A NEW PROOF OF THE GERRITZEN-GRAUERT THEOREM

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Introduction

The Gerritzen-Grauert theorem ([GG], [BGR, 7.3.5/1]) is one of the most important foundational results of rigid analytic geometry. It describes so called locally closed immersions between affinoid varieties, and this description implies the fact that any affinoid subdomain of an affinoid variety is a finite union of rational domains. In its turn, the latter fact allowed one to extend Tate's theorem (see [Tate], [BGR, 8.2.1/1]) on acyclicity of the Čech complex associated to a finite rational covering of an affinoid variety to finite covering by arbitrary affinoid domains. The same fact also plays an important role in foundations of non-Archimedean analytic geometry developed by V. Berkovich in [Ber1] and [Ber2]. Recall that building blocks of the latter are affinoid spaces associated to a class of affinoid algebras broader than that considered in rigid analytic geometry (the latter were called in [Ber1] strictly affinoid) and, besides, the valuation on the ground field is not assumed to be nontrivial. In the recent papers by A. Ducros [Duc, 2.4] and the author [Tem, 3.5], the above fact on the structure of affinoid domains was extended to arbitrary affinoid spaces, but its proof was based on the case of strictly affinoid ones (i.e., affinoid varieties).

The original proof of the Gerritzen-Grauert theorem is not easy, and since then the only different proof was found by M. Raynaud in the framework of his approach to rigid analytic geometry (see [Ray], [BL]). Although that proof is more conceptual, it is based on a complicated algebraic technics.

The purpose of this paper is to give a new proof of the Gerritzen-Grauert theorem which uses basic properties of affinoid algebras in a standard way. The only novelty is in using the whole spectrum $\mathcal{M}(\mathcal{A})$ of an affinoid algebra $\mathcal{A}$, introduced in [Ber1], instead of the maximal spectrum $\mathrm{Max}(\mathcal{A})$, considered in rigid analytic geometry. The use of the whole spectrum allows one to apply additional but standard compactness arguments.

In §§1-2, we work in the setting of rigid analytic geometry, i.e., the valuation on the ground field is assumed to be nontrivial and only the class of strictly affinoid algebras is considered. In §1, we recall basic definitions of an affinoid algebra, an affinoid domain, all notions necessary for the formulation of the Gerritzen-Grauert theorem, and formulate it (Theorem 1.1). The only new fact is Proposition 1.2 which establishes the simple fact that a morphism of affinoid varieties is a locally closed immersion if and only if it is a monomorphism in the category of affinoid varieties. It is only the latter property of locally closed immersions which is used in our proof of Theorem 1.1. In §2, before giving the proof we recall the definition and basic properties of the spectrum $\mathcal{M}(\mathcal{A})$ and establish a property of monomorphisms (whose particular case was proven in [Ber1] with the use of the Gerritzen-Grauert theorem). We also recall basic properties of affinoid algebras which are used in our proof. The most important of them is the fact that a homomorphism of affinoid algebras $\mathcal{A} \to \mathcal{B}$ is finite if and only if the induced homomorphism between their reductions $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is also finite ([BGR, 6.3.5/1]). In §3, we work in the general setting, i.e., the valuation on the ground field is not assumed to be nontrivial and the whole class of affinoid algebras is considered. We give a definition of an affinoid domain in an affinoid space which is slightly different from that given in [Ber1, 2.2] but whose equivalence to it is established as a consequence of the generalized Gerritzen-Grauert theorem (Theorem 3.1). In the formulation of the latter one uses the notion of a monomorphism (instead of that of a locally closed immersion which does not work in the general setting). The proof of Theorem 3.1 is the same as that of Theorem 1.1 with the only difference that one should use the notion of the reduction $\widetilde{\mathcal{A}}_{\mathrm{gr}}$ of an affinoid algebra $\mathcal{A}$, introduced in [Tem], instead of the usual reduction $\widetilde{\mathcal{A}}$ (which does not work for arbitrary affinoid algebras).

