ON LOCAL PROPERTIES OF NON-ARCHIMEDEAN ANALYTIC SPACES II

M. TEMKIN

Department of Mathematics, The Weizmann Institute of Science, P.O.B. 26, 76100 Rehovot, Israel
(e-mail: temkin@wisdom.weizmann.ac.il; fax: 972-8-9343216)

Abstract

The non-Archimedean analytic spaces are studied. We extend to the general case notions and results defined earlier only for strictly analytic spaces. In particular we prove that any strictly analytic space admits a unique rigid model.

Introduction

Rigid geometry was introduced by Tate at 1960's due to Grothendieck's idea of attaching generic fiber to certain formal schemes. Since then the theory became classic and found various applications in algebraic geometry and representation theory. Another approach was suggested later by Berkovich, see [Ber1] and [Ber2], where for a non-Archimedean field k a category k-An of k-analytic spaces was introduced. The first success of the new theory was in constructing etale cohomology theory for k-analytic spaces (see [Ber2]). An interesting feature of k-analytic spaces, which has no rigid counterpart, is that one is not restricted to convergence radii from $\sqrt{|k^*|}$, in particular the trivial valuation on k is allowed as well. Namely, analytic spaces are built from spectra $\mathcal{M}(\mathcal{A})$ of k-affinoid algebras, which in their turn are defined as quotients of the algebras $k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}$ of convergent power series on the polydisc of radii $(r_1,\dots,r_n)\in\mathbf{R}_+^n$. An affinoid algebra is called strictly affinoid if it may be represented in such a way, that all radii r_i are taken from $\sqrt{|k^*|}$. A strictly analytic space is an analytic space with a chosen strictly affinoid atlas and the corresponding category modulo natural equivalence of strictly affinoid atlases is denoted st-k-An. Strictly analytic spaces are exactly the analytic spaces admitting a rigid model, but it is not clear that this model is unique. So the question, whether the natural embedding functor st-k-$An \longrightarrow k$-An is fully faithful, is a basic question of the theory. It was one of the main motivations of this work, and the positive answer is obtained at Sect. 4. In order to prove this result we generalize to all k-analytic spaces the notions and constructions which were introduced in our previous work [T] for strictly k-analytic spaces. We define notions of reduction of a k-affinoid space and reduction of a germ of a k-analytic space at a point and establish analogs of results from [Ber1], 2.4, 2.5, and [T], in particular we obtain criterions, for a space to be good at a point and for a morphism to be closed at a point. In [T] it was shown, that in the case of strictly analytic spaces the latter criterion implies that formal properness is equivalent to properness due to Kiehl.

One of the basic tools in rigid geometry is the notion of reduction of algebras and spaces. The reduction $\widetilde{\mathcal{A}}_1$ of a strictly affinoid algebra $\mathcal{A}$ is defined as the quotient of the subalgebra $\mathcal{A}_1^\circ = \{f \in \mathcal{A} | \ |f|_{sup} \le 1\}$ by the ideal $\mathcal{A}_1^{\circ\circ} = \{f \in \mathcal{A} | \ |f|_{sup} < 1\}$. It reflects well the properties of the algebra, for example a homomorphism of strictly affinoid algebras is finite if and only if its reduction is finite. This notion is used in analytic geometry only for strictly analytic spaces because, for example, the reduction of a non-trivial affinoid algebra, which is not strictly affinoid, may coincide with $\widetilde{k}_1$. In the present paper we propose the following modification. Define reduction $\widetilde{\mathcal{A}}$ of an affinoid algebra $\mathcal{A}$ as the $\mathbf{R}_+^*$-graded algebra associated to the spectral seminorm filtration on $\mathcal{A}$ ($\widetilde{\mathcal{A}}_1$ is the trivially graded component of $\widetilde{\mathcal{A}}$). The new definition is closely related to the old one in the strictly affinoid case (cf. below) and it behaves well on the whole class of affinoid algebras. For example, the reduction $\widetilde{k}$ of the base field k may be a non-noetherian ring. However, it has no non-trivial homogeneous ideals and any non-zero homogeneous element is invertible. Thus as a graded ring it behaves like a field. The necessary definitions are given in Sect. 1, where we introduce graded analogs of the following notions: field, prime ideal, local ring, valuation ring, etc. (for example the graded spectrum of a graded ring is the set of its prime homogeneous ideals). In Sect. 3 we prove that a homomorphism of affinoid algebras is finite if and only if its reduction is finite. Then for an affinoid space $X = \mathcal{M}(\mathcal{A})$ the reduction map $\pi_X \colon X \longrightarrow \widetilde{X}$ to the set $\widetilde{X}$ of all prime homogeneous ideals of $\widetilde{\mathcal{A}}$ is constructed and analogs of statements from [Ber1], 2.4, 2.5, are proved. In particular, π_X is surjective and maps bijectively the Shilov boundary of X onto the set of generic points of $\widetilde{X}$. Note also, that in the strictly affinoid case $\widetilde{X}$ and π_X coincide with the usual reduction on the level of sets.

In Sect. 2, we define for a graded field k a category bir_k analogously to [T], Sect. 1. Its object is a local homeomorphism from a connected quasi-compact and quasi-separated topological space to a Riemann-Zariski space over k, while the latter space is defined in terms of graded valuation rings analogously to the usual Riemann-Zariski space. We define notions of affine objects of bir_k and separated and proper morphisms of bir_k. In Sect. 4, we construct a reduction functor $X_x \mapsto \widetilde{X}_x$ from the category of germs of k-analytic spaces at a point to the category $bir_{\widetilde{k}}$. We prove that the reduction functor induces a bijection between analytic subdomains of a k-germ X_x and open quasi-compact subsets of its reduction $\widetilde{X}_x$. Another useful result of the section states that a space X is strictly analytic at a point x if and only if its reduction $\widetilde{X}_x$ is $|k^*|$-strict, i.e. may be defined using homogeneous elements of $\widetilde{\mathcal{H}(x)}$ graded by $|k^*|$. The desired full faithfulness of the embedding functor st-k-$An \longrightarrow k$-An follows easily. The last section follows closely the sections 3 and 4 of [T]. We prove that a germ X_x is good if and only if its reduction $\widetilde{X}_x$ is affine, and that a morphism $f\colon X_x \longrightarrow Y_y$ of germs is closed if and only if its reduction $\widetilde{f}\colon \widetilde{X}_x \longrightarrow \widetilde{Y}_y$ is proper. As a corollary one obtains that properness is a G-local on the base property of morphisms of analytic spaces. It follows also, that in the strictly analytic case Kiehl's definition of proper morphism is equivalent to the definition from [Ber2], 1.5 (in [T] it was proved for a slightly weaker definition). The author thanks Prof. V. Berkovich for attention to work and encouragement. He thanks also Prof. B. Conrad for discussing his previous work [T] and pointing out some mistakes.

§1. G-graded rings

Let G be an abelian group, we write it multiplicatively and fix until the end of the next section. By a G**-graded ring** we mean a commutative ring A with 1 endowed with a G-graduation $A = \oplus_{g\in G} A_g$ by additive subgroups A_g with $A_g A_h \subset A_{gh}$. Note, that in a G-graded ring A one has $1 \in A_1$. The aim of this section is to extend to G-graded rings very basic algebraic notions (as usual it suffices to replace the word "element" by the words "homogeneous element"). Usually we shall omit the letter G and say graded ring, homogeneous ideal, etc.

For a non-zero homogeneous element $x \in A_g$ we use notation $\rho(x) = g$ and say that x is of order g, we set also $\rho(0) = 0$. A **homomorphism** of G-graded rings is a homomorphism $\phi\colon A \longrightarrow B$ such that $\phi(A_g) \subset B_g$ for any element g of G. A kernel of such a homomorphism is a G**-homogeneous ideal** (i.e. an ideal generated by homogeneous elements). If A is a G-graded ring and I is its G-homogeneous ideal, then the quotient ring A/I has a natural G-graduation. More generally, given a G-graded ring A, the category of G-graded A-modules is an abelian category with tensor products, and the forgetful functor to the category of A-modules is an exact functor commuting with tensor products. Let $B = A[T_1, \dots, T_n]$ be the polynomial algebra over a graded ring A and $g_1, \dots, g_n$ elements of G, then B has a unique G-graduation extending that of A such that $T_i \in B_{g_i}$. This graded algebra will be denoted $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n]$ or simply $A[g^{-1}T]$, in particular the notation $A[T_1, \dots, T_n]$ is used in the case, when all T_i are of order 1. Given a graded ring A and a multiplicative set $S \in A$ of homogeneous elements, the localization ring A_S has a natural G-graduation (it suffices to consider only the case of $S = \{1, f, f^2, ...\}$ and $f \in A_g$, but then $A_S = A[gT]/(Tf - 1)$ as graded rings). Given a homomorphism $\phi\colon A \longrightarrow B$ of graded rings, we say that ϕ is **finite** (resp. **integral**, resp. **finitely generated**) if it is finite (resp. integral, resp. finitely generated) as a homomorphism of usual rings. It is easily seen, that ϕ is finite if and only if there exists an epimorphism $A^n \longrightarrow B$ of graded A-modules, that ϕ is integral if and only if any homogeneous element of B satisfies an integral equation over A with homogeneous coefficients, and that ϕ is finitely generated if and only if there exists an epimorphism $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \longrightarrow B$.

We define **graded field** to be a graded ring A with no non-zero homogeneous ideals, it is equivalent to the condition that all non-zero homogeneous elements of A are invertible. For example, starting from a trivially graded field K, the graded ring $K[g^{-1}T, gS]/(TS-1)$ is a graded field for any $g \in G$ of infinite order. A graded ring A is said to be **integral** if the product of any two non-zero homogeneous elements of A does not vanish. We notice, that in an integral graded ring A any non-zero homogeneous element is not a zero divisor (however A may have non-homogeneous zero divisors). Any integral graded ring A may be embedded into graded field, the minimal such field is obtained by inverting all non-zero homogeneous elements. We call this graded field the **quotient field** of A and denote it $Q(A)$. We say that a homogeneous ideal I is **prime**, if the graded ring A/I is integral. Note that I is prime if and only if the set of all homogeneous elements of $A \setminus I$ is closed under multiplication.

Let k be a G-graded field, we fix it until the end of the section.

1.1. Lemma. *Let A be a graded k-algebra, such that $\rho(A) = \rho(k)$, then:*
(i) $A \widetilde{\to} A_1 \otimes_{k_1} k$,
(ii) $I \mapsto I \cap A_1$ defines a one-to-one correspondence between homogeneous ideals (resp. prime homogeneous ideals) of A and ideals (resp. prime ideals) of A_1. □

1.2. Lemma. *Any graded k-module M is free.*

Proof. By $k(g)$ we denote the module k with graduation shifted by g (i.e. $\rho_{k(g)} = g\rho_k$). Let $\{f_i\}_{i \in I}$ be a maximal set of non-zero homogeneous elements of M such that the corresponding homomorphism $F = \oplus_{i \in I} k(\rho(f_i)) \longrightarrow M$ is injective. If $f \in M/F$ is a non-zero homogeneous element, then we obtain a homomorphism ϕ: $k(\rho(f)) \longrightarrow M/F$. Since the modules $k(g)$ have no non-trivial graded submodules, ϕ is injective. But then the set $\{f_i\}_{i \in I}$ may be enlarged by any preimage of f. The contradiction shows that $M/F = 0$ and we are done. □

1.3. Corollary. *Let C be a graded k-algebra, then a homomorphism f: $A \longrightarrow B$ of graded k-algebras is finite (resp. finitely generated) if and only if the homomorphism $f_C = f \otimes_k C$ is.*

Proof. Choose elements $\sum_{j=1}^{n_i} b_{ij} \otimes c_{ij} \in B \otimes_k C$ generating $B \otimes_k C$ over $A \otimes_k C$ as a module (resp. as an algebra) and such that the elements b_{ij} are homogeneous, set $r_{ij} = \rho(b_{ij})$. The elements b_{ij} define a homomorphism f': $\oplus_{i,j} A(r_{ij}) \longrightarrow B$ (resp. f': $A[r_{ij}^{-1}T_{ij}] \longrightarrow B$) of graded k-modules which is an epimorphism because $f' \otimes C$ is an epimorphism and C is a free k-module. □

A graded ring A is a **local graded ring** if it has a unique maximal homogeneous ideal m, equivalently, all non-invertible homogeneous elements of A

are contained in a proper homogeneous ideal. For any prime homogeneous ideal $p \subset A$ let A_p denote the localization of A by homogeneous elements of $A \setminus p$. We call A_p the **localization** of A at p, it is obviously a local graded ring. Let K be a graded field, we say that a graded subring $A \subset K$ (we require, that the embedding is a graded homomorphism) is a **graded valuation ring**, if it is a local graded ring whose graded quotient field coincides with K and for any homogeneous element $f \in K$, at least one of the two elements f and f^{-1} belongs to A. Any local graded subring $B \subset K$ is dominated by a graded valuation subring of K (by usual argument using the Zorn's lemma).

