NON-ARCHIMEDEAN ANALYTIFICATION OF ALGEBRAIC SPACES

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1. Introduction

1.1. **Motivation.** This paper is largely concerned with constructing quotients by étale equivalence relations. We are inspired by questions in classical rigid geometry, but to give satisfactory answers in that category we have to first solve quotient problems within the framework of Berkovich's k-analytic spaces. One source of motivation is the relationship between algebraic spaces and analytic spaces over $\mathbf{C}$, as follows. If X is a reduced and irreducible proper complex-analytic space then the meromorphic functions on X form a field $\mathscr{M}(X)$ and this field is finitely generated over $\mathbf{C}$ with transcendence degree at most $\dim X$ [CAS, 8.1.3, 9.1.2, 10.6.7]. A proper complex-analytic space X is called *Moishezon* if $\mathrm{trdeg}_{\mathbf{C}}(\mathscr{M}(X_i)) = \dim X_i$ for all irreducible components X_i of X (endowed with the reduced structure). Examples of such spaces are analytifications of proper $\mathbf{C}$-schemes, but Moishezon found more examples, and Artin found "all" examples by analytifying proper algebraic spaces over $\mathbf{C}$. To be precise, the analytification $\mathscr{X}^{\mathrm{an}}$ of an algebraic space $\mathscr{X}$ locally of finite type over $\mathbf{C}$ [Kn, Ch. I, 5.17*ff*] is the unique solution to an étale quotient problem that admits a solution if and only if $\mathscr{X}$ is locally separated over $\mathbf{C}$ (in the sense that $\Delta_{\mathscr{X}/\mathbf{C}}$ is an immersion). The functor $\mathscr{X} \rightsquigarrow \mathscr{X}^{\mathrm{an}}$ from the category of proper algebraic spaces over $\mathbf{C}$ to the category of Moishezon spaces is fully faithful, and it is a beautiful theorem of Artin [A2, Thm. 7.3] that this is an equivalence of categories.

It is natural to ask if a similar theory works over a non-archimedean base field k (i.e., a field k that is complete with respect to a fixed nontrivial non-archimedean absolute value). This is a surprisingly nontrivial question. One can carry over the definition of analytification of locally finite type algebraic spaces $\mathscr{X}$ over k in terms of uniquely solving a rigid-analytic étale quotient problem; when the quotient problem has a solution we say that $\mathscr{X}$ is *analytifiable* in the sense of rigid geometry. (See §2 for a general discussion of definitions, elementary results, and functorial properties of analytification over such k.) In Theorem 2.3.4 we show that local separatedness is a necessary condition for analytifiability over non-archimedean base fields, but in contrast with the complex-analytic case it is not sufficient; there are smooth 2-dimensional counterexamples over any k (even arising from algebraic spaces over the prime field), as we shall explain in Example 3.1.1. In concrete terms, the surprising dichotomy between the archimedean and non-archimedean worlds is due to the lack of a Gelfand–Mazur theorem over non-archimedean fields. (That is, any non-archimedean field k admits nontrivial non-archimedean extension fields with a compatible absolute value, even if k is algebraically closed.) Since local separatedness fails to be a sufficient criterion for the existence of non-archimedean analytification of an algebraic space, it is natural to seek a reasonable salvage of the situation. We view separatedness as a reasonable additional hypothesis to impose on the algebraic space.

1.2. **Results.** Our first main result is the following (recorded as Theorem 4.2.1):

Theorem 1.2.1. *Any separated algebraic space locally of finite type over a non-archimedean field is analytifiable in the sense of rigid geometry.*

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For technical reasons related to admissibility of coverings and the examples of non-analytifiable algebraic spaces in Example 3.1.1, we do not think it is possible to prove this theorem via the methods of rigid geometry. The key to our success is to study quotient problems in the category of k-analytic spaces in the sense of Berkovich, and only *a posteriori* translating such results back into classical rigid geometry. In this sense, the key theorem in this paper is an existence result for quotients in the k-analytic category (recorded as Theorem 4.2.2):

Theorem 1.2.2. *Let $R \rightrightarrows U$ be an étale equivalence relation in the category of k-analytic spaces. The quotient U/R exists whenever the diagonal $\delta : R \to U \times U$ is a closed immersion. In such cases U/R is separated. If U is good (resp. strictly k-analytic) then so is U/R.*

The quotient $X = U/R$ in this theorem represents a suitable quotient sheaf for the étale topology, and the natural map $U \to X$ is an étale surjection with respect to which the natural map $R \to U \times_X U$ is an isomorphism. (See §4.1 for a discussion of basic notions related to quotients in the k-analytic category.) The hypothesis on δ in Theorem 1.2.2 is never satisfied by the étale equivalence relations arising from k-analytification of étale charts $\mathscr{R} \rightrightarrows \mathscr{U}$ of *non-separated* locally separated algebraic spaces $\mathscr{X}$ locally of finite type over k.

The locally separated algebraic spaces in Example 3.1.1 that are not analytifiable in the sense of rigid geometry are also not analytifiable in the sense of k-analytic spaces (by Remark 3.1.2). Thus, some assumption on δ is necessary in Theorem 1.2.2 to avoid such examples. We view quasi-compactness assumptions on δ as a reasonable way to proceed. Note that since δ is separated (it is a monomorphism), it is quasi-compact if and only if it topologically proper (i.e., a *compact* map in the terminology of Berkovich).

Remark 1.2.3. There are two senses in which it is not possible to generalize Theorem 1.2.2 by just assuming δ is compact (i.e., topologically proper). First, if we assume that U is locally separated in the sense that each $u \in U$ has a separated open neighborhood then compactness of δ forces it to be a closed immersion. That is, if an étale equivalence relation on a locally separated k-analytic space U has a compact diagonal then this diagonal is automatically a closed immersion. For example, a locally separated k-analytic space X has compact diagonal Δ_X if and only if it is separated. This is shown in [CT, 2.2].

Second, with no local separatedness assumptions on U it does not suffice to assume that δ is compact. In fact, in Example 5.1.4 we give examples of (non-separated) compact Hausdorff U and a free right action on U by a finite group G such that U/G does not exist. In such cases the action map $\delta : R = U \times G \to U \times U$ defined by $(u, g) \mapsto (u, u.g)$ is a compact map. We also give analogous such examples of non-existence in the context of rigid geometry. Recall that in the k-analytic category, separatedness is stronger than the Hausdorff property since the theory of fiber products is finer than in topology.

The weaker Hausdorff property of U/R in Theorem 1.2.2 can also be deduced from the general fact (left as an exercise) that a locally Hausdorff and locally compact topological space X is Hausdorff if and only if it is quasi-separated (in the sense that the overlap of any pair of quasi-compact subsets of X is quasi-compact).

Beware that étaleness in rigid geometry is a weaker condition than in the category of k-analytic spaces (aside from maps arising from algebraic scheme morphisms), so our existence result for U/R in the k-analytic category does not yield a corresponding existence result for étale equivalence relations in rigid geometry when the diagonal map is a closed immersion: we do not have a satisfactory analogue of Theorem 1.2.2 in rigid geometry.

Example 1.2.4. Let U be a separated k-analytic space and let G be an abstract group equipped with a right action on U that is free in the sense that the action map $\delta : U \times G \to U \times U$ is a monomorphism. Equivalently, G acts freely on $U(K)$ for each algebraically closed analytic extension K/k. Assume that the action is properly discontinuous with finite physical stabilizers in the sense that for each $u \in U$ the group $G_u = \{g \in G \,|\, g(u) = u\}$ is finite and u admits an open neighborhood V such that $g(V) \cap V = \emptyset$ for all $g \in G - G_u$. In such cases δ is a closed immersion, so by Theorem 1.2.2 the quotient U/G exists as a separated k-analytic space; it is good (resp. strictly k-analytic) if U is. If U is paracompact and strictly k-analytic and U_0 denotes the associated separated rigid space in the sense of [Ber2, 1.6.1] then the separated strictly k-analytic quotient U/G is paracompact if either G is finite or U is covered by countably many compacts, and when U/G is paracompact then the associated separated rigid space $(U/G)_0$ is a quotient U_0/G in the sense of rigid geometry (as defined in §2.1). We leave the details to the reader. As a special case, if k'/k is a finite Galois extension then descent data relative to k'/k on separated k'-analytic spaces is always effective. (This allows one to replace G with an étale k-group above, provided that its geometric fiber is finitely generated.)

1.3. **Strategy of proof.** To construct U/R under our assumptions in Theorem 1.2.2, we first observe that the problem can be localized because representable functors on the category of k-analytic spaces are sheaves for the étale topology (proved in [Ber2] under some goodness/strictness hypotheses that we remove). We are thereby able to use an étale localization argument and the nice topological properties of k-analytic spaces (via compactness and connectedness arguments) to reduce to the case of a free right action by a finite group G on a possibly non-affinoid space U. Since the action map $\delta : U \times G \to U \times U$ is a closed immersion (due to the initial hypotheses), such U must at least be separated.

Our existence result for U/G requires a technique to construct coverings of U by affinoid domains that have good behavior with respect to the G-action. We do not know any technique of this type in rigid geometry (with admissibility of the covering), but in the k-analytic category there is a powerful tool to do this, namely the theory of reduction of germs of k-analytic spaces that was developed in [T2]. Briefly, this technique often reduces difficult local construction problems in non-archimedean geometry to more tractable problems in an algebro-geometric setting (with Zariski–Riemann spaces). Here we use the separatedness of U: assuming only that U is Hausdorff (and making no assumptions on δ) we formulate a condition on reduction of germs that is always satisfied when U is separated and is sufficient for the existence of U/G.

A special case of Theorem 1.2.2 is the k-analytification of an étale chart of a separated algebraic space locally of finite type over k. This yields an analogue of Theorem 1.2.1 using analytification in the sense of k-analytic spaces. But in Theorem 1.2.2 if U is strictly k-analytic (resp. good) then the quotient space U/R is as well. Hence, we can make a link with quotient constructions in classical rigid geometry, thereby deducing Theorem 1.2.1 from Theorem 1.2.2.

1.4. **Overview of paper.** In §2 we gather some basic definitions, preliminary results, and formalism related to flat equivalence relations and analytification in the sense of rigid geometry. Examples of non-analytifiable (but locally separated) algebraic spaces are given in §3, where we also discuss some elementary examples of étale quotients in the affinoid case and we review GAGA for proper algebraic spaces over a non-archimedean field. This is all preparation for §4, where we adapt §2 to the k-analytic category and then carry out the preceding strategy to prove the two theorems stated above. More precisely, in §4 we reduce ourselves to proving the existence of U/G when G is a finite group with a free right action on a separated k-analytic space U, and in §5 we solve this existence problem by using the theory from [T2].

Since we can now always analytify proper algebraic spaces, as an application of our results it makes sense to try to establish a rigid-analytic analogue of Artin's equivalence between proper complex-analytic spaces and Moishezon spaces. This involves new difficulties (especially in positive characteristic), and so will be given in a subsequent paper [C4].

1.5. **Conventions.** The ground field k in the rigid-analytic setting is always understood to be a non-archimedean field, and a field extension K/k is called *analytic* if K is non-archimedean with respect to a fixed absolute value extending the one on k. When we work with Berkovich's theory of analytic spaces we also allow the possibility that the absolute value on the ground field is trivial, and we define the notion of *analytic* extension similarly (allowing trivial absolute values). By abuse of notation, if x is a point in a k-analytic space X then we write (X, x) to denote the associated germ (denoted X_x in [T1], [T2]).

We require algebraic spaces to have quasi-compact diagonal over $\operatorname{Spec} \mathbf{Z}$, and an *étale chart* $\mathscr{R} \rightrightarrows \mathscr{U}$ for an algebraic space $\mathscr{X}$ is always understood to have $\mathscr{U}$ (and hence $\mathscr{R} = \mathscr{U} \times_{\mathscr{X}} \mathscr{U}$) be a scheme. Also, throughout this paper, it is tacitly understood that "algebraic space" means "algebraic space locally of finite type over k" (and maps between them are k-maps) unless we explicitly say otherwise (e.g., sometimes we work over $\mathbf{C}$).

Contents

2. Étale equivalence relations and algebraic spaces

In this section we develop basic concepts related to analytification for algebraic spaces $\mathscr{X}$ that are locally of finite type over k. An existence theorem for such analytifications in the separated case is given in §4.2.

2.1. **Topologies and quotients.** We give the category of rigid spaces (over k) the *Tate-fpqc topology* that is generated by the Tate topology and the class of faithfully flat maps that admit local *fpqc* quasi-sections in the sense of [C2, Def. 4.2.1]; that is, a covering of X is a collection of flat maps $X_i \to X$ such that locally on X for the Tate topology there exist sections to $\coprod X_i \to X$ after a faithfully flat and quasi-compact base change. If we instead work with étale surjections and require the existence of local étale quasi-sections in the sense of [C2, Def. 4.2.1] then we get the *Tate-étale topology.* By [C2, Cor. 4.2.5], all representable functors are sheaves for the Tate-fpqc topology.

Example 2.1.1. If $\{X_i\}$ is a set of admissible opens in X that is a set-theoretic cover then it is a cover of X for the Tate-fpqc topology if and only if it is an admissible covering for the usual Tate topology.

Example 2.1.2. Let $h : \mathscr{X}' \to \mathscr{X}$ be a faithfully flat map of algebraic k-schemes. By [C2, Thm. 4.2.2] (whose proof uses k-analytic spaces), the associated map $h^{\mathrm{an}} : X' \to X$ between analytifications is a covering map for the Tate-fpqc topology. Likewise, if h is an étale surjection then h^{an} is a covering map for the Tate-étale topology.

Example 2.1.3. We will later need GAGA for proper algebraic spaces, but algebraic spaces have only an étale topology rather than a Zariski topology. Thus, to make the comparison of coherent cohomologies, on both the algebraic and rigid-analytic sides we want to use an étale topology. In particular, as preparation for this we need to compare coherent cohomology for the Tate topology and for the Tate-étale topology on a rigid space. We now explain how this goes.

For any rigid space S, let S_{Tate} denote the site defined by the Tate topology (objects are admissible opens in S) and let $S_{\text{ét}}$ denote the site defined by the Tate-étale topology (objects are rigid spaces étale over S). The evident left-exact pushforward map from sheaves on sets on $S_{\text{ét}}$ to sheaves of sets on S_{Tate} has an exact left adjoint that is constructed by sheafification in the usual manner, so the continuous map of sites $S_{\text{ét}} \to S_{\mathrm{Tate}}$ may be uniquely enhanced to a map of topoi $\widetilde{S_{\text{ét}}} \to \widetilde{S_{\mathrm{Tate}}}$. Descent theory for coherent sheaves on rigid spaces (see [C2, Thm. 4.2.8] for the formulation we need) shows that $\mathscr{O}_{S_{\text{ét}}}(U \to S) := \mathscr{O}_U(U)$ is a sheaf on $S_{\text{ét}}$. Using this sheaf of k-algebras makes $\widetilde{S_{\text{ét}}} \to \widetilde{S_{\mathrm{Tate}}}$ a map of ringed topoi in the evident manner. Let $\mathscr{F} \rightsquigarrow \mathscr{F}_{\text{ét}}$ denote the associated pullback operation on sheaves of modules. This is an exact functor because for any étale map $h : U \to V$ between affinoid spaces the image $h(U)$ is covered by finitely many admissible affinoid opens $V_i \subseteq V$ and the pullback map of affinoids $h^{-1}(V_i) \to V_i$ is flat on coordinate rings. It is then easy to check via descent theory (as for schemes with the Zariski and étale topologies) that $\mathscr{F} \rightsquigarrow \mathscr{F}_{\text{ét}}$ establishes an equivalence between the full subcategories of coherent sheaves on $S_{\text{ét}}$ and S_{Tate}.

Exactness of pullback provides a canonical δ-functorial $\mathscr{O}_{Y_{\text{ét}}}$-linear comparison morphism

$$(\mathrm{R}^j h_*(\mathscr{F}))_{\text{ét}} \to \mathrm{R}^j (h_{\text{ét}})_*(\mathscr{F}_{\text{ét}})$$

for any map of rigid spaces $h : X \to Y$ and any $\mathscr{O}_X$-module $\mathscr{F}$, and we claim that this is an isomorphism when h is proper and $\mathscr{F}$ is coherent. By coherence of $\mathrm{R}^j h_*(\mathscr{F})$ on Y for such h and $\mathscr{F}$, this immediately reduces to the general claim that for any rigid space S the canonical δ-functorial comparison map $\mathrm{H}^i(S, \mathscr{F}) \to \mathrm{H}^i(S_{\text{ét}}, \mathscr{F}_{\text{ét}})$ for $\mathscr{O}_S$-modules $\mathscr{F}$ is an isomorphism when $\mathscr{F}$ is coherent. Using a Čech-theoretic spectral sequence exactly as in the comparison of étale and Zariski cohomology for quasi-coherent sheaves on schemes, this reduces the problem to checking that if $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ is an étale surjection of affinoids and M is a finite A-module then the habitual complex

$$0 \to M \to M \widehat{\otimes}_A B \to M \widehat{\otimes}_A (B \widehat{\otimes}_A B) \to \dots$$

is an exact sequence. By the theory of formal models over the valuation ring R of k, it suffices to consider the case when there is a faithfully flat map $\mathrm{Spf}(\mathscr{B}) \to \mathrm{Spf}(\mathscr{A})$ of admissible formal R-schemes and a finitely presented $\mathscr{A}$-module $\mathfrak{M}$ giving rise to $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ and M on generic fibers. Thus, it is enough to show that the complex

$$0 \to \mathfrak{M} \to \mathfrak{M} \widehat{\otimes}_{\mathscr{A}} \mathscr{B} \to \mathfrak{M} \widehat{\otimes}_{\mathscr{A}} (\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{B}) \to \dots$$

is exact. For any fixed $\pi \in k$ with $0 < |\pi| < 1$, this is a complex of π-adically separated and complete R-modules, so by a simple diagram chase with compatible liftings it suffices to prove exactness modulo π^n for all $n \geq 1$. This in turn is a special case of ordinary faithfully flat descent theory since $\mathscr{A}/(\pi^n) \to \mathscr{B}/(\pi^n)$ is faithfully flat for all n.

Let $X' \to X$ be a flat surjection of rigid spaces and assume that it admits local *fpqc* quasi-sections. The maps $R = X' \times_X X' \rightrightarrows X'$ define a monomorphism $R \to X' \times X'$, and we have an isomorphism $X'/R \simeq X$ as sheaves of sets on the category of rigid spaces with the Tate-fpqc topology since the maps $R \rightrightarrows X'$ are faithfully flat and admit local *fpqc* quasi-sections in such cases.

Conversely, given a pair of flat maps $R \rightrightarrows X'$ admitting local *fpqc* quasi-sections such that $R \to X' \times X'$ is functorially an equivalence relation (in which case we call $R \to X' \times X'$ a *flat equivalence relation*), consider the sheafification of the presheaf $Z \mapsto X'(Z)/R(Z)$ with respect to the Tate-fpqc topology. If this sheaf is represented by some rigid space X then we call X (equipped with the map $X' \to X$) the *flat quotient* of X' modulo R and we denote it X'/R. By the very definition of the Tate-fpqc topology that is used to define the quotient sheaf X'/R, if a flat quotient X exists then the projection map $p : X' \to X$ admits local *fpqc* quasi-sections. Moreover, p is automatically faithfully flat. Indeed, arguing as in the case of schemes, choose a faithfully flat map $z : Z \to X$ such that there is a quasi-section $z' : Z \to X'$ over X. The map p is faithfully flat if and only if the projection $q_2 : X' \times_X Z \to Z$ is faithfully flat, and via the isomorphism $X' \times_X Z \simeq X' \times_X X' \times_{X', z'} Z = R \times_{p_2, X', z'} Z$ the map q_2 is identified with a base change of the projection $p_2 : R \to X'$ that is faithfully flat.

When the flat quotient $X = X'/R$ exists, the map $R \to X' \times_X X'$ is an isomorphism and so for every property $\mathbf{P}$ in [C2, Thm. 4.2.7] the map $X' \to X$ satisfies $\mathbf{P}$ if and only if the maps $R \rightrightarrows X'$ satisfy $\mathbf{P}$.

Likewise, X is quasi-separated (resp. separated) if and only if the map $R \to X' \times X'$ is quasi-compact (resp. a closed immersion). By descent theory for morphisms, the diagram of sets

$$\mathrm{Hom}(X, Z) \to \mathrm{Hom}(X', Z) \rightrightarrows \mathrm{Hom}(R, Z) \tag{2.1.1}$$

is left-exact for any rigid space Z when $X = X'/R$ is a flat quotient.

Definition 2.1.4. An *étale equivalence relation* on a rigid space X' is a functorial equivalence relation $R \to X' \times X'$ such that the maps $R \rightrightarrows X'$ are étale and admit local étale quasi-sections in the sense of [C2, Def. 4.2.1]. If the flat quotient X'/R exists, it is called an *étale quotient* in such cases.

Example 2.1.5. If $X' \to X$ is an étale surjection that admits local étale quasi-sections then the étale quotient of X' modulo the étale equivalence relation $R = X' \times_X X'$ exists: it is X. Thus, the preceding arguments with flat quotients work with "faithfully flat" replaced by "étale surjective" to show that in the definition of étale quotient it does not matter if we form X'/R only with respect to the Tate-étale topology.

Lemma 2.1.6. *Let $R \to X' \times X'$ be a flat equivalence relation on a rigid space X', and assume that the flat quotient X'/R exists. The equivalence relation $R \to X' \times X'$ is étale if and only if the map $X' \to X'/R$ is étale and admits local étale quasi-sections.*

Proof. Let $X = X'/R$. Since $R = X' \times_X X'$ and the map $X' \to X$ is faithfully flat with local *fpqc* quasi-sections, we may use [C2, Thm. 4.2.7] for the property $\mathbf{P}$ of being étale with local étale quasi-sections. ■

Let $\mathscr{X}$ be an algebraic space and let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$. By Example 2.1.2, the maps $\mathscr{R}^{\mathrm{an}} \rightrightarrows \mathscr{U}^{\mathrm{an}}$ admit local étale quasi-sections. Since a map in any category with fiber products is a monomorphism if and only if its relative diagonal is an isomorphism, analytification of algebraic k-schemes carries monomorphisms to monomorphisms. Thus, the morphism $\mathscr{R}^{\mathrm{an}} \to \mathscr{U}^{\mathrm{an}} \times \mathscr{U}^{\mathrm{an}}$ is a monomorphism and so $\mathscr{R}^{\mathrm{an}}$ is an étale equivalence relation on $\mathscr{U}^{\mathrm{an}}$. It therefore makes sense to ask if the étale quotient $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ exists.