1. Formulation of the Gerritzen-Grauert theorem

Let k be a non-Archimedean field with a nontrivial valuation. Recall that a k-affinoid algebra is a commutative Banach k-algebra isomorphic to a quotient of the algebra of convergent power series (on the closed unit polydisc) $k\{T_1, \dots, T_n\}$, $n \geq 1$. (Such an algebra is called in [Ber1] and will be called in §3 strictly k-affinoid.) Recall also that any k-algebra homomorphism between affinoid algebras is bounded, as a map of Banach spaces (see [BGR, 6.1.3/1]). The category of affinoid varieties is, by definition, the category opposite to that of affinoid algebras (see [BGR, 7.1.4]). For brevity, the affinoid variety that corresponds to an affinoid algebra $\mathcal{A}$ will be mentioned by its maximal spectrum $X_0 = \mathrm{Max}(\mathcal{A})$, and the morphism that corresponds to a homomorphism of affinoid algebras $\mathcal{A} \to \mathcal{B}$ will be mentioned by the induced map of maximal spectra $Y_0 = \mathrm{Max}(\mathcal{B}) \to X_0 = \mathrm{Max}(\mathcal{A})$. (The letters X and Y without subscript 0 will be reserved for the whole spectra $\mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{B})$.) The category of affinoid varieties admits fiber products (which correspond to complete tensor products of affinoid algebras).

Recall ([BGR, 7.2.2]) that an affinoid subdomain of a k-affinoid variety $X_0 = \mathrm{Max}(A)$ is a subset $V_0 \subset X_0$ such that there is a morphism $\varphi : \mathrm{Max}(\mathcal{A}_V) \to X_0$ with $\mathrm{Im}(\varphi) \subset V_0$ and, for any morphism $\psi : \mathrm{Max}(\mathcal{B}) \to X_0$ with $\mathrm{Im}(\psi) \subset V_0$, there exists a unique morphism $\mathrm{Max}(\mathcal{B}) \to \mathrm{Max}(\mathcal{A}_V)$ whose composition with φ is ψ. One shows that, for such a subset V_0, the morphism φ is unique up to a unique isomorphism and the induced map $\mathrm{Max}(\mathcal{A}_V) \to V_0$ is a bijection. In what follows, we identify V_0 with the

k-affinoid variety $\mathrm{Max}(\mathcal{A}_V)$ and call the morphism φ an *affinoid domain embedding*. The class of such morphisms is preserved by compositions and any base change.

For example, let $f_1, \dots, f_n, g$ be elements of $\mathcal{A}$ without common zeros in $X_0 = \mathrm{Max}(\mathcal{A})$. Then the subset $X\{\frac{f}{g}\} = \{x \in X_0 \big| |f_i(x)| \le |g(x)|\}$ is an affinoid subdomain, called *rational*, that corresponds to the homomorphism $\mathcal{A} \to \mathcal{A}\{\frac{f}{g}\} = \mathcal{A}\{T_1, \dots, T_n\}/(gT_i - f_i)$. If $g = 1$, the affinoid subdomain is called *Weierstrass* and denoted by $X\{f\}$. Another particular example of a rational subdomain is a *Laurent* one $X\{f, g^{-1}\} = \{x \in X_0 \big| |f_i(x)| \le 1, |g_j(x)| \ge 1\}$ defined by elements $f_1, \dots, f_n, g_1, \dots, g_m \in \mathcal{A}$ and the homomorphism $\mathcal{A} \to \mathcal{A}\{f, g^{-1}\} = \mathcal{A}\{T_1, \dots, T_n, S_1, \dots, S_m\}/(T_i - f_i, S_j g_j - 1)$. The above three classes of affinoid subdomains are preserved by intersection and any base change, and the property to be Weierstrass or rational is a transitive one.

For a point $x \in X_0 = \mathcal{M}(\mathcal{A})$, the inductive limit $\varinjlim \mathcal{A}_V$, taken over all affinoid subdomains V_0 that contain x, is a local Noetherian ring (see [BGR, 7.3.2]). This ring is denoted by $\mathcal{O}_{X,x}$. Its residue field, denoted by $\kappa(x)$, is a finite extension of k, and it coincides with the quotient of $\mathcal{A}$ by the maximal ideal that corresponds to x.