1.4. Lemma. *Let K be a graded field over k and $A \subset B$ its graded k-subalgebras, then:*

(i) B is integral over A if and only if there exists no graded valuation subring of K containing A and not containing B;

(ii) if A and B are finitely generated k-algebras, then B is integral over A if and only if it is finite over A.

Proof. The proof of (i) is absolutely analogous to the classical case. Let $x \in B$ be a homogeneous element integral over A and $\mathcal{O} \supset A$ a graded valuation subring of K. If x does not belong to $\mathcal{O}$, then its inverse belongs to the maximal homogeneous ideal m of $\mathcal{O}$. Since x satisfies an equality $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$ with homogeneous coefficients $a_i \in A$, we obtain $1 = -1/x \sum_{i=0}^{n-1} a_i (1/x)^{n-i-1} \in m$. The contradiction shows, that actually $x \in \mathcal{O}$. Conversely, suppose $x \in B$ is not integral over A. By [Bou], ch. VI, Sect. 1, Lemma 1, $1/x$ is not invertible in $C = A[1/x]$, hence there exists a prime homogeneous ideal $p \subset C$ containing $1/x$. Let $\mathcal{O}$ be any graded valuation ring dominating the local graded ring C_p, then $1/x \in m_{\mathcal{O}}$ and hence $x \notin \mathcal{O}$ and $B \not\subset \mathcal{O}$.

(ii) Only direct implication is not obvious, so suppose B is integral over A. Let k' be a graded field over k, such that $\rho(B) \subset \rho(k')$, and set $A' = A \otimes_k k'$ and $B' = B \otimes_k k'$. Then A' and B' are finitely generated k' algebras and by Lemmas 1.1 (i) and 1.3, A'_1 and B'_1 are finitely generated k'_1 algebras. Obviously, B' is integral over A', and it follows, that B'_1 is integral over A'_1. Since B'_1 and A'_1 are usual algebras finitely generated over the field k_1, we obtain that B'_1 is finite over A'_1. Therefore B' is finite over A' and it remains to use Lemma 1.3. □

Finally, we define the spectrum $\mathrm{Spec}_G(A)$ of a graded ring A as the set X of prime homogeneous ideals of A provided with topology whose basis consists of sets of the form $D(f) = \{p \in X | \ f \notin p\}$, where f is a homogeneous element of A. As in usual scheme theory one proves, that $\mathrm{Spec}_G(A)$ is quasi-compact.

1.5. Lemma. *If A is a graded ring finitely generated over k, then the set of generic points of $\mathrm{Spec}_G(A)$ is finite.*

Proof. Again, let k' be a graded field over k, such that $\rho(A) \subset \rho(k')$, and set $A' = A \otimes_k k'$. Lemma 1.1 implies, that the statement of the Lemma holds for A', but it is easily seen, that the set of generic points of $\mathrm{Spec}_G(A')$ is mapped surjectively to that of $\mathrm{Spec}_G(A)$. $\square$

1.6. Remark. The spectrum of a graded ring can be provided with a (structure) sheaf of graded rings. Then one can define a notion of a graded scheme as a locally graded ringed space locally isomorphic to affine graded scheme. The theory of such objects seems to be parallel to the usual scheme theory. In the case of finitely generated group G, these objects may be identified also with usual schemes provided with an action of a commutative algebraic group of multiplicative type. For example, the category of $\mathbf{Z}$-graded affine schemes is equivalent to the category of affine schemes provided with a G_m-action, and on the level of topological spaces $\mathrm{Spec}_{\mathbf{Z}}(A)$ coincides with the set of scheme orbits of the corresponding action of G_m on $\mathrm{Spec}(A)$.

§2. The category bir_k

Throughout this section k is a fixed G-graded field. Given a G-graded field K over k, let $\mathbf{P}_K = \mathbf{P}_{K/k}$ denote the set of all graded valuation subrings $\mathcal{O}$ of K such that $K = Q(\mathcal{O})$ and $k \subset \mathcal{O}$. One endows $\mathbf{P}_K$ with the weakest topology with respect to which the set $\{\mathcal{O} \in \mathbf{P}_K \big| f \in \mathcal{O}\}$ is open for any homogeneous element f of K. Given subsets $X \subset \mathbf{P}_K$ and $A \subset K$ we set $X\{A\} = \{\mathcal{O} \in X \big| f \in \mathcal{O}$ for all homogeneous $f \in A\}$.

Lemma 2.1. *Let A be a subset of K, then the set $X = \mathbf{P}_K\{A\}$ is quasi-compact.*

Proof. (Compare [Mat], 10.5.) We can replace A by the graded subring of K generated by 1 and the homogeneous elements of A. Given a subsets $X \subset \mathbf{P}_K$ and $F \subset K$, set $X\{\{F\}\} = \{\mathcal{O} \in X \big| f \in m_{\mathcal{O}}$ for all homogeneous $f \in F\}$. The topology on X is given by the condition that any subset $X\{f\}$ is open, or that is equivalent, any subset $X\{\{f^{-1}\}\}$ is closed. Therefore it suffices to check, that if $X\{\{F\}\} = \emptyset$ for some set $F \subset K$ then $X\{\{F_0\}\} = \emptyset$ already for some finite subset F_0 of F. Let B be the graded subring of K generated by A and the homogeneous elements of F and $m \subset B$ the graded ideal generated by the homogeneous elements of F, then $X\{\{F\}\} = \emptyset$ if and only if $m = B$ (otherwise we can enlarge it to a prime homogeneous ideal p of B and find a graded valuation ring $\mathcal{O} \in \mathbf{P}_K$ which dominates p, then obviously $\mathcal{O} \in X\{\{F\}\}$). But obviously one uses only finitely many elements of F to represent 1 as an element of m, so F may be replaced by its finite subset. $\square$

The particular case of the Lemma concerning trivial graduations was used in [T], §1. However the proof given there has a gap pointed out by Brian Conrad, the reason is that projective limit of topological spaces does not preserve quasi-compactness in general. So, this proof (or referencing to

[Mat]) should be used instead. Subsets of $\mathbf{P}_K$ of the form $\mathbf{P}_K\{f_1,\dots,f_n\}$ are said to be *affine*. By Lemma 1.4 the following statement holds.

2.2. Lemma. *Given graded algebras $A \subset B \subset K$ finitely generated over k, one has $\mathbf{P}_K\{A\} = \mathbf{P}_K\{B\}$ if and only if B is finite over A.* $\square$

Let now bir_k be the category whose objects are triples $\overline{X} = (X, K, \phi)$, where X is a connected quasi-compact and quasi-separated topological space, K is a graded field over k, and ϕ is a local homeomorphism $X \longrightarrow \mathbf{P}_K$. A morphism $\overline{X} = (X, K, \phi) \longrightarrow \overline{Y} = (Y, L, \psi)$ is a pair (h, i), where h is a continuous map $X \longrightarrow Y$ and $i\colon L \hookrightarrow K$ is an embedding of graded fields, such that $\psi \circ h = i^{\#} \circ \phi$, where $i^{\#}\colon \mathbf{P}_K \longrightarrow \mathbf{P}_L$ is the induced map. A morphism $(h, i)\colon \overline{X} \longrightarrow \overline{Y}$ is said to be **separated** (resp. **proper**) if the map $X \longrightarrow Y \times_{\mathbf{P}_L} \mathbf{P}_K$ is injective (resp. bijective). In this case the above map is an open embedding (resp. a homeomorphism). An object $\overline{X} = (X, K, \phi)$ of bir_k is said to be **affine** if ϕ induces a homeomorphism of X with an affine subset of $\mathbf{P}_K$. If $\overline{X} = (X, K, \phi) \in Ob(bir_k)$, then for any open quasi-compact subset $X' \subset X$ the triple $(X', K, \phi\big|_{X'})$ is an object of bir_k. If the latter object is affine, X' is said to be an **affine** subset of X.

2.3. Remark. Exactly as in [T], 1.4., one can define also a category Var_k of irreducible graded schemes with a fixed morphism from spectrum of a graded field to its generic point. It may be shown then, that bir_k is equivalent to Var_k localized by some system of proper morphisms.

Let K be a graded field over k and X an open subset of $\mathbf{P}_K$. A **Laurent covering** of X is a covering of the form $\{X\{f_1^{\varepsilon_1},\dots,f_n^{\varepsilon_n}\}\}_{(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n}$, where $f_1,\dots,f_n$ are non-zero homogeneous elements of K. The proof of the following lemma is a word by word repetition of the proof of [T], 1.5.

2.4. Lemma. *Any finite covering of X by opens of the form $X\{f_1,\dots,f_n\}$ has a Laurent refinement.* $\square$

Let H be a subgroup of G such that $H \supset \rho(k^*)$. For a graded field K over k set $K_H = \oplus_{h\in H} K_h$ (it is a graded field over k too) and let $\psi_K\colon \mathbf{P}_K \longrightarrow \mathbf{P}_{K_H}$ be the natural map. A separated object $X \hookrightarrow \mathbf{P}_K$ of bir_k is called **H-strict** if it is the preimage under ψ_K of an open quasi-compact subset X_H of $\mathbf{P}_{K_H}$. It is equivalent to the condition that X has a covering $\{\mathbf{P}_K\{f_{i1},\dots,f_{in_i}\}\}_{1\le i\le m}$, where f_{ij} are homogeneous elements of K_H. Let H' be a subgroup of G containing H such that H'/H is a torsion group (we denote the maximal such subgroup as $\sqrt{H}$). Since $\mathbf{P}_K\{f\} = \mathbf{P}_K\{f^n\}$, the notions of H and H'-strictness coincide.

2.5. Proposition.
(i) An affine set $\mathbf{P}_K\{f_1,\dots,f_n\}$ is H-strict if and only if $\rho(f_i) \in \sqrt{H} \cup \{0\}$ for any $1 \le i \le n$.
(ii) The map ψ_K is open.
(iii) The fibers of ψ_K are connected.

Proof. Let $A \supset k$ be a graded subalgebra of K, prove that $\psi_K(\mathbf{P}_K\{A\}) = \mathbf{P}_{K_H}\{A_H\}$, where $A_H = \oplus_{h\in H} A_h$. The direct inclusion is obvious. Conversely, let $\mathcal{O} \supset A_H$ be a graded valuation subring of K_H. Since the graded subring $B = A\mathcal{O}$ of K generated by A and $\mathcal{O}$ may be expressed as $\oplus_{g\in G/H} A_{gH}\mathcal{O}$, we obtain that $B \cap K_H = \mathcal{O}$ and $m_{\mathcal{O}} B \cap \mathcal{O} = m_{\mathcal{O}}$. Therefore there exists a prime homogeneous ideal $p \subset B$ such that $p \cap \mathcal{O} = m_{\mathcal{O}}$, and then a graded valuation ring $\mathcal{O}' \in \mathbf{P}_K$ dominating B_p satisfies $\mathcal{O}' \cap K_H = \mathcal{O}$. In particular, $\mathcal{O}' \subset \mathbf{P}_K\{A\}$ and $\psi_K(\mathcal{O}') = \mathcal{O}$.

The above observation immediately implies (ii). Moreover, it follows that given a graded k-subalgebra $A \subset K$ the set $\mathbf{P}_K\{A\}$ is H-strict if and only if it coincides with $\mathbf{P}_K\{A_H\}$. But by Lemma 2.2, it happens if and only if A is finite over A_H and (i) follows.