We will show in Lemma 2.2.1 that such existence and the actual quotient $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ (when it exists!) are independent of the chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$ in a canonical manner, in which case we define $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ to be the *analytification* of $\mathscr{X}$. The rigid-analytic étale equivalence relations $\mathscr{R}^{\mathrm{an}} \rightrightarrows \mathscr{U}^{\mathrm{an}}$ that arise in the problem of analytifying algebraic spaces are rather special, and so one might hope that in such cases the required quotient exists whenever the algebraic space $\mathscr{X}$ is locally separated over k (as is necessary and sufficient for the existence of analytifications of algebraic spaces over $\mathbf{C}$ in the complex-analytic sense). However, we will give counterexamples in Example 3.1.1: locally separated algebraic spaces that are not analytifiable in the above sense defined via quotients. In the positive direction, the quotient $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ will be shown to always exist when $\mathscr{X}$ is separated.

2.2. **Analytification of algebraic spaces.** Let $\mathscr{X}$ be an algebraic space, and $\mathscr{R} \rightrightarrows \mathscr{U}$ an étale chart for $\mathscr{X}$. We now address the "independence of choice" and canonicity issues for $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ in terms of $\mathscr{X}$. These will go essentially as in the complex-analytic case except that we have to occasionally use properties related to local étale quasi-sections for the Tate topology. In the complex-analytic case it does not seem that the relevant arguments are available in the literature, so for this reason and to ensure that the Tate topology presents no difficulties we have decided to give the arguments in detail (especially so we can see that they carry over to k-analytic spaces, as we shall need later).

Let $\mathscr{R}_1 \rightrightarrows \mathscr{U}_1$ and $\mathscr{R}_2 \rightrightarrows \mathscr{U}_2$ be two étale charts for $\mathscr{X}$. Let $\mathscr{U}_{12} = \mathscr{U}_1 \times_{\mathscr{X}} \mathscr{U}_2$ and let $\mathscr{R}_{12} = \mathscr{R}_1 \times_{\mathscr{X}} \mathscr{R}_2$, so $\mathscr{R}_{12} \rightrightarrows \mathscr{U}_{12}$ is an étale chart dominating each chart $\mathscr{R}_i \rightrightarrows \mathscr{U}_i$.

Lemma 2.2.1. *If $\mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}}$ exists then so do $\mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ and $\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$, and the natural maps*

$$\pi_i : \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}} \to \mathscr{U}_i^{\mathrm{an}}/\mathscr{R}_i^{\mathrm{an}}$$

are isomorphisms. The induced isomorphism $\phi = \pi_2 \circ \pi_1^{-1} : \mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}} \simeq \mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ is transitive with respect to a third choice of étale chart for $\mathscr{X}$.

Proof. The natural composite map $\mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_1^{\mathrm{an}} \to \mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}}$ is étale with local étale quasi-sections (as each step in the composite has this property, due to Example 2.1.2 applied to $\mathscr{U}_{12} \to \mathscr{U}_1$ and the defining properties for $\mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}}$ as an étale quotient). We claim that this composite map serves as the étale quotient for $\mathscr{U}_{12}^{\mathrm{an}}$ by $\mathscr{R}_{12}^{\mathrm{an}}$ (so the étale quotient $\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ exists and π_1 is an isomorphism). The problem is to prove

$$\mathscr{R}_{12}^{\mathrm{an}} \stackrel{?}{=} \mathscr{U}_{12}^{\mathrm{an}} \times_{\mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}}} \mathscr{U}_{12}^{\mathrm{an}}$$

as subfunctors of $\mathscr{U}_{12}^{\mathrm{an}} \times \mathscr{U}_{12}^{\mathrm{an}}$, and this is easily proved via two ingredients: the analytic isomorphism

$$\mathscr{R}_1^{\mathrm{an}} \simeq \mathscr{U}_1^{\mathrm{an}} \times_{\mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}}} \mathscr{U}_1^{\mathrm{an}}$$

(that arises from the assumption of existence for the étale quotient $\mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}}$) and the analytification of the algebraic isomorphism

$$\mathscr{R}_{12} \simeq (\mathscr{U}_{12} \times \mathscr{U}_{12}) \times_{\mathscr{U}_1 \times \mathscr{U}_1} \mathscr{R}_1$$

as algebraic k-schemes and as subfunctors of $\mathscr{U}_{12} \times \mathscr{U}_{12}$.

Now we address the existence of $\mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ and the isomorphism property for π_2. The map $\mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}}$ is an étale surjection with local étale quasi-sections, and so by rigid-analytic descent theory the étale quotient map $\mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ admits at most one factorization through the map $\mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}}$, in which case the resulting map $h : \mathscr{U}_2^{\mathrm{an}} \to \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ is an étale surjection with local étale quasi-sections. To prove that h exists, by (2.1.1) it is necessary and sufficient to check that the two maps $\mathscr{U}_{12}^{\mathrm{an}} \times_{\mathscr{U}_2^{\mathrm{an}}} \mathscr{U}_{12}^{\mathrm{an}} \rightrightarrows \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ coincide, and this equality follows from the inclusion

$$\mathscr{U}_{12} \times_{\mathscr{U}_2} \mathscr{U}_{12} \subseteq \mathscr{U}_{12} \times_{\mathscr{X}} \mathscr{U}_{12} = \mathscr{R}_{12}$$

as subfunctors of $\mathscr{U}_{12} \times \mathscr{U}_{12}$. Thus, we have an étale surjection $h : \mathscr{U}_2^{\mathrm{an}} \to \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ with local étale quasi-sections.

Since $\mathscr{R}_{12} = (\mathscr{U}_{12} \times \mathscr{U}_{12}) \times_{\mathscr{U}_2 \times \mathscr{U}_2} \mathscr{R}_2$ as subfunctors of $\mathscr{U}_{12} \times \mathscr{U}_{12}$, the natural map $\mathscr{R}_{12} \to \mathscr{R}_2$ is an étale surjection and hence the two composite maps

$$\mathscr{R}_2^{\mathrm{an}} \rightrightarrows \mathscr{U}_2^{\mathrm{an}} \xrightarrow{h} \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$$

are equal if and only if equality holds after composition with the map $\mathscr{R}_{12}^{\mathrm{an}} \to \mathscr{R}_2^{\mathrm{an}}$. Such equality holds after composition because $\mathscr{R}_{12} \rightrightarrows \mathscr{U}_{12}$ is co-commutative over $\mathscr{R}_2 \rightrightarrows \mathscr{U}_2$. Hence, we have

$$\mathscr{R}_2^{\mathrm{an}} \subseteq \mathscr{U}_2^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_2^{\mathrm{an}} \tag{2.2.1}$$

as subfunctors of $\mathscr{U}_2^{\mathrm{an}} \times \mathscr{U}_2^{\mathrm{an}}$. To conclude that $\mathscr{U}_2^{\mathrm{an}} \to \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ is an étale quotient by $\mathscr{R}_2^{\mathrm{an}}$, we need the inclusion of subfunctors (2.2.1) to be an equality, which is to say that the natural map

$$g : \mathscr{U}_2^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_2^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}} \times \mathscr{U}_2^{\mathrm{an}}$$

factors through the subfunctor $\mathscr{R}_2^{\mathrm{an}}$. Since the map

$$f : \mathscr{R}_{12}^{\mathrm{an}} \simeq \mathscr{U}_{12}^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_2^{\mathrm{an}}$$

is an étale surjection with local étale quasi-sections (as $\mathscr{U}_{12} \to \mathscr{U}_2$ is an étale surjection of schemes, so Example 2.1.2 applies), by descent theory for rigid-analytic morphisms it suffices to check that the composite map $g \circ f$ factors through $\mathscr{R}_2^{\mathrm{an}}$. Clearly $g \circ f$ equals the canonical composite map

$$\mathscr{R}_{12}^{\mathrm{an}} \to \mathscr{U}_{12}^{\mathrm{an}} \times \mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}} \times \mathscr{U}_2^{\mathrm{an}}$$

that is the analytification of the composite map $\mathscr{R}_{12} \to \mathscr{U}_{12} \times \mathscr{U}_{12} \to \mathscr{U}_2 \times \mathscr{U}_2$. This latter composite map obviously factors through the inclusion $\mathscr{R}_2 \to \mathscr{U}_2 \times \mathscr{U}_2$, thereby completing the proof that $\mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ exists and that π_2 is an isomorphism.

To check that the isomorphism $\pi_2 \circ \pi_1^{-1} : \mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}} \simeq \mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ is transitive with respect to a third choice of étale chart for $\mathscr{X}$, it suffices to note that in the preceding considerations we only needed that the étale chart $\mathscr{R}_{12} \rightrightarrows \mathscr{U}_{12}$ dominates the other two charts, and not that it is specifically their "fiber product". ■

Lemma 2.2.1 permits us to make the following definition.

Definition 2.2.2. An algebraic space $\mathscr{X}$ is *analytifiable* if the étale quotient $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ exists for some (and hence any) étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$.

For an analytifiable $\mathscr{X}$, the *analytification* $\mathscr{X}^{\rm an}$ is defined to be $\mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$. Up to unique isomorphism, this étale quotient is independent of the specific choice of étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ (by Lemma 2.2.1). We now express the functoriality of analytification when analytifications exist.

Theorem 2.2.3. *Let $\mathscr{X}$ and $\mathscr{X}'$ be analytifiable algebraic spaces and let $f : \mathscr{X}' \to \mathscr{X}$ be a k-morphism. Let $\mathscr{R} \rightrightarrows \mathscr{U}$ and $\mathscr{R}' \rightrightarrows \mathscr{U}'$ be respective étale charts such that f lifts to a map $F : \mathscr{U}' \to \mathscr{U}$ for which $F \times F$ carries $\mathscr{R}'$ into $\mathscr{R}$ (such a pair of charts always exists). The map of rigid spaces*

$$f^{\rm an} : (\mathscr{X}')^{\rm an} \simeq (\mathscr{U}')^{\rm an}/(\mathscr{R}')^{\rm an} \to \mathscr{U}^{\rm an}/\mathscr{R}^{\rm an} \simeq \mathscr{X}^{\rm an}$$

induced by $F^{\rm an}$ depends only on f and not on the étale charts or the map F lifting f, and this procedure enhances the construction $\mathscr{X} \rightsquigarrow \mathscr{X}^{\rm an}$ to be a functor from the category of analytifiable algebraic spaces over k to the category of rigid spaces over k. Moreover:

- *the category of analytifiable algebraic spaces is stable under the formation of fiber products and passage to open and closed subspaces,*
- *the functor $\mathscr{X} \rightsquigarrow \mathscr{X}^{\rm an}$ is compatible with the formation of fiber products and carries open/closed immersions to Zariski-open/closed immersions,*
- *if $f : \mathscr{X}' \to \mathscr{X}$ is a morphism between analytifiable algebraic spaces then $f^{\rm an}$ is separated if and only if f is separated.*

Proof. Any two étale charts are dominated by a third, and any two lifts of f with respect to a fixed choice of charts are $\mathscr{R}$-equivalent. Thus, the well-definedness of $f^{\rm an}$ is an immediate consequence of Lemma 2.2.1. The compatibility with composition of morphisms follows from the independence of the choice of charts, so analytification is indeed a functor on analytifiable algebraic spaces over k.

Let $\mathscr{X}$ be an analytifiable algebraic space over k and let $\mathscr{X}' \to \mathscr{X}$ be an open (resp. closed) immersion. We let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$, and let $\mathscr{U}' \to \mathscr{U}$ denote the pullback of $\mathscr{X}'$. The analytification of this inclusion is a Zariski-open (resp. closed) immersion. Rigid-analytic descent theory with respect to coherent ideal sheaves trivially implies that the analytification of $\mathscr{U}'$ descends to a Zariski-open (resp. closed) immersion into $\mathscr{X}^{\rm an}$, and this descent is easily seen to be an analytification of $\mathscr{X}'$.

Now we consider fiber products. Let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic spaces over an algebraic space $\mathscr{Z}$, and assume that all three are analytifiable. We need to check that $\mathscr{P} = \mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$ is analytifiable and that the natural map $\mathscr{P}^{\rm an} \to \mathscr{X}^{\rm an} \times_{\mathscr{Z}^{\rm an}} \mathscr{Y}^{\rm an}$ is an isomorphism. We choose étale charts $\mathscr{X} = \mathscr{U}'/\mathscr{R}'$, $\mathscr{Y} = \mathscr{U}''/\mathscr{R}''$, and $\mathscr{Z} = \mathscr{U}/\mathscr{R}$, and we consider the fiber product

$$\mathscr{R}' \times_{\mathscr{R}} \mathscr{R}'' \rightrightarrows \mathscr{U}' \times_{\mathscr{U}} \mathscr{U}''$$

as an étale chart for $\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$. There is an evident étale surjective map

$$\pi : (\mathscr{U}' \times_{\mathscr{U}} \mathscr{U}'')^{\rm an} \simeq (\mathscr{U}')^{\rm an} \times_{\mathscr{U}^{\rm an}} (\mathscr{U}'')^{\rm an} \to \mathscr{X}^{\rm an} \times_{\mathscr{Z}^{\rm an}} \mathscr{Y}^{\rm an},$$

and this admits local étale quasi-sections because $\mathscr{X}^{\rm an}$, $\mathscr{Y}^{\rm an}$, and $\mathscr{Z}^{\rm an}$ are étale quotients associated to the analytifications of the chosen étale charts for $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ respectively. The quotient properties for $\mathscr{X}^{\rm an}$, $\mathscr{Y}^{\rm an}$, and $\mathscr{Z}^{\rm an}$ in the category of rigid spaces permit us to identify the fiber square of π with $\mathscr{R}'^{\rm an} \times_{\mathscr{R}^{\rm an}} \mathscr{R}''^{\rm an} = (\mathscr{R}' \times_{\mathscr{R}} \mathscr{R}'')^{\rm an}$ in the evident manner, and so analytification is compatible with fiber products.

Finally, we show that if $f : \mathscr{X}' \to \mathscr{X}$ is a morphism between analytifiable algebraic spaces then $f^{\rm an}$ is separated if and only if f is separated. The compatibility of analytification and fiber products identifies $\Delta_{f^{\rm an}}$ and $\Delta_f^{\rm an}$, so upon replacing f with Δ_f we are reduced to checking that if $f : \mathscr{X}' \to \mathscr{X}$ is a quasi-compact immersion between algebraic spaces then f is a closed immersion if and only if $f^{\rm an}$ is a closed immersion. Let $h : \mathscr{U} \to \mathscr{X}$ be an étale covering by a scheme and let $F : \mathscr{U}' \to \mathscr{U}$ be the base change of f along h. Since f is a quasi-compact immersion, so is F. In particular, $\mathscr{U}'$ is a scheme and $F^{\rm an}$ is identified with the base change of $f^{\rm an}$ by means of $h^{\rm an}$. The map $h^{\rm an}$ is an étale surjection with local étale quasi-sections, so by rigid-analytic descent theory with respect to the property of being a closed immersion we see that $f^{\rm an}$ is a closed immersion if and only if $F^{\rm an}$ is a closed immersion. Likewise, on the algebraic side we see that f is a closed immersion if and only if F is a closed immersion. Thus, we may replace f with F to reduce to the case when f is a quasi-compact immersion between algebraic k-schemes. This case is handled by [C1, 5.2.1(2)]. ∎

Corollary 2.2.4. *Let $\mathscr{X}$ be an algebraic space and let $\{\mathscr{X}_i\}$ be an open covering. Analytifiability of $\mathscr{X}$ is equivalent to that of all of the $\mathscr{X}_i$'s.*

Proof. By Theorem 2.2.3, if $\mathscr{X}^{\rm an}$ exists then so does $\mathscr{X}_i^{\rm an}$ (as a Zariski-open in $\mathscr{X}^{\rm an}$) for all i. Conversely, assume that $\mathscr{X}_i^{\rm an}$ exists for all i. The algebraic space $\mathscr{X}_{ij} = \mathscr{X}_i \cap \mathscr{X}_j$ is identified with a Zariski-open subspace of both $\mathscr{X}_i$ and $\mathscr{X}_j$, so by Theorem 2.2.3 the rigid space $\mathscr{X}_{ij}^{\rm an}$ exists and is identified with a Zariski-open locus in $\mathscr{X}_i^{\rm an}$ and $\mathscr{X}_j^{\rm an}$. Since $\mathscr{X}_{ij} \cap \mathscr{X}_{ij'} = \mathscr{X}_{ij} \times_{\mathscr{X}_i} \mathscr{X}_{ij'}$ for any i, j, j', the fiber-product compatibility in Theorem 2.2.3 provides the triple-overlap compatibility that is required to glue the $\mathscr{X}_i^{\rm an}$'s to construct a rigid space X having the $\mathscr{X}_i^{\rm an}$'s as an admissible covering with $\mathscr{X}_i^{\rm an} \cap \mathscr{X}_j^{\rm an} = \mathscr{X}_{ij}^{\rm an}$ inside of X for all i and j. In particular, $X - \mathscr{X}_i^{\rm an}$ meets every $\mathscr{X}_j^{\rm an}$ in an analytic set, and hence $\mathscr{X}_i^{\rm an}$ is Zariski-open in X.

We now check that X serves as an analytification of $\mathscr{X}$. Let $p_1, p_2 : \mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$ and let $\mathscr{U}_i$ be the preimage of $\mathscr{X}_i$ in $\mathscr{U}$. Let $\mathscr{R}_i = p_1^{-1}(\mathscr{U}_i) \cap p_2^{-1}(\mathscr{U}_i)$. Clearly $\mathscr{U}_i$ is $\mathscr{R}$-saturated in $\mathscr{U}$ and $\mathscr{R}_i \rightrightarrows \mathscr{U}_i$ is an étale chart for $\mathscr{X}_i$ via the natural map $\mathscr{U}_i \to \mathscr{X}_i$. Let $\mathscr{U}_{ij} = \mathscr{U}_i \cap \mathscr{U}_j$ and $\mathscr{R}_{ij} = p_1^{-1}(\mathscr{U}_{ij}) \cap p_2^{-1}(\mathscr{U}_{ij})$, so $\mathscr{U}_{ij}$ is $\mathscr{R}$-saturated and $\mathscr{R}_{ij} \rightrightarrows \mathscr{U}_{ij}$ is an étale chart for $\mathscr{X}_{ij}$. The gluing construction of X implies that the maps

$$f_i : \mathscr{U}_i^{\rm an} \to \mathscr{X}_i^{\rm an} \hookrightarrow X$$

satisfy $f_i|_{\mathscr{U}_{ij}^{\rm an}} = f_j|_{\mathscr{U}_{ij}^{\rm an}}$, so the f_i's uniquely glue to define an étale surjection $f : \mathscr{U}^{\rm an} \to X$ that restricts to the canonical map $\mathscr{U}_i^{\rm an} \to \mathscr{X}_i^{\rm an}$ over each Zariski-open $\mathscr{X}_i^{\rm an} \subseteq X$ and hence (by the definition of $\mathscr{X}_i^{\rm an}$ as the quotient $\mathscr{U}_i^{\rm an}/\mathscr{R}_i^{\rm an}$) the map f admits local étale quasi-sections. Since the $\mathscr{R}_i^{\rm an}$'s form a Zariski-open covering of $\mathscr{R}^{\rm an}$, it is easy to check that the composite maps $\mathscr{R}^{\rm an} \rightrightarrows \mathscr{U}^{\rm an} \to X$ coincide. Thus, $\mathscr{R}^{\rm an}$ is naturally a rigid space over X and we obtain a canonical X-map $h : \mathscr{R}^{\rm an} \to \mathscr{U}^{\rm an} \times_X \mathscr{U}^{\rm an}$. The restriction of h over $\mathscr{X}_i^{\rm an}$ is the canonical map $\mathscr{R}_i^{\rm an} \to \mathscr{U}_i^{\rm an} \times_{\mathscr{X}_i^{\rm an}} \mathscr{U}_i^{\rm an}$ that is an isomorphism (due to the definition of $\mathscr{X}_i^{\rm an}$), so h is an isomorphism. Hence, X indeed serves as an analytification of $\mathscr{X}$. ∎

2.3. **Properties of analytification.** We now summarize some basic observations concerning properties of analytification of algebraic spaces, especially in connection with properties of morphisms between algebraic spaces.

If $f : \mathscr{X} \to \mathscr{Y}$ is a faithfully flat map between analytifiable algebraic spaces over k, then we claim that the induced faithfully flat map $f^{\rm an} : \mathscr{X}^{\rm an} \to \mathscr{Y}^{\rm an}$ has local *fpqc* quasi-sections, and if f is an étale surjection then we claim that $f^{\rm an}$ has local étale quasi-sections. This generalizes Example 2.1.2 to the case of algebraic spaces. To prove these claims, we pick a chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{Y}$, so $\mathscr{U}^{\rm an} \to \mathscr{Y}^{\rm an}$ has local étale quasi-sections (as $\mathscr{Y}^{\rm an}$ is the étale quotient $\mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$). Hence, we may replace $\mathscr{X} \to \mathscr{Y}$ with its base change by the étale surjection $\mathscr{U} \to \mathscr{Y}$, so we can assume that $\mathscr{Y}$ is an algebraic k-scheme. Running through a similar argument with an étale chart for $\mathscr{X}$ reduces us to the case when $\mathscr{X}$ is also an algebraic k-scheme, and so we are brought to the settled scheme case.

Theorem 2.3.1. *If $f : \mathscr{X} \to \mathscr{Y}$ is a map between analytifiable algebraic spaces over k, then f has property* **P** *if and only if $f^{\rm an}$ has property* **P**, *where* **P** *is any of the following properties: separated, monomorphism, surjective, isomorphism, open immersion, flat, smooth, and étale. Likewise, if f is finite type then we may take* **P** *to be: closed immersion, finite, proper, quasi-finite* (*i.e., finite fibers*).

Proof. By [C2, Thm. 4.2.7] and descent theory for schemes, we may work étale-locally on $\mathscr{Y}$ and so we can assume that $\mathscr{Y}$ is a scheme of finite type over k. Since flat maps locally of finite type between algebraic spaces are open, the essential properties to consider are isomorphism and properness; the rest then follow exactly as in the case of schemes. By Chow's lemma for algebraic spaces and [C2, §A.1] (for properness), the proper case is reduced to the case of quasi-compact immersions of schemes (more specifically, a quasi-compact immersion into a projective space over $\mathscr{Y}$), and this case follows from [C1, 5.2.1(2)]. If $f^{\rm an}$ is an isomorphism then f is quasi-finite, flat, and (by Theorem 2.2.3) separated, so $\mathscr{X}$ is necessarily a scheme. Thus, we may use [C1, 5.2.1(1)] to infer that f is an isomorphism. ∎

Example 2.3.2. Let $\mathscr{X}$ be an analytifiable algebraic space. Recall that a *point* of an algebraic space is a monic morphism from the spectrum of a field, and the underlying topological space $|\mathscr{X}|$ of an algebraic space is its set of (isomorphism classes of) points. Every map to $\mathscr{X}$ from the spectrum of a field factors through a unique point of $\mathscr{X}$ [Kn, II, 6.2], so distinct points have empty fiber product over $\mathscr{X}$. Moreover, every point of an algebraic space admits a residually-trivial pointed étale scheme neighborhood [Kn, II, 6.4].