A morphism of affinoid varieties $\varphi : Y_0 \to X_0$ is said to be a *locally closed immersion* if it is injective and, for every point $y \in Y_0$, the induced homomorphism $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ is surjective (see [BGR, 7.3.3]). Locally closed immersions are evidently preserved by compositions. An example of such a morphism is a *Runge immersion* which is defined as a composition of a closed immersion and a Weierstrass domain embedding ([BGR, 7.3.4]). Any Runge immersion which is an affinoid domain embedding is a Weierstrass domain embedding ([BGR, 7.3.4/6]).

Theorem 1.1 *(Gerritzen-Grauert). Let $\varphi : Y_0 \to X_0$ be a locally closed immersion of affinoid varieties. Then there is a finite covering of X by rational subdomains X_0^i such that all of the morphisms $\varphi_i : \varphi^{-1}(X_0^i) \to X_0^i$ are Runge immersions.*

A simple proof of Theorem 1.1 will be given in §2. We finish this section with a category theoretical characterization of locally closed immersions.

Recall that a morphism $\varphi : Y \to X$ in a category $\mathcal{C}$ is called a *monomorphism* if, for all objects Z of $\mathcal{C}$, the induced maps $\mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ are injective. Assume that $\mathcal{C}$ admits fiber products. Then a morphism $\varphi : Y \to X$ is a monomorphism if and only if the diagonal morphism $\Delta : Y \to Y \times_X Y$ is an isomorphism (see [EGAI, 5.3.8]).

Proposition 1.2. *A morphism of affinoid varieties $\varphi : Y_0 \to X_0$ is a locally closed immersion if and only if it is a monomorphism in the category of affinoid varieties.*

Proof. Let $X_0 = \mathrm{Max}(\mathcal{A})$ and $Y_0 = \mathrm{Max}(\mathcal{B})$. Assume first that φ is a locally closed immersion. Then for each point $y \in Y_0$ the induced homomorphism $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ is surjective. Since the canonical homomorphism from $\mathcal{B}$ to the product of all $\mathcal{O}_{Y,y}$ is injective ([BGR, 7.3.2/4]), it follows that φ is a monomorphism.

Conversely, assume that φ is a monomorphism. Then the co-diagonal homomorphism $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B} \to \mathcal{B}$ is an isomorphism and, therefore, for every point $x \in X_0$ from the image of φ one has $\mathcal{B}_x\widehat{\otimes}_{\kappa(x)}\mathcal{B}_x \widetilde{\to} \mathcal{B}_x$, where $\mathcal{B}_x = \mathcal{B}\otimes_{\mathcal{A}}\kappa(x)$. Since the canonical map from a tensor product of Banach spaces over a non-Archimedean field to their complete tensor product is injective (see [Gru, 3.2.1(4)]), it follows that $\kappa(x)\widetilde{\to}\mathcal{B}_x$. The latter implies that φ is injective and, by [BGR, 7.2.5/2], φ is a locally closed immersion. ∎

2. Proof of Theorem 1.1

Recall ([Ber1, Ch. 1]) that the spectrum $\mathcal{M}(\mathcal{A})$ of a commutative Banach ring $\mathcal{A}$ is the space of all non-zero bounded multiplicative semi-norms $|\ | : \mathcal{A} \to \mathbf{R}_+$. Every point $x \in \mathcal{M}(\mathcal{A})$, defines a norm on the quotient ring of $\mathcal{A}/\mathrm{Ker}(|\ |_x)$ and, therefore, it extends to a norm on its fraction field. The completion of the latter is denoted by $\mathcal{H}(x)$, the character $\mathcal{A} \to \mathcal{H}(x)$ is denoted by χ_x, and the image of an element $f \in \mathcal{A}$ under χ_x is denoted by $f(x)$. The space $\mathcal{M}(\mathcal{A})$ is provided with the weakest topology with respect to which all real functions of the form $x \mapsto |f(x)|$ with $f \in \mathcal{A}$ are continuous, and it is always a non-empty compact space.

In what follows, the k-affinoid variety corresponding to a k-affinoid algebra $\mathcal{A}$ will be mentioned by its spectrum $X = \mathcal{M}(\mathcal{A})$. The maximal spectrum $X_0 = \mathrm{Max}(\mathcal{A})$ coincides with the set of all points $x \in X$ with $[\mathcal{H}(x) : k] < \infty$ (for such a point $\mathcal{H}(x) = \kappa(x)$), and X_0 is dense in X ([Ber1, 2.1.15]). Notice that every point of X has a fundamental system of neighborhoods consisting of Laurent domains.