It remains to establish (iii). Let $\mathcal{O} \in \mathbf{P}_{K_H}$ be a graded valuation ring, set $X = \psi_K^{-1}(\mathcal{O})$. Suppose oppositely that X is disconnected, say $X = U \sqcup V$, where U and V are open and non-empty. Let Y (resp. X_Y) denote the set of all graded valuation subrings of K_H (resp. K) that contain $\mathcal{O}$. The spaces Y and X_Y are quasi-compact by Lemma 2.1. Furthermore, $\mathcal{O}$ is the closed point of Y and $X_Y = \psi_K^{-1}(Y)$. Hence X is a closed subset of X_Y and in particular X is quasi-compact. Then U and V are quasi-compact too and therefore admit finite coverings by sets of the form $X\{f_1, \dots, f_n\}$. Since U and V are determined by finite number of elements of K, decreasing K we can assume, that $N = \rho(K^*)/H$ is finitely generated. We can also replace H by $\sqrt{H}$, because $\mathbf{P}_{K_H} \widetilde{\to} \mathbf{P}_{K_{\sqrt{H}}}$, and then N is a lattice of a finite rank n. We shall prove by induction on n, that our assumption on X can not hold.

In the case of $n = 1$ we have $K = K_H[g^{-1}T, gT^{-1}]$, where $g \notin \sqrt{H}$. Note that $\mathcal{O}$ induces a preorder on K_H, namely $a \geq b$ if $a/b \in \mathcal{O}$ (then $a > b$ if $a/b \in m_{\mathcal{O}}$ and $a \sim b$ if $a/b \in \mathcal{O}^*$). Since $X\{cT^l\} = X\{c^m T^{lm}\}$, for any elements $a, b \in K_H$ and integral numbers i, j of the same sign we have $X\{aT^i, bT^j\} = X\{\min(b^i, a^j)T^{ij}\}$. Thus, open sets of the form $X\{aT^i, bT^{-j}\}$ form a basis of topology of X. Since X is quasi-compact, U and V admit finite coverings by sets $W_1, \dots, W_l$ and $W_{l+1}, \dots, W_m$, respectively, such that $W_i = X\{a_i T^k, b_i T^{-k}\}$ for some natural k. Since U and V are contained in $X\{\max(a_i)T^k\}$, at least one of the elements a_i is zero, say $a_1 = 0$. If $W_i \cap W_1 \neq \emptyset$, then $a_i b_1 \in \mathcal{O}$ and it follows, that $W_1' = W_1 \cup W_i = X\{\max(b_1, b_i)T^{-k}\}$. Remove W_i, replace W_1 by W_1', decrease m by one and continue this process until W_1 is disjoint with all other W_i's, by our assumption it happens for some $m > 1$. Then W_1 is closed and does not coincide with X, in particular $b_1 \neq 0$. However a set $W = X\{bT^{-k}\}$ with non-zero b is not closed by the following reason. Consider the graded valuation subrings $\mathcal{O}_1' = \{\mathcal{O}[xT^k, yT^{-k}] |\ y > b, x \geq b^{-1}\}$ and $\mathcal{O}_2' = \{\mathcal{O}[xT^k, yT^{-k}] |\ y \geq b, x \geq b^{-1}\}$ of $K_H[g^{-k}T^k, g^k T^{-k}]$, and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be their continuations to K. Then $\mathcal{O}_1 \notin W$ and $\mathcal{O}_2 \in W$, but $\mathcal{O}_1$ belongs to the closure of $\mathcal{O}_2$ in $\mathbf{P}_K$, because $\mathcal{O}_2 \supset \mathcal{O}_1$. The contradiction proves, that X is connected for $n = 1$.

Consider now the case of an arbitrary n, assuming that the case of smaller n is already established. Choose a subgroup L of $\rho(K^*)$ such that $L \supset H$ and $\rho(K^*)/L \widetilde{\rightarrow} \mathbf{Z}$, and let K_L be the corresponding graded subfield of K. By the previous case, the fibers of the map $\phi: \mathbf{P}_K \longrightarrow \mathbf{P}_{K_L}$ are connected, hence $U = \phi^{-1}(U_L)$ and $V = \phi^{-1}(V_L)$ for some disjoint sets $U_L, V_L \subset \mathbf{P}_{K_L}$ and the union $X_L = U_L \cup V_L$ coincides with the preimage of $\mathcal{O}$ in $\mathbf{P}_{K_L}$. Let U_0 be an open subset of $\mathbf{P}_K$ whose intersection with X is U. Its image in $\mathbf{P}_{K_L}$ is open by (ii) and obviously $\phi(U_0) \cap X_L = U_L$. We obtain, that U_L is open in X_L, and the same reasoning applies to V_L. Thus X_L is disconnected, but it contradicts the induction assumption. $\square$

Given an object $\overline{X} = (X, K, \phi)$ of bir_k, by an H-strict structure on $\overline{X}$ we mean a morphism $\psi_X: \overline{X} \longrightarrow \overline{X}_H = (X_H, K_H, \phi_H)$ such that $X \widetilde{\rightarrow} X_H \times_{\mathbf{P}_{K_H}} \mathbf{P}_K$. An object which admits such a structure is called H-**strict**.

2.6. Proposition. *If $\psi_X: \overline{X} \longrightarrow \overline{X}_H$ and $\psi_Y: \overline{Y} \longrightarrow \overline{Y}_H$ are H-strict structures on objects $\overline{X}$ and $\overline{Y}$ of bir_k, then any morphism $f: \overline{X} \longrightarrow \overline{Y}$ is induced from a morphism $f_H: \overline{X}_H \longrightarrow \overline{Y}_H$. In particular all H-strict structures on an H-strict object $\overline{X}$ of bir_k are isomorphic.*

Proof. Our aim is to find a continuous map $f_H: X_H \longrightarrow Y_H$ making the following diagram commutative

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_H \arrow[r] \arrow[d, dashed] & \mathbf{P}_{K_H} \arrow[d] \\ Y \arrow[r] & Y_H \arrow[r] & \mathbf{P}_{L_H} \end{tikzcd}$$

Let $Z \subset X$ be a fiber of ψ_X, by Proposition 2.5 (iii), Z is connected. The image of Z in $\mathbf{P}_{L_H}$ consists of one point, hence its image in Y_H is contained in a discrete set. Since Z is connected, its image in Y_H is actually a single point. It follows, that there exists a map $f_H: X_H \longrightarrow Y_H$ making the diagram commutative. Finally, since by Proposition 2.5 (ii), ψ_X is a surjective open map, the map f_H is continuous. $\square$

2.7. Corollary. *Let $f: \overline{X} \longrightarrow \overline{Y}$ be a morphism of H-strict objects and let $\overline{U}, \overline{V}$ be H-strict subobjects of $\overline{Y}$, then:*

(i) the union $\overline{U} \cup \overline{V}$ and the intersection $\overline{U} \cap \overline{V}$ are H-strict;

(ii) the preimage $f^{-1}(\overline{U})$ is H-strict. $\square$

It may be proved using 2.6 and 1.1, that the category of $\rho(k^*)$-strict objects of bir_k is equivalent to bir_{k_1}. On the other side, "absolutely" non-strict objects admit a combinatorial interpretation very close to toric geometry. Consider for example the graded field $K = k[g_1^{-1}T, g_1T^{-1}, \ldots, g_n^{-1}T, g_nT^{-1}]$, where the elements $g_1, \ldots, g_n$ are linearly independent over $\rho(k^*)$, in particular $\Lambda = \rho(K^*)/\rho(k^*) \widetilde{\rightarrow} \mathbf{Z}^n$. The space $\mathbf{P}_K$ has no non-trivial $\rho(k^*)$-strict subspaces, its points may be interpreted as saturated semigroups $M \subset \Lambda$ such that the intersection of M with any non-trivial subgroup of Λ is not trivial. Inclusion relation induces a partial order on the set of such semigroups. The maximal semigroup $M = \Lambda$ corresponds to the trivial valuation on K (height is zero). A semigroup M dominated only by Λ corresponds to a valuation of height one, it is defined uniquely by an element $x \in \mathbf{R}^n = \Lambda \otimes \mathbf{R}$ such that $(x,x)=1$ and $M = \{y \in \Lambda |\ (x,y) \geq 0\}$ with respect to the standard scalar product. Thus, the subset S of $\mathbf{P}_K$ consisting of the elements of height 1 is homeomorphic to an $n-1$-dimensional sphere $S^{n-1} \subset \mathbf{R}^n$. It is easily seen also, that if X is a good subspace of $\mathbf{P}_K$, then $X \cap S$ corresponds to a subset of S^{n-1} cutoff by a convex polyhedral cone defined over $\mathbf{Q}$.

§3. Reduction of affinoid algebras and spaces

Starting from this section, we denote by k a non-Archimedean field (may be with trivial valuation). Our aim in this section is to generalize the notion of reduction to arbitrary affinoid algebras and spaces. We establish analogs of [BGR], 6.3.5/1, [Ber1], 2.4 and 2.5, and prove the Gerritzen-Grauert theorem for arbitrary affinoid spaces.

Let $\mathcal{D}$ be a Banach k-algebra, following [Ber1], by $\rho_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathbf{R}_+$ we denote the spectral seminorm of $\mathcal{D}$. Given a positive real number r, set $\mathcal{D}_r^\circ = \{x \in \mathcal{D} |\ \rho_{\mathcal{D}}(x) \leq r\}$, $\mathcal{D}_r^{\circ\circ} = \{x \in \mathcal{D} |\ \rho_{\mathcal{D}}(x) < r\}$ and $\widetilde{\mathcal{D}}_r = \mathcal{D}_r^\circ / \mathcal{D}_r^{\circ\circ}$. Let $\widetilde{\mathcal{D}} = \oplus_{r \in \rho_{\mathcal{D}}(\mathcal{D}) \setminus \{0\}} \widetilde{\mathcal{D}}_r$ be the $\mathbf{R}_+^*$-graded ring associated to the filtration on $\mathcal{D}$ induced by the spectral seminorm $\rho_{\mathcal{D}}$. We call $\widetilde{\mathcal{D}}$ the reduction ring of $\mathcal{D}$ (usually one uses these notion and notation for $\widetilde{\mathcal{D}}_1$). Any bounded homomorphism $\phi : \mathcal{D} \longrightarrow \mathcal{D}'$ of Banach k-algebras induces a homomorphism $\widetilde{\phi} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{\mathcal{D}}'$ of $\mathbf{R}_+^*$-graded $\widetilde{k}$-algebras. Note also, that if $\mathcal{D}$ is a field, then the ring $\widetilde{\mathcal{D}}$ is an $\mathbf{R}_+^*$-graded field (as a ring however it may be even non-noetherian).

In the sequel, we consider only $\mathbf{R}_+^*$-graduation and omit as a rule the group $\mathbf{R}_+^*$ in all notations. Given a set $r_1, \ldots, r_n$ of positive numbers, consider the quotient field of $k\{r^{-1}T\}$ provided with the valuation extending the spectral norm of $k\{r^{-1}T\}$, and let K_r be its completion (K_r is a non-Archimedean field and its definition generalizes the one of [Ber1], 2.1).

3.1. Proposition. *The reduction functor satisfies the following properties:*

(i) for a k-affinoid algebra $\mathcal{A}$ and a set $r_1, \ldots, r_n$ of positive numbers the natural isomorphisms $\widetilde{\mathcal{A}}[r^{-1}T] \widetilde{\to} \widetilde{\mathcal{A}\{r^{-1}T\}}$ and $\widetilde{\mathcal{A}} \otimes_{\widetilde{k}} \widetilde{K_r} \widetilde{\to} \widetilde{\mathcal{A}\widehat{\otimes}_k K_r}$ hold;

(ii) for a k-affinoid algebra $\mathcal{A}$ and an element $f \in \mathcal{A}$ with $r = \rho(f) > 0$ one has $\widetilde{\mathcal{A}}_{\widetilde{f}} \widetilde{\to} \widetilde{\mathcal{A}\{rf^{-1}\}}$;

(iii) a bounded homomorphism $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ of k-affinoid algebras is finite and admissible if and only if the homomorphism $\widetilde{\phi} : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{B}}$ is finite;

(iv) given a k-affinoid algebra $\mathcal{A}$ and $\mathcal{A}$-affinoid algebras $\mathcal{B}$ and $\mathcal{C}$, the natural homomorphism $\widetilde{\mathcal{B}} \otimes_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}}$ is finite;

(v) for a k-affinoid algebra $\mathcal{A}$ and a non-Archimedean field K over k the natural homomorphism $\widetilde{\phi} : \widetilde{\mathcal{A}} \otimes \widetilde{K} \longrightarrow \widetilde{\mathcal{A}\widehat{\otimes}K}$ is finite.