With respect to the usual Zariski topology on $|\mathscr{X}|$, it follows by working with an étale scheme cover that a point $\mathrm{Spec}(k') \to \mathscr{X}$ is closed in $|\mathscr{X}|$ if and only if $[k' : k]$ is finite. In particular, if $j : \mathrm{Spec}(k') \to \mathscr{X}$ is a closed point then analytification defines a map $j^{\mathrm{an}} : \mathrm{Sp}(k') \to X := \mathscr{X}^{\mathrm{an}}$ that is a monomorphism (since $\Delta_{j^{\mathrm{an}}} = \Delta_j^{\mathrm{an}}$ is an isomorphism). The monic map j^{an} is easily seen to be an isomorphism onto an ordinary point of X, so in this way we have defined a map of sets $|\mathscr{X}|_0 \to X$, where $|\mathscr{X}|_0$ is the set of closed points of $\mathscr{X}$. By a fiber product argument we see that this map is bijective, and by construction it preserves residue fields over k.

Recall that at any point of an algebraic space there is a naturally associated henselian local ring with the same residue field (use the limit of residually-trivial pointed étale scheme neighborhoods; for schemes this is the henselization of the usual local ring). Thus, for any coherent sheaf $\mathscr{F}$ on $\mathscr{X}$ and closed point $x_0 \in |\mathscr{X}|_0$, we get a stalk $\mathscr{F}_{x_0}$ that is a finite module over the henselian local noetherian k-algebra $\mathscr{O}_{\mathscr{X},x_0}$. If $x \in X$ is the corresponding point of X then every residually trivial pointed étale neighborhood $(\mathscr{U}, x_0) \to (\mathscr{X}, x_0)$ with analytifiable $\mathscr{U}$ (e.g., a scheme) induces an étale map $\mathscr{U}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ that is an isomorphism between small admissible opens around the canonical copy of x in each (due to residual triviality). Hence, we get a natural map of k-algebras $\mathscr{O}_{\mathscr{X}}(\mathscr{U}) \to \mathscr{O}_{X,x}$ compatible with the $k(x)$-points on each, and passage to the direct limit thereby defines a local k-algebra map $\mathscr{O}_{\mathscr{X},x_0} \to \mathscr{O}_{X,x}$. There is likewise a map $\mathscr{F}_{x_0} \to \mathscr{F}_x^{\mathrm{an}}$ that is linear over this map of rings, and so we get a natural $\mathscr{O}_{X,x}$-linear *comparison map*

$$\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{\mathscr{X},x_0}} \mathscr{F}_{x_0} \to \mathscr{F}_x^{\mathrm{an}}.$$

This is functorial in $(\mathscr{X}, x_0)$, so it can be computed on a residually-trivial pointed étale scheme neighborhood of x_0. Hence, by reduction to the scheme case we see that this comparison map between finite $\mathscr{O}_{X,x}$-modules is an isomorphism because the induced map on completions is an isomorphism (due to the isomorphism between completions of algebraic and analytic local rings at a common point of an algebraic k-scheme).

Example 2.3.3. Since reducedness is inherited under analytification of reduced algebraic k-schemes, by using an étale scheme cover we see that if $\mathscr{X}$ is an analytifiable reduced algebraic space then $\mathscr{X}^{\mathrm{an}}$ is reduced. Thus, if $\mathscr{X}$ is any analytifiable algebraic space (so the closed subspace $\mathscr{X}_{\mathrm{red}}$ is also analytifiable) then there is a natural closed immersion $(\mathscr{X}_{\mathrm{red}})^{\mathrm{an}} \hookrightarrow (\mathscr{X}^{\mathrm{an}})_{\mathrm{red}}$ whose formation is compatible with étale base change on $\mathscr{X}$, so it is an isomorphism by reduction to the scheme case.

As an application of this compatibility, if $\mathscr{X}$ is any analytifiable algebraic space and $\{\mathscr{X}_i\}$ is its set of irreducible components with reduced structure then we get closed immersions $\mathscr{X}_i^{\mathrm{an}} \hookrightarrow \mathscr{X}^{\mathrm{an}}$ which we claim are the irreducible components of $\mathscr{X}^{\mathrm{an}}$ endowed with their reduced structure. To check this, since $(\mathscr{X}_i \times_{\mathscr{X}} \mathscr{X}_j)^{\mathrm{an}} \simeq \mathscr{X}_i^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} \mathscr{X}_j^{\mathrm{an}}$ we see by dimension reasons that it suffices to show that $\mathscr{X}^{\mathrm{an}}$ is irreducible when $\mathscr{X}$ is irreducible and reduced. There is a dense Zariski-open subspace $\mathscr{X}_0 \subseteq \mathscr{X}$ that is a scheme, and by [C1, 2.3.1] the rigid space $\mathscr{X}_0^{\mathrm{an}}$ is irreducible. Thus, by the existence of global irreducible decomposition for rigid spaces, it suffices to show that the Zariski-open locus $\mathscr{X}_0^{\mathrm{an}}$ in $\mathscr{X}^{\mathrm{an}}$ is everywhere dense. Consideration of points valued in finite extensions of k shows that the analytic set in $\mathscr{X}^{\mathrm{an}}$ complementary to $\mathscr{X}_0^{\mathrm{an}}$ is $\mathscr{Z}^{\mathrm{an}}$, where $\mathscr{Z} = \mathscr{X} - \mathscr{X}_0$ (say, with its reduced structure). Thus, we just need that the coherent ideal of $\mathscr{Z}^{\mathrm{an}}$ in the reduced $\mathscr{X}^{\mathrm{an}}$ is nowhere zero, and this is clear by comparing its completed stalks with those of $\mathscr{Z}$ in the irreducible and reduced algebraic space $\mathscr{X}$. (Here we use the end of Example 2.3.2 and the fact that a proper closed subspace of an irreducible algebraic space has nowhere-dense pullback to any étale scheme cover, even though irreducibility is not local for the étale topology.)

If X is a complex-analytic space then since X rests on an ordinary topological space, we can find an open subset $V \subseteq X \times X$ such that the diagonal $X \to X \times X$ factors through a closed immersion into V. (Take V to be a union $\cup(U_i \times U_i)$ where the U_i are Hausdorff open sets that cover X.) That is, Δ_X factors like an immersion in algebraic geometry. This suggests that local separatedness of $\mathscr{X}$ (i.e., the diagonal $\Delta_{\mathscr{X}}$ being an immersion) should be a necessary condition for the analytifiability of an algebraic space $\mathscr{X}$ over $\mathbf{C}$. Due to lack of a reference, we now give a proof of this fact and its non-archimedean counterpart.

Theorem 2.3.4. *Let $\mathscr{X}$ be an algebraic space locally of finite type over either $\mathbf{C}$ or a non-archimedean field k. If $\mathscr{X}^{\mathrm{an}}$ exists then $\mathscr{X}$ must be locally separated.*

See Theorem 4.1.4 for an analogous result for k-analytic spaces.

Proof. Choose a scheme $\mathscr{U}$ equipped with an étale surjection $\mathscr{U} \twoheadrightarrow \mathscr{X}$. Let $\mathscr{R} = \mathscr{U} \times_{\mathscr{X}} \mathscr{U}$, so our aim is to prove that the monomorphism $i : \mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is an immersion. The map i is separated since it is a monomorphism. Note that i is a base change of the quasi-compact diagonal $\Delta_{\mathscr{X}}$. By the compatibility of analytification and fiber products (for algebraic spaces in both the complex-analytic and non-archimedean cases), $\mathscr{X}^{\mathrm{an}} \times \mathscr{X}^{\mathrm{an}}$ is identified with an analytification of $\mathscr{X} \times \mathscr{X}$, and in this way $\Delta_{\mathscr{X}}^{\mathrm{an}}$ is identified with $\Delta_{\mathscr{X}^{\mathrm{an}}}$. Thus, the map i^{an} is a base change of $\Delta_{\mathscr{X}^{\mathrm{an}}}$.

We first treat the complex-analytic case, and then we adapt the argument to work in the non-archimedean case. The diagonal map of a complex-analytic space is a topological embedding. Indeed, if S is such a space then for each $s \in S$ there is a Hausdorff open neighborhood $U_s \subseteq S$ around s, so $U = \cup(U_s \times U_s)$ is an open subset of $S \times S$ through which Δ_S factors as a closed immersion $S \hookrightarrow U$. Hence, Δ_S is a topological embedding. Applying this to $S = \mathscr{X}^{\mathrm{an}}$ gives that $\Delta_{\mathscr{X}^{\mathrm{an}}}$ is a topological embedding, so its topological base change i^{an} is also a topological embedding. It therefore suffices to show that if $f : \mathscr{Y} \to \mathscr{S}$ is a quasi-compact monomorphism between algebraic $\mathbf{C}$-schemes and f^{an} is a topological embedding then f is an immersion. Since f is monic, so Δ_f is an isomorphism, f is certainly separated. We can assume that $\mathscr{S}$ is separated and quasi-compact (e.g., affine), so $\mathscr{Y}$ is separated. By Zariski's Main Theorem [EGA, IV_3, 8.12.6], the quasi-finite separated map f factors as a composition

$$\mathscr{Y} \overset{j}{\hookrightarrow} \mathscr{T} \overset{\pi}{\to} \mathscr{S}$$

where j is an open immersion and π is finite. We can replace $\mathscr{T}$ with the schematic closure of $\mathscr{Y}$ to arrange that j has dense image. Thus, j^{an} is an open immersion with dense image with respect to the analytic topology [SGA1, XII, §2]. We can also replace $\mathscr{S}$ with the natural closed subscheme structure on the closed set $\pi(\mathscr{T})$ to arrange that π is surjective.

We now show that $j^{\mathrm{an}}(\mathscr{Y}^{\mathrm{an}})$ is the preimage of its image under π^{an}, and that the restriction of π^{an} to $j^{\mathrm{an}}(\mathscr{Y}^{\mathrm{an}})$ is injective. Choose $y \in \mathscr{Y}^{\mathrm{an}}$ and a point $t \in \mathscr{T}^{\mathrm{an}}$ such that $\pi^{\mathrm{an}}(t) = \pi^{\mathrm{an}}(j^{\mathrm{an}}(y)) = f^{\mathrm{an}}(y)$ in $\mathscr{Y}^{\mathrm{an}}$. We will prove $t = j^{\mathrm{an}}(y)$. By denseness of $j^{\mathrm{an}}(\mathscr{Y}^{\mathrm{an}})$ in the Hausdorff analytic space $\mathscr{T}^{\mathrm{an}}$, there is a sequence of points $y_n \in \mathscr{Y}^{\mathrm{an}}$ such that $j^{\mathrm{an}}(y_n) \to t$, so applying π^{an} gives that $f^{\mathrm{an}}(y_n) \to f^{\mathrm{an}}(y)$ in the Hausdorff analytic space $\mathscr{S}^{\mathrm{an}}$. But f^{an} is a topological embedding by hypothesis, so $y_n \to y$ in the Hausdorff analytic space $\mathscr{Y}^{\mathrm{an}}$. Applying j^{an} gives $j^{\mathrm{an}}(y_n) \to j^{\mathrm{an}}(y)$, but t is the limit of this sequence, so $t = j^{\mathrm{an}}(y)$ as required. Returning to the algebraic setting, the Zariski-open set $j(\mathscr{Y}) \subseteq \mathscr{T}$ is the preimage of its image under the finite map π because this asserts an equality of constructible sets and it suffices to check such an equality on closed points. Since π is a finite surjection, hence a topological quotient map, it follows that $f(\mathscr{Y}) = \pi(j(\mathscr{Y}))$ is an open subset of $\mathscr{S}$ since its preimage $j(\mathscr{Y})$ in $\mathscr{T}$ is a Zariski-open subset. We can replace $\mathscr{S}$ with the open subscheme structure on this open set and replace $\mathscr{T}$ with its preimage under π, so the open immersion j is now surjective on closed points. Thus, j is an isomorphism, so the monomorphism $f = \pi \circ j$ is finite and therefore is a closed immersion.

Next we consider the non-archimedean case. We need to exercise some more care with respect to topological arguments, due to the role of the Tate topology and the fact that fiber products of rigid spaces are generally not fiber products on underlying sets (unless the base field is algebraically closed). As a first step, if X is a rigid space and $\Delta : X \to X \times X$ is its diagonal, then for any rigid-analytic morphism $h : Z \to X \times X$ we claim that the induced map of rigid spaces $h^*(\Delta) : Z \times_{X \times X} X \to Z$ is an embedding with respect to the canonical (totally disconnected) topology. To prove this, let $\{X_j\}$ be an admissible affinoid open covering of X and let $U = \cup_j(X_j \times X_j)$, so U is an open subset of $X \times X$ with respect to the canonical topology (though it is unclear if U must be an admissible open subset). The map Δ factors through a continuous map $\Delta' : X \to U$ under which the preimage of $X_j \times X_j$ is $X_j \subseteq X$, and which induces the diagonal Δ_{X_j} that is a closed immersion (since X_j is affinoid). Hence, Δ' is a closed embedding with respect to the canonical topologies. The preimage $h^{-1}(U) \subseteq Z$ is open with respect to the canonical topology on Z, and $h^*(\Delta)$ factors continuously through $h^{-1}(U)$ (as a map with respect to the canonical topologies). The resulting continuous map $Z \times_{X \times X} X \to h^{-1}(U)$ with respect to the canonical topologies is a closed embedding because on each of the opens $h^{-1}(X_j \times X_j) \subseteq h^{-1}(U)$ that cover $h^{-1}(U)$ it restricts to the rigid-analytic morphism $h^{-1}(X_j \times X_j) \times_{X_j \times X_j} X_j \to h^{-1}(X_j \times X_j)$ that is a base change of the closed immersion Δ_{X_j}. Hence, for any h as above, $h^*(\Delta)$ is an embedding with result to the canonical topologies, as claimed. As a special case, the analytification of the monic map $i : \mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is a topological embedding with respect to the canonical topologies since it is a base change of $\Delta_{\mathscr{X}^{\mathrm{an}}}$.

Thus, exactly as in the complex-analytic case, it suffices to prove that if $f : \mathscr{Y} \to \mathscr{S}$ is a quasi-compact monomorphism between algebraic k-schemes such that f^{an} is a topological embedding with respect to the canonical topologies then f is an immersion. The denseness result cited above from [SGA1, XII, §2] remains valid in the non-archimedean case (with the same proof) when using the canonical topology, so the preceding argument in the complex-analytic case carries over *verbatim* once we show that a separated rigid space is Hausdorff with respect to its underlying canonical topology. (The preceding argument with sequences works in the rigid-analytic case because any point in a rigid space has a countable base of open neighborhoods, though this countability property is not needed; we could instead use nets.) Let S be a separated rigid space, and let $|S|$ denote the underlying set with the canonical topology. To prove that it is Hausdorff, we have to overcome the possibility that $|S \times S| \to |S| \times |S|$ may not be bijective (let alone a homeomorphism with respect to the canonical topologies). Choose $s, s' \in |S|$ and let U and U' be admissible affinoid opens that contain s and s' respectively. The overlap $U \cap U'$ is affinoid and the union $U \cup U'$ is an admissible open subspace with $\{U, U'\}$ as an admissible cover (since S is separated). We can replace S with $U \cup U'$, so we can assume that S is quasi-compact and separated. Let S^{an} denote the associated separated strictly k-analytic space (in the sense of Berkovich), so the underlying topological space $|S^{\mathrm{an}}|$ (with its "spectral" topology) is compact Hausdorff. The subset of classical points (i.e., those $s \in S^{\mathrm{an}}$ with $[\mathscr{H}(s) : k] < \infty$) is $|S|$, and its subspace topology is the canonical topology because this is true when S is replaced with each of finitely many (compact Hausdorff) strictly k-analytic affinoid domains that cover S^{an}. Hence, $|S|$ is also Hausdorff. ∎

In view of the known converse to Theorem 2.3.4 in the complex-analytic case, it is natural to ask if the property of being locally separated is sufficient for the analytifiable of an algebraic space over k. We will see that this is not true, via counterexamples in §3. Such counterexamples will be explained in two ways, using rigid-analytic methods and using k-analytic spaces. To carry out a rigid-analytic approach, we need to consider how analytifiability behaves with respect to change in the base field. A relevant notion for this was used in [C2]: a *pseudo-separated* map $f : X \to S$ between rigid spaces is a map whose diagonal $\Delta_f : X \to X \times_S X$ factors as the composite of a Zariski-open immersion followed by a closed immersion. The reason that we choose this order of composition is that in the scheme case it is available in a canonical manner (via scheme-theoretic closure) and hence behaves well with respect to étale localization and descent for schemes (so it generalizes to define the notion of quasi-compact immersion for algebraic spaces). We note that a map of rigid spaces is pseudo-separated and quasi-separated (i.e., has quasi-compact diagonal) if and only if it is separated.

Lemma 2.3.5. *Let $\mathscr{X}$ be an analytifiable algebraic space. The analytification $\mathscr{X}^{\mathrm{an}}$ is pseudo-separated. In particular, $k'\widehat{\otimes}_k \mathscr{X}^{\mathrm{an}}$ makes sense as a rigid space over k' for any analytic extension field k'/k.*

Proof. Since $\mathscr{X}$ is analytifiable, so is $\mathscr{X} \times \mathscr{X}$. Clearly $\Delta_{\mathscr{X}}^{\mathrm{an}} = \Delta_{\mathscr{X}^{\mathrm{an}}}$. By Theorem 2.3.4, $\mathscr{X}$ is locally separated. Thus, by étale descent, the quasi-compact immersion $\Delta_{\mathscr{X}}$ uniquely factors as a (schematically dense) Zariski-open immersion followed by a closed immersion, and these intervening locally closed subspaces of $\mathscr{X}$ must be analytifiable (by Theorem 2.2.3). It follows that $\mathscr{X}^{\mathrm{an}}$ must be pseudo-separated. ∎

Now we can address the question of compatibility of analytification for algebraic spaces and extension of the base field, generalizing the known case of analytification of algebraic k-schemes. This compatibility will be useful for giving a rigid-analytic justification of our counterexamples to analytifiability.

Theorem 2.3.6. *Let $\mathscr{X}$ be an analytifiable algebraic space over k, and let k'/k be an analytic extension field. The algebraic space $k' \otimes_k \mathscr{X}$ over k' is analytifiable and there is a natural isomorphism $k'\widehat{\otimes}_k \mathscr{X}^{\mathrm{an}} \simeq (k' \otimes_k \mathscr{X})^{\mathrm{an}}$ which is the usual isomorphism when $\mathscr{X}$ is an algebraic k-scheme. These natural isomorphisms are transitive with respect to further analytic extension of the base field and are compatible with the formation of fiber products.*

Proof. Let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$. Since $\mathscr{U}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is étale and admits local étale quasi-sections, the same holds for the map $k'\widehat{\otimes}_k\mathscr{U}^{\mathrm{an}} \to k'\widehat{\otimes}_k\mathscr{X}^{\mathrm{an}}$ because (as we see via formal models) a faithfully flat map between k-affinoids induces a faithfully flat map after applying $k'\widehat{\otimes}_k(\cdot)$. Moreover, the natural map

$$k'\widehat{\otimes}_k\mathscr{R}^{\mathrm{an}} \to (k'\widehat{\otimes}_k\mathscr{U}^{\mathrm{an}}) \times_{k'\widehat{\otimes}_k\mathscr{X}^{\mathrm{an}}} (k'\widehat{\otimes}_k\mathscr{U}^{\mathrm{an}})$$

is an isomorphism because it is identified with the extension of scalars of the map $\mathscr{R}^{\mathrm{an}} \to \mathscr{U}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} \mathscr{U}^{\mathrm{an}}$ that is necessarily an isomorphism (due to the defining property of the étale quotient $\mathscr{X}^{\mathrm{an}}$ that we are assuming to exist). Thus, we conclude that $k'\widehat{\otimes}_k\mathscr{X}^{\mathrm{an}}$ serves as an étale quotient for the diagram

$$k'\widehat{\otimes}_k\mathscr{R}^{\mathrm{an}} \rightrightarrows k'\widehat{\otimes}_k\mathscr{U}^{\mathrm{an}} \tag{2.3.1}$$

that is an étale equivalence relation, due to its identification with the analytification $(k'\otimes_k\mathscr{R})^{\mathrm{an}} \rightrightarrows (k'\otimes_k\mathscr{U})^{\mathrm{an}}$ of an étale chart for the algebraic space $k' \otimes_k \mathscr{X}$ over k'. This shows that $k'\widehat{\otimes}_k\mathscr{X}^{\mathrm{an}}$ naturally serves as an analytification for the algebraic space $k' \otimes_k \mathscr{X}$ over k'. Moreover, it is clear that this identification $k'\widehat{\otimes}_k\mathscr{X}^{\mathrm{an}} \simeq (k' \otimes_k \mathscr{X})^{\mathrm{an}}$ is independent of the choice of étale chart $\mathscr{U} \rightrightarrows \mathscr{R}$ for $\mathscr{X}$ and that it is therefore functorial in the analytifiable $\mathscr{X}$. The compatibility with fiber products (when the relevant analytifications exist over k) is now obvious. ■

We can define an exact analytification functor from coherent $\mathscr{O}_{\mathscr{X}}$-modules to coherent $\mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$-modules in two (equivalent) ways. One method is to use an étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ and descent theory for coherent sheaves on rigid spaces [C2, 4.2.8]; it is easy to see that this approach gives a functor that is independent of the choice of étale chart and that is natural in $\mathscr{X}$ with respect to pullback along maps of algebraic spaces $\mathscr{X}' \to \mathscr{X}$. An alternative method that avoids the crutch of an étale chart will be explained above Example 3.3.1.

3. Analytification counterexamples and constructions

To show that the theory in §2 is not vacuous, we need to prove the analytifiability of an interesting class of algebraic spaces that are not necessarily schemes. We also explain how analytification interacts with étale topoi, as is required for GAGA on algebraic spaces over k as well as for a natural definition of analytification for coherent sheaves on algebraic spaces.

3.1. **Non-analytifiable surfaces.** By Theorem 2.3.4, it is necessary to restrict attention to those algebraic spaces (locally of finite type over k) that are locally separated. It will be proved in Theorem 4.2.1 that such algebraic spaces are analytifiable in the separated case, but we first give locally separated examples where analytifiability fails. Our construction will provide locally separated smooth algebraic spaces $\mathscr{S}$ over $\mathbf{Q}$ such that $\mathscr{S}_k^{\mathrm{an}}$ does not exist for any non-archimedean field $k/\mathbf{Q}$, though of course $\mathscr{S}_{\mathbf{C}}^{\mathrm{an}}$ does exist by local separatedness!