Proposition 2.1. *Let $\varphi : Y \to X$ be a monomorphism of affinoid varieties. Then for every point $y \in Y$ with $x = \varphi(y)$ one has $\varphi^{-1}(x) = \{y\}$ and $\mathcal{H}(x)\widetilde{\to}\mathcal{H}(y)$.*

Proof. Let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$. It suffices to show that the canonical map $\mathcal{H}(x) \to \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{H}(x)$ is an isomorphism. By the assumption, the co-diagonal homomorphism $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B} \to \mathcal{B}$ is an isomorphism. It follows that its base change $\mathcal{B}_K\widehat{\otimes}_K\mathcal{B}_K \to \mathcal{B}_K = \mathcal{B}\widehat{\otimes}_{\mathcal{A}}K$ with respect to the homomorphism $\mathcal{A} \to K = \mathcal{H}(x)$ is also an isomorphism. The same fact, used in the proof of Proposition 1.3, implies that the canonical homomorphism $\mathcal{B}_K\otimes_K\mathcal{B}_K \to \mathcal{B}_K\widehat{\otimes}_K\mathcal{B}_K \to \mathcal{B}_K$ is injective and, therefore, $\mathcal{H}(x)\widetilde{\to}\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{H}(x)$. ∎

For an affinoid algebra $\mathcal{A}$, one sets $\widetilde{\mathcal{A}} = \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$, where $\mathcal{A}^\circ = \{f \in \mathcal{A} \big| \rho(f) \le 1\}$, $\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} \big| \rho(f) < 1\}$ and $\rho(f) = \max\{|f(x)| \big| x \in \mathcal{M}(\mathcal{A})\}$ (the spectral seminorm of f), and one denotes by $\widetilde{g}$ the image of an element

$g \in \mathcal{A}^\circ$ in $\widetilde{\mathcal{A}}$. By [BGR, 6.3.4/3], $\widetilde{\mathcal{A}}$ is a finitely generated $\widetilde{k}$-algebra. Every point $x \in X = \mathcal{M}(\mathcal{A})$ defines a character $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$ whose kernel is a prime ideal of $\widetilde{\mathcal{A}}$, and the correspondence $x \mapsto \mathrm{Ker}(\widetilde{\chi}_x)$ defines a reduction map $\pi : X \to \widetilde{X} = \mathrm{Spec}(\widetilde{\mathcal{A}})$, which is anti-continuous in the sense that the preimage of an open subset is closed (see [Ber1, §2.4]). It is very easy to show (see [Ber1, 2.4.3]) that the image of π contains all of the generic points of irreducible components of $\widetilde{X}$. (By [Ber1, 2.4.4], the reduction map is even surjective.)

Lemma 2.2. *Let $X' = \mathcal{M}(\mathcal{A}')$ be a Laurent subdomain of $X = \mathcal{M}(\mathcal{A})$ of the form $X\{f, g^{-1}\}$ with $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{A}$. Then*

(i) the algebra $\widetilde{\mathcal{A}}'$ is finite over the subalgebra generated by the elements $\widetilde{f}_1, \ldots, \widetilde{f}_n$, $\widetilde{g}_1^{-1}, \ldots, \widetilde{g}_m^{-1}$ over the image of $\widetilde{\mathcal{A}}$;

(ii) if X' is a neighborhood of a point $x \in X$, then $\widetilde{\chi}_x(\widetilde{\mathcal{A}}')$ is finite over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$ (i.e., it is finitely generated as a $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$-module).

Proof. (i) Consider the surjective homomorphism $\mathcal{A}\{T, S\} \to \mathcal{A}'$ that takes T_i to f_i and S_j to g_j^{-1}. By [BGR, 6.3.5/1], it gives rise to a finite homomorphism $\widetilde{\mathcal{A}}[T, S] \to \widetilde{\mathcal{A}}'$, and the required fact follows.