Recall, that a homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of Banach algebras is called admissible, if the residue norm on $\mathrm{Im}(\mathcal{A})$ is equivalent to the restriction of the norm of $\mathcal{B}$ ([Ber1], 1.1). Note, that if the valuation on k is non-trivial, then Banach open map theorem implies easily, that any finite homomorphism is admissible. Conversely, if the valuation is trivial, then there exist finite (and even bijective) non-admissible homomorphisms, for example $\phi : k\{r^{-1}T\} \longrightarrow k\{s^{-1}T\}$, where $0 < s < r < 1$. Such a homomorphism may become not finite after taking reduction or completed tensor product.

Proof. Note, that for an element $f = \sum_i f_i T^i \in \mathcal{A}\{r^{-1}T\}$ one has $\rho(f) = \max_i \rho(f_i) r^i$, where r^i denotes the product $r_1^{i_1} \dots r_n^{i_n}$. Therefore for any positive number s we obtain

$$(\widetilde{\mathcal{A}\{r^{-1}T\}})_s \widetilde{\longrightarrow} \oplus_{i \in \mathbf{N}^n} \widetilde{\mathcal{A}}_{s \cdot r^{-i}}$$

It establishes the first isomorphism of (i). The second isomorphism follows from the first one and (ii). Our proof of (ii) follows closely [BGR], 7.2.6, and consists of two steps.

Step 1. Let $\mathcal{A}$ be a k-affinoid algebra, $f \in \mathcal{A}$ an element with $r = \rho(f) > 0$ and $\mathcal{B} = \mathcal{A}\{rf^{-1}\}$. Given an element $a \in \mathcal{A}$, one has $\rho_{\mathcal{B}}(a) = \lim_n r^{-n} \rho_{\mathcal{A}}(f^n a)$. If $\rho_{\mathcal{B}}(a) > 0$, then for $n \geq n_0$ one has $\rho_{\mathcal{B}}(a) = r^{-n} \rho_{\mathcal{A}}(f^n a)$ and $\rho_{\mathcal{B}}(f^n a) = \rho_{\mathcal{A}}(f^n a)$.

Since $\rho_{\mathcal{B}}(f) = r$ and $\rho_{\mathcal{B}}(f^{-1}) = r^{-1}$, one has $\rho_{\mathcal{B}}(f^i b) = r^i \rho_{\mathcal{B}}(b)$ for any $i \in \mathbf{Z}$ and $b \in \mathcal{B}$. Thus the last two statements are equivalent. Pick up a field K_r for which the algebras $\mathcal{A}' = \mathcal{A} \widehat{\otimes} K_r$ and $\mathcal{B}' = \mathcal{B} \widehat{\otimes} K_r$ are strictly k-affinoid. Since the embeddings $\mathcal{A} \longrightarrow \mathcal{A}'$ and $\mathcal{B} \longrightarrow \mathcal{B}'$ preserve the spectral seminorm and $\mathcal{A}'\{rf^{-1}\} \widetilde{\longrightarrow} \mathcal{B}'$, we are reduced to the case of strictly affinoid algebras which is proved in [BGR], 7.2.6/2.

Step 2. The end of the proof .

Set $\mathcal{B} = \mathcal{A}\{rf^{-1}\}$ and consider the natural homomorphisms $\widetilde{\phi} : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}_{\widetilde{f}}$ and $\widetilde{\psi} : \widetilde{\mathcal{A}}_{\widetilde{f}} \longrightarrow \widetilde{\mathcal{B}}$. Let $\widetilde{a} \in \widetilde{\mathcal{A}}$ be a non-zero element such that $\widetilde{\psi}(\widetilde{\phi}(\widetilde{a})) = 0$, then for its lifting $a \in \mathcal{A}$ we have $\rho_{\mathcal{B}}(a) < \rho_{\mathcal{A}}(a)$. By the first step, $\rho_{\mathcal{A}}(f^n a) < r^n \rho_{\mathcal{A}}(a)$ for some n, i.e. $\widetilde{f}^n \widetilde{a} = 0$ in $\widetilde{\mathcal{A}}$. Thus already $\widetilde{\phi}(\widetilde{a})$ vanishes, and therefore $\widetilde{\psi}$ is injective. Furthermore, given a non-zero element $\widetilde{b} \in \widetilde{\mathcal{B}}$ we can lift it to an element $b \in \mathcal{B}$ of the form $f^{-n}a$ where $a \in \mathcal{A}$. By the first step, enlarging n we can assume that $\rho_{\mathcal{B}}(a) = \rho_{\mathcal{A}}(a)$. It follows, that the element $\widetilde{a} = \widetilde{f}^n \widetilde{b} \in \widetilde{\mathcal{B}}$ has a preimage in $\widetilde{\mathcal{A}}$, and therefore $\widetilde{b}$ comes already from $\widetilde{\mathcal{A}}_{\widetilde{f}}$.

(iii) Pick up r for which the algebras $\mathcal{A}' = \mathcal{A} \widehat{\otimes} K_r$ and $\mathcal{B}' = \mathcal{B} \widehat{\otimes} K_r$ satisfy the following condition, $\rho(\mathcal{A}') = \rho(\mathcal{B}') = |K_r|$ (in particular the algebras $\mathcal{A}'$ and $\mathcal{B}'$ are strictly K_r-affinoid).

3.2. Lemma. *Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a bounded homomorphism of k-affinoid algebras and $r_1, \dots, r_n$ positive numbers, then ϕ is finite and admissible if and only if $\phi_r = \phi \widehat{\otimes} K_r$ is finite and admissible.*

Proof. The direct implication is obvious. It is easily seen also, that ϕ is admissible if and only if ϕ_r is. It remains to prove, that ϕ is finite if ϕ_r is finite (we do not need admissibility in this direction). Since K_r is the completed tensor product of K_{r_i}, we can assume $n=1$. Now the case of r not belonging to $\sqrt{|k^*|}$ follows from [Ber1], 2.1.11. Since for a positive number $r \in \sqrt{|k^*|}$ the field K_r is finite over a subfield isomorphic to K_1, we can restrict ourselves to the case of $r=1$. Set $K=K_1$, one has $K=\widehat{Q(k\{T\})}$ and $\widetilde{K}_1=\widetilde{k}_1(T)$. Let $\{\widetilde{f}_i(T)\}_{i\in I}$ be the set of irreducible monic polynomials of $\widetilde{k}_1[T]$ and f_i some monic liftings to $k^\circ[T]\subset K$. It is easily seen then, that given an affinoid algebra $\mathcal{A}$, any element $a\in\mathcal{A}_K=\mathcal{A}\widehat{\otimes}K$ admits a unique representation $a=\sum_{n=0}^\infty a_nT^n+\sum_{i,j,k}a_{ijk}\frac{T^k}{f_i^j}$, where $i\in I$, $1\le j$, $0\le k<\deg(f_i(T))$ and the set of coefficients $\{a_n,a_{ijk}\}\subset\mathcal{A}$ is such that for any positive ε only finitely many coefficients have norm larger than ε. Now, if $\mathcal{B}_K$ is finite over $\mathcal{A}_K$, then any element $b\in\mathcal{B}$ satisfies an equality $b^m+\sum_0^{m-1}a_lb^l=0$, where $a_l\in\mathcal{A}_K$. But obviously replacing a_l by their free coefficients $(a_l)_0$ preserves the equality, hence $\mathcal{B}$ is finite over $\mathcal{A}$ as well. $\square$

By Lemma 3.2, the homomorphism ϕ is finite and admissible if and only if the induced homomorphism $\phi':\mathcal{A}'\longrightarrow\mathcal{B}'$ is finite and admissible. Since the algebras $\mathcal{A}'$ and $\mathcal{B}'$ are strictly affinoid, it is the same as to require that ϕ' is finite. By [BGR], 6.3.5/1, the homomorphism ϕ' is finite if and only if the homomorphism $\widetilde{\phi}'_1:\widetilde{\mathcal{A}}'_1\longrightarrow\widetilde{\mathcal{B}}'_1$ is finite and by Lemmas 1.1 (i) and 1.3, it is equivalent to finiteness of the homomorphism $\widetilde{\phi}':\widetilde{\mathcal{A}}'\longrightarrow\widetilde{\mathcal{B}}'$. Finally we note, that by the second isomorphism of (i), $\widetilde{\phi}'$ is obtained from $\widetilde{\phi}$ by tensoring with $\widetilde{K}_r$. So, by Lemma 1.3, $\widetilde{\phi}'$ is finite if and only if $\widetilde{\phi}$ is finite.

The last two parts of the proposition are proved in the same way, so we restrict ourselves to (v). Fix an admissible epimorphism $f:k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}\longrightarrow\mathcal{A}$ and set $f_K:K\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}\longrightarrow\mathcal{A}\widehat{\otimes}K$. Obviously f_K is epimorphic and by part (iii) of the proposition, its reduction $\widetilde{f}_K:\widetilde{K}[r^{-1}T]\longrightarrow\widetilde{\mathcal{A}\widehat{\otimes}K}$ is finite. Note, that $\widetilde{f}_K(T_i)\neq 0$ if and only if $\rho(f_K(T_i))=r_i$, the latter implies that $\rho(f(T_i))=r_i$ and therefore $\widetilde{f}_K(T_i)\in\widetilde{\phi}(\widetilde{\mathcal{A}})$. We see, that $\widetilde{\phi}(\widetilde{\mathcal{A}})$ contains $\widetilde{f}_K(T_i)$ and obviously $\widetilde{\phi}(\widetilde{K})=\widetilde{f}_K(\widetilde{K})$. So, $\mathrm{Im}(\widetilde{\phi})\supset\mathrm{Im}(\widetilde{f}_K)$ and $\widetilde{\phi}$ is finite. $\square$

Note, that one can not expect to have an isomorphism at (v), as the following simple example shows: take $k=\mathbf{Q}_p$ and let $\mathcal{A}$ and K be its ramified extensions of degree p which do not coincide.

Given an affinoid k-space $X=\mathcal{M}(\mathcal{A})$, we define its reduction as $\widetilde{X}=\mathrm{Spec}_{\mathbf{R}_+^*}(\widetilde{\mathcal{A}})$. Let $x\in X$ be a point, consider the character $\mathcal{A}\longrightarrow\mathcal{H}(x)$. It induces a homomorphism $\widetilde{\mathcal{A}}\longrightarrow\widetilde{\mathcal{H}(x)}$ of graded rings. Since $\widetilde{\mathcal{H}(x)}$ is a graded field, we obtain a point $\widetilde{x}\in\widetilde{X}$. Thus, the reduction map $\pi_X:X\longrightarrow\widetilde{X}$ is defined. Moreover, any morphism $f:Y\longrightarrow X$ of affinoid k-spaces gives rise to the commutative square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\pi_X"] \\ \widetilde{Y} \arrow[r, "\widetilde{f}"'] & \widetilde{X} \end{tikzcd} \tag{1}$$

Note, that π_X is "anticontinuous", i.e. the preimage of an open set is closed and vice versa.

3.3. Proposition. *Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space and $\widetilde{X}_{gen}$ the set of generic points of its reduction.*

(i) The reduction map $\pi_X : X \longrightarrow \widetilde{X}$ is surjective.

(ii) Any point $\widetilde{x} \in \widetilde{X}_{gen}$ has a unique preimage in X.

(iii) The set $\pi_X^{-1}(\widetilde{X}_{gen})$ is the Shilov boundary of X.