Example 3.1.1. Let k be an abstract field, let $\mathscr{T} \subseteq \mathbf{A}_k^2$ be a dense open subset of the x-axis, and let $\mathscr{T}' \to \mathscr{T}$ be the geometrically connected finite étale covering $\{u^d = f(x)\}$ with degree $d > 1$ given by extracting the dth root of a monic separable polynomial $f \in k[x]$ whose zeros are away from $\mathscr{T}$; we assume d is not divisible by the characteristic of k. By shrinking $\mathscr{T}$ near its generic point, we can find an open $\mathscr{X} \subseteq \mathbf{A}_k^2$ in which $\mathscr{T}$ is closed and over which there is a quasi-compact étale cover $\mathscr{U} \to \mathscr{X}$ restricting to $\mathscr{T}' \to \mathscr{T}$ over $\mathscr{T}$. There is a locally separated algebraic space $\mathscr{X}'$ that is étale over $\mathscr{X}$ (hence k-smooth and 2-dimensional) and obtained from the open $\mathscr{X} \subseteq \mathbf{A}_k^2$ by replacing the curve $\mathscr{T}$ with the degree-d covering $\mathscr{T}'$.

More generally, in [Kn, Intro., Ex. 2, pp.10–12] Knutson gives the following construction. For any quasi-separated scheme $\mathscr{X}$ equipped with a closed subscheme $\mathscr{T}$ and a quasi-compact étale surjection $\pi : \mathscr{U} \to \mathscr{X}$, he builds a locally separated algebraic space $\mathscr{X}'$ equipped with a quasi-compact étale surjection $i : \mathscr{X}' \to \mathscr{X}$ such that i is an isomorphism over $\mathscr{X} - \mathscr{T}$ but has pullback to $\mathscr{T}$ given by the étale covering $\mathscr{T}' = \pi^{-1}(\mathscr{T}) \to$

$\mathscr{T}$. When $\mathscr{T}'$ and $\mathscr{T}$ are irreducible and $\mathscr{T}'$ has generic degree $d > 1$ over $\mathscr{T}$ then the behavior of fiber-rank for the quasi-finite map $\mathscr{X}' \to \mathscr{X}$ is opposite to what happens for quasi-finite separated étale maps of schemes (via the structure theorem for such maps [EGA, IV$_4$, 18.5.11]) in the sense that the fiber-degree goes up (rather than down) at special points. Hence, $\mathscr{X}'$ cannot have an open scheme neighborhood (or equivalently, a separated open neighborhood) around any point of $\mathscr{T}'$ in such cases, since if it did then such a neighborhood would contain the 1-point generic fiber over $\mathscr{T}$, yet no fiber over $\mathscr{T}$ can have a separated open neighborhood in $\mathscr{X}'$ (e.g., an affine open subscheme).

We now consider the construction over $\mathscr{X} \subseteq \mathbf{A}^2_k$ that replaces an open subset $\mathscr{T}$ in the x-axis with a degree-d finite étale covering $\{u^d = f(x)\}$ as above. For $k = \mathbf{C}$, an analytification of $\mathscr{X}'$ does exist (since $\mathscr{X}'$ is locally separated) and its local structure over an open neighborhood of a point $t \in \mathscr{T}(\mathbf{C})$ is very easy to describe: it is a product of an open unit disc with a gluing of d copies of an open disc to itself via the identity map on the complement of the origin. In particular, this analytification is non-Hausdorff over such a neighborhood. In the non-archimedean setting, if $\mathscr{T}' \to \mathscr{T}$ has a non-split fiber over some $t \in \mathscr{T}(k)$ then we will see that no analogous such local gluing can be done over an open neighborhood of t in $\mathscr{X}^{\mathrm{an}}$. If k is algebraically closed then the local gluing can be done but there are global admissibility problems with the gluing. We shall show that the admissibility problems are genuine. The key is that there is an obstruction to analytifiability caused by the failure of the Gelfand–Mazur theorem over non-archimedean fields: k admits analytic extension fields k'/k such that the étale cover $\mathscr{T}' \to \mathscr{T}$ has a non-split fiber over some $t \in \mathscr{T}(k')$, even if k is algebraically closed.

Let us now prove that if k is a non-archimedean field then the smooth 2-dimensional locally separated algebraic space $\mathscr{X}'$ is not analytifiable. Assume that an analytification X' of $\mathscr{X}'$ exists. By Lemma 2.3.5 the rigid space X' must be pseudo-separated, and by Theorem 2.3.6 if k'/k is any analytic extension field then $k' \otimes_k \mathscr{X}'$ is analytifiable with analytification $k' \widehat{\otimes}_k X'$. Hence, to get a contradiction it suffices to consider the situation after a preliminary analytic extension of the base field $k \to k'$ (which is easily checked to automatically commute with the formation of $\mathscr{X}'$ in terms of $\mathscr{X}$ and $\mathscr{T}' \to \mathscr{T}$). The extension $K' = k(\mathscr{T}')$ of $K = k(\mathscr{T}) = k(x)$ is defined by adjoining a root to the irreducible polynomial $u^d - f \in K[u]$. We first increase k a finite amount so that f splits completely in $k[x]$. We then make a linear change of variable on x so that $f = x \prod_{i>1}(1 - r_i x)$ with $|r_i| < 1$ for each i. In case of mixed characteristic we also require $|r_i|$ to be so small that $1 - r_i x$ has a dth root as a power series (a condition that is automatic in case of equicharacteristic k). Thus, f lies in the valuation ring of the Gauss norm on $k(x)$ and its image in the residue field $\widetilde{k}(x)$ for the Gauss norm (with $\widetilde{k}$ the residue field of k) is x, which is not a dth power in $\widetilde{k}(x)$ since $d > 1$. Hence, $u^d - f$ has no root over the completion $\widehat{K}$ of $K = k(x)$ with respect to the Gauss norm. By taking $k' = \widehat{K}$ and working with the canonical point in $\mathscr{T}(\widehat{K})$ arising from the generic point of $\mathscr{T}$ we put ourselves in the situation (upon renaming k' as k) where there exists $t_0 \in \mathscr{T}(k)$ with no k-rational point in its non-empty fiber in $\mathscr{T}'$. All we shall actually require is that there is some $t_0' \in \mathscr{T}'_{t_0}$ with $k(t_0') \neq k$.

Letting X, T, and T' denote the analytifications of $\mathscr{X}$, $\mathscr{T}$, and $\mathscr{T}'$, by Theorem 2.2.3 the analytified map $X' \to X$ is an isomorphism over $X - T$ and restricts to a degree-d finite étale covering $h : T' \to T$ over T. Consider the fiber over $t_0 \in \mathscr{T}(k) = T(k)$. Since $T'_{t_0} = (\mathscr{T}'_{t_0})^{\mathrm{an}}$, we can choose $t_0' \in T'$ over t_0 with $k(t_0') \neq k$. Since $X' \to X$ is locally quasi-finite, by the local structure theorem for quasi-finite maps of rigid spaces [C2, Thm. A.1.3] there are connected admissible opens $U' \subseteq X'$ around t_0' and $U \subseteq X$ around t_0 such that U' lands in U and the induced map $U' \to U$ is finite étale with fiber $\{t_0'\}$ over t_0. Connectedness of U forces $U' \to U$ to have constant fiber-degree, so this degree must equal $[k(t_0') : k(t_0)] > 1$. But the non-empty 2-dimensional admissible open U in X obviously cannot be contained in the 1-dimensional analytic set T in X, and the fiber-degree of the finite étale covering $U' \to U$ over any point of U not in T has to be 1 since $X' \to X$ restricts to an isomorphism over $X - T$. This contradicts the constancy of fiber-degree over U, and so shows that no such analytification X' can exist.

Remark 3.1.2. The preceding example can be carried out in the category of k-analytic spaces without requiring an extension of the base field in the argument. The reason is that on k-analytic spaces there are generally many non-rational points even if k is algebraically closed. More specifically, the k-analytic space associated to T has a point ξ with $\mathscr{H}(\xi) = \widehat{K}$, and over this point ξ there is a unique point ξ' in the k-analytic space associated to T' and $\mathscr{H}(\xi')$ is a separable degree-d extension of $\widehat{K}$.

3.2. **Finite étale quotients for affinoid spaces.** In §4 we will take up a general study of the existence of analytic étale quotients, and in §5.1 we will give further examples of the failure of existence for certain kinds of analytic étale quotient problems (in both classical rigid geometry and in the category of k-analytic spaces). Since Example 3.1.1 shows that the necessary condition for analytifiability in Theorem 2.3.4 is not sufficient in the non-archimedean case, for a general study of analytifiability of algebraic spaces over non-archimedean fields we are now motivated to focus attention on the problem of analytifying separated algebraic spaces. The strategy in the proof that separated algebraic spaces are analytifiable will be to show that locally (in the rigid sense) we can describe the quotient problem in such cases as that of forming the quotient of an affinoid by a *finite* étale equivalence relation. This is difficult to carry out, for two reasons that have no counterpart in the complex-analytic theory: products for rigid spaces (and k-analytic spaces) cannot be easily described set-theoretically, and saturation with respect to an equivalence relation is a problematic operation with respect to the property of admissibility for subsets of a rigid space. In fact, we do not know how to carry out the reduction to the finite étale case without leaving the rigid-analytic category.

In the finite étale case with affinoid spaces, the construction of quotients goes as in algebraic geometry except that there is the additional issue of checking that various k-algebras are also k-affinoid:

Lemma 3.2.1. *Let $f : U \to X$ be a finite étale surjective map of rigid spaces. The rigid space X is affinoid if and only if the rigid space U is affinoid. Moreover, if $R' \rightrightarrows U'$ is a finite étale equivalence relation on an affinoid rigid space U' then the étale quotient $X' = U'/R'$ exists and $U' \to X'$ is a finite étale cover.*

The étale hypothesis in the first part of the lemma is essential, in contrast with a theorem of Chevalley [EGA, II, 6.7.1] in the case of finite surjections of schemes. Indeed, in [Liu] there is an example of a non-affinoid quasi-compact separated surface (over any k) such that the normalization is affinoid. The proof of Lemma 3.2.1 carries over verbatim to the case of affinoid k-analytic spaces that are not necessarily strictly k-analytic, the key point being that if A is a k-affinoid algebra in the sense of Berkovich and it is endowed with a continuous action by a finite group G then the closed subalgebra A^G is k-affinoid and A is finite and admissible as an A^G-module [Ber1, 2.1.14(*ii*)]. We will return to this issue in §4 when we address the problem of étale quotients of k-analytic spaces (as the foundation of our approach to analytifying separated algebraic spaces via rigid spaces).

Proof. Let $R = U \times_X U$, so the two projections $R \rightrightarrows U$ are finite étale covers. If X is affinoid then certainly U is affinoid, so now assume that U is affinoid. Hence, the U-finite R is affinoid and we have to prove that U/R is affinoid when it exists. More generally, we suppose that we are given a finite étale equivalence relation $R' \rightrightarrows U'$ with R' and U' affinoid rigid spaces over k, and we seek to prove that the étale quotient U'/R' exists as an affinoid rigid space with $U' \to U'/R'$ a finite étale covering.

We have $U' = \mathrm{Sp}(A')$ for some k-affinoid A', and likewise $R' = \mathrm{Sp}(A'')$ for some k-affinoid A''. Since the maps $p_1, p_2 : R' \rightrightarrows U'$ are finite, the groupoid conditions may be expressed in opposite terms using k-affinoid algebras with only ordinary tensor products intervening in the description. The resulting pair of maps of affine k-schemes $\mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$ is therefore a finite étale equivalence relation in the category of k-schemes provided that the natural map

$$\mathrm{Spec}(A'') \to \mathrm{Spec}(A') \times_{\mathrm{Spec}\, k} \mathrm{Spec}(A')$$

is a monomorphism. We claim that it is a closed immersion. By hypothesis, the map $\delta : R' \to U' \times U'$ is a monomorphism between rigid spaces, yet the first projection $U' \times U' \to U'$ is separated and has composite with δ that is finite, so δ is finite. A finite monomorphism of rigid spaces is a closed immersion (by Nakayama's lemma), so δ is a closed immersion. Hence, the natural map $A' \widehat{\otimes}_k A' \to A''$ is surjective. The image of $a'_1 \widehat{\otimes} a'_2$ is $p_1^*(a'_1) p_2^*(a'_2)$, so every element of the Banach A'-module A'' (say via p_1^*) is a convergent linear combination of elements in the image of p_2^*. Since A'' is in fact A'-finite via p_1^*, so all A'-submodules are closed, it follows that $p_2^*(A')$ algebraically spans A'' over A' (via p_1). Hence, the natural map $A' \otimes_k A' \to A''$ is indeed surjective, as desired.

By [SGA3, Exp. V, 4.1], the étale quotient of $\mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$ exists as an affine scheme $\mathrm{Spec}(A)$ over k, with $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ a finite étale covering (and $A' \otimes_A A' \to A''$ an isomorphism). If we can show that A is a k-affinoid algebra, then for $X' = \mathrm{Sp}(A)$ the finite étale covering $U' \to X'$ yields *equal* composites $R' \rightrightarrows X'$ and the induced map $R' \to U' \times_{X'} U'$ is an isomorphism since $A' \otimes_A A' = A' \widehat{\otimes}_A A'$. Thus, X' would serve as the étale quotient U'/R' in the category of rigid spaces.

To show that such an A must be k-affinoid, consider more generally an affine k-scheme $\mathrm{Spec}\, A$ equipped with a finite étale covering $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ with A' a k-affinoid algebra. We claim that A must be k-affinoid. Since A' has only finitely many idempotents, the same must hold for A, and so we may assume $\mathrm{Spec}(A)$ is connected. The map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a finite étale covering, so each of the finitely many connected components of $\mathrm{Spec}(A')$ is a finite étale cover of $\mathrm{Spec}(A)$. Thus, we may also assume that $\mathrm{Spec}(A')$ is connected. By the theory of the étale fundamental group, the connected finite étale covering $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is dominated by a Galois finite étale covering $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, say with Galois group G. The faithful G-action on B is A-linear, hence continuous, and $A = B^G$. Since B is A'-finite, B is a k-affinoid algebra. Thus, by [BGR, 6.3.3/3] the invariant subalgebra $A = B^G$ is k-affinoid. ∎

To generalize beyond the case of finite étale equivalence relations on affinoids as in Lemma 3.2.1, a fundamental issue is the possibility that the rigid-analytic morphism $R \to U \times U$ may not be quasi-compact. For example, if $\mathscr{X}$ is a locally separated algebraic space then its diagonal is a quasi-compact immersion that is not a closed immersion if $\mathscr{X}$ is not separated, and so when working over an étale chart of the algebraic space the pullback of this diagonal morphism has analytification that is not quasi-compact in the sense of rigid geometry when $\mathscr{X}$ is not separated. Lack of such quasi-compactness on the rigid side presents a difficulty because forming saturations under the equivalence relation thereby involves the image of a non-quasi-compact admissible open under a flat morphism of rigid spaces, and the admissibility of such images is difficult to control (even when the flat morphism is quasi-compact). This is what happens in Example 3.1.1 if we try to use gluing to build the non-existent analytification there. We are therefore led to restrict our attention to the analytic quotient problem when $\delta : R \to U \times U$ is quasi-compact. For reasons explained in Remark 1.2.3, we will focus on the case when δ is a closed immersion.

3.3. **GAGA for algebraic spaces.** We conclude this section with a discussion of cohomological issues related to the Tate-étale topology. The GAGA theorems aim to compare cohomology of coherent sheaves on (analytifiable) proper algebraic spaces and proper rigid spaces, so a basic fact that we must confront before contemplating such theorems is that algebraic spaces have only an étale topology rather than a Zariski topology whereas rigid spaces have a Tate topology with respect to which étale maps are *not* generally local isomorphisms. This contrasts with the situation over $\mathbf{C}$, where étale analytic maps are local isomorphisms. Thus, it seems appropriate to sketch how the GAGA formalism is to be set up for analytifiable algebraic spaces over a non-archimedean field k.

Let $\mathscr{X}$ be an analytifiable algebraic space, say with $X = \mathscr{X}^{\mathrm{an}}$, and let $\mathscr{X}_{\mathrm{\acute{e}t}}$ denote the étale site whose objects are schemes étale over $\mathscr{X}$. For any sheaf of sets $\mathscr{F}$ on $X_{\mathrm{\acute{e}t}}$, define the pushforward $(f_{\mathscr{X}})_*(\mathscr{F})$ on $\mathscr{X}_{\mathrm{\acute{e}t}}$ by the formula $((f_{\mathscr{X}})_*(\mathscr{F}))(\mathscr{U}) = \mathscr{F}(\mathscr{U}^{\mathrm{an}})$. This is a sheaf, due to Example 2.1.2. It is easy to construct an exact left adjoint in the usual manner, and this gives a map of ringed topoi $f_{\mathscr{X}} : \widetilde{X_{\mathrm{\acute{e}t}}} \to \widetilde{\mathscr{X}_{\mathrm{\acute{e}t}}}$ that is natural in $\mathscr{X}$.

The pullback operation $f_{\mathscr{X}}^*$ on sheaves of modules is exact because if $U \to \mathscr{U}^{\mathrm{an}}$ is an étale map from an affinoid space over k to the analytification of an affine algebraic k-scheme then the induced map on coordinate rings is flat (due to the induced map on completed stalks at maximal ideals being finite étale). We call this the *analytification* functor on sheaves of modules. By exactness, this functor preserves coherence. At the end of §2.3 we noted that for any coherent sheaf $\mathscr{G}$ on $\mathscr{X}_{\mathrm{\acute{e}t}}$ one can use descent theory to naturally construct a coherent sheaf $\mathscr{G}^{\mathrm{an}}$ on X_{Tate} whose formation is compatible with pullback in $\mathscr{X}$. We claim that the associated coherent sheaf $(\mathscr{G}^{\mathrm{an}})_{\mathrm{\acute{e}t}}$ on $X_{\mathrm{\acute{e}t}}$ is naturally isomorphic to $f_{\mathscr{X}}^*(\mathscr{G})$. This follows from the commutative diagram of ringed sites

$$\begin{tikzcd} U_{\text{ét}} \arrow[r] \arrow[d] & \mathscr{U}_{\text{ét}} \arrow[d] \\ U_{\text{Tate}} \arrow[r] & \mathscr{U}_{\text{Zar}} \end{tikzcd}$$

for any scheme $\mathscr{U}$ étale over $\mathscr{X}$. Since pullback along $X_{\text{ét}} \to X_{\text{Tate}}$ defines an equivalence between categories of coherent sheaves, we may therefore write $\mathscr{G}^{\text{an}}$ to denote $f^*_{\mathscr{X}}(\mathscr{G})$ for any $\mathscr{O}_{\mathscr{X}_{\text{ét}}}$-module $\mathscr{G}$ without creating confusion.

Example 3.3.1. We can now state GAGA for algebraic spaces. Let $h : \mathscr{X} \to \mathscr{Y}$ be a map between analytifiable algebraic spaces, with associated map $h^{\text{an}} : X = \mathscr{X}^{\text{an}} \to \mathscr{Y}^{\text{an}} = Y$. Since the commutative diagram of ringed topoi

$$\begin{tikzcd} \widetilde{X_{\text{ét}}} \arrow[r] \arrow[d] & \widetilde{\mathscr{X}_{\text{ét}}} \arrow[d] \\ \widetilde{Y_{\text{ét}}} \arrow[r] & \widetilde{\mathscr{Y}_{\text{ét}}} \end{tikzcd}$$

has exact pullback operations along the horizonal direction, there is a natural δ-functorial map of $\mathscr{O}_{Y_{\text{ét}}}$-modules

$$(\mathrm{R}^j h_*(\mathscr{F}))^{\text{an}} \to \mathrm{R}^j h^{\text{an}}_*(\mathscr{F}^{\text{an}}).$$

GAGA for algebraic spaces is the assertion that this comparison morphism is an isomorphism when h is proper and $\mathscr{F}$ is coherent, from which the usual GAGA results concerning full faithfulness on proper objects over k and equivalences between categories of coherent sheaves on such objects follow exactly as over $\mathbf{C}$ (using Chow's Lemma for algebraic spaces).

To prove GAGA, first note that by Theorem 2.3.1, h^{an} is proper. Also, recall from Example 2.1.3 that via the pullback equivalence between categories of coherent sheaves for the Tate and Tate-étale topologies, we get a δ-functorial compatibility between higher direct images for coherent sheaves in the proper case. Consequently, we can argue exactly as over $\mathbf{C}$ to reduce GAGA for proper maps between analytifiable algebraic spaces to the known case of schemes with the Zariski (rather than étale) topology and rigid spaces with the Tate (rather than Tate-étale) topology.

4. Analytification via k-analytic spaces

4.1. **Preliminary considerations.** We are going to now study analytification in the category of k-analytic spaces, and then use such spaces to overcome admissibility problems in the rigid case. In order to make sense of this, we briefly digress to discuss how the methods in §2 carry over to the category of k-analytic spaces, endowed with their natural étale topology. (As usual in the theory of k-analytic spaces, we allow the possibility that k has trivial absolute value.) An *étale equivalence relation* in the category of k-analytic spaces is a pair of étale morphisms $R \rightrightarrows U$ such that the map $R \to U \times U$ (called the *diagonal*) is a functorial equivalence relation; in particular, it is a monomorphism. As one example, if $\mathscr{R} \rightrightarrows \mathscr{U}$ is an étale chart for an algebraic space $\mathscr{X}$ over k then the analytification functor [Ber2, 2.6.1] to the category of good strictly k-analytic spaces yields an étale equivalence relation $R \rightrightarrows U$ on k-analytic spaces. (By [T2, 4.10], the category of strictly k-analytic spaces is a full subcategory of the category of k-analytic spaces, so there is no ambiguity about where the morphisms $R \rightrightarrows U$ take place when R and U are strictly k-analytic.)

Definition 4.1.1. Let $R \rightrightarrows U$ be an étale equivalence relation on k-analytic spaces. A *quotient* of $R \rightrightarrows U$ is a k-analytic space X equipped with an étale surjection $U \to X$ such that the composite maps $R \rightrightarrows U \to X$ coincide and the resulting map $R \to U \times_X U$ is an isomorphism.

In order to check that the quotient (when it exists) is unique up to unique isomorphism (and in fact represents a specific sheaf functor), we can use the usual descent theory argument as in the case of schemes provided that representable functors on the category of k-analytic spaces are étale sheaves. This sheaf property is true within the full subcategories of good k-analytic spaces and strictly k-analytic spaces by [Ber2, 4.1.5], according to which the general case holds once we prove the next result.