(ii) We can replace X' by a smaller Laurent neighborhood of x which is represented in the same form but with other elements f_i and g_j such that $|f_i(x)| < 1$ and $|g_j(x)| > 1$ for all $1 \le i \le n$ and $1 \le j \le m$. In this case the required fact follows from (i). ∎

We are now ready to prove Theorem 1.1. Let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$. *We claim that it suffices to show that, for every point $y \in Y$, there is a rational domain U which contains the point $x = \varphi(y)$ and such that $V = \varphi^{-1}(U)$ is a neighborhood of y in Y and the induced morphism $V \to U$ is a closed immersion.* Indeed, assume this is true, and let $U = X\{\frac{f}{g}\}$ for some $f_1, \ldots, f_n, g \in \mathcal{A}$. Since g is invertible on U, we can find a sufficiently small rational neighborhood W of x in X such that g is invertible on W and $\varphi^{-1}(W) \subset \varphi^{-1}(U)$. Then $U \cap W$ is a Weierstrass subdomain of W and, therefore, the induced morphism $\varphi^{-1}(W) \to W$ is a Runge immersion. Since Y is compact, we can find in this way rational subdomains $X_1, \ldots, X_m$ of X such that each morphism $\varphi^{-1}(X_i) \to X_i$ is a Runge immersion and $X_1 \cup \cdots \cup X_m$ contains an open neighborhood $\mathcal{U}$ of $\varphi(Y)$. Since X is compact, we can find rational subdomains $X_{m+1}, \ldots, X_n$ which do not intersect $\varphi(Y)$ and such that $X \backslash \mathcal{U} \subset \cup_{i=m+1}^n X_i$. Then $\{X_i\}_{1 \le i \le n}$ is a finite covering of X by rational domains and the morphisms $\varphi^{-1}(X_i) \to X_i$ are Runge immersions for all $1 \le i \le n$.

A construction of U as above is done as follows by shrinking X, i.e., replacing it by a rational domain X' that contains x and such that $Y' = \varphi^{-1}(X')$ is a neighborhood of y in Y. (Notice that the induced morphism $Y' \to X'$ is a base change of φ and, therefore, it is also a monomorphism.)

Step 1. *Shrinking X, one may assume that $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$.* Indeed, let $h_1,\dots,h_n$ be elements of $\mathcal{B}^\circ$ such that $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$ is finite over the $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$-subalgebra generated by $\widetilde{h}_1,\dots,\widetilde{h}_n$. By Proposition 2.1, $\mathcal{H}(x)\widetilde{\to}\mathcal{H}(y)$ and, therefore, we can find elements $f_1,\dots,f_n,g\in\mathcal{A}$ with $|g(x)|=1$ and $|(\frac{f_i}{g}-h_i)(y)|<1$ for all $1\le i\le n$. Let $X'=X\{\frac{a}{g}\}$ and $Y'=\varphi^{-1}(X')$ for some $a\in k^*$ with $|a|<1$. By Lemma 2.2(ii), $\widetilde{\chi}_y(\widetilde{\mathcal{B}'})$ is finite over $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$ and, therefore, we can replace X and Y by X' and Y', respectively, and assume that g is invertible in $\mathcal{A}$. Replacing each f_i by $\frac{f_i}{g}$, we may assume that $g=1$. Furthermore, if $X'=X\{f_1,\dots,f_n\}$, then $Y'=\varphi^{-1}(X')$ is a neighborhood of y and, therefore, we can replace X and Y by X' and Y', respectively, and assume that $f_i\in\mathcal{A}^\circ$ for all $1\le i\le n$. But in this case $\widetilde{h}_i=\widetilde{f}_i$ and, therefore, $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\chi}_x(\widetilde{\mathcal{A}})$.