Proof. We deduce (i) from [Ber1], 2.4.4, while the proof of (ii) and (iii) follows the *loc. cit.* closely. Pick up a field K_r such that $\rho(\mathcal{A}\widehat{\otimes}K_r) = |K_r|$. Set $\mathcal{B} = \mathcal{A}\widehat{\otimes}K_r$ and $Y = \mathcal{M}(\mathcal{B})$ and let f denote the natural morphism $Y \longrightarrow X$, then the diagram (1) holds. By Lemma 1.1 (ii), on the level of sets $\widetilde{Y} = \mathrm{Spec}_{\mathbf{R}_+^*}(\widetilde{\mathcal{B}})$ is isomorphic to the reduction of Y in the sense of [Ber1] (i.e. $\mathrm{Spec}(\widetilde{\mathcal{B}}_1)$). It is a trivial check, that our reduction map and the reduction map from [Ber1] are compatible due to this isomorphism. Thus, by [Ber1], 2.4.4, the space Y satisfies (i). By 3.1 (v), $\widetilde{Y} = \widetilde{X} \otimes_{\widetilde{k}} \widetilde{K}_r$, since the map $\widetilde{X} \otimes_{\widetilde{k}} \widetilde{K}_r \longrightarrow \widetilde{X}$ is surjective, we obtain (i).

Next we prove (ii). Suppose, first that the space $\widetilde{X}$ is irreducible, and let η denote its generic point. Note, that the induced character $\widetilde{\mathcal{A}} \longrightarrow \widetilde{k}(\eta)$ is injective ($\widetilde{\mathcal{A}}$ has no non-trivial homogeneous nilpotents, because $\rho_{\mathcal{A}}$ is power multiplicative). Hence for any point $x \in \pi_X^{-1}(\eta)$ one has $|f(x)| \geq \rho_{\mathcal{A}}(f)$ for any $f \in \mathcal{A}$. It means that actually the equality holds, and $\rho_{\mathcal{A}}$ defines the only point of $\pi_X^{-1}(\eta)$. In the general case we choose a point $\eta \in \widetilde{X}_{gen}$ and use Lemma 1.5, to find an element $\widetilde{f} \in \widetilde{\mathcal{A}}$ vanishing on all generic points of $\widetilde{X}$ except η. Let f be a lifting of $\widetilde{f}$, $r = \rho(\widetilde{f})$ and $\mathcal{B} = \mathcal{A}\{r^{-1}f\}$, then $\pi_X^{-1}(\eta) \subset X\{r^{-1}f\} = \mathcal{M}(\mathcal{B})$ and by Proposition 3.1 (ii), $\widetilde{\mathcal{B}} = \widetilde{\mathcal{A}}_{\widetilde{f}}$. Since the statement holds for $X\{r^{-1}f\}$ it holds also for X.

Finally, let $f \in \mathcal{A}$ be an element with $\rho(f) = r > 0$. Its reduction $\widetilde{f} \in \widetilde{\mathcal{A}}$ does not vanish on some point $\widetilde{x} \in \widetilde{X}_{gen}$, hence for the point $x = \pi_X^{-1}(\widetilde{x})$ we have $|f(x)| = r$. Thus any element of $\mathcal{A}$ takes its maximum on $\Gamma = \pi_X^{-1}(\widetilde{X}_{gen})$. Conversely, let $x \in \Gamma$ be a point and U its open neighborhood. Since $x = \cap_{\widetilde{f}(\widetilde{x})\neq 0}\pi_X^{-1}(D(\widetilde{f}))$, for some $f \in \mathcal{A}$ we have $\widetilde{f}(\widetilde{x}) \neq 0$ and $X\{rf^{-1}\} \subset U$, where $r = \rho(f)$. Therefore already for some $\varepsilon > 0$ we have $X\{(r-\varepsilon)f^{-1}\} \subset U$. It follows, that x belongs to any boundary of X, as claimed. $\square$

In the following proposition we generalize [Ber1], 2.5.2 (d). See [Ber1], 2.5, for the definition of inner homomorphisms.

3.4. Proposition. *Let $\mathcal{A}$ be a k-affinoid algebra and $\phi : \mathcal{B} \longrightarrow \mathcal{D}$ a bounded $\mathcal{A}$-homomorphism from an $\mathcal{A}$-affinoid algebra to a Banach $\mathcal{A}$-algebra, then ϕ is inner with respect to $\mathcal{A}$ if and only if $\widetilde{\phi}(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\phi}(\widetilde{\mathcal{A}})$.*

Proof. Suppose ϕ is inner, and let $\pi : \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \longrightarrow \mathcal{B}$ be an admissible epimorphism such that $\rho(\phi(\pi(T))) < r$. By Proposition 3.1 (iii), the induced homomorphism $\widetilde{\mathcal{A}}[r^{-1}T] \longrightarrow \widetilde{\mathcal{B}}$ is finite. Since $\widetilde{\phi}(\widetilde{\pi}(T)) = 0$, we obtain that $\widetilde{\phi}(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\phi}(\widetilde{\mathcal{A}})$. Conversely, suppose $\widetilde{\phi}(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\phi}(\widetilde{\mathcal{A}})$. Let $\psi : \mathcal{A}\{r^{-1}T\} \longrightarrow \mathcal{B}$ be a bounded $\mathcal{A}$-homomorphism, such that $\rho(\psi(T)) = r$. Consider its reduction $\widetilde{\psi} : \widetilde{\mathcal{A}}[r^{-1}T] \longrightarrow \widetilde{\mathcal{B}}$ and set $\widetilde{b} = \widetilde{\psi}(T)$. Find an element $\widetilde{b}' = \sum_{i=0}^{n} \widetilde{a}_i \widetilde{b}^{n-i}$, where $\widetilde{a}_i \in \widetilde{\mathcal{A}}$ and $\widetilde{a}_0 = 1$, such that $\widetilde{\phi}(\widetilde{b}') = 0$. We can assume, that $\widetilde{a}_i \widetilde{b}^{n-i} \in \widetilde{\mathcal{B}}_{r^n}$, or equivalently $\widetilde{a}_i \in \widetilde{\mathcal{B}}_{r^i}$. Then for liftings $a_n \in \mathcal{A}^\circ_{r^n}, \dots, a_1 \in \mathcal{A}^\circ_r, a_0 = 1$ and the element $P = \sum_{i=0}^{n} a_i T^{n-i}$ we have $\rho(\phi(\psi(P))) < r^n$. Now, by [Ber1], 2.5.2 (c), ϕ is inner. □

At the end of the section we prove an analog of the Gerritzen-Grauert theorem.

3.5. Proposition. *Given an affinoid space $X = \mathcal{M}(\mathcal{A})$, any its affinoid domain $Y = \mathcal{M}(\mathcal{B})$ may be represented as a finite union of rational subdomains.*

Proof. Since Y is compact, it suffices to prove, that any point $y \in Y$ has a neighborhood Y' which is a rational subdomain of X. In order to find such a neighborhood we can replace Y by a neighborhood of y in Y and X by a rational subdomain containing Y. Consider the natural homomorphism $\widetilde{\psi} : \widetilde{\mathcal{B}} \longrightarrow \widetilde{\mathcal{H}(y)}$. Let $\widetilde{f}_1, \dots, \widetilde{f}_n$ be homogeneous generators of $\widetilde{\mathcal{C}} = \widetilde{\psi}(\widetilde{\mathcal{B}})$ over $\widetilde{k}$ and $r_i = \rho(\widetilde{f}_i)$. Choose liftings $f_i = \frac{g_i}{h_i} \in \mathcal{H}(y)$ of $\widetilde{f}_i$, such that g_i and h_i come from $\mathcal{A}$. The rational domain $X' = X\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\}$ contains an affinoid neighborhood Y' of y in Y. Let $X' = \mathcal{M}(\mathcal{A}')$ and $Y' = \mathcal{M}(\mathcal{B}')$, consider the induced characters $\widetilde{\phi}' : \widetilde{\mathcal{A}}' \longrightarrow \widetilde{\mathcal{H}(y)}$ and $\widetilde{\psi}' : \widetilde{\mathcal{B}}' \longrightarrow \widetilde{\mathcal{H}(y)}$. By [Ber1], 2.5.13 (ii), $y \in \mathrm{Int}(Y'/Y)$, hence by Proposition 3.4, $\widetilde{\psi}'(\widetilde{\mathcal{B}}')$ is finite over $\widetilde{\mathcal{C}}$. But by our construction $\widetilde{\phi}'(\widetilde{\mathcal{A}}')$ contains $\widetilde{\mathcal{C}}$. Therefore $\widetilde{\psi}'(\widetilde{\mathcal{B}}')$ is finite over $\widetilde{\phi}'(\widetilde{\mathcal{A}}')$, and using Proposition 3.4 again we obtain that $y \in \mathrm{Int}(Y'/X')$. By [Ber1], 2.5.13 (ii), Y' is a neighborhood of y in X' and shrinking Y' we can make it rational (and even Laurent) in X'. □

3.6. Remark. We do not define reduction of a non-affinoid space X. However, the notion of a formal affinoid covering $\mathcal{U}$ due to Bosch (see [Bo] or [Ber1], 4.3) may be generalized straightforwardly to non-strictly analytic spaces. Then the reduction map $\pi_{\mathcal{U}} : X \longrightarrow \widetilde{X}_{\mathcal{U}}$ arises (it is glued from affinoid reduction maps), where $\widetilde{X}_{\mathcal{U}}$ is an $\mathbf{R}^*_+$-graded scheme (see Remark 1.6). The author conjectures, that some part of Raynaud's theory may be done in such a framework, for example that any compact analytic space admits a formal covering.

§4. The reduction functor

By *Germs* we denote the category of germs of k-analytic spaces at a point (see [T], §2). The germ corresponding to a punctual k-analytic space (X, x) will be denoted by X_x. A germ X_x is said to be **good** if the point x has an affinoid neighborhood in X. A morphism of germs $\varphi\colon X_x \longrightarrow Y_y$ is said to be **separated** (resp. **closed**) if it is induced by a separated (resp. closed) morphism $X' \longrightarrow Y$, where X' is an open neighborhood of x in X. By a subdomain X'_x of X_x we mean an equivalence class of subdomains X' of X, whose germs at x are isomorphic. Furthermore, if X_x is a good germ, $f = (f_1, \dots, f_n) \subset \mathcal{O}_{X,x}$ is a family of elements and $r = (r_1, \dots, r_n)$ are positive numbers, then one defines a germ $X_x\{r^{-1}f\}$ in a natural way (see [T], §2, for details).

Our next purpose is to define a functor Red: $Germs \longrightarrow bir_{\widetilde{k}}$. In the case of a good germ one can proceed analogously to [T] (§2 and Lemma 2.2). Namely, given an affinoid space $X = \mathcal{M}(\mathcal{A})$, its point $x \in X$ and the corresponding character $\chi_x\colon \mathcal{A} \longrightarrow \mathcal{H}(x)$, set $\widetilde{(X, x)} = \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{\chi}_x(\widetilde{\mathcal{A}})\}$. In this way we obtain a functor from punctual affinoid spaces to $bir_{\widetilde{k}}$. Let $X = \mathcal{M}(\mathcal{A})$ be an affinoid space, $Y = \mathcal{M}(\mathcal{B})$ its affinoid subdomain and $x \in Y$ a point such that Y is its neighborhood in X, then the natural homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$ is inner with respect to $\mathcal{H}(x)$ and by Proposition 3.4, $\widetilde{\chi}_x(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$. Therefore the induced morphism $\widetilde{(Y, x)} \longrightarrow \widetilde{(X, x)}$ is an isomorphism and we obtain a functor Red from the category of good germs to $bir_{\widetilde{k}}$, which will be denoted $X_x \mapsto \widetilde{X}_x$. For a good germ X_x and its subdomain Y_x the reduction $\widetilde{Y}_x \longrightarrow \widetilde{X}_x$ of the inclusion morphism is an open embedding, we identify $\widetilde{Y}_x$ with its image in $\widetilde{X}_x$. Since we did not develop the notion of reduction for non-affinoid spaces (see Remark 3.6), we can not extend the above definition of the reduction functor to non-good germs. Therefore we propose a more straightforward and technical construction than in [T], §2. Its main advantage is that it makes no use of the Raynaud's theory.

