = 0$ for $j \ne n$, the only non-zero summands in the expression

$$|C| = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{l=1}^n c_{l,\sigma(l)}$$

correspond to the permutations that satisfy $\sigma(n) = n$, i.e. $\sigma = (\sigma_1,\dots,\sigma_{n-1},n)$. Giving such a permutation is the same as giving a permutation $\sigma' = (\sigma_1,\dots,\sigma_{n-1}) \in$

S_{n-1}. Thus

$$|C| = \sum_{\sigma' \in S_{n-1}} \operatorname{sign}(\sigma) c_{nn} \prod_{l=1}^{n-1} c_{l,\sigma'(l)}.$$

Furthermore, $c_{nn} = 1$ and $\operatorname{sign}(\sigma) = \operatorname{sign}(\sigma')$ because both permutations have the same inversions. So, we obtain that

$$|C| = \sum_{\sigma' \in S_{n-1}} \operatorname{sign}(\sigma') \prod_{l=1}^{n-1} c_{l,\sigma'(l)}$$

but the latter is precisely the formula for the minor $C_{nn} = (-1)^{n+n} C_{nn}$.

Next, assume that $i = n$ but k is arbitrary. We will use decreasing induction on k, so assume that $1 \le k < n$ and the case of $k+1$ is already established. Let D be obtained from C switching columns k and $k+1$. Then we have the equality of minors $C_{nk} = D_{n,k+1}$ because the corresponding submatrices are equal. The n-th row of D equals to ε_{k+1} hence $|D| = (-1)^{n+k+1} D_{n,k+1}$ by the induction assumption. Since det is antisymmetric in columns, $|C| = -|D|$, and we obtain that $|C| = (-1)^{n+k} D_{n,k+1} = (-1)^{n+k} C_{nk}$, as claimed.

Finally, the case of an arbitrary i is obtained in the same way: one uses decreasing induction on i, and establishes the induction step by switching the rows i and $i+1$ and using that det is antisymmetric in rows. □

1.2. **The adjoint matrix.**

Definition 1.2.1. The matrix $B = \operatorname{Adj}(A)$ is defined by the rule $b_{ij} = (-1)^{i+j} A_{ji}$.

Thus, in order to obtain $\operatorname{Adj}(A)$ we form the matrix of minors with the appropriate sign and transpose it. Its importance is explained by the following theorem.

Theorem 1.2.2. *For any $n \times n$ matrix A the equality $A \cdot \operatorname{Adj}(A) = |A| I_n$ holds.*

Proof. Set $B = A \cdot \operatorname{Adj}(A)$. Then $b_{ij} = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} A_{jk}$ and by the i-th row decomposition of $\det(A)$ we have that $b_{ii} = |A|$. It remains to show that $b_{ij} = 0$ when $i \ne j$. Consider the matrix C obtained from A by replacing the j-th row with the i-th row. Then C has two equal rows and so $|C| = 0$. On the other hand $C_{jk} = A_{jk}$ for any $1 \le k \le n$ because A and C are equal outside of the j-th row. Using the j-th row decomposition for C we obtain that

$$0 = |C| = \sum_{k=1}^{n} (-1)^{j+k} c_{jk} C_{jk} = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} A_{jk} = b_{ij}.$$

□

As an immediate corollary we obtain the following extremely important theorem.

Theorem 1.2.3. *Let A be an $n \times n$ matrix. Then,*

(a) A is invertible if and only if $|A| \neq 0$, and in this case $A^{-1} = |A|^{-1}\mathrm{Adj}(A)$.

(b) A is not invertible if and only if $|A| = 0$, and in this case $A \cdot \mathrm{Adj}(A) = 0$.

Proof. If A is invertible then $|A^{-1}| = |A|^{-1}$, in particular, $|A|$ is invertible. Conversely, if $|A|$ is invertible then the matrix $|A|^{-1}\mathrm{Adj}(A)$ is defined and by the previous theorem its product with A equals to $|A|^{-1}(|A|I_n) = I_n$. In particular, $|A|$ is invertible and its inverse is given by the asserted formula. This proves that A is invertible (resp. not invertible) if and only if $|A|$ is so. Finally, if $|A| = 0$ then $A \cdot \mathrm{Adj}(A) = |A|I_n = 0$. □

1.3. **Cramer's rule.** As an application of the inversion formula, one can write down the formula for solution of the system of equations of the form $Ax = b$ when A is invertible. One should not expect for a simple formula in a more general situation because the solution does not have to exist or be unique.

Theorem 1.3.1. *Assume that $Ax = b$ is a system of n linear equations in n variables such that A is invertible. Then the solution λ satisfies $\lambda_i = \frac{|B_i|}{|A|}$ where the matrix B_i is obtained from A by replacing its i-column with b.*

Proof. Since $\lambda = A^{-1}b = |A|^{-1}\mathrm{Adj}(A)b$ we have that

$$\lambda_i = |A|^{-1}\sum_{k=1}^{n}(-1)^{k+i}A_{ki}b_k.$$

On the other hand, if $C = B_i$ then $|C| = \sum_{k=1}^{n}(-1)^{i+k}b_kC_{ki}$ by the i-th column decomposition. It remains to note that $A_{ki} = C_{ki}$ because the corresponding submatrices are equal. So, the righthand side of the above formula simplifies as $|A|^{-1}|C| = |A|^{-1}|B_i|$. □

1.4. **Determinant and Gauss elimination.** The direct formula for $\det(A)$ involves $n!$ summands, so even for $n = 30$ its application is not so realistic even for computers. The inductive formulas with row or column decompositions do not improve the situation unless many entries of A vanish. The reason for this is that we have to compute n minors, i.e., determinants of size $(n-1)\times(n-1)$, computing them involves $n(n-1)$ computations of determinants of the size $(n-2)\times(n-2)$, etc. However, using that $\det(A)$ is linear and antisymmetric, it is very simple to describe how it is changed by elementary transformations: (i) $\det(A) = \det(B)$ if B is obtained from A by an operation $R_i := R_i + aR_j$, (ii) $\det(A)$ changes sign when we switch two rows, and (iii) $\det(A)$ is multiplied by c when we multiply a row by c. As a consequence, we can compute $\det(A)$ very effectively by the following procedure: apply Gauss elimination to A so that in a sequence $A = A_0, A_1, A_2, \dots, A_n$ the last matrix is triangular and each A_{i+1} is obtained from A_i by an elementary transformation. At each stage write $|A_i| = \alpha_{i+1}|A_{i+1}|$ where $\alpha_{i+1} = 1$

in (i), $\alpha_{i+1} = -1$ in (ii), and $\alpha_{i+1} = c^{-1}$ in (iii). Then $|A| = |A_n| \prod_{i=1}^{n} \alpha_i$ and, as we know, $|A_n|$ is the product of the diagonal entries of the triangular matrix A_n.

Remark 1.4.1. In combinatorics one also studies a quantity called permanent and defined as $\mathrm{Perm}(A) = \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{k,\sigma(k)}$ (without signs). One can easily establish an analog of row or column decomposition for $\mathrm{Perm}(A)$, but there is no analogs of Gauss elimination for it. The permanent is conjectured to be a quantity that cannot be effectively computed (e.g. better then in an exponential time).