Step 2. *Shrinking X, one may assume that $\widetilde{\chi}_{y'}(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\chi}_{x'}(\widetilde{\mathcal{A}})$ for all points $y'\in Y$, where $x'=\varphi(y')$.* Indeed, if $\mathcal{Y}$ denotes the Zariski closure of the point $\pi(y)$ in $\widetilde{Y}$, then for every point $y'\in Y$ with $\pi(y')\in\mathcal{Y}$ one has $\mathrm{Ker}(\widetilde{\chi}_y)\subset\mathrm{Ker}(\widetilde{\chi}_{y'})$ and, therefore, $\widetilde{\chi}_{y'}(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\chi}_{x'}(\widetilde{\mathcal{A}})$. Since the set $\pi^{-1}(\mathcal{Y})$ is open in Y, we can find a Laurent neighborhood $X'=X\{f,g^{-1}\}$ of x with $Y'=\varphi^{-1}(X')\subset\pi^{-1}(\mathcal{Y})$. Let $X'=\mathcal{M}(\mathcal{A}')$ and $Y'=\mathcal{M}(\mathcal{B}')$. Then, for every point $y'\in Y'$, $\chi_{y'}(\widetilde{\mathcal{B}'})$ is finite over $\chi_{x'}(\widetilde{\mathcal{A}'})$. Indeed, by Lemma 2.2(i), $\chi_{y'}(\widetilde{\mathcal{B}'})$ and $\chi_{x'}(\widetilde{\mathcal{A}'})$ are finite over the subalgebras generated by the elements $\widetilde{f}_1,\dots,\widetilde{f}_n,\widetilde{g}_1^{-1},\dots,\widetilde{g}_m^{-1}$ over $\chi_{y'}(\widetilde{\mathcal{B}})$ and $\chi_{x'}(\widetilde{\mathcal{A}})$, respectively. Since $\widetilde{\chi}_{y'}(\widetilde{\mathcal{B}})$ is finite over $\widetilde{\chi}_{x'}(\widetilde{\mathcal{A}})$, the required claim follows.

Step 3. *In the situation of Step 2 the morphism φ is a closed immersion.* Indeed, for every minimal prime ideal $\wp$ of $\widetilde{\mathcal{B}}$ there exists a point $y'\in Y$ with $\mathrm{Ker}(\chi_{y'})=\wp$ and, by the assumption, $\widetilde{\mathcal{B}}/\wp$ is finite over $\widetilde{\mathcal{A}}$. If $\wp_1,\dots,\wp_n$ is the set of minimal prime ideals of $\widetilde{\mathcal{B}}$, the homomorphism $\widetilde{\mathcal{B}}\to C=\oplus_{i=1}^n\widetilde{\mathcal{B}}/\wp_i$ is injective because $\widetilde{\mathcal{B}}$ has no nilpotent elements. Since $\widetilde{\mathcal{A}}$ is Noetherian and C is a finite $\widetilde{\mathcal{A}}$-module, its submodule $\widetilde{\mathcal{B}}$ is also finite. By [BGR,6.3.5/1], $\mathcal{B}$ is finite over $\mathcal{A}$ and, in particular, $\mathcal{B}\otimes_{\mathcal{A}}\mathcal{B}\widetilde{\to}\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B}$. Since φ is a monomorphism, it follows that $\mathcal{B}\otimes_{\mathcal{A}}\mathcal{B}\widetilde{\to}\mathcal{B}$, i.e., the finite morphism of affine schemes $\mathrm{Spec}(\mathcal{B})\to\mathrm{Spec}(\mathcal{A})$ is a monomorphism. Thus, it remains to prove the following simple fact: given a ring A and a finite A-algebra B such that $B\otimes_A B\widetilde{\to}B$, the canonical homomorphism $A\to B$ is surjective. Indeed, replacing A by the localization of a prime ideal, we may assume that A is a local ring. By the Nakayama Lemma, we may replace A by its quotient by the maximal ideal, and so we may assume that A is a field. In that case the required fact is trivial.

3. A Gerritzen-Grauert theorem for affinoid spaces

In this section k is a non-Archimedean field whose valuation is not assumed to be nontrivial, and the class of k-affinoid algebras is that introduced in [Ber1]. Namely, let $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ be the commutative Banach k-algebra of convergent power series on the closed polydisc of radii $r_1, \dots, r_n > 0$. (It consists of formal power series $f = \sum_\nu a_\nu T^\nu$ with $|a_\nu| r^\nu \to 0$ as $|\nu| \to \infty\}$ and is provided with the norm $||f|| = \max |a_\nu| r^\nu)$. A k-affinoid algebra is a commutative Banach k-algebra $\mathcal{A}$ isomorphic to a quotient Banach algebra of some $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$. (The algebras which are affinoid in the usual sense as in §1-2, i.e., for which such a representation can be found with $r_i = 1$, $1 \le i \le n$, are called strictly k-affinoid.)