4.1. Proposition. *The functor Red may be extended in a unique way to a functor Red: Germs $\longrightarrow bir_{\widetilde{k}}$, such that the following conditions are satisfied:*

(i) if $\phi\colon Y_x \longrightarrow X_x$ is an embedding of a subdomain, then $\widetilde{\phi}$ is an open embedding;

(ii) given subdomains Y_x and Z_x of a germ X_x, one has $\widetilde{Y_x \cap Z_x} = \widetilde{Y}_x \cap \widetilde{Z}_x$ and $\widetilde{Y_x \cup Z_x} = \widetilde{Y}_x \cup \widetilde{Z}_x$;

(iii) given a morphism of germs $\phi\colon Y_y \longrightarrow X_x$, a subdomain X'_x of X_x and its preimage Y'_y in Y_y, one has $\widetilde{Y}'_y = \widetilde{\phi}^{-1}(\widetilde{X}'_x)$.

Proof. First we check, that the needed properties are satisfied on good germs.

4.2. Lemma.
(i) Let X_x be a good germ and $X_x^1, \dots, X_x^m$ a covering of X by good subdomains, then $\cup \widetilde{X}_x^i = \widetilde{X}_x$.
(ii) Let $\phi\colon Y_y \longrightarrow X_x$ be a morphism of good germs, X'_x a good subdomain of X_x and $Y'_y = \phi^{-1}(X'_x)$, then $\widetilde{Y}'_y = \widetilde{\phi}^{-1}(\widetilde{X}'_x)$.

4.3. Lemma. *Given a good germ X_x, any its subdomain Y_x may be covered by finite number of subdomains of the form $X_x\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\}$, where $|f_i(x)| = r_i$.*

Proof. Choose an affinoid representative X of X_x. By Proposition 3.5, we can assume that Y is a rational subdomain of X, i.e. $Y = X\{r_1^{-1}\frac{g_1}{h}, \dots, r_n^{-1}\frac{g_n}{h}\}$. Since $x \in Y$, we can assume also that h does not vanish on X, and thus $Y = X\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\}$. Finally, shrinking X we can remove all f_i's with $|f_i(x)| < r_i$. □

4.4. Lemma. *Given a good germ X_x and a family $f = (f_1, \dots, f_n) \subset \mathcal{O}_{X,x}$ of elements, one has $\widetilde{X_x\{r^{-1}f\}} = \widetilde{X}_x\{\widetilde{f}\}$, where $r_i = |f_i(x)|$ and $\widetilde{f}_i$ is the image of f_i in $\widetilde{\mathcal{H}(x)}$.*

Proof. We may assume, that $X = \mathcal{M}(\mathcal{A})$ and $n = 1$. Applying Proposition 3.1 (iii) to the admissible epimorphism $\mathcal{A}\{r^{-1}T\} \longrightarrow \mathcal{A}\{r^{-1}f\}$ that takes T to f, we obtain that the induced homomorphism $\widetilde{\mathcal{A}}[r^{-1}T] \longrightarrow \widetilde{\mathcal{A}\{r^{-1}f\}}$ is finite. Therefore the graded algebra $\widetilde{\chi}_x(\widetilde{\mathcal{A}\{r^{-1}f\}})$ is finite over the graded subalgebra $\widetilde{\chi}_x(\widetilde{\mathcal{A}})[\widetilde{f}]$ of $\widetilde{\mathcal{H}(x)}$ generated by $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$ and $\widetilde{f}$. The lemma follows. □

Proof of Lemma 4.2.
(i) In order to prove the lemma we can replace the covering $\{X_x^i\}_{1\le i\le m}$ by any of its refinements. By Lemma 4.3, we can assume, that $X^i = X\{r_{i1}^{-1}f_{i1}, \dots, r_{in_i}^{-1}f_{in_i}\}$ and $|f_{ij}(x)| = r_{ij}$. Shrink X so that f_{ij} have no zeros on X. Adding ones we can achieve $n = n_1 = \dots = n_m$ and $f_{in} = 1$. Let J be the set of all sequences $j = (j_1, \dots, j_m)$ such that $1 \le j_i \le n$ for all $1 \le i \le m$ and $\max\limits_{1\le i\le m}\{j_i\} = n$. For any $j \in J$ set $f_j = f_{1j_1}\dots f_{mj_m}$ and $r_j = r_{1j_1}\dots r_{mj_m}$, then analogously to [T], 1.5, one proves, that the rational covering $\{X\{(\frac{r_{j'}}{r_j})^{-1}\frac{f_{j'}}{f_j}\}_{j'\in J}\}_{j\in J}$ refines the covering $\{X^i\}_{1\le i\le m}$. By Lemma 4.4, applying functor Red to the elements of the last covering, we obtain the open subsets $\widetilde{X}_x\{\frac{\widetilde{f}_{j'}}{\widetilde{f}_j}\}_{j'\in J}$ of $\widetilde{X}_x$ which form a rational covering of $\widetilde{X}_x$.
(ii) By Lemma 4.3, $X'_x = \cup_i X\{r_{i1}^{-1}g_{i1}, \dots, r_{in_i}^{-1}g_{in_i}\}$ and then obviously $Y'_x = \cup_i Y\{r_{i1}^{-1}g'_{i1}, \dots, r_{in_i}^{-1}g'_{in_i}\}$, where g'_{ij} are the images of g_{ij}. By Lemmas

4.4 and 4.2 (i), $\widetilde{X}'_x = \cup_i \widetilde{X}_x\{\widetilde{g}_{i1}, \dots, \widetilde{g}_{in_i}\}$ and the same representation may be written for $\widetilde{Y}'_y$, thus $\widetilde{Y}'_y = \widetilde{\phi}^{-1}(\widetilde{X}'_x)$ as claimed. □

We deal first with the extension of Red to the category *Gsep* of all separated germs. For an arbitrary separated germ X_x the following construction may be applied. Fix a finite good covering $\mathcal{V} = \{V^i_x\}_{i\in I}$ of X_x, then the germs $V^{ij}_x = V^i_x \cap V^j_x$ are good and we get a gluing data $\{\widetilde{V}^i_x\}_I, \{\widetilde{V}^{ij}_x \longrightarrow \widetilde{V}^i_x\}_{I^2}$ for the category $bir_{\widetilde{k}}$ (the data is consistent, because by Lemma 4.2 (ii), $\widetilde{V}^{ij}_x \cap \widetilde{V}^{ik}_x = \widetilde{V}^{ijk}_x$, where $V^{ijk}_x = V^i_x \cap V^j_x \cap V^k_x$). Let $\widetilde{X}_{\mathcal{V},x}$ be the corresponding element of $bir_{\widetilde{k}}$. For any functor Red: $Gsep \longrightarrow bir_{\widetilde{k}}$ satisfying conditions (i) and (ii) of the proposition, one necessarily has $\mathrm{Red}(X_x) \widetilde{\rightarrow} \widetilde{X}_{\mathcal{V},x}$, it proves uniqueness. To prove existence one should show, that $\widetilde{X}_{\mathcal{V},x}$ does not depend on the covering $\mathcal{V}$. The latter reduces to the case of a covering and its refinement and then may be deduced from Lemma 4.2 (i). The needed properties of the functor are obtained from the good case (intersection is a particular case of the preimage). An arbitrary germ X_x may be covered by separated germs X^i_x and the intersections $X^i_x \cap X^j_x$ are obviously separated. Using gluing procedure again we can extend Red to a functor Red: *Germs* $\longrightarrow bir_{\widetilde{k}}$ which satisfies (i), (ii) and (iii). □

4.5. Theorem. *Given a germ X_x, the reduction functor establishes a one-to-one correspondence between subdomains of X_x and open quasi-compact subsets of $\widetilde{X}_x$.*

Proof. In Steps 1-3 we assume that the germ X_x is good, and prove the theorem in this case. The general case is deduced from the particular one in Step 4.

Step 1. *Let $f = (f_1, \dots, f_l)$ and $g = (g_1, \dots, g_m)$ be two families of elements of $\mathcal{O}_{X,x}$, $r_i = |f_i(x)|$, $s_j = |g_j(x)|$ and suppose, that $\widetilde{X}_x\{\tilde{f}\} \subset \widetilde{X}_x\{\tilde{g}\}$, then $X_x\{r^{-1}f\} \subset X_x\{s^{-1}g\}$.* We can assume, that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid and $f_i, g_j \in \mathcal{A}$. Let χ_x: $\mathcal{A} \longrightarrow \mathcal{H}(x)$ be the character of x and set $B = \widetilde{\chi_x}(\widetilde{\mathcal{A}}) \subset \widetilde{\mathcal{H}(x)}$, then Lemma 4.4 implies, that $\widetilde{X}_x \widetilde{\rightarrow} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{B\}$, $\widetilde{X_x\{f\}} \widetilde{\rightarrow} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{B[\tilde{f}]\}$ and $\widetilde{X_x\{g\}} \widetilde{\rightarrow} \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{B[\tilde{g}]\}$. Since $\widetilde{X}_x\{\tilde{f}\} = \widetilde{X}_x\{\tilde{f}, \tilde{g}\}$, Lemma 2.2 implies, that $B[\tilde{f}, \tilde{g}]$ is finite over $B[\tilde{f}]$. Therefore, each element $\widetilde{g}_j$ satisfies an equation of the form $\tilde{g}^n_j + \sum_{k=0}^{n-1} \tilde{a}_{n-k}\tilde{g}^k_j = 0$, where $\tilde{a}_k \in B[\tilde{f}]$. We can assume also, that $\tilde{a}_k \in B[\tilde{f}]_{s^k_j}$. The coefficients $\tilde{a}_k$ may be lifted to elements $a_k \in (\mathcal{A}\{r^{-1}f\})^\circ_{s^k_j}$ and obviously $|(g^n_j + \sum_{k=0}^{n-1} a_{n-k} g^k_j)(x)| < s^n_j$. The last inequality holds also in a neighborhood V of x in $X\{r^{-1}f\}$ and $|a_k| \le s^k_j$ in $X\{r^{-1}f\}$, therefore $|g_j| \le s_j$ in V, i.e. $X_x\{r^{-1}f\} \subset X_x\{s^{-1}g\}$.

Step 2. *Let Y_x be a subdomain of X_x such that $\widetilde{Y}_x \widetilde{\rightarrow} \widetilde{X}_x$, then $Y_x = X_x$.* By Lemma 4.3, Y_x has a finite covering of the form $\{X_x\{r_i^{-1}f_i\}\}_{1\le i\le m}$ (where $|f_i(x)| = r_i$), say $Y_x = \cup_{i=1}^m V_i$. Our assumption implies, that

$\{\widetilde{V}_i\}_{1\le i\le m}$ is a covering of $\widetilde{X}_x$. By Lemma 2.4, this covering has a Laurent refinement $\{\widetilde{U}_j\}_{j\in\{\pm1\}^l}=\{\widetilde{X}_x\{\tilde{g}_1^{j_1},\dots,\tilde{g}_l^{j_l}\}\}_j$ (i.e. for any $j\in\{\pm1\}^l$, $\widetilde{U}_j\subset\widetilde{V}_{i(j)}$). Let $s_q=\rho(\tilde{g}_q)$ and $g_q\in\mathcal{O}_{X,x}$ are liftings of $\tilde{g}_q$, then the sets $U_j=X_x\{s_1^{-j_1}g_1^{j_1},\dots,s_l^{-j_l}g_l^{j_l}\}$, $j\in\{\pm1\}^l$, form a Laurent covering of X_x, whose reduction coincides with $\{\widetilde{U}_j\}_j$. By the previous step, for any $j\in\{\pm1\}^l$ we have $U_j\subset V_{i(j)}$, hence $\{V_i\}_i$ is also a covering of X_x, i.e. $X_x=Y_x$.