Theorem 4.1.2. *Let $f : X' \to X$ be a finite étale map between k-analytic spaces. If $V' \subseteq X'$ is a quasi-compact k-analytic subdomain then $f(V') \subseteq X$ is a finite union of k-affinoid subdomains in X. In particular, $f(V')$ is a k-analytic domain in X. If X, X', and V' are strictly k-analytic then so is $f(V')$.*

The fiber product $R = X' \times_X X'$ is finite over X' (under either projection), so it is good (resp. strictly k-analytic) when X' is. Since $X = X'/R$, it will follow from Theorem 4.2.2 below that if X' is separated then X is necessarily good (resp. strictly k-analytic) if X' is so.

Proof. The image of f is open and closed in X, so we may and do assume that f is surjective. Since X is locally Hausdorff and V' is compact, there is a finite collection of Hausdorff open subsets $U_1, \dots, U_n$ in X that cover $f(V')$. The open cover $\{f^{-1}(U_i)\}$ of the quasi-compact V' has a finite refinement consisting of k-affinoid subdomains $V'_j \subseteq V'$, so if we can settle the case of a Hausdorff target then applying this to $f^{-1}(U_i) \to U_i$ and each V'_j mapping into U_i gives the result for $f(V')$. Hence, we now may and do assume that X is Hausdorff, so X' is also Hausdorff.

Let $W_1, \dots, W_m \subseteq X$ be a finite collection of k-affinoid subdomains whose union contains $f(V')$ (with all W_j strictly k-analytic when X', X, and V are so). The pullback subdomains $W'_j = f^{-1}(W_j)$ are k-affinoid in X', and are strictly k-analytic when X', X, and V' are so. Moreover, $V' \cap f^{-1}(W_j)$ is quasi-compact since the graph morphism $\Gamma_f : X' \to X' \times X$ is quasi-compact (as it is a base change of the diagonal morphism $\Delta_X : X \to X \times X$ that is topologically proper since $|X|$ is Hausdorff and $|X \times X| \to |X| \times |X|$ is proper). Hence, we may reduce to the case when $f(V') \subseteq W$ for some k-affinoid subdomain $W \subseteq X$. It is harmless to make the base change by $W \to X$, so we can assume that X and X' are k-affinoid and even connected. Say $X' = \mathscr{M}(A')$ and $X = \mathscr{M}(A)$.

By the theory of the étale fundamental group as in the proof of Lemma 3.2.1, now applied to $\operatorname{Spec} A' \to \operatorname{Spec} A$, we may find a connected finite étale cover $X'' \to X'$ that is Galois over X. In particular, if X' is strict then so is X''. The preimage of V' in X'' is quasi-compact (and strict when X' and V' are strict), so we may assume that X' is Galois over X, say with Galois group G. The union $W' = \cup_{g \in G} g(V')$ is a quasi-compact k-analytic subdomain whose image in X is the same as that of V', so we can rename it as V' to get to the case when V' is G-stable.

For each point $x' \in V'$ we let $G_{x'} \subseteq G$ denote the stabilizer group of the physical point x', so by the Hausdorff property of our spaces we can find a k-affinoid neighborhood $W' \subseteq V'$ around x' in V' that is disjoint from its g-translate for each $g \in G - G_{x'}$. Replacing W' with the k-affinoid overlap $\cap_{g \in G_{x'}} g(W')$ allows us to assume that W' is $G_{x'}$-stable and that the subdomains $g(W')$ for $g \in G/G_{x'}$ are pairwise disjoint. Hence, $Y' = \coprod g(W')$ is a G-stable k-affinoid subdomain in X' that is a neighborhood of x' in V'. By quasi-compactness of V', finitely many such subdomains $Y'_1, \dots, Y'_n$ cover V'. Thus, we can replace V' with each of the Y'_i's separately, so we can assume that $V' = \mathscr{M}(B')$ is k-affinoid. By [Ber1, 2.1.14(ii)], the closed k-subalgebra $B = B'^G$ is k-affinoid. It is moreover a strict k-affinoid algebra if V' is strict [BGR, 6.3.3]. The map $V' \subseteq X' \to X$ factors through the surjection $V' = \mathscr{M}(B') \to \mathscr{M}(B)$, so it suffices to check that the natural map $V = \mathscr{M}(B) \to \mathscr{M}(A) = X$ is a k-analytic subdomain. This amounts to the property that if $Z = \mathscr{M}(C)$ is k-affinoid and a morphism $h : Z \to X$ factors through V set-theoretically then it uniquely does so in the category of k-analytic spaces. It is therefore equivalent to prove that the projection $Z \times_X V \to Z$ is an isomorphism. But this is a map of k-affinoids, so it suffices to check the isomorphism assertion after an analytic extension of the base field (an operation which commutes with the formation of B from B'). We may therefore put ourselves in the strictly k-analytic case (with $|k^\times| \neq \{1\}$), in which case the image $f(V')$ is a k-analytic subdomain by Raynaud's theory, and $V' \to f(V')$ is a finite mapping because V' is the full preimage of $f(V')$ in X' (due to the G-stability of V' in X'). Hence, Lemma 3.2.1 gives that $f(V')$ is k-affinoid, and then its coordinate ring is forced to be $B'^G = B$ since $X' \to X$ corresponds to $A = A'^G \to A'$. That is, the k-analytic subdomain $f(V') \subseteq X$ is precisely V equipped with its natural map to X, so V is a k-analytic subdomain of X as desired. ∎

Example 4.1.3. In the setup of Theorem 4.1.2, if $V' \subseteq X'$ is a quasi-compact k-analytic subdomain whose two pullbacks to X'' coincide then it descends uniquely to a k-analytic subdomain $V \subseteq X$. Indeed, if we let V be the quasi-compact k-analytic subdomain $f(V') \subseteq X$ then to check that the preimage of V in X' is no larger than (and hence is equal to) V' it suffices check this after base change on X by geometric points of V. This case is trivial.

By Theorem 4.1.2, if $R \rightrightarrows U$ is an étale equivalence relation on k-analytic spaces and X is a quotient for this equivalence relation in the sense that we have defined for k-analytic spaces, then X represents the quotient sheaf of sets U/R on the étale site for the category of k-analytic spaces. Thus, such an X is unique up to unique isomorphism. We can also use descent arguments as in the classical rigid case to run this in reverse: if the quotient sheaf U/R on the étale site for the category of k-analytic spaces is represented by a k-analytic space X then the natural map $U \to X$ is automatically an étale surjection that equalizes the maps $R \rightrightarrows U$ and yields an isomorphism $R \simeq U \times_X U$. In particular, the formation of the quotient is compatible with arbitrary analytic extension of the base field (when the quotient exists over the initial base field).

In the k-analytic category, if the diagonal $R \to U \times U$ of an étale equivalence relation on a locally separated k-analytic space U is compact then it must be a closed immersion [CT, 2.2]. This is why in Theorem 1.2.2 we impose the requirement that δ be a closed immersion rather than the apparently weaker condition that it be compact. We do not have an analogous such result in the rigid-analytic case because étaleness and local separatedness in k-analytic geometry are stronger conditions than in rigid geometry.

The arguments in §2 carry over essentially *verbatim* to show that if $R \rightrightarrows U$ arises from an étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for an algebraic space $\mathscr{X}$ then whether or not an analytic quotient $X = U/R$ exists is independent of the choice of étale chart for $\mathscr{X}$, and its formation (when it does exist) is Zariski-local on $\mathscr{X}$. In particular, when X exists it is canonically independent of the chart and is functorial in $\mathscr{X}$ in a manner that respects the formation of fiber products and Zariski-open and Zariski-closed immersions. We call such an X (when it exists) the *analytification* of $\mathscr{X}$ in the sense of k-analytic spaces, and we say that $\mathscr{X}$ is analytifiable (in the sense of k-analytic spaces); we write $\mathscr{X}^{\mathrm{an}}$ to denote this k-analytic space if there is no possibility of confusion with respect to the analogous notion for rigid spaces. In principle analytifiability in the sense of rigid spaces is weaker than in the sense of k-analytic spaces since étaleness is a weaker condition in rigid geometry than in k-analytic geometry. It seems likely (when $|k^\times| \neq \{1\}$) that analytifiability in the sense of k-analytic spaces implies it in the sense of rigid spaces over k, but we have not considered this matter seriously because in the separated case we will prove analytifiability in both senses (and the deduction of the rigid case from the k-analytic case will use separatedness).

Since change of the base field is a straightforward operation for k-analytic spaces (unlike for general rigid spaces), it is easy to see that if K/k is an analytic extension field and $\mathscr{X}$ is analytifiable in the sense of k-analytic spaces then $K \otimes_k \mathscr{X}$ is analytifiable in the sense of K-analytic spaces with $K \widehat{\otimes}_k \mathscr{X}^{\mathrm{an}}$ as its analytification.

Theorem 4.1.4. *If $\mathscr{X}$ is analytifiable in the sense of k-analytic spaces then $\mathscr{X}$ is locally separated.*

Proof. We wish to carry over the method used to prove Theorem 2.3.4 for rigid spaces, so we need to recall several properties of k-analytic spaces that are relevant to this method. A separated k-analytic space has Hausdorff underlying topological space, and by using rigid-analytic techniques we see that a dense open immersion of algebraic k-schemes induces an open immersion of k-analytic spaces with dense image. Also, any base change of the diagonal map $\Delta_S : S \to S \times S$ of a k-analytic space S is a topological embedding. To see this, since S is locally Hausdorff we may (as in the complex-analytic case) reduce to the case when S is Hausdorff. Thus, $|S| \to |S| \times |S|$ is a closed embedding. Letting $h : Z \to S \times S$ be any map of k-analytic spaces, we want to show that the natural map $h^*(\Delta_S) : Z \times_{S \times S} S \to Z$ is a topological embedding. We will show that it is even a closed embedding. On geometric points this map is clearly injective, so it suffices to prove that it is topologically a proper map. But the map from the underlying topological space of a k-analytic fiber product to the fiber product of the underlying topological spaces is always proper, so we are reduced to showing that

$$|Z| \times_{|S \times S|} |S| \to |Z|$$

is a closed embedding. Thus, it is enough to prove that Δ_S is topologically a closed embedding. Since $|S \times S| \to |S| \times |S|$ is separated (even a proper surjection) and $|S| \to |S| \times |S|$ is a closed embedding (as $|S|$ is Hausdorff), Δ_S is indeed a closed embedding.

To use the proof of Theorem 2.3.4 in the setting of k-analytic spaces, it remains to show that a finite type map $f : \mathscr{V} \to \mathscr{W}$ of algebraic k-schemes is injective (resp. surjective) if its analytification $f^{\mathrm{an}} : V \to W$ (in the sense of k-analytic spaces) is injective (resp. surjective). For surjectivity we use that the natural map $W = \mathscr{W}^{\mathrm{an}} \to \mathscr{W}$ is surjective. For injectivity, recall that a map of algebraic k-schemes is injective if and only if it is injective on underlying sets of closed points, and the closed points of an algebraic k-scheme are functorially identified with the set of points of the analytification with residue field of finite degree over k. Hence, we get the desired inheritance of injectivity from k-analytic spaces to schemes. ■

4.2. **Main results.** Here is the main existence result in the rigid-analytic setting; the proof will occupy the rest of §4, and will largely be taken up with the proof of an existence result for étale quotients in the k-analytic category (modulo a crucial existence theorem for quotients by free actions of finite groups, to be treated in Theorem 5.1.1).

Theorem 4.2.1. *Assume $|k^\times| \neq \{1\}$. If $\mathscr{X}$ is a separated algebraic space over k then $\mathscr{X}$ is analytifiable in the sense of rigid spaces. Moreover, the rigid space $\mathscr{X}^{\mathrm{an}}$ is separated.*

Once $\mathscr{X}^{\mathrm{an}}$ is proved to exist, it must be separated since $\Delta_{\mathscr{X}^{\mathrm{an}}} = \Delta_{\mathscr{X}}^{\mathrm{an}}$ is a closed immersion (as $\mathscr{X}$ is separated). For a separated algebraic space, we will prove analytifiability in the sense of rigid spaces by deducing it from a stronger existence theorem for étale quotients in the setting of k-analytic spaces (allowing $|k^\times| = \{1\}$). Consider an étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$. Note that we can take $\mathscr{U}$ to be separated. Let U and R be the good strictly k-analytic spaces associated to $\mathscr{U}$ and $\mathscr{R}$ (so U is separated when $\mathscr{U}$ is). The dictionary relating k-analytic spaces and algebraic schemes [Ber2, 3.3.11] ensures that $R \rightrightarrows U$ is an étale equivalence relation on U and that $R \to U \times U$ is a closed immersion. At the end of §4.3, Theorem 4.2.1 will be deduced from the following purely k-analytic result (allowing $|k^\times| = \{1\}$).

Theorem 4.2.2. *Let $R \rightrightarrows U$ be an étale equivalence relation on k-analytic spaces such that $R \to U \times U$ is a closed immersion. The quotient U/R exists and is a separated k-analytic space. If U is strictly k-analytic* (*resp. good*) *then so is U/R.*

In Example 5.1.3 we will show that it is insufficient in Theorem 4.2.2 to weaken the hypothesis on δ to compactness. Before we proceed to global considerations, let us first show that the existence problem for U/R is local on U (setting aside for now the matter of proving separatedness of U/R). To this end, suppose U is covered by open subsets $\{U_i\}$ such that for $R_i = R \times_{U \times U} (U_i \times U_i) = R \cap (U_i \times U_i)$ the quotient $X_i = U_i/R_i$ exists (with X_i strictly k-analytic when U_i is, and likewise for the property of being good); note that $R_i \to U_i \times U_i$ is a closed immersion. (We do not assume that each X_i is known to be separated.) We need to define "overlaps" along which we shall glue the X_i's to build a k-analytic quotient U/R. The open overlap $R_{ij} = p_1^{-1}(U_i) \cap p_2^{-1}(U_j)$ in R classifies equivalence among points of U_i and U_j, so its open image U_{ij} in U_i under the étale morphism $p_1 : R \to U$ classifies points of U_i that are equivalent to points of U_j. Let $X_{ij} \subseteq X_i$ be the open image of U_{ij}, so $p_1 : R_{ij} \to X_{ij}$ is an étale surjection. Geometrically, the points of X_{ij} are the R-equivalence classes that meet U_i and U_j (viewed within $X_i = U_i/R_i$).

The canonical involution $R \simeq R$ restricts to an isomorphism $\phi_{ij} : R_{ij} \simeq R_{ji}$ such that $\phi_{ji} = \phi_{ij}^{-1}$, and it is easy to check that the resulting isomorphism $R_{ij} \times R_{ij} \simeq R_{ji} \times R_{ji}$ restricts to an isomorphism of subfunctors $R_{ij} \times_{X_{ij}} R_{ij} \simeq R_{ji} \times_{X_{ji}} R_{ji}$. Hence, since representable functors on the category of k-analytic spaces are étale sheaves (due to Theorem 4.1.2 and [Ber2, 4.1.5]), the isomorphisms ϕ_{ij} uniquely descend to isomorphisms $X_{ij} \simeq X_{ji}$ between open subsets $X_{ij} \subseteq X_i$ and $X_{ji} \subseteq X_j$. These descended isomorphisms among opens in X satisfy the triple overlap condition, and so we can glue the X_i's along these isomorphisms to build a k-analytic space X. Moreover, if U is strictly k-analytic (resp. good) and the U_i's can be chosen to be strictly k-analytic (resp. good) then so are all X_i and hence so is the space X that has an open covering by the X_i's. The étale composites $U_i \to X_i \subseteq X$ glue to define an étale morphism $U \to X$ such that the two composite maps $R \rightrightarrows U \to X$ coincide and $R \to U \times_X U$ is an isomorphism (as it is an étale monomorphism that is surjective on geometric points). It follows that as an étale sheaf of sets on the category of k-analytic spaces, X represents the sheafified quotient U/R.

To finish the localization argument, we now check the global property that diagonal map $X \to X \times X$ is a closed immersion (i.e., X is separated) when $X = U/R$ exists. Étale surjective base change by $U \times U \to X \times X$ yields the map $R \to U \times U$ that is a closed immersion by hypothesis. To deduce that Δ_X is a closed immersion it remains to show that the property of a k-analytic morphism being a closed immersion is étale-local on the target. That is, if $f : Y' \to Y$ is a map of k-analytic spaces and $V \to Y$ is an étale cover such that the base change $F : V' \to V$ of f is a closed immersion then we want to prove that f is a closed immersion. To prove this we require a straightforward descent theory for coherent sheaves with respect to the G-topology S_G [Ber2, §1.3] on k-analytic spaces S:

Lemma 4.2.3. *Let $f : S' \to S$ be a flat quasi-finite surjection of k-analytic spaces, and let $p_1, p_2 : S'' = S' \times_S S' \to S'$ be the canonical projections. For any coherent sheaf $\mathscr{F}$ on S_G define $\mathscr{F}'$ to be the coherent pullback $f^*(\mathscr{F})$ on S'_G and define $\varphi_{\mathscr{F}} : p_1^*(\mathscr{F}') \simeq p_2^*(\mathscr{F}')$ to be the evident isomorphism. The functor $\mathscr{F} \rightsquigarrow (\mathscr{F}', \varphi_{\mathscr{F}})$ from the category $\mathrm{Coh}(S_G)$ of coherent sheaves on S_G to the category of pairs consisting of an object $\mathscr{F}' \in \mathrm{Coh}(S'_G)$ equipped with a descent datum $\varphi : p_1^*(\mathscr{F}') \simeq p_2^*(\mathscr{F}')$ relative to f is an equivalence of categories.*

The case of interest to use is when f is étale. The general notions of quasi-finite and flat quasi-finite maps are discussed in [Ber2, §3.1-§3.2].

Proof. For faithfulness it suffices to work locally on S' and S, so we can assume f is a flat finite surjection. Once faithfulness is proved, for full faithfulness we can work locally on S' since flat quasi-finite maps are open, so we can again reduce to the case when f is a flat finite map. Similarly, once full faithfulness is proved the essential surjectivity holds in general if it holds for flat finite f. Hence, we can assume that f is is a flat finite map. Since the coherent sheaves are taken with respect to the G-topology, for the proof of full faithfulness we can work locally for the G-topology on S and so we can assume that S is k-affinoid. Similarly, once full faithfulness is proved we can work locally for the G-topology on S for the proof of essential surjectivity. Hence, it suffices to prove the lemma when $S = \mathscr{M}(\mathscr{A})$ for a k-affinoid algebra $\mathscr{A}$ and $S' = \mathscr{M}(\mathscr{A}')$ for a finite and faithfully flat $\mathscr{A}$-algebra $\mathscr{A}'$ that is admissible as an $\mathscr{A}$-module. (The surjectivity of $\mathrm{Spec}(\mathscr{A}') \to \mathrm{Spec}(\mathscr{A})$ follows from the surjectivity of f and the surjectivity of $\mathscr{M}(\mathscr{B}) \to \mathrm{Spec}(\mathscr{B})$ for any k-affinoid algebra $\mathscr{B}$. This is why the flat map of algebras $\mathscr{A} \to \mathscr{A}'$ is faithfully flat.) But in this special case coherent sheaves correspond to finite modules over the coordinate ring and completed tensor products are ordinary tensor products, so the required result is a special case of faithfully flat descent for quasi-coherent sheaves on schemes. ■

To use this lemma, we first recall that there is a natural bijection between (isomorphism classes of) closed immersions into a k-analytic space S and coherent ideal sheaves for the G-topology S_G of S (in the sense of [Ber2, §3]). More specifically, any closed immersion gives rise to such an ideal sheaf via [Ber2, 1.3.7] and conversely any coherent ideal sheaf arises from a unique closed immersion (up to unique isomorphism) by [Ber2, 1.3.7] in the k-affinoid case and a standard gluing procedure [Ber2, 1.3.3(b)] in the general case. (Strictly speaking there is a local finiteness condition in this general gluing procedure, but it is easily bypassed in the case of intended application by local compactness considerations on the ambient space S.) In this way, the closed immersion $F : V' \hookrightarrow V$ arising by base change from f naturally corresponds to a coherent ideal sheaf $\mathscr{I}$ on V_G, and on $(V' \times_V V')_G = ((Y' \times_Y Y') \times_V V')_G$ the pullback coherent ideal sheaves $p_j^*(\mathscr{I})$ are equal since there is an equality of closed immersions $p_1^*(F) = p_2^*(F)$ into $(Y' \times_Y Y' \times_Y Y') \times_Y V$. By Lemma 4.2.3, $\mathscr{I}$ is therefore the pullback of a unique coherent ideal sheaf $\mathscr{I}_0$ on Y_G. For the associated closed immersion $\iota : Y_0 \hookrightarrow Y$ there is a unique V-isomorphism $\phi : Y_0 \times_Y V \simeq V' = Y' \times_Y V$ (since F and ι_V are closed immersions into V with the same associated coherent ideal sheaves). The two pullback isomorphisms induced over $V' \times_V V'$ from ϕ coincide because closed immersions have no nontrivial automorphisms, and since representable functors in the k-analytic category are sheaves for the étale topology it follows that ϕ uniquely descents to a Y-isomorphism $Y_0 \simeq Y'$. Hence, $f : Y' \to Y$ is a closed immersion due to how $Y_0 \to Y$

was constructed. This completes the proof that the étale quotient $X = U/R$ is separated when it exists (given that we are assuming that $\delta : R \to U \times U$ is a closed immersion).

The next lemma, which is an analogue of Lemma 3.2.1, will be useful for analyzing properties of the map $U \to U/R$ when the quotient has been constructed.

Lemma 4.2.4. *Let $f : X' \to X$ be a finite étale surjection between k-analytic spaces. If X' is k-affinoid then so is X, and if in addition X' is strictly k-analytic then so is X.*

Proof. Since $X'' = X' \times_X X'$ is finite over X' under either projection, it is k-affinoid (and strict when X' is so). Also, the map $X'' \to X' \times X'$ between k-affinoid spaces is a closed immersion because a finite monomorphism between k-analytic spaces is a closed immersion (as we may check after first using analytic extension of the base field to reduce to the strict case; the monomorphism property is preserved by such extension since it is equivalent to the relative diagonal map being an isomorphism). The method of proof of Lemma 3.2.1 therefore carries over (using [Ber1, 2.1.14(i)] to replace [BGR, 6.3.3]) to construct a k-affinoid quotient for the finite étale equivalence relation $X'' \rightrightarrows X'$, and this quotient is (by construction) even strict when X' is strict. But X is also such a quotient, so it must be k-affinoid. ∎

Lemma 4.2.5. *Assume that $R \rightrightarrows U$ is a finite étale equivalence relation on k-analytic spaces such that the quotient $X = U/R$ exists. The map $U \to U/R$ must be finite, and if U is Hausdorff then X is Hausdorff. Moreover, if U is k-affinoid then so is X.*

Proof. The base change of $\pi : U \to X$ by the étale covering $\pi : U \to U/R$ is a finite map (it is a projection $R \to U$), so to prove that π is finite we just have to show that if a map $h : Y' \to Y$ between k-analytic spaces becomes finite after an étale surjective base change on Y then it is finite. We can work locally on Y, so since étale maps are open and are finite locally on the source we can assume that there is a finite étale cover $Z \to Y$ such that $Y' \times_Y Z \to Z$ is finite. To prove finiteness of h from this we can easily reduce to the case when Y is k-affinoid, so Z is k-affinoid and hence $Y' \times_Y Z$ is k-affinoid. The map $Y' \times_Y Z \to Y'$ is a finite étale cover with k-affinoid source, so Y' is k-affinoid by Lemma 4.2.4. Thus, $h : Y' \to Y$ is a map between k-affinoids which becomes finite after the finite étale base change by $Z \to Y$. The desired finiteness of h is therefore clear.