The category of k-affinoid spaces k-$\mathcal{A}ff$ is, by definition, the category opposite to that of k-affinoid algebras with bounded homomorphisms between them. The k-affinoid space corresponding to a k-affinoid algebra $\mathcal{A}$ is mentioned by the spectrum $X = \mathcal{M}(\mathcal{A})$. If $\mathcal{A}$ is strictly k-affinoid, X is also called strictly k-affinoid. The full subcategory of the latter is denoted by *st-k-*$\mathcal{A}ff$. If the valuation on k is nontrivial, *st-k-*$\mathcal{A}ff$ is equivalent to the category of k-affinoid varieties (considered in §§1-2).

An *affinoid domain* in a k-affinoid space $X = \mathcal{M}(\mathcal{A})$ is a subset $V \subset X$ such that there is a morphism $\mathcal{M}(\mathcal{A}_V) \to X$ with $\mathrm{Im}(\varphi) = V$ and, for any morphism $\psi : \mathcal{M}(\mathcal{B}) \to X$ with $\mathrm{Im}(\psi) \subset V$, there exists a unique morphism $\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A}_V)$ whose composition with φ is ψ. This definition is different from that of [Ber1, §2.2] in which instead of the first equality one only required the inclusion $\mathrm{Im}(\varphi) \subset V$ but the second condition was required to hold for K-affinoid algebras $\mathcal{B}$, where K is an arbitrary non-Archimedean field over k. (The equivalence of both definitions is shown in Corollary 3.2.) Furthermore, our definition, applied to the category *st-k-*$\mathcal{A}ff$, gives rise to the notion of a *strictly affinoid domain.* If the valuation on k is nontrivial, the latter is equivalent to the notion of an affinoid domain in the category of k-affinoid varieties (because the maximal spectrum of a strictly k-affinoid algebra is dense in its whole spectrum). It follows easily from Theorem 1.1 that any strictly affinoid domain (in a strictly k-affinoid space) is an affinoid domain (cf. [Ber1, 2.2.3(i)] or the proof of Corollary 3.2).

If V is an affinoid domain in $X = \mathcal{M}(\mathcal{A})$, the corresponding morphism $\varphi : \mathcal{M}(\mathcal{A}_V) \to X$ is called an *affinoid domain embedding.* It follows immediately from the definition, that any affinoid domain embedding is a monomorphism. The proof of Proposition 2.1 is applicable to arbitrary monomorphism of affinoid spaces and, in particular it gives rise to a bijection $\mathcal{M}(\mathcal{A}_V) \widetilde{\to} V$, which allows us to identify V with the k-affinoid space $\mathcal{M}(\mathcal{A}_V)$. Notice also that, if for an affinoid subdomain $Y \subset X$ the canonical morphism $Y \to X$ is a closed immersion, then Y is identified with a union of connected components of X and, in particular, Y is a Weierstrass domain in X (see the following paragraph).

The classes of rational, Weierstrass and Laurent affinoid domains in a k-affinoid space $X = \mathcal{M}(\mathcal{A})$ are broader than those considered in §1. Namely, a rational domain is the set $X\{p^{-1}\frac{f}{g}\} = \{x \in X \big| |f_i(x)| \le p_i|g(x)|\}$ associated to elements $f_1, \dots, f_n, g \in \mathcal{A}$ without common zeros in X and a tuple of positive numbers $p = (p_1, \dots, p_n)$ (it corresponds to the homomorphism $\mathcal{A} \to \mathcal{A}\{p^{-1}\frac{f}{g}\} = \mathcal{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}/(gT_i - f_i))$. If $g = 1$, the affinoid subdomain is called Weierstrass and denoted by $X\{p^{-1}f\}$. Also, a Laurent domain is the set $X\{p^{-1}f, qg^{-1}\} = \{x \in X \big| |f_i(x)| \le p_i, |g_j(x)| \ge q_j\}$ associated to tuples $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_m)$ of elements of $\mathcal{A}$ and tuples of positive numbers $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_m)$. It is easy to see that every point of X has a fundamental system of neighborhoods formed by Laurent domains. Finally, a Runge immersion is defined in the same way as in §1, namely, as a composition of a closed immersion with a Weierstrass domain embedding. The definition of a locally closed immersion is not applicable in the general case, and we work with monomorphisms instead.