Step 3. *The theorem holds if the germ X_x is good.* It suffices to prove the following two statements: (1) any open quasi-compact subset $\widetilde{Y}_x$ of $\widetilde{X}_x$ is a reduction of some subdomain of X_x, (2) if the reductions of two subdomains Y_x and Z_x of X_x coincide, then the subdomains are equal. To prove the first statement, find a representation $\widetilde{Y}_x=\cup_i\widetilde{X}_x\{\tilde{f}_{i1},\dots,\tilde{f}_{in_i}\}$ and let $f_{ij}\in\mathcal{O}_{X,x}$ be some liftings and $r_{ij}=|f_{ij}(x)|$, then $\cup_iX_x\{r_{i1}^{-1}f_{i1},\dots,r_{in_i}^{-1}f_{in_i}\}$ is a lifting of $\widetilde{Y}_x$. Suppose now, that $\widetilde{Y}_x=\widetilde{Z}_x$. Find representations $Y_x=\cup_{i=1}^pY_x^i$ and $Z_x=\cup_{j=1}^qZ_x^j$, where Y_x^i and Z_x^j are good, then $\widetilde{Y}_x=\cup_i\widetilde{Y}_x^i=\cup_j\widetilde{Z}_x^j$. Therefore for any fixed $i\in\{1,\dots,p\}$, the sets $\widetilde{Y_x^i\cap Z_x^j}$ $(1\le j\le q)$ form a covering of $\widetilde{Y}_x^i$. By the previous step, the sets $Y_x^i\cap Z_x^j$ cover Y_x^i. It follows, that $Y_x\subset Z_x$, and by the symmetry the converse inclusion is also satisfied.

Step 4. *The general case.* Let $\{X_x^i\}_{i\in I}$ be a good covering of X_x. Again, it suffices to check the conditions (1) and (2) from the previous step. Let $\widetilde{Y}_x$ be an open quasi-compact subset of $\widetilde{X}_x$. By the previous step we can lift all sets $\widetilde{Y}_x\cap\widetilde{X}_x^i$ to subdomains Y_x^i, then the union of all Y_x^i's is the required lifting of $\widetilde{Y}_x$. Suppose now, that for subdomains Y_x and Z_x of X_x we have $\widetilde{Y}_x=\widetilde{Z}_x$. Then for any $i\in I$ we have $\widetilde{Y_x\cap X_x^i}=\widetilde{Y}_x\cap\widetilde{X}_x^i=\widetilde{Z}_x\cap\widetilde{X}_x^i=\widetilde{Z_x\cap X_x^i}$. By the case of a good germ, the liftings $Y_x\cap X_x^i$ and $Z_x\cap X_x^i$ coincide, and since X_x^i cover X_x, we obtain, that $Y_x=Z_x$. □

4.6. Proposition. *Let $Y\longrightarrow X$ and $Z\longrightarrow X$ be morphisms of k-analytic spaces, $T=Y\times_XZ$ and $t\in T$ a point whose images in Z, Y and X are z, y and x, respectively. Set $Z_0=\widetilde{Z}_z\times_{\mathbf{P}_{\widetilde{\mathcal{H}(z)}}}\mathbf{P}_{\widetilde{\mathcal{H}(t)}}$ and define Y_0 and X_0 analogously, then the induced map ε: $T_0=Z_0\times_{X_0}Y_0\longrightarrow\mathbf{P}_{\widetilde{\mathcal{H}(t)}}$ is a local homeomorphism and the corresponding object $(T_0,\widetilde{\mathcal{H}(t)},\varepsilon)$ of $bir_{\tilde{k}}$ is isomorphic to $\widetilde{T}_t$.*

Proof. The general case reduces to the case of k-affinoid spaces X, Y and Z by considering affinoid coverings $\mathcal{X}=\{X_i\}$, $\mathcal{Y}=\{Y_j\}$ and $\mathcal{Z}=\{Z_k\}$ of X, Y and Z, respectively, such that $\mathcal{Y}$ (resp. $\mathcal{Z}$) refine the preimage of $\mathcal{X}$. So we can assume that $X=\mathcal{M}(\mathcal{A})$, $Y=\mathcal{M}(\mathcal{B})$ and $Z=\mathcal{M}(\mathcal{C})$. Let $\widetilde{\mathcal{A}}_0$ denote the image of $\widetilde{\mathcal{A}}$ in $\widetilde{\mathcal{H}(t)}$ (we have the homomorphisms of reductions $\widetilde{\mathcal{A}}\longrightarrow\widetilde{\mathcal{H}(x)}$ and $\widetilde{\mathcal{H}(x)}\longrightarrow\widetilde{\mathcal{H}(t)}$) and define $\widetilde{\mathcal{B}}_0$ and $\widetilde{\mathcal{C}}_0$ analogously, then $X_0\widetilde{\to}\mathbf{P}_{\widetilde{\mathcal{H}(t)}}\{\widetilde{\mathcal{A}}_0\}$, $Y_0\widetilde{\to}\mathbf{P}_{\widetilde{\mathcal{H}(t)}}\{\widetilde{\mathcal{B}}_0\}$ and $Z_0\widetilde{\to}\mathbf{P}_{\widetilde{\mathcal{H}(t)}}\{\widetilde{\mathcal{C}}_0\}$. It follows that

$T_0 \widetilde{\to} Y_0 \cap Z_0 \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(t)}}\{\widetilde{\mathcal{B}}_0, \widetilde{\mathcal{C}}_0\}$. From other side, consider a homomorphism $\phi\colon \widetilde{\mathcal{B}} \otimes_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{H}(t)}$ induced by $\psi\colon \widetilde{\mathcal{B}} \otimes_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C}}$. Since ψ is finite by 3.1 (iv), we obtain that $\widetilde{T}_t \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(t)}}\{\mathrm{Im}(\phi)\}$. But obviously $\mathrm{Im}(\phi)$ is generated by $\widetilde{\mathcal{B}}_0$ and $\widetilde{\mathcal{C}}_0$, hence $\widetilde{T}_t \widetilde{\to} \mathbf{P}_{\widetilde{\mathcal{H}(t)}}\{\widetilde{\mathcal{B}}_0, \widetilde{\mathcal{C}}_0\} \widetilde{\to} T_0$ as claimed. $\square$

4.7. Corollary. *Under assumptions of the previous proposition suppose also, that the morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper. Then the morphism $\widetilde{T}_t \longrightarrow \widetilde{Z}_z$ is proper too.*

Proof. By our assumption $\widetilde{Y}_y \widetilde{\to} \widetilde{X}_x \times_{\mathbf{P}_{\widetilde{\mathcal{H}(x)}}} \mathbf{P}_{\widetilde{\mathcal{H}(y)}}$, hence the induced map $Y_0 \longrightarrow X_0$ is also bijective. It follows, that the map $T_0 \longrightarrow Z_0$ is a bijection, or that is equivalent, the morphism $\widetilde{T}_t \longrightarrow \widetilde{Z}_z$ is proper. $\square$

4.8. Proposition. *Let $f\colon Y \longrightarrow X$ be a morphism of k-analytic spaces, $y \in Y$ a point and $x = f(y)$.*

(i) If f is a closed embedding at y, then the reduction morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is an isomorphism.

(ii) Suppose that f is a G-locally closed embedding, then f is a closed embedding at y if and only if the reduction morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is an isomorphism.

(iii) The morphism f is separated at y if and only if the reduction morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is separated.

Proof. The first statement reduces easily to the case, when f is a closed embedding of k-affinoid spaces, but the latter is obvious. Suppose next, that f is a G-locally closed embedding, then there exists an analytic subdomain X' of X containing the image of a neighborhood Y' of y and such that the restriction morphism $Y' \longrightarrow X'$ is a closed embedding. Furthermore, f is locally closed at y if and only if $X'_x \widetilde{\to} X_x$, that is equivalent by Theorem 4.5 to $\widetilde{X}_x \widetilde{\to} \widetilde{X}'_x$. From the other side, $\widetilde{X}'_x \widetilde{\to} \widetilde{Y}'_y$ by the first statement of the proposition and obviously $\widetilde{Y}'_y \widetilde{\to} \widetilde{Y}_y$.

Finally we prove (iii). By the definition (see [Ber2], 1.4) the morphism f is separated at y if and only if the diagonal morphism $\Delta_{Y/X}\colon Y \longrightarrow Y \times_X Y = Z$ is locally closed at y. By *loc. cit.*, $\Delta = \Delta_{Y/X}$ is a G-locally closed embedding, hence it is closed at y if and only if $\widetilde{\Delta}\colon \widetilde{Y}_y \longrightarrow \widetilde{Z}_z$ is an isomorphism, where $z = \Delta(y)$. Proposition 4.6 implies, that $\widetilde{Z}_z \widetilde{\to} \widetilde{Y}_y \times_{X'} \widetilde{Y}_y$, where $X' = \widetilde{X}_x \times_{\mathbf{P}_{\widetilde{\mathcal{H}(x)}}} \mathbf{P}_{\widetilde{\mathcal{H}(y)}}$. So $\widetilde{\Delta}$ is an isomorphism if and only if the natural map $\widetilde{Y}_y \longrightarrow X'$ is injective, but the last condition means that the morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is separated. $\square$

At the end of the section we assume, that the valuation on k is non-trivial. We shall use the reduction functor to prove that the natural functor st-k-An $\longrightarrow k$-An is fully faithful. In particular, an analytic space cannot have two non-isomorphic structures of strictly analytic space. Let X be a strictly analytic space and $x \in X$ a point. A subdomain $Y_x \subset X_x$ is called strictly analytic if it has a representative Y which is a strictly analytic domain in X.

4.9. Lemma. *Let X be a strictly analytic space and $x \in X$ a point. A subdomain $Y_x \subset X_x$ is strictly analytic if and only if its reduction $\widetilde{Y}_x$ is $|k^*|$-strict.*

Proof. Reduction of a good strictly analytic germ is $|k^*|$-strict, hence reduction of an arbitrary strictly analytic germ can be glued from $|k^*|$-strict objects along $|k^*|$-strict subobjects, so it is $|k^*|$-strict too. Conversely, suppose $\widetilde{Y}_x$ is $|k^*|$-strict. We assume first, that X_x is good. Let $\{\widetilde{Y}_x^j\}_{1\le j\le m}$ be a $|k^*|$-strict affine covering of $\widetilde{Y}_x$ and Y_x^j the liftings of $\widetilde{Y}_x^j$. Find representations $\widetilde{Y}_x^j = \widetilde{X}_x\{\widetilde{f}_{j1},\dots,\widetilde{f}_{jn_j}\}$, then $Y_x^j = X_x\{r_{j1}^{-1}f_{j1},\dots,r_{jn_j}^{-1}f_{jn_j}\}$, where $f_{jk}\in\mathcal{O}_{X,x}$ are some liftings of $\widetilde{f}_{jk}$ and $r_{jk}=\rho(\widetilde{f}_{jk})$. By Proposition 2.5 (i), $r_{jk}\in\sqrt{|k^*|}$, hence Y_x^j are strictly analytic. So, the subdomain Y_x is strictly analytic as well.

In the general case, let X_x^i be a covering of X_x by good strictly analytic domains. The objects $\widetilde{Y}_x\cap\widetilde{X}_x^i$ of $bir_{\widetilde{\mathcal{H}(x)}}$ are $|k^*|$-strict by Corollary 2.7 (i), hence by the case of a good germ proved above, the subdomains $Y_x\cap X_x^i$ are strictly analytic. It finishes the proof. $\square$

4.10. Corollary. *The functor st-k-An $\longrightarrow$ k-An is fully faithful.*

Proof. Let $f\colon Y\longrightarrow X$ be a morphism of analytic spaces and suppose, that X and Y are strictly analytic. In order to prove that f is a morphism of *st-k-An* it suffices to show, that for a strictly analytic subdomain X' of X its preimage Y' in Y is strictly analytic too. Given a point $y\in Y'$, let $x=f(y)$ and let $\widetilde{f}\colon\widetilde{Y}_y\longrightarrow\widetilde{X}_x$ be the reduction morphism, then $\widetilde{Y}'_y=\widetilde{f}^{-1}(\widetilde{X}'_x)$. Since $\widetilde{X}_x$, $\widetilde{Y}_y$ and $\widetilde{X}'_x$ are $|k^*|$-strict, we can apply Corollary 2.7 (ii) to obtain that $\widetilde{Y}'_y$ is $|k^*|$-strict too. Now, the lemma implies, that Y'_y is strictly analytic and the corollary follows. $\square$

§5. Good germs and closed morphisms

The following Theorem is proved exactly as its analog in [T] (see [T], 3.1), the only difference is that one should allow arbitrary radii of convergence. That is why in this paper we only formulate the main steps of the proof, while concrete technical details may be found in [T].

5.1. Theorem. *A germ X_x is good if and only if its reduction $\widetilde{X}_x$ is affine.*

Proof (sketch). The direct implication is obvious, prove the inverse one.