With π now known to be finite étale (and surjective) in general, if U is k-affinoid then Lemma 4.2.4 ensures that X must be k-affinoid. To see that X must be Hausdorff when U is Hausdorff, the finite surjection $\pi \times \pi : U \times U \to X \times X$ induces a closed map on topological spaces, so properness and surjectivity of $|T_1 \times T_2| \to |T_1| \times |T_2|$ for k-analytic spaces T_1 and T_2 implies that $|U| \times |U| \to |X| \times |X|$ is closed. The diagonal $|U| \subseteq |U| \times |U|$ is closed since U is Hausdorff, so we conclude that $|X|$ has closed diagonal image in $|X| \times |X|$ as desired. That is, X must be Hausdorff when it exists. ∎

4.3. **Étale localization and reduction to group quotients.** Now we return to the global construction problem for $p_1, p_2 : R \rightrightarrows U$ as in Theorem 4.2.2. The aim of this section is to reduce the problem to the particular case when $p : R \rightrightarrows U$ is induced by a free right action of a finite group G on U; i.e., $R = U \times G$ with p_1 the canonical projection and $p_2(u, g) = u.g$. We emphasize that the hypothesis that $\delta : R \to U \times U$ is a closed immersion will be preserved under this reduction step. The existence result for U/R in this special case is proved in Theorem 5.1.1.

We have already seen that it suffices to work locally on U to solve the existence problem for U/R. By "work locally" we mean that we work with opens V that cover U and the étale equivalence relation $R_V = p_1^{-1}(V) \cap p_2^{-1}(V) = \delta^{-1}(V \times V)$ on V (for $\delta : R \to U \times U$). Note that the map $R_V \to V \times V$ is still a closed immersion since it is a base change of the map δ that we assume is a closed immersion. Localizing in this way does not lose the property of the new U being strictly k-analytic or good when the original U is so.

It is possible to first use topological arguments (especially compactness and connectedness considerations) to reduce the problem to the case when the maps $R \rightrightarrows U$ are finite étale, and to then use étale localization to split the equivalence relation, thus passing to the group action case. However, it turns out that a shorter way to the same goal is to first apply étale localization to split the equivalence relation étale-locally around points of U and to then use compactness and connectedness considerations. Given an étale morphism $f : U' \to U$

we define

$$R' = R \times_{U \times U} (U' \times U') = U' \times_{U,p_1} R \times_{p_2,U} U',$$

obtaining an étale equivalence relation $p_1', p_2' : R' \rightrightarrows U'$ induced from R. Beware that even if the maps $R \rightrightarrows U$ are quasi-compact, the maps $R' \rightrightarrows U'$ may fail to be quasi-compact. (In the intended applications to algebraic spaces, such quasi-compactness properties for the projections $R \rightrightarrows U$ are often not satisfied even when the algebraic space is separated.)

Lemma 4.3.1. *Let $R \rightrightarrows U$ be an étale equivalence relation on k-analytic spaces and let $f : U' \to U$ be an étale surjection. The quotient $X = U/R$ exists if and only if the quotient $X' = U'/R'$ exists, and then $X' \simeq X$.*

Proof. We will need the following set-theoretic analogue of the lemma: if $\mathscr{R} \rightrightarrows \mathscr{U}$ is an equivalence relation on a set $\mathscr{U}$ and there is given a surjective map of sets $\mathscr{U}' \twoheadrightarrow \mathscr{U}$ then for the induced equivalence relation

$$\mathscr{R}' = \mathscr{R} \times_{\mathscr{U} \times \mathscr{U}} (\mathscr{U}' \times \mathscr{U}') \rightrightarrows \mathscr{U}'$$

on $\mathscr{U}'$ we have that the natural map $\mathscr{U}'/\mathscr{R}' \to \mathscr{U}/\mathscr{R}$ is bijective. To apply this, we will work with étale sheaves of sets on the category of k-analytic spaces; examples of such sheaves are representable functors $\underline{Z} = \mathrm{Hom}(\cdot, Z)$ for k-analytic spaces Z (as we indicated above Theorem 4.1.2). Consider the equivalence relations $\underline{R} \rightrightarrows \underline{U}$ and $\underline{R}' \rightrightarrows \underline{U}'$, and let $\mathscr{X}$ and $\mathscr{X}'$ be the corresponding quotients in the category of étale sheaves of sets. It suffices to prove that $\mathscr{X}' \simeq \mathscr{X}$. By surjectivity of f, the corresponding morphism of sheaves $\underline{U}' \to \underline{U}$ is surjective. In particular, this latter map of sheaves induces surjections on stalks at geometric points, so by the above set-theoretic result we conclude that the natural map $\mathscr{X}' \to \mathscr{X}$ induces a bijection on geometric stalks. Hence, by [Ber2, 4.2.3] this natural map is an isomorphism of sheaves of sets. ∎

Corollary 4.3.2. *If $R \rightrightarrows U$ is an étale equivalence relation on k-analytic spaces then the quotient U/R exists if and only if for any point $u \in U$ there exists an étale morphism $U' \to U$ whose image contains u and such that the quotient U'/R' exists, where $R' = R \times_{U \times U} (U' \times U')$.*

Now we are in position to prove Theorem 4.2.2, assuming that it is true in the case of a free action by a finite group (with action map $U \times G \to U \times U$ a closed immersion!), a case that we will settle in Theorem 5.1.1. We just have to prove the following lemma.

Lemma 4.3.3. *Let $R \rightrightarrows U$ be an étale equivalence relation on k-analytic spaces, and assume that $R \to U \times U$ is a closed immersion. For any point $u \in U$ there exists an étale morphism $W \to U$ whose image contains u and such that the induced equivalence relation $R_W \rightrightarrows W$ is split; i.e., induced by a free right action of a finite group G on W.*

Note that if U is strictly analytic (resp. good) then so is W.

Proof. By working locally on the topological space of U we can assume that U is Hausdorff (so $U \times U$ and R are Hausdorff, as $|U \times U| \to |U| \times |U|$ is topologically proper and $R \to U \times U$ is monic and hence separated).

To construct W and G we first want to split R over u. By hypothesis the diagonal $R \to U \times U$ is a topologically proper map, so for any compact analytic domain $K \subseteq U$ that is a neighborhood of u in U we have that $R_K = R \cap (K \times K)$ is compact. But the closed subset $p_1^{-1}(u) \subseteq R$ is étale over $\mathscr{H}(u)$ and hence is discrete, so it has finite overlap with R_K. Thus, u has an open neighborhood $\mathrm{int}_U(K)$ in U that contains only finitely many points which are R-equivalent to u, so by localizing U and using the Hausdorff property we can arrange that u is not R-equivalent to any other points of U. That is, now we have $p_1^{-1}(u) = p_2^{-1}(u)$ and this common set is finite. For each r in this finite set, the pullback maps $\mathscr{H}(u) \rightrightarrows \mathscr{H}(r)$ with respect to p_1^* and p_2^* are finite separable. Hence, we can choose a finite Galois extension $\mathscr{H}/\mathscr{H}(u)$ that splits each $\mathscr{H}(r)$ with respect to both of its $\mathscr{H}(u)$-structures (via p_1 and p_2). By [Ber2, 3.4.1] we can find an étale morphism $U' \to U$ so that u has a single preimage $u' \in U'$ and $\mathscr{H}(u')$ is $\mathscr{H}(u)$-isomorphic to $\mathscr{H}$. Shrinking U' around u' also allows us to suppose that $U' \to U$ is separated (even finite over an open neighborhood of u in U). In particular, U' is Hausdorff, so R' is Hausdorff.

Since u' is the only point in U' over u and no point in $U-\{u\}$ is R-equivalent to u, no point in $U'-\{u'\}$ is R'-equivalent to u'. That is, ${p'_1}^{-1}(u') = {p'_2}^{-1}(u')$ as (finite) subsets of R'. Let $\{r'_1, \dots, r'_n\}$ be an enumeration of this set, so the natural pullback maps $\mathscr{H} = \mathscr{H}(u') \rightrightarrows \mathscr{H}(r'_j)$ via p'_1 and p'_2 are isomorphisms for each j, due to the formula $R' = U' \times_{U,p_1} R \times_{p_2,U} U'$ and the way that $\mathscr{H}(u')/\mathscr{H}(u)$ was chosen. It follows from these residual isomorphisms and [Ber2, 3.4.1] that the étale maps p'_1 and p'_2 are local isomorphisms near each r'_j; i.e., for $i = 1, 2$ there exist open neighborhoods U''_i of u' and open neighborhoods R'_{ij} of r'_j for each j such that p'_i induces isomorphisms $R'_{ij} \to U''_i$ for each j. Since our original problem only depends on the étale neighborhood of (U, u), we can replace $R \rightrightarrows U$ and u with $R' \rightrightarrows U'$ and u' so reduce to the case when $p_1^{-1}(u) = p_2^{-1}(u) = \{r_1, \dots, r_n\}$ and there are open subspaces $U_i \subseteq U$ around u such that $p_i^{-1}(U_i)$ contains an open neighborhood R_{ij} around r_j mapping isomorphically onto U_i under p_i. However, there may be overlaps among the R_{ij} for a fixed i and $p_i^{-1}(U_i)$ may contain points not in any R_{ij}.

Choose a compact k-analytic domain $\overline{U} \subseteq U$ that is a neighborhood of u in U, and let $\overline{R} = \overline{U} \times_{U,p_1} R \times_{p_2,U} \overline{U}$. Note that $\overline{R}$ is compact since $R \to U \times U$ is compact. Thus, $p : R \rightrightarrows U$ induces an equivalence relation $\overline{p} : \overline{R} \rightrightarrows \overline{U}$ that may not be étale (though it is rig-etale in the strictly analytic case). Since $\overline{p}$ is induced by p, $\overline{R} = p_1^{-1}(\overline{U}) \cap p_2^{-1}(\overline{U})$ is a compact neighborhood of $p_1^{-1}(u) = p_2^{-1}(u)$ in R. Make an initial choice of U_1 and let $\overline{R}_{1j} \subseteq \overline{R} \cap p_1^{-1}(U_1)$ be an open neighborhood around r_j in R that is small enough so that it maps isomorphically onto an open subspace of U_1 and such that the $\overline{R}_{1j}$'s are pairwise disjoint. The image $p_1(\overline{R} - (\cup \overline{R}_{1j}))$ is compact in $\overline{U}$ and does not contain u, so for any open subspace $U' \subseteq U_1$ around u with $U' \subseteq \overline{U} - p_1(\overline{R} - \cup \overline{R}_{1j})$ we have that $p_1^{-1}(U')$ contains pairwise disjoint open neighborhoods R'_{1j} around the r_j's such that each R'_{1j} maps isomorphically onto U' under p_1 and $p_2^{-1}(U')$ meets $p_1^{-1}(U')$ inside of $\overline{R}$. Thus, $p_1^{-1}(U')$ is the disjoint union of the open subspaces R'_{1j} and $p_1^{-1}(U') \cap (R - \overline{R})$. We can likewise choose U_2 and $\overline{R}_{2j}$ adapted to p_2, so by choosing

$$U' \subseteq \overline{U} - (p_1(\overline{R} - \cup \overline{R}_{1j}) \cup p_2(\overline{R} - \cup \overline{R}_{2j}))$$

we also have that $p_2^{-1}(U')$ is the disjoint union of open subspaces R'_{2j} and $p_2^{-1}(U') \cap (R - \overline{R})$ with $r_j \in R'_{2j}$ and $p_2 : R'_{2j} \simeq U'$.

Let $R' = p_1^{-1}(U') \cap p_2^{-1}(U') \subseteq \overline{R}$ be the étale equivalence relation on U' induced from R. Thus, the excess open sets $p_i^{-1}(U') \cap (R - \overline{R})$ are disjoint from R', so R' is the disjoint union of the overlaps $R'_{ij} \cap R'_{ij'}$ for $1 \le j, j' \le n$. The initial choices of $\overline{R}_{ij}$ above (prior to the choice of U') should be made so that not only are these pairwise disjoint for fixed i but also so that $\overline{R}_{1j} \cap \overline{R}_{2j'} = \emptyset$ for $j \ne j'$. (This can be done since $r_j \ne r_{j'}$ and R is Hausdorff.) Thus, $R'_{ij} \cap R'_{ij'} = \emptyset$ for $j \ne j'$, so R' is the disjoint union of the n open subspaces $R'_j = R'_{1j} \cap R'_{2j}$ with $r_j \in R'_j$. Each p_i induces an open immersion $p'_{ij} : R'_j \to U'$ whose image is a neighborhood of u. We will prove that a careful choice of U' leads to a split equivalence relation R', so in particular the p'_{ij}'s are isomorphisms in such cases.

Let V be the connected component of $\cap_{i,j} p'_{ij}(R'_j)$ that contains u, so it is an open neighborhood of u in U' and applying p'_{2j} to ${p'_{1j}}^{-1}(V)$ for each $1 \le j \le n$ yields open immersions

$$\phi_j : V \simeq {p'_{1j}}^{-1}(V) \subseteq R'_j \to U'$$

that fix u. Consider the equivalence relation of sets

$$\mathscr{P} : \mathscr{R} = \mathrm{Hom}(V, R') \rightrightarrows \mathscr{U} = \mathrm{Hom}(V, U').$$

Since V is connected and $R' = \coprod R'_j$ with $p'_{1j} : R'_j \to U'$ an open immersion that hits $u \in V$, there exist exactly n ways to lift the canonical open immersion $i_V : V \to U'$ with respect to p'_1 to a morphism $V \to R'$ (just liftings with respect to p'_{1j}). Therefore, $\{\phi_j\}_{1 \le j \le n}$ is the set of all elements of $\mathscr{U}$ that are $\mathscr{P}$-equivalent to i_V. In particular, for any pair $1 \le a, b \le n$ the morphisms ϕ_a and ϕ_b are $\mathscr{P}$-equivalent, and therefore (again using the connectedness of V) the morphism $\phi_b \circ \phi_a^{-1} : \phi_a(V) \to \phi_b(V)$ is induced from $p'_{2c} \circ {p'_{1c}}^{-1}$ for some $1 \le c \le n$. Let ψ_j denote the automorphism of the germ (V, u) induced from ϕ_j, so the above argument implies that $\psi_b \circ \psi_a^{-1} = \psi_c$. Since the set $G = \{\psi_j\}_{1 \le j \le n}$ includes the identity (which corresponds to the identity point r_{j_0} over u in the identity part of the equivalence relation), it follows that G is actually a group of automorphisms of (V, u). Note that composition of these automorphisms corresponds to a right action on this germ due to how each ϕ_j is defined in the form "$p_2 \circ p_1^{-1}$".

We identify the index set $\{1, \dots, n\}$ for the R'_j's with G via $j \mapsto \psi_j$. Since G is a finite group of automorphisms of (V, u), there exists an open neighborhood V' of u in V such that for any $g \in G$ we have $g(V') \subseteq V$. It is clear that the open subspace $W = \cap_{g \in G} g(V') \subseteq V'$ is taken to itself by each $g \in G$, so G thereby acts on W. (This is a right action.) It follows that the equivalence relation R' induces on W the split equivalence relation corresponding to this right action of G. Thus, W and G are as required. ■

Granting Theorem 4.2.2 (whose proof rests on Theorem 5.1.1 that is proved below), we return to the étale equivalence relation $\mathscr{R}^{\mathrm{an}} \rightrightarrows \mathscr{U}^{\mathrm{an}}$ on rigid spaces for proving Theorem 4.2.1 for an algebraic space $\mathscr{X}$ (with $|k^\times| \neq \{1\}$). By Corollary 2.2.4 we may assume that $\mathscr{X}$ is quasi-compact, so we can and do take $\mathscr{U}$ to be affine. This forces $\mathscr{R}$ to be quasi-compact and separated since $\mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is a quasi-compact monomorphism. Hence, the associated k-analytic spaces U and R are paracompact. By using analytification with values in the category of good strictly k-analytic spaces, we conclude from Theorem 4.2.2 that $\mathscr{X}$ admits an analytification X in the sense of k-analytic spaces (as opposed to the sense of rigid spaces, which is what we want), and X is separated, good, and strictly k-analytic. Let us check that X is also paracompact. The Hausdorff space U is covered by a rising sequence $\{K_j\}_{j \geq 1}$ of compact subsets such that $K_j \subseteq \mathrm{int}_U(K_{j+1})$ for all j because U arises from an affine $\mathscr{U}$. (Choose a closed immersion of $\mathscr{U}$ into an affine space over k and intersect U with closed balls of increasing radius centered at the origin in the analytified affine space.) The images of the K_j's under the open surjective map $U \to X$ are therefore an analogous sequence of compact subsets in the Hausdorff space X, so X is indeed paracompact.

It follows that under the equivalence of categories in [Ber2, 1.6.1] there is a quasi-separated rigid space X_0 uniquely associated to X, and the étale surjective map $U \to X$ (which is in the full subcategory of strictly k-analytic spaces) arises from a unique morphism $\mathscr{U}^{\mathrm{an}} \to X_0$ that is necessarily étale (as we may check using complete local rings) and surjective (due to k-strictness). The two maps $\mathscr{R}^{\mathrm{an}} \rightrightarrows \mathscr{U}^{\mathrm{an}}$ are equalized by the map $\mathscr{U}^{\mathrm{an}} \to X_0$ because this equality holds back in the k-analytic setting (since $X = U/R$). The resulting map $\mathscr{R}^{\mathrm{an}} \to \mathscr{U}^{\mathrm{an}} \times_{X_0} \mathscr{U}^{\mathrm{an}}$ is likewise an isomorphism due to the isomorphism $R \simeq U \times_X U$ and the functoriality and compatibility with fiber products for the functor from paracompact strictly k-analytic spaces to quasi-separated rigid spaces.

Hence, by Example 2.1.5 we see that X_0 represents $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ as long as the map $\mathscr{U}^{\mathrm{an}} \to X_0$ admits local étale quasi-sections. Since X is paracompact and Hausdorff, by [Ber2, 1.6.1] the rigid space X_0 has an admissible covering arising from a locally finite collection of strictly k-analytic affinoid subdomains D that cover X. The étale surjection $U \to X$ gives an étale cover $U \times_X D \to D$, so by quasi-compactness of D and the local finiteness of étale maps we get a finite collection of strictly k-analytic subdomains $Y_i \subseteq D$ that cover D and over which there is a finite étale cover $Y'_i \to Y_i$ that maps into $U \times_X Y_i$ over Y_i. The rigid space associated to $\coprod Y'_i$ is then a quasi-compact étale cover of $D_0 \subseteq X_0$ over which $\mathscr{U}^{\mathrm{an}} \to X_0$ acquires a section.

5. Group quotients

5.1. **Existence result and counterexamples.** We shall now use the theory of reduction of germs as developed in [T1] for strictly analytic spaces and in [T2] in general to prove an existence result for group-action quotients to which the proof of Theorem 4.2.2 was reduced in the previous section.

Theorem 5.1.1. *Let G be a finite group equipped with a free right action on a k-analytic space U. Assume that the action map $\alpha : U \times G \to U \times U$ defined by $(u, g) \mapsto (u, u.g)$ is a closed immersion. The quotient U/G exists as a separated k-analytic space. If U is strictly k-analytic* (*resp. good, resp. k-affinoid*) *then so is U/G.*

The inheritance of strict k-analyticity (resp. goodness) from that of U will follow from our method of construction of U/G. Note also that the action map $\alpha : U \times G \to U \times U$ is the diagonal of an étale equivalence relation since the action is free, and since it is a closed immersion the space U must be separated (since Δ_U is the restriction of α to the open and closed subspace $U \times \{1\}$ in $U \times G$). The separatedness of U/G (once it exists) follows by the same étale descent argument as we used for a general quotient $X = U/R$

in the discussion following Theorem 4.2.2. Observe that if we assume U is Hausdorff then α is a compact topological map (since $|U \times U| \to |U| \times |U|$ is compact) but it is not a closed immersion if U is not separated, and Example 5.1.4 shows that U/G can fail to exist in such cases.

Example 5.1.2. Let us see how the formation of U/G in Theorem 5.1.1 interacts with passage to G-stable k-analytic subdomains in U. In the setup of Theorem 5.1.1, assume that U/G exists and let $\pi : U \to U/G$ be the quotient map, which must be finite étale (by Lemma 4.2.5). Let $N \subseteq U$ be a G-stable quasi-compact k-analytic subdomain, so $\pi(N) \subseteq U/G$ is a k-analytic domain by Theorem 4.1.2 and it is good (resp. strictly k-analytic) if U, N, and U/G are good (resp. strictly k-analytic). The inclusion of analytic domains $N \to \pi^{-1}(\pi(N))$ in U is bijective on geometric points due to G-stability of N, so it is an isomorphism. Thus, the map $\pi : N \to \pi(N)$ serves as a quotient N/G. Likewise, if N, U, and U/G are good (resp. strictly k-analytic) then so is N/G. Note in particular that the natural map $N/G \to U/G$ identifies N/G with a k-analytic domain in U/G.

Before we prove Theorem 5.1.1, we give an example that shows that the closed immersion hypothesis on α in Theorem 5.1.1 cannot be replaced with a compactness hypothesis for the purposes of ensuring the existence of U/G.

Example 5.1.3. We give an example of a 1-dimensional compact Hausdorff (and non-separated) strictly k-analytic space U equipped with a free action of the group $G = \mathbf{Z}/2\mathbf{Z}$ such that U/G does not exist, assuming that there is a separable quadratic extension k'/k that is ramified in the sense that $k \to k'$ induces an isomorphism on residue fields (so in particular, $|k^\times| \neq \{1\}$). We also give an analogous such non-existence result in the rigid-analytic category (with quasi-compact quasi-separated rigid spaces that are not separated). In Example 5.1.4 this will be adapted to work in both the rigid-analytic and k-analytic categories with an arbitrary k (perhaps algebraically closed or, in the k-analytic case, with trivial absolute value) using 2-dimensional compact Hausdorff strictly k-analytic spaces.

Let U be the strictly k'-analytic gluing of the closed unit ball $\mathscr{M}(k'\langle t\rangle)$ to itself along the identity map on the k'-affinoid subdomain $\{|t| = 1\}$. Let B_1 and B_2 be the two canonical copies of $\mathscr{M}(k'\langle t\rangle)$ in U, so $B_1 \cap B_2$ is the affinoid space $\{|t| = 1\}$ over k' whose (diagonal) map into $B_1 \times B_2$ is not finite (since this map of strict affinoids over k' has reduction over $\widetilde{k'}$ that is not finite: it is the diagonal inclusion of $\mathbf{G}_m$ into $\mathbf{A}^2_{\widetilde{k'}}$). In particular, this map is not a closed immersion, so U is not separated. However, U is compact Hausdorff since it is a topological gluing of compact Hausdorff spaces along a closed subset. Under the equivalence of categories between compact Hausdorff strictly k'-analytic spaces and quasi-compact quasi-separated rigid spaces over k', U corresponds to the non-separated gluing U_0 of two copies of $\mathbf{B}^1_{k'} = \mathrm{Sp}(k'\langle t\rangle)$ along the admissible open $\mathrm{Sp}(k'\langle t, 1/t\rangle)$ via the identity map. Now view this gluing in the k-analytic category. Let the nontrivial element in $G = \mathbf{Z}/2\mathbf{Z}$ act on U over k by swapping B_1 and B_2 and then acting via the nontrivial element of $\mathrm{Gal}(k'/k)$ on each B_j. This is seen to be a free action by computing on geometric points. (The action would not be free if we used only the swap.)

There is an analogous such action of G on the rigid space U_0 over k. Let us check that non-existence of U/G as a k-analytic space implies non-existence of U_0/G as a rigid space. Assume that U_0/G exists. Since U_0 is quasi-compact and quasi-separated, the natural map $U_0 \times G \to U_0 \times U_0$ is quasi-compact. Thus, U_0/G is quasi-separated, so any set-theoretic union of finitely many affinoid opens in U_0/G is a quasi-compact admissible open subspace of U_0/G. It follows that U_0/G must be quasi-compact. In particular, U_0/G corresponds to a compact Hausdorff k-analytic space. Since the étale quotient map $\pi : U_0 \to U_0/G$ has local *fpqc* quasi-sections and $U_0 \times G \simeq U_0 \times_{U_0/G} U_0$, by [C2, Thm. 4.2.7] the map π must be finite étale (of degree 2). The map of k-analytic spaces $U \to \overline{U}$ corresponding to π is therefore finite étale (of degree 2), G-invariant, and gives rise to a map $U \times G \to U \times_{\overline{U}} U$ that is an isomorphism. Hence, $\overline{U}$ serves as a quotient U/G in the category of k-analytic spaces. Once it is proved that U/G does not exist, it therefore follows that U_0/G does not exist.

To see that U/G does not exist as a k-analytic space, assume to the contrary that such a quotient does exist. Let $\xi \in B_1 \cap B_2$ be the common Gauss point in the gluing U of the two unit disks B_j over k', and let $\xi' = \pi(\xi) \in U/G$. Since U/G is a k-analytic space, there exist finitely many k-affinoid domains $V'_1, \dots, V'_n$

in U/G containing ξ' with $\cup V'_j$ a neighborhood of ξ' in U/G. The preimage $V_j = \pi^{-1}(V'_j)$ is a G-stable k-affinoid in U containing ξ with $\cup V_j$ a neighborhood of ξ in U.

Choose some j and let $V = V_j$. Since G physically fixes the point $\xi \in U$, the action of G on U induces a G-action on the strictly k-affinoid germ (V, ξ). Let $\widetilde{V}_\xi$ be the reduction of (V, ξ) in the sense of [T1]; this is a birational space over the residue field $\widetilde{k}$ (not to be confused with the theory of reduction of germs and birational spaces over $\mathbf{R}^\times_{>0}$-graded fields as developed in [T2]). This reduction is a separated birational space over $\widetilde{k}$ since V is k-affinoid, and it inherits a canonical G-action from the one on (V, ξ). By separatedness of the birational space, the G-action on this reduction is uniquely determined by its effect on the residue field $\widetilde{\mathscr{H}(\xi)}$. But this latter residue field is identified with $\widetilde{k}'(t)$ compatibly with the natural action of $G = \mathrm{Gal}(k'/k)$, so since $\widetilde{k} = \widetilde{k}'$ (by hypothesis) we see that this G-action is trivial. Hence, G acts trivially on $\widetilde{V}_\xi$. Letting Y' (resp. Y'') denote the k-analytic domain $V \cap B_1$ (resp. $V \cap B_2$) in U, (V, ξ) is covered by (Y', ξ) and (Y'', ξ), so by [T1, 2.3(*ii*)] the reductions $\widetilde{Y'}_\xi$ and $\widetilde{Y''}_\xi$ are an open cover of $\widetilde{V}_\xi$ and are swapped by the G-action. But we have seen that this G-action is trival, so we obtain the equality $\widetilde{Y'}_\xi = \widetilde{V}_\xi$. Thus, by [T1, 2.4] it follows that $(Y', \xi) = (V, \xi)$ as germs at ξ. Likewise, $(Y'', \xi) = (V, \xi)$, so $(Y' \cap Y'', \xi) = (V, \xi)$. Hence, for a sufficiently small open $W \subseteq U$ around ξ we have $V \cap W \subseteq Y' \cap Y'' \subseteq B_1 \cap B_2$. But $V = V_j$ for an arbitrary choice among the finitely many j's, so for any sufficiently small open W around ξ in U we have that $W \cap (\cup V_j) \subseteq B_1 \cap B_2$. We can take W so small that it is contained in the neighborhood $\cup V_j$ of ξ in U, so $W \subseteq B_1 \cap B_2$. Hence, $B_1 \cap B_2$ is a neighborhood of ξ in U, so $\mathscr{M}(k'\langle t, 1/t\rangle)$ is a (closed) neighborhood of the Gauss point in $\mathscr{M}(k'\langle t\rangle)$. This is a contradiction, since the closure in $\mathscr{M}(k'\langle t\rangle)$ of the open residue disc around any k'-point (such as the origin) contains the Gauss point. Thus, U/G does not exist.

Example 5.1.4. The preceding example can be adapted in both the rigid-analytic and k-analytic categories to give similar non-existence examples in the 2-dimensional case without restriction on k (e.g., k may be algebraically closed or, in the k-analytic case, have trivial absolute value). Rather than work with a ramified quadratic extension of the constant field, consider a finite étale degree-2 map of connected smooth affinoid strictly k-analytic curves $C' \to C$ such that for some $c \in C$ there is a unique point $c' \in C'$ over c and $\mathscr{H}(c')/\mathscr{H}(c)$ is a quadratic ramified extension. Such examples are easily constructed from algebraic curves (even if k is algebraically closed or has trivial absolute value) by taking c to correspond to a suitable completion of the function field of an algebraic curve. Let $D = \mathscr{M}(k\langle t\rangle)$, and let U be the gluing of $S_1 = C' \times D$ to $S_2 = C' \times D$ along $C' \times \{|t| = 1\}$ via the identity map. This is a compact Hausdorff k-analytic space of dimension 2, and it is not separated. Let the nontrivial element in $G = \mathbf{Z}/2\mathbf{Z}$ act on U by swapping S_1 and S_2 and then applying the nontrivial automorphism of C' over C. This is a free action. As in the previous example, if the analogous rigid space situation admits a quotient then so does the k-analytic situation, so to show that no quotient exists in either case it suffices to prove that U/G does not exist as a k-analytic space. Assume U/G exists. The natural map $U \to C$ is invariant by G, so it uniquely factors through a map $U/G \to C$. The fiber U_c equipped with its G-action is an instance of Example 5.1.3 with k'/k replaced with $\mathscr{H}(c')/\mathscr{H}(c)$. Moreover, the finite étale map $U_c \to (U/G)_c$ clearly serves as a quotient U_c/G in the category of $\mathscr{H}(c)$-analytic spaces, contradicting that such a quotient does not exist (by Example 5.1.3).

5.2. **Local existence criteria.** We now turn to the task of proving Theorem 5.1.1. It has already been seen that the closed immersion hypothesis on the action map $\alpha : U \times G \to U \times U$ forces U to be separated, and conversely it is clear that if U is separated then for any finite group equipped with a free right action on U the action map is a closed immersion. To keep the role of the separatedness conditions clear, we will initially consider free right actions without any assumption on the action map, and assuming only that U is Hausdorff rather than separated. For any point $u \in U$, with $G_u \subseteq G$ its isotropy group, say that the G-action on U is *locally effective* at u if there is a G_u-stable analytic domain $N_u \subseteq U$ that is a neighborhood of u such that N_u/G_u exists; obviously G_u acts freely on N_u.

Lemma 5.2.1. *Assume that U is Hausdorff. The quotient U/G exists if and only if the G-action is locally effective at all points of U. If the quotient N_u/G_u exists and is good for each $u \in U$ and some G_u-stable*

k-analytic domain $N_u \subseteq U$ that is a neighborhood of u then U/G is also good. The same holds for the property of being strictly k-analytic.

Proof. First assume that U/G exists. By Lemma 4.2.5, the étale surjective quotient map $\pi : U \to U/G$ must be finite. It is clearly a right G-torsor, so for each $\overline{u} \in U/G$ the fiber $\pi^{-1}(\overline{u}) = \{u_1, \dots, u_n\}$ has transitive G-action with $\pi^{-1}(\overline{u}) = \coprod \mathscr{M}(\mathscr{H}(u_i))$ over $\mathscr{M}(\mathscr{H}(\overline{u}))$ for a finite Galois extension $\mathscr{H}(u_i)$ of $\mathscr{H}(\overline{u})$ with Galois group $G_{u_i} \subseteq G$. The equivalence of categories of finite étale covers of the germ $(U/G, \overline{u})$ and finite étale covers of $\mathrm{Spec}(\mathscr{H}(\overline{u}))$ [Ber2, 3.4.1] thereby provides a connected open $\overline{V} \subseteq U/G$ around $\overline{u}$ such that $\pi^{-1}(\overline{V}) = \coprod V_i$, with V_i the connected component through u_i and $V_i \to \overline{V}$ a finite étale G_{u_i}-torsor. Hence, V_i/G_{u_i} exists and equals $\overline{V}$. Varying $\overline{u}$, this implies that G acts locally effectively on U.

Conversely, assume that the G-action is locally effective at every $u \in U$. Let $N_u \subseteq U$ be a G_u-stable analytic domain that is a neighborhood of u in U and for which N_u/G_u exists. The quotient map $\pi_u : N_u \to N_u/G_u$ is a finite étale G_u-torsor, so u is the only point over $\overline{u} = \pi_u(u)$ and any G_u-stable open subset $W_u \subseteq N_u$ admits the open subset $\pi_u(W_u) \subseteq N_u/G_u$ as a quotient W_u/G_u. In particular, any open set around u in N_u contains $\pi_u^{-1}(\overline{V})$ for some open $\overline{V} \subseteq N_u/G_u$ around $\overline{u}$. Since N_u is a neighborhood of u in U, it follows that u has a base of G_u-stable open neighborhoods in U that admit a quotient by their G_u-action. Since U is Hausdorff and G is finite, we can therefore shrink the choice of N_u so that $g(N_u) \cap N_u = \emptyset$ for all $g \in G$ with $g \notin G_u$. Hence, the overlap R_{N_u} of the analytic domain $R = U \times G \subseteq U \times U$ with $N_u \times N_u$ is $N_u \times G_u$. Let $N_u^0 = \mathrm{int}_U(N_u)$, an open subset of U containing u. This is G_u-stable by functoriality of the topological interior with respect to automorphisms, and the open overlap $R_{N_u^0} = R \cap (N_u^0 \times N_u^0)$ inside of $R = U \times G$ is exactly $N_u^0 \times G_u$. Thus, the quotient N_u^0/G_u serves as the quotient $N_u^0/R_{N_u^0}$. We can therefore use the locality argument with open sets following the statement of Theorem 4.2.2 to glue the N_u^0/G_u's to get a global quotient U/G (which is good when every N_u/G_u is good, and likewise for strict k-analyticity, since N_u^0/G_u is an open subspace of N_u/G_u for every $u \in U$). ■

By Lemma 5.2.1, the global existence of U/G for a free right action of a finite group G on a Hausdorff k-analytic space U is reduced to the local effectivity of the G-action (and keeping track of the properties of goodness and strict k-analyticity for the local quotients by isotropy groups). We now use this to give a criterion for the existence of U/G in terms of good analytic domains when U is Hausdorff.

Lemma 5.2.2. *With notation and hypotheses as above, G is locally effective at $u \in U$ if and only if there are finitely many k-affinoid domains $V \subseteq U$ containing u such that the germs $(V, u) \subseteq (U, u)$ cover (U, u) and are G_u-stable as germs. It is equivalent to take the V's to merely be good analytic domains in U with $u \in V$. In such cases, if U has a strictly k-analytic* (*resp. good*) *open neighborhood of u then N_u/G_u exists and is strictly k-analytic* (*resp. good*) *for some G_u-stable k-analytic domain N_u in U containing u that we may take to be compact and as small as we please.*

Note that G_u-stability of a germ (V, u) is a weaker condition than G_u-stability of the k-analytic domain V in U.

Proof. First assume that G is locally effective at u, so N_u/G_u exists for some G_u-stable k-analytic domain $N_u \subseteq U$ that is a neighborhood of u in U. We can replace U and G with N_u and G_u so that $G = G_u$ and U/G exists. Let $\pi : U \to U/G$ be the quotient map and let $\overline{u} = \pi(u)$. Let $\overline{V}_1, \dots, \overline{V}_n$ be k-affinoid domains in U/G containing $\overline{u}$ such that $\cup \overline{V}_i$ is a neighborhood of $\overline{u}$ in U/G. Each $V_i = \pi^{-1}(\overline{V}_i)$ is a k-affinoid domain (since π is finite), contains u, and is G-stable. The finite collection of germs (V_i, u) therefore satisfies the required conditions.

Conversely, assume that there are good k-analytic domains $V_1, \dots, V_n$ in U containing u such that $(V_i, u) \subseteq (U, u)$ is G_u-stable as a germ and such that these germs cover (U, u). We may and do shrink each good domain V_i around u so that it is a separated (e.g., k-affinoid) domain. Thus, $\cup V_i$ is a neighborhood of u in U and $gV_i \cap V_i$ is a neighborhood of u in both V_i and gV_i for all $g \in G_u$. In particular, $V_i' = \cap_{g \in G_u} gV_i$ is a G_u-stable k-analytic domain with $(V_i', u) = (V_i, u)$ as domains in the germ (U, u). We may therefore assume that each V_i' is separated, good, and G_u-stable. Let $U_i \subseteq V_i$ be a k-affinoid neighborhood of u in the good k-analytic space V_i. The overlap $U_i' = \cap_{g \in G_u} gU_i$ is a k-affinoid neighborhood of u in V_i (hence in U) since

V_i is separated. But $(U_i, u) = (V_i, u)$ is a G_u-stable germ domain in (U, u), so $(U'_i, u) = (V_i, u)$ as well. Thus, we can assume that each $V_i = \mathscr{M}(\mathscr{A}_i)$ is k-affinoid and G_u-stable. By [Ber1, 2.1.14(*ii*)], $\mathscr{A}_i^{G_u}$ is a k-affinoid algebra over which $\mathscr{A}_i$ is finite and admissible, so $\mathscr{M}(\mathscr{A}_i^{G_u})$ makes sense and $V_i \to \mathscr{M}(\mathscr{A}_i^{G_u})$ is a finite map. Moreover, if $\mathscr{A}_i$ is strictly k-analytic then so is $\mathscr{A}_i^{G_u}$. The action of G_u on $\mathrm{Spec}(\mathscr{A}_i)$ is free because $V_i = \mathscr{M}(\mathscr{A}_i) \to \mathrm{Spec}(\mathscr{A}_i)$ is surjective and G_u acts freely on V_i. Hence, by [SGA3, Exp. V, 4.1], the finite admissible map $\mathscr{A}_i^{G_u} \to \mathscr{A}_i$ is étale and the natural map $\mathscr{A}_i \otimes_{\mathscr{A}_i^{G_u}} \mathscr{A}_i \to \prod_{g \in G_u} \mathscr{A}_i$ defined by $a \otimes a' \mapsto (ag(a'))_g$ is an isomorphism. This latter tensor product coincides with the corresponding completed tensor product, so it exhibits $\mathscr{M}(\mathscr{A}_i^{G_u})$ as the quotient V_i/G_u for all i.

Let $N_u = \cup V_i$, a compact G_u-stable k-analytic domain in U that is a neighborhood of u. Observe that by our initial shrinking of the V_i's we can take N_u to be as small as we please. If U is strictly k-analytic then it is clear that we could have taken each V_i to be strictly k-analytic, and every overlap $V_i \cap V_j = V_i \times_U V_j$ is then strictly k-analytic. If U is good then we could have shrunk U around u to be separated, so there is a G_u-stable k-affinoid neighborhood $V \subseteq U$ around u, and hence $N_u = \cap_{g \in G_u} gV$ is a G_u-stable k-affinoid neighborhood of u in U. It suffices to prove that N_u/G_u exists (and that it is strictly k-analytic, resp. k-affinoid, when N_u is). Thus, we can replace U and G with N_u and G_u. That is, we may assume that U is covered by G-stable k-affinoid domains $V_1, \dots, V_n$, and that V_i/G exists as a k-affinoid domain (even strictly k-analytic when V_i is so). Our aim is to construct U/G in this case, with U/G strictly k-analytic when U and all V_i are strictly k-analytic, and with U/G a (strictly) k-affinoid space when U is (strictly) k-affinoid. This latter (strictly) k-affinoid case was already settled in the preceding paragraph. In general, since U is Hausdorff, each overlap $V_{ij} = V_i \cap V_j$ is a G-stable compact k-analytic domain in U. Moreover, if U and all V_i are strictly k-analytic then so is each V_{ij}. Applying Example 5.1.2 to the finite étale quotient maps $\pi_i : V_i \to V_i/G$ and $\pi_j : V_j \to V_j/G$, $\pi_i(V_{ij})$ and $\pi_j(V_{ij})$ are each identified as a quotient V_{ij}/G and each is strictly k-analytic when U, V_i, and V_j are strictly k-analytic. Thus, $\pi_i(V_{ij})$ and $\pi_j(V_{ij}) = \pi_j(V_{ji})$ are uniquely isomorphic (say via an isomorphism $\phi_{ji} : \pi_i(V_{ij}) \simeq \pi_j(V_{ji})$) in a manner that respects the maps from V_{ij} onto each, and the analogous quotient conclusions hold for the triple overlaps among the V_i's.

It is easy to check (by chasing G-actions) that the triple overlap cocycle condition holds. Thus, we can define a k-analytic space $\overline{U}$ that is covered by the k-analytic domains V_i/G with overlaps V_{ij}/G (using the ϕ_{ji}'s), and this gluing $\overline{U}$ is strictly k-analytic when U and all V_i are strictly k-analytic. By computing on geometric points we see that the maps $V_i \to V_i/G_i \to \overline{U}$ uniquely glue to a morphism $\pi : U \to \overline{U}$ whose pullback over each k-analytic domain $V_i/G \subseteq \overline{U}$ is the k-analytic domain $V_i \subseteq U$. Hence, π is finite étale since each π_i is finite étale, and π is G-invariant since every π_i is G-invariant. The resulting natural map $U \times G \to U \times_{\overline{U}} U$ over $U \times U$ restricts to the natural map $V_{ji} \times G \to V_{ji} \times_{V_{ji}/G, \phi_{ji} \circ \pi_i} V_{ij}$ over $V_j \times V_i$ for all i and j, and these latter maps are isomorphisms. Hence, $\pi : U \to \overline{U}$ serves as a quotient U/G. By construction, if U is strictly k-analytic then so is $\overline{U}$. ■

By Lemma 5.2.2 and the gluing used in the proof of Lemma 5.2.1, it remains to show that if U is separated then every germ (U, u) is covered by finitely many G_u-stable good subdomain germs $(V, u) \subseteq (U, u)$ with $V \subseteq U$ a k-analytic domain. In particular, the finer claims concerning inheritance of goodness and strict k-analyticity of U/G are immediate corollaries of such an existence result in the general (separated) case. As above, we postpone the separatedness hypothesis on U until we need it, assuming at the outset just that U is Hausdorff. We fix $u \in U$, and by Lemma 5.2.1 we may rename G_u as G, so we can assume $G = G_u$. To construct the required collection of subdomains around u we will use the theory of reduction of germs developed in [T2]. For ease of notation, we now write U_u rather than (U, u), and we write $\widetilde{U}_u$ to denote the reduction of this germ (an object in the category $\mathrm{bir}_{\widetilde{k}}$ of birational spaces over the $\mathbf{R}^\times_{>0}$-graded field $\widetilde{k}$; see [T2, §1–§3] for terminology related to graded fields and birational spaces over them, such as the quasi-compact Zariski–Riemann space $\mathbf{P}_{L/\ell}$ of graded valuation rings associated to an extension $\ell \to L$ of graded fields). The reduction of germs is a useful technique for studying the local structure of a k-analytic space. For example, in [T2, 4.5] it is shown that $V_u \mapsto \widetilde{V}_u$ is a bijection from the set of subdomains of U_u to the set of quasi-compact open subspaces of the birational space $\widetilde{U}_u$, and by [T2, 4.1(*ii*)] this bijection is compatible with inclusions and finite unions and intersections. Moreover, by [T2, 4.8(*iii*)] (resp. [T2, 5.1]) the germ V_u is separated (resp. good) if and only if the reduction $\widetilde{V}_u$ is a separated birational space (resp. an affine birational space) over $\widetilde{k}$.

At this point we assume that U is separated. To find a finite collection of G-stable good subdomains $(V_j)_u$ in U_u that cover U_u, it is equivalent to cover the separated birational space $\widetilde{U}_u$ over $\widetilde{k}$ by G-stable affine open subspaces. Note that any intersection of finitely many affine subspaces of $\widetilde{U}_u$ is affine (because separatedness of $\widetilde{U}_u$ allows us to identify $\widetilde{U}_u$ with an open subspace of the birational space $\mathbf{P}_{\widetilde{K}/\widetilde{k}}$). Choose a point $z \in \widetilde{U}_u$ and let Gz denote its G-orbit. If Gz is contained in an affine open subspace $V \subseteq \widetilde{U}_u$ then the intersection of all G-translates of V is a G-stable affine neighborhood of z. Thus, by quasi-compactness of $\widetilde{U}_u$ our problem reduces to proving the following statement.

Theorem 5.2.3. *Let H be a commutative group and let k be an H-graded field and U a separated object of bir_k provided with an action of a finite group G. Any G-orbit $S \subseteq U$ admits an affine open neighborhood.*

In the intended application of this theorem we have $H = \mathbf{R}^\times_{>0}$, so the group law on H will be denoted multiplicatively below. By the separatedness hypothesis, we can identify U with an open subspace of a Zariski–Riemann space of H-graded valuations $\mathbf{P}_K = \mathbf{P}_{K/k}$, where K/k is an extension of H-graded fields. In the classical situation when K/k is finitely generated and gradings are trivial (i.e., $H = \{1\}$) one can use Chow's lemma to prove the generalization of Theorem 5.2.3 in which there is no G-action and $S \subseteq U$ is an arbitrary finite subset of $\mathbf{P}_K$: (i) find a model $\mathrm{Spec}(K) \to X$ in Var_k (see [T1, §1]), (ii) using Chow's lemma, replace X by a modification that is quasi-projective over k, and (iii) use that any finite set in a quasi-projective k-scheme is contained in an open affine subscheme. Surprisingly, one has to be very careful in the general graded case. For example, Theorem 5.2.3 without the G-action is false for an arbitrary finite S.

5.3. **Graded valuations on graded fields.** To prove Theorem 5.2.3, we first need to generalize some classical results to the graded case. Throughout this section, we assume all gradings are taken with respect to a fixed commutative group H, and "graded" always means "H-graded". We start with a certain portion of Galois theory that is very similar to the theory of tamely ramified extensions of valued fields. Let K/L be an extension of graded fields. Since K is a free L-module by [T2, 2.1], we can define the *extension degree* $n = [K : L]$ to be the L-rank of K; we say that K/L is a *finite* extension when the extension degree is finite. Two more invariants of the extension K/L are analogues of the residual degree and ramification index in the classical theory, defined as follows. Writing $\oplus_{h\in H} K_h$ and $\oplus_{h\in H} L_h$ for the decompositions of K and L into their graded components, the components K_1 and L_1 indexed by the identity of H are ordinary fields and we define $f = f_{K/L} = [K_1 : L_1]$. For any nonzero graded ring A we let $A^\times$ denote the homogeneous unit group of A (i.e., the ordinary units of A that are homogeneous with respect to the grading); note that $A_1 \cap A^\times$ is the unit group of the ring A_1. We let $\rho : A^\times \to H$ denote the multiplicative map that sends each homogeneous unit to its uniquely determined grading index, so $\rho(A^\times) \subseteq H$ is a subgroup. Define $e = e_{K/L} = \#(\rho(K^\times)/\rho(L^\times))$. The invariants n, e, and f of K/L may be infinite.

Lemma 5.3.1. *The equality $n = ef$ holds, where we use the conventions $\infty \cdot d = \infty \cdot \infty = \infty$ for any $d \geq 1$.*

Proof. Let B be a basis of K_1 over L_1 and $T \subseteq K^\times$ be any set of representatives for the fibers of the surjective homomorphism $K^\times \to \rho(K^\times) \to \rho(K^\times)/\rho(L^\times)$. Since K is a graded field, in the decomposition $K = \oplus_{r\in\rho(K^\times)} K_r$ the K_r's are all 1-dimensional as K_1-vector spaces. The same holds for L, so the products bt for $(b, t) \in B \times T$ are pairwise distinct and the set of these products is a basis of K over L. The lemma now follows. ■

For any extension K/L of graded fields, the set of intermediate graded fields F satisfying $e_{F/L} = 1$ is in natural bijection with the set of intermediate fields F_1 in K_1/L_1 via the recipe $F_1 \mapsto F_1 \otimes_{L_1} L$, so $F = K_1 \otimes_{L_1} L$ is such an intermediate graded field and it contains all others. It is clear that this particular F is the unique intermediate graded field in K/L satisfying $e_{F/L} = 1$ and $f_{K/F} = 1$. For a fixed K the graded subfields $L \subseteq K$ such that $f_{K/L} = 1$ are in natural bijection with subgroups of $\rho(K^\times)$ via the recipe $L = \oplus_{r\in\rho(L^\times)} K_r$, and in such cases K/L is finite if and only if $e_{K/L}$ is finite. We say that an extension K/L is *totally ramified* if it is a finite extension and $f_{K/L} = 1$.

Let K be a graded field and $G \subseteq \mathrm{Aut}(K)$ a finite subgroup. The graded subring $L = K^G$ is obviously a graded field, and $L_1 = K_1^G$. The *inertia subgroup* $I \subseteq G$ is the subgroup of elements that act trivially on K_1, so $G/I \subseteq \mathrm{Aut}(K_1)$. An element $\sigma \in I$ acts K_1-linearly on the 1-dimensional K_1-vector space K_r for each $r \in \rho(K^\times)$, so this action must be multiplication by an element $\xi_{\sigma,r} \in K_1^\times$ that is obviously a root of unity. In particular, I is always abelian, just like for tamely ramified extensions in classical valuation theory. By a lemma of Artin, K_1/L_1 is a finite Galois extension with $G/I \simeq \mathrm{Gal}(K_1/L_1)$. In particular, $[K_1 : L_1] | \#G$. Here is an analogue of the lemma of Artin for K/L.

Lemma 5.3.2. *Let K be a graded field and $G \subseteq \mathrm{Aut}(K)$ a finite subgroup. The degree of K over the graded subfield $L = K^G$ is finite and equal to $\#G$.*

Proof. Let I be the inertia subgroup of G. The graded subfield $F = K^I$ has $F_1 = K_1$ (i.e., $f_{K/F} = 1$), so F is uniquely determined by its value group $\rho(F^\times) \subseteq \rho(K^\times)$. Clearly $\rho(F^\times)$ is the set of r's such that I acts trivially on K_r, and this is a subgroup $V \subseteq \rho(K^\times)$. Since $f_{K/F} = 1$, we have $[K : F] = e_{K/F} = \#(\rho(K^\times)/V)$. For any $r \in \rho(K^\times)$, the ratio $\sigma(x)/x \in K_1^\times$ for nonzero $x \in K_r$ is independent of x and only depends on $r \bmod V$. Thus, we can denote this ratio $\xi_{\sigma,\overline{r}}$ for $\overline{r} = r \bmod V$. The pairing $\xi : I \times (\rho(K^\times)/V) \to K_1^\times$ defined by $(\sigma, \overline{r}) \mapsto \xi_{\sigma,\overline{r}}$ is biadditive and nondegenerate in each variable. In particular, $\rho(K^\times)/V$ is finite and ξ takes values in a finite subgroup of $K_1^\times$ that is necessarily cyclic with order that annihilates I and $\rho(K^\times)/V$. Thus, by duality for finite abelian groups we get $\rho(K^\times)/V = \#I$, so $[K : F]$ is finite and equal to $\#I$. That is, the result holds with I in the role of G.

The group $G' = G/I$ naturally acts on F. Since $F^{G'} = L$, it suffices to prove $[F : L] = \#G'$. Viewing F as a vector space over $F_1 = K_1$, its G'-action is semilinear. Artin's lemma in the classical case gives that F_1 is a finite Galois extension of L_1 with Galois group G'. Hence, Galois descent for vector spaces provides an identification $F = F_1 \otimes_{L_1} L$, so $[F : L] = [F_1 : L_1] = \#G'$ (and $e_{F/L} = 1$). ∎

We now study extensions of graded valuation rings. Let L be a graded field, and let R be a graded valuation ring of L, meaning that $R \subseteq L$ is a graded subring that is a graded valuation ring with L equal to its graded fraction field. Let K/L be an extension of graded fields. For any graded valuation ring A of K, A_1 is a valuation ring of K_1 and $A \cap L$ is a graded valuation ring of L whose inclusion into A is a graded-local map. If $A \cap L = R$ then we say that A *extends* R (with respect to K/L). Note that in such cases, the valuation ring A_1 of the field K_1 extends the valuation ring R_1 of the field L_1, and $R_1 \to A_1$ is a local map (i.e., $A_1^\times \cap L_1 = R_1^\times$).

Lemma 5.3.3. *Let K/L be a finite extension of graded fields and fix a graded valuation ring R of L. The correspondence $A \mapsto A_1$ is a bijection between the set of extensions of R to a graded valuation ring of K and the set of extensions of R_1 to a valuation ring of K_1. Moreover, if $R \subseteq R'$ is an inclusion of graded valuation rings of L, and A and A' are graded valuation rings of K that respectively extend R and R', then $A \subseteq A'$ if and only if $A_1 \subseteq A'_1$.*

Proof. The intermediate graded field $F = K_1 \otimes_{L_1} L$ satisfies $e_{F/L} = 1$ and $f_{K/F} = 1$, so it suffices to consider separately the cases $e = 1$ and $f = 1$. First assume $f = 1$, so we have to show that R admits a unique extension to K (and that inclusions among such extensions over an inclusion $R \subseteq R'$ in L can be detected in K_1). An extension of R to K exists without any hypotheses on the graded field extension K/L (by Zorn's Lemma), so the main issue is to prove uniqueness. Due to the grading, we just have to check that the homogeneous elements in such an $A \subseteq K$ are uniquely determined. Since $L_1 = K_1$, we have $L = \oplus_{r \in \rho(L^\times)} K_r$ with $\rho(L^\times)$ an index-e subgroup of $\rho(K^\times)$. Hence, any homogeneous element $a \in K$ is an eth root of a homogeneous element of L. If $a \in K^\times$ satisfies $a \notin A$ then $1/a$ lies in the unique graded-maximal ideal of A, so $(1/a)^e \in A \cap L = R$ lies in the unique graded-maximal ideal of R (since $R \to A$ is a graded-local map). Hence, $a^e \notin R$ in such cases, so we have the characterization that a homogeneous $a \in K$ lies in A if and only if $a^e \in R$. This gives the desired uniqueness, and also shows that if $R \subseteq R'$ is a containment of graded valuation rings of L then their unique respective extensions $A, A' \subseteq K$ satisfy $A \subseteq A'$; observe also that in this case $A_1 = R_1$ and $A'_1 = R'_1$.

It remains to analyze the case $e = 1$, so $K = K_1 \otimes_{L_1} L$. In particular, $K_r = K_1 \cdot L_r$ for all $r \in \rho(K^\times) = \rho(L^\times)$. We have to prove that for any extension $A_1 \subseteq K_1$ of $R_1 \subseteq L_1$, there exists a unique graded valuation ring $B \subseteq K$ that extends R and satsifies $B_1 = A_1$, and that if A_1' is an extension of R_1' where $R' := R_1' \cap L$ is a graded valuation ring of L containing R then the corresponding B' in K satisfies $B \subseteq B'$ if and only if $A_1 \subseteq A_1'$. As a preliminary step, we recall a fact from classical valuation theory: if k'/k is a degree-d extension of fields and V' is a valuation ring of k' then $(x')^{d!} \in k \cdot V'^{\times}$ for all $x' \in k'$. This follows from the fact that all ramification indices are finite and bounded above by the field degree. In our initial situation, we conclude that for $N = f!$ and any homogeneous element $x \in K_r = K_1 \cdot L_r$, we have

$$x^N \in (K_1 \cdot L_r)^N \subseteq K_1^N \cdot L_{r^N} \subseteq (L_1 \cdot A_1^{\times}) \cdot L_{r^N} = A_1^{\times} \cdot L_{r^N}.$$

Given such a factorization $x^N = a_1 \ell$ with $a_1 \in A_1^{\times}$ and (necessarily) homogeneous $\ell \in L$, the element ℓ is uniquely determined up to a factor lying in $L \cap A_1^{\times} = R_1^{\times}$. In particular, it is an intrinsic property of x whether or not $\ell \in R$, so if B exists then it is contained in the set A of $x \in K$ each of whose homogeneous components $x_h \in K_h$ lies in the subset

$$A_h := \{y \in K_h \,|\, y^N \in A_1^{\times} \cdot R_{h^N}\}.$$

We will show that this set A is a graded valuation ring of K, so its homogeneous units are those homogeneous $x \in K^{\times}$ such that $x^N \in A_1^{\times} \cdot R^{\times}$, and moreover $A \cap L = R$ and $A \cap K_1 = A_1$. Hence, this A works as such a B, and any valuation ring B of K that extends R and satisfies $B \cap K_1 = A_1$ must have graded-local inclusion into A, thereby forcing $B = A$. This would give the required uniqueness, and also implies the desired criterion for containment of extensions (over a containment $R \subseteq R'$ in L) by checking in K_1.

Since it is clear that $A_h \cdot A_{h'} \subseteq A_{hh'}$ for all $h, h' \in H$, to show that A is a graded subring of K we just need to prove that if x, y are two nonzero elements of $A_h = A \cap K_h$ for some $h \in H$ then $x + y \in A_h$. The ratios x/y and y/x in K both lie in K_1 and are inverse to each other, so at least one of them lies in the valuation ring A_1 of K_1. Thus, by switching the roles of x and y if necessary we may assume $x/y \in A_1$. We then have $(x+y)/y = 1 + x/y \in A_1$, so $((x+y)/y)^N \in A_1^{\times} \cdot R_1$ by the classical valuation-theoretic fact recalled above. Hence,

$$(x+y)^N = \left(\frac{x+y}{y}\right)^N \cdot y^N \in (A_1^{\times} \cdot R_1) \cdot (A_1^{\times} \cdot R_{h^N}) = A_1^{\times} \cdot R_{h^N},$$

so $x + y \in A$ as desired. Finally, to show that A is a graded valuation ring of K it suffices to prove that for any homogeneous nonzero $t \in K$ either t or $1/t$ lies in A. We have $t^N = a_1 \ell$ with $a_1 \in A_1^{\times}$ and a homogeneous nonzero $\ell \in L$. But L is the graded fraction field of the graded valuation ring R, so $\ell \in R$ or $1/\ell \in R$, and hence $t \in A$ or $1/t \in A$ respectively. ∎

Corollary 5.3.4. *Let K be a graded field, $G \subseteq \mathrm{Aut}(K)$ a finite subgroup, and R a graded valuation ring of $L = K^G$. Then*

(i) G acts transitively on the non-empty set of extensions of R to K;

(ii) if R' is a graded valuation ring of L containing R as a graded subring and A' is an extension of R' to K, then there exists an extension A of R to K with $A \subseteq A'$ as graded subrings of K.

Proof. Let $I \subseteq G$ be the inertia subgroup, so $G/I = \mathrm{Gal}(K_1/L_1)$. By classical valuation theory, G/I acts transitively on the set of extensions of R_1 to K_1, so (i) follows from Lemma 5.3.3. For (ii), classical valuation theory also gives that R_1 admits an extension A_1 to K_1 with $A_1 \subseteq A_1'$, so consider the corresponding A extending R to K. To see $A \subseteq A'$ we can use the containment criterion in Lemma 5.3.3. ∎

Corollary 5.3.5. *If K/k is an extension of graded fields, $G \subseteq \mathrm{Aut}_k(K)$ is a finite subgroup, and $L = K^G$, then*

(i) G acts transitively on the fibers of the natural surjective map $\psi : \mathbf{P}_{K/k} \to \mathbf{P}_{L/k}$;

(ii) if $x \in \mathbf{P}_{L/k}$ is a point, $S = \psi^{-1}(x)$ is the fiber, and $\overline{x}$ is the set of generalizations of x in $\mathbf{P}_{L/k}$ then $\psi^{-1}(\overline{x})$ is the set of generalizations of the points of S.

Proof. The first part is Corollary 5.3.4(i). To prove the second part we note that a point $y \in \mathbf{P}_{L/k}$ is a generalization of x (i.e. is contained in every open neighborhood of x) if and only if its associated graded valuation ring of L contains the one associated to x. Hence (ii) follows from Corollary 5.3.4(ii). ∎

Another classical notion whose graded analogue will be used in the proof of Theorem 5.2.3 is the constructible topology on a Zariski–Riemann space (also sometimes called the patching topology). Let K/k be an extension of graded fields, and let $\mathbf{P}_{K/k}$ be the associated Zariski–Riemann space. Consider subsets

$$\mathbf{P}_{K/k}\{F\}\overline{\{G\}} = \{\mathcal{O} \in \mathbf{P}_{K/k} \mid F \subseteq \mathcal{O}, G \cap \mathcal{O} = \emptyset\} \subseteq \mathbf{P}_{K/k}$$

for $F, G \subseteq K^\times$. Such subsets with finite F and empty G form a basis of the usual topology on $\mathbf{P}_{K/k}$. Since $\cap_i \mathbf{P}_{K/k}\{F_i\}\overline{\{G_i\}} = \mathbf{P}_{K/k}\{\cup F_i\}\overline{\{\cup G_i\}}$, the subsets $\mathbf{P}_{K/k}\{F\}\overline{\{G\}}$ with finite F and G satisfy the requirements to be a basis of open sets for a finer topology on $\mathbf{P}_{K/k}$ called the *constructible topology*.

Lemma 5.3.6. *Let $F, G \subseteq K^\times$ be arbitrary subsets. The subset $X = \mathbf{P}_{K/k}\{F\}\overline{\{G\}} \subseteq \mathbf{P}_{K/k}$ is compact and Hausdorff with respect to the constructible topology on $\mathbf{P}_{K/k}$.*

Proof. Let S_K be the set of all subsets of $K^\times$. Any graded valuation ring of K is uniquely determined by its intersection with $K^\times$, so there is a natural injection $i : \mathbf{P}_{K/k} \to S_K$ given by $i(\mathcal{O}) = \mathcal{O} \cap K^\times$. Consider the bijection $S_K \to \{0,1\}^{K^\times}$ that assigns to each $\Sigma \in S_K$ the characteristic function of Σ. The discrete topology on $\{0,1\}$ thereby endows S_K with a compact Hausdorff product topology, and a basis of open sets for this topology is given by the sets

$$S_K\{F'\}\overline{\{G'\}} = \{\Sigma \in S_K \mid F' \subseteq \Sigma, G' \cap \Sigma = \emptyset\}$$

for finite subsets $F', G' \subseteq K^\times$. The sets $S_K\{F'\}\overline{\{\emptyset\}}$ and $S_K\{\emptyset\}\overline{\{G'\}}$ will be denoted $S_K\{F'\}$ and $S_K\overline{\{G'\}}$ respectively. Clearly the subspace topology induced on $\mathbf{P}_{K/k}$ via i is the constructible topology, so it suffices to prove that $i(X)$ is closed in S_K. We will show that $S_K - i(X)$ is open in S_K by covering it by members of the above basis of open sets in S_K.

Choose a point $\Sigma \in S_K$. The condition that $\Sigma \notin i(X)$ means that the subset $\Sigma \subseteq K^\times$ cannot be expressed as $\mathcal{O} \cap K^\times$ for a graded valuation ring $\mathcal{O}$ of K containing k. The only possibility for such an $\mathcal{O}$ is the graded additive subgroup $\mathcal{O}_\Sigma \subseteq K$ generated by Σ. Thus, $\Sigma \notin i(\mathbf{P}_{K/k})$ if and only if either $\mathcal{O}_\Sigma \cap K^\times$ is strictly larger than Σ or $\mathcal{O}_\Sigma \cap K^\times = \Sigma$ but $\mathcal{O}$ fails to be a graded valuation ring of K containing k. If $\Sigma \in i(\mathbf{P}_{K/k})$ (i.e., $\mathcal{O}_\Sigma$ is a graded valuation ring of K containing k with $\mathcal{O}_\Sigma \cap K^\times = \Sigma$) then the finer condition $\Sigma \notin i(X)$ says exactly that $F \not\subseteq \mathcal{O}_\Sigma$ or $G \cap \mathcal{O}_\Sigma \neq \emptyset$. In other words, $\Sigma \notin i(X)$ if and only if Σ satisfies at least one of the following seven properties: (i) $a + b \notin \Sigma$ for some $a, b \in \Sigma$ lying in the same graded component of K with $a + b \in K^\times$ (i.e., with $a + b \neq 0$), (i′) $-a \notin \Sigma$ for some $a \in \Sigma$, (ii) $a \notin \Sigma$ for some $a \in k^\times$, (iii) $ab \notin \Sigma$ for some $a, b \in \Sigma$, (iv) $a \notin \Sigma$ and $1/a \notin \Sigma$ for some $a \in K^\times$, (v) $f \notin \Sigma$ for some $f \in F$, (v′) $g \in \Sigma$ for some $g \in G$. Hence, it suffices to show that if $\Sigma \subseteq K^\times$ is a subset satisfying one of these conditions then it has an open neighborhood in S_K satisfying the same condition. For each respective condition, use the following neighborhood (with notation as above): (i) $S_K\{a,b\}\overline{\{a+b\}}$, (i′) $S_K\{a\}\overline{\{-a\}}$, (ii) $S_K\overline{\{a\}}$, (iii) $S_K\{a,b\}\overline{\{ab\}}$, (iv) $S_K\overline{\{a, 1/a\}}$, (v) $S_K\overline{\{f\}}$, (v′) $S_K\{g\}$. ∎

Now we have all of the necessary tools to prove Theorem 5.2.3.

Proof. By definition, the data of the separated birational space U consists of the specification of a connected quasi-compact and quasi-separated topological space equipped with an open embedding into $\mathbf{P}_K = \mathbf{P}_{K/k}$. Thus, although (by abuse of notation) we shall write U to denote the open subspace, the action of G on U as a birational space really means an action α of G on both the open subspace and on the graded field K over k such that the induced action on $\mathbf{P}_K$ carries the open subspace back to itself via α. That is, the action by G on $\mathbf{P}_K$ arising from the G-action on K over k induces the original action of G on $U \subseteq \mathbf{P}_K$. Let $L = K^G$, so L/k is a graded field extension as well. By Corollary 5.3.5(i), S is a fiber of the induced map $\psi : \mathbf{P}_K \to \mathbf{P}_L$ over a point $x \in \mathbf{P}_L$. Our problem is therefore to find a G-stable affine open neighborhood of S in $\mathbf{P}_K$ that contains $\psi^{-1}(x)$ and is contained in U.

Let $\{U_i\}_{i\in I}$ be the family of all affine open neighborhoods of x, so $\{V_i = \psi^{-1}(U_i)\}_{i\in I}$ is a family of G-stable affine open neighborhoods of S. Clearly $\overline{x} := \cap_{i\in I} U_i$ is the set of all generalizations of x, and by Corollary 5.3.5(ii) we see that $\overline{S} := \cap_{i\in I} V_i$ is the set of generalizations of points of S. In particular, $\overline{S} \subseteq U$. Note that U is open and the V_i's are closed in the constructible topology on $\mathbf{P}_K$, so U is an open neighborhood of the intersection of closed subsets V_i in the compact Hausdorff space $\mathbf{P}_K$ (provided with the constructible topology). It follows that U contains the intersection of finitely many V_i's, and that intersection is the required G-stable affine open neighborhood of S. ■

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