Theorem 3.1. *Let $\varphi : Y \to X$ be a monomorphism of affinoid spaces. Then there is a finite covering of X by rational subdomains X_i such that all of the morphisms $\varphi_i : \varphi^{-1}(X_i) \to X_i$ are Runge immersions.*

The proof follows the proof of Theorem 1.1 word for word with the only modification that instead of the reduction $\widetilde{\mathcal{A}}$ of a strictly k-affinoid algebra $\mathcal{A}$ one uses the reduction $\widetilde{\mathcal{A}}_{gr}$ of an arbitrary k-affinoid algebra $\mathcal{A}$ as introduced in [Tem]. Recall that $\widetilde{\mathcal{A}}_{\rm gr}$ is the $\mathbf{R}_+^*$-graded algebra $\oplus\widetilde{\mathcal{A}}_r^\circ/\widetilde{\mathcal{A}}_r^{\circ\circ}$, where $\widetilde{\mathcal{A}}_r^\circ = \{f \in \mathcal{A} \big| \rho(f) \le r\}$ and $\widetilde{\mathcal{A}}_r^{\circ\circ} = \{f \in \mathcal{A} \big| \rho(f) < r\}$. (Notice that the usual reduction $\widetilde{\mathcal{A}}$ coincides with $\widetilde{\mathcal{A}}_1$.) One shows in [Tem,§3] that $\widetilde{\mathcal{A}}_{\rm gr}$ is a finitely generated graded $\widetilde{k}_{\rm gr}$-algebra, and a bounded homomorphism of k-affinoid algebras $\mathcal{A} \to \mathcal{B}$ is finite if and only if the homomorphism $\widetilde{\mathcal{A}}_{\rm gr} \to \widetilde{\mathcal{B}}_{\rm gr}$ is finite. Furthermore, one denotes by $\widetilde{X}_{\rm gr}$ the graded spectrum of $\widetilde{\mathcal{A}}_{\rm gr}$, i.e., the space of all homogeneous ideals $\wp$ with the property that, if $ab \in \wp$ for homogeneous elements a and b, then one of them is contained in $\wp$. Every point $x \in X$ defines a graded character $\widetilde{\chi}_x : \widetilde{\mathcal{A}}_{\rm gr} \to \widetilde{\mathcal{H}(x)}_{\rm gr}$, and the correspondence $x \mapsto {\rm Ker}(\widetilde{\chi}_x)$ defines a surjective anti-continuous map $\pi : X \to \widetilde{X}_{\rm gr}$. The formulation and proof of Lemma 2.2 take place with the reduction map $\mathcal{A} \to \widetilde{\mathcal{A}}_{\rm gr} : f \mapsto \widetilde{f}$, where $\widetilde{f}$ is the image of f in $\widetilde{\mathcal{A}}_r^\circ/\widetilde{\mathcal{A}}_r^{\circ\circ}$ with $r = \rho(f)$, and the remaining part of the proof of Theorem 1.1 works in the general situation without any changes.

Theorem 3.1 straightforwardly implies that any affinoid domain in an affinoid space is a finite union of rational domains.

3.2. Corollary. *A subset Y of a k-affinoid space X is an affinoid domain in the sense of [Ber1, §2.2] if and only if it is an affinoid domain (in the sense of the above definition).*

Proof. The direct implication is trivial. Assume that Y is an affinoid domain, and let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$. We have to show that any bounded homomorphism $\mathcal{A} \to \mathcal{C}$, such that the image of $Z = \mathcal{M}(\mathcal{C})$ in X is contained in Y with $\mathcal{C}$ a K-affinoid algebra for a non-Archimedean field K over k, can be extended in a unique way to a bounded homomorphism $\mathcal{B} \to \mathcal{C}$. This is obviously true if Y is a rational domain. In the general case, we take a finite covering of Y by rational subdomains $Y_i = \mathcal{M}(\mathcal{B}_i)$ of X and use the exact admissible sequences $0 \to \mathcal{B} \to \prod_i \mathcal{B}_i \to \prod_{i,j} \mathcal{B}_{i,j}$ and $0 \to \mathcal{C} \to \prod_i \mathcal{C}_i \to \prod_{i,j} \mathcal