Step 1. One can assume, that X is a union of two affinoid subdomains $Y=\mathcal{M}(\mathcal{B})$ and $Z=\mathcal{M}(\mathcal{C})$, such that $x\in Y\cap Z$, $\widetilde{Y}_x=\widetilde{X}_x\{\lambda\}$ and $\widetilde{Z}_x=\widetilde{X}_x\{\lambda^{-1}\}$ for a non-zero homogeneous element $\lambda\in\widetilde{\mathcal{H}(x)}$.

Step 2. Let $t=\rho(\lambda)$. One can assume, that $Y\cap Z=\mathcal{M}(\mathcal{A})$, $\mathcal{A}=\mathcal{B}\{tf^{-1}\}=\mathcal{C}\{t^{-1}g\}$, $\rho_{\mathcal{A}}(f-g)<t$ and $\lambda=\widetilde{f(x)}$.

Step 3. One can shrink X so that the following is true. There exist admissible epimorphisms

$$k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n,t^{-1}S_1,pS_2\}\longrightarrow\mathcal{B}:\ T_i\mapsto f_i,\ S_1\mapsto f,\ S_2\mapsto f^{-1}$$

$$k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n,q^{-1}S_1,tS_2\}\longrightarrow\mathcal{C}:\ T_i\mapsto g_i,\ S_1\mapsto g,\ S_2\mapsto g^{-1}$$

such that $p<t<q$, $||f_i-g_i||<r_i$ and $||f-g||<t$, where $||\ ||$ denotes the quotient norm on $\mathcal{A}$ induced from the canonical norm of the algebra $k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n,t^{-1}S_1,tS_2\}$ with respect to the admissible epimorphism $k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n,t^{-1}S_1,tS_2\}\longrightarrow\mathcal{A}:T_i\mapsto f_i,\,S_1\mapsto f,\,S_2\mapsto f^{-1}$.

Step 4. An analytic space X satisfying conditions of the previous step is affinoid. ☐

As a by-product one can construct simple examples of not good separated analityc space. The first such example is due to Huber, see [H], 8.3.8. We generalize it to a family containing also not strictly analytic examples. Let $E=\mathcal{M}(k\{r^{-1}T_1,s^{-1}T_2\})$ be a closed two dimensional disc of radii r and s, x its maximal point, D an open two dimensional disc of the same radii and $X=E\setminus D$. The reduction $\widetilde{X}_x=\mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{T}_1,\widetilde{T}_2,\widetilde{T}_2^{-1}\}\cup\mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{T}_1,\widetilde{T}_1^{-1},\widetilde{T}_2\}$ is easily seen to be not affine, and hence X is not good at x. If r and s are linearly independent over $|k^*|$, then one has also a toric picture of $\widetilde{X}_x$ as explained at the end of Sect. 2. The corresponding polyhedral cone is a union of two rays $(t_1=0,t_2\ge 0)$ and $(t_1\ge 0,t_2=0)$, in particular it is not convex. The second main result is the following theorem, whose proof occupies the rest of the section.

5.2. Theorem. *A morphism of germs $\phi_y\colon Y_y\longrightarrow X_x$ is closed if and only if the induced morphism between their reductions $\widetilde{\phi}_y\colon\widetilde{Y}_y\longrightarrow\widetilde{X}_x$ is proper.*

For arbitrary $\mathbf{R}_+^*$-graded fields $l\subset L$ let $bir_l(L)$ denote the subcategory of bir_l formed by objects of the form (X,L,ϕ). Given an extension m of l and an epimorphism $\psi\colon m\otimes_l L\longrightarrow M$ onto a graded field, one has a map $\mathbf{P}_{M/m}\longrightarrow\mathbf{P}_{L/l}$ (for a graded valuation m-subring $\mathcal{O}$ of M, the graded ring $\mathcal{O}\cap\mathrm{Im}(L)$ is a graded valuation l-subring of L). Now the correspondence $X\mapsto X\times_{\mathbf{P}_{L/l}}\mathbf{P}_{M/m}$ gives rise to a functor $\mathcal{E}_\psi\colon bir_l(L)\longrightarrow bir_m(M)$. (This definition corrects the analogous one at [T], §4.) The application of $\mathcal{E}_\psi$ to affine objects can be described also as follows. Given a graded l-subalgebra A of L, we have $\mathcal{E}_\psi(\mathbf{P}_{L/l}\{A\})=\mathbf{P}_{M/m}\{B\}$, where B denotes the image of $A\otimes_l m$ in M.

5.3. Lemma. *Let X be a k-analytic space, K a non-Archimedean field over k, $y\in Y=X\widehat{\otimes}K$ a point whose image in X is x and $\psi\colon\widetilde{\mathcal{H}(x)}\otimes\widetilde{K}\longrightarrow\widetilde{\mathcal{H}(y)}$ the induced homomorphism. Then $\widetilde{Y}_y\widetilde{\rightarrow}\mathcal{E}_\psi(\widetilde{X}_x)$.*

Proof. It suffices to check the case of an affinoid space X, which follows from Proposition 3.1 (v). ☐

5.4. Lemma. *Given a Cartesian diagram of morphisms of k-analytic spaces X and Y and K-analytic spaces X' and Y' (where $K \supset k$)*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

and a point $y' \in Y'$ whose images in X', Y and X are x', y and x, respectively, assume that the morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper. Then the morphism $\widetilde{Y}'_{y'} \longrightarrow \widetilde{X}'_{x'}$ is also proper.

Proof. The diagram appearing in the statement may be factored as follows

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y\widehat{\otimes}K \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X\widehat{\otimes}K \arrow[r] & X \end{tikzcd}$$

(left and right squares are Cartesian). Thus we should prove the lemma in the two particular cases: (1) all spaces X, Y, X' and Y' are defined over the same field, (2) the diagram is the natural diagram

$$\begin{tikzcd} Y\widehat{\otimes}K \arrow[r] \arrow[d] & Y \arrow[d] \\ X\widehat{\otimes}K \arrow[r] & X \end{tikzcd}$$

The second case follows from the previous lemma, while the first case was proven in Corollary 4.7. $\square$

5.5. Lemma. *Let $\phi\colon Y \longrightarrow X$ be a morphism of k-analytic spaces, $y \in Y$ a point and $x = f(y)$. Suppose, that the morphism $\widetilde{\phi}_y\colon \widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper, then there exists an open neighborhood Z of y, such that for any point $y' \in Z$ and $x' = \phi(y')$ the morphism $\widetilde{\phi}_{y'}\colon \widetilde{Y}_{y'} \longrightarrow \widetilde{X}_{x'}$ is proper.*

Proof. Suppose first, that $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ are affinoid. Note, that ϕ is closed at a point $y' \in Y$ if and only if the induced morphism $\widetilde{\phi}_{y'}\colon \widetilde{Y}_{y'} \longrightarrow \widetilde{X}_{x'}$ is proper. Really, $\widetilde{\phi}_{y'}$ is proper if and only if $\mathbf{P}_{\widetilde{\mathcal{H}(y)}}\{\widetilde{\chi}_y(\widetilde{\mathcal{B}})\} = \mathbf{P}_{\widetilde{\mathcal{H}(y)}}\{\widetilde{\chi}_x(\widetilde{\mathcal{A}})\}$, where $\chi_x\colon \mathcal{A} \longrightarrow \mathcal{H}(x) \subset \mathcal{H}(y)$ and $\chi_y\colon \mathcal{B} \longrightarrow \mathcal{H}(y)$ are the characters corresponding to the points x and y, respectively. By Lemma 2.2, it is equivalent to finiteness of $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$ over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$, and by Proposition 3.4, the latter is equivalent to ϕ being close at the point y. It remains to notice, that the points of Y, where ϕ is closed, form an open set.

Consider now the general case. Let $X^1, \dots, X^n$ be affinoid subdomains of X containing x and such that their union is a neighborhood of x, set $Y^i = \phi^{-1}(X^i)$. The induced morphisms $\widetilde{\phi}^i_y\colon \widetilde{Y}^i_y \longrightarrow \widetilde{X}^i_x$ are proper and the objects $\widetilde{X}^i_x$ are affine, therefore the objects $\widetilde{Y}^i_y$ are affine too. By Theorem 5.1, the germs Y^i_y are good, and shrinking Y^i we can assume that they are affinoid. Using the first part of the proof we can find open neighborhoods Z^i of y in Y^i such that for any point $y' \in Z^i$ the morphism $\widetilde{\phi}_{y'}\colon \widetilde{Z}^i_{y'} \longrightarrow \widetilde{X}^i_{x'}$ is proper. Let Z be an open neighborhood of y in Y such that for any i one has $Z \cap Y^i \subset Z^i$. Then for any point $y' \in Z$ and its image x' in X

the morphism $\widetilde{Y}_{y'} \longrightarrow \widetilde{X}_{x'}$ is glued from proper morphisms $\widetilde{Y}^i_{y'} \longrightarrow \widetilde{X}^i_{x'}$ and therefore is proper itself (we set $\widetilde{Y}^i_{y'} = \emptyset$ if $y' \notin Y^i$). □

Proof of Theorem 5.2. Assume first that the two germs are good. In this case the morphism of germs is induced by a morphism of affinoid spaces $\phi\colon Y = \mathcal{M}(\mathcal{B}) \longrightarrow X = \mathcal{M}(\mathcal{A})$ and, as we saw in the proof of Lemma 5.5, the morphism $\widetilde{\phi}_y$ is proper if and only if ϕ is closed at y.

Consider now the general case. The direct implication is easily reduced to the case of good germs. Assume that the morphism $\widetilde{\phi}_y$ is proper. Let $\phi\colon Y \longrightarrow X$ be a morphism of k-analytic spaces that induces ϕ_y. By Lemma 5.5, shrinking Y we can assume that for each point $y' \in Y$ the induced morphism $\widetilde{\phi}_{y'}\colon \widetilde{Y}_{y'} \longrightarrow \widetilde{X}_{x'}$ is proper. We shall prove that in this case the morphism ϕ is closed. Let $X' \longrightarrow X$ be a morphism from a good K-analytic space X', where K is a non-Archimedean field over k. We have to show that the K-analytic space $Y' = Y \times_X X'$ is also good and that the induced morphism $\phi'\colon Y' \longrightarrow X'$ is closed. Let y' be a point in Y', and let x', y and x be its images in X', Y and X, respectively. (The points y and x here are not necessarily the original y and x.) The morphism $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper and therefore, by Lemma 5.4, the morphism $\widetilde{Y}'_{y'} \longrightarrow \widetilde{X}'_{x'}$ is also proper. Since $\widetilde{X}'_{x'}$ is affine, it follows that $\widetilde{Y}'_{y'}$ is affine too and, by Theorem 5.1, the germ $Y'_{y'}$ is good. Thus, the required claim follows from the case of good germs. ∎

5.6. Corollary. *A morphism $f\colon Y \longrightarrow X$ is closed (resp. proper) if and only if there exists a covering $\{X_i\}$ of X by analytic subdomains, such that the induced morphisms $f^{-1}(X_i) \longrightarrow X_i$ are closed (resp. proper).* □

In particular, the definition of properness from [T], §4, which uses only strictly analytic spaces is equivalent to the definition we worked with here. One can also generalize the Kiehl's definition of properness to non-strictly analytic spaces and it is easily seen that it is equivalent too.

5.7. Corollary. *Given two morphisms $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$, assume that φ is locally separated. Then* $\mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X))$. □

5.8. Remark. Using language of formal coverings (see Remark 3.6) and graded schemes (see Remark 1.6) one can also generalize Corollary 4.4 of [T]. Namely, if $f\colon Y \longrightarrow X$ is a morphism of analytic spaces, $\mathcal{X} = \{X_i\}$ and $\mathcal{Y} = \{Y_j\}$ are formal coverings of X and Y, respectively, such that $\mathcal{Y}$ refines $f^{-1}(\mathcal{X})$ and $\widetilde{f}\colon \widetilde{Y}_{\mathcal{Y}} \longrightarrow \widetilde{X}_{\mathcal{X}}$ is the corresponding reduction morphism, then f is closed (resp. proper) if and only if $\widetilde{f}$ is locally proper (resp. proper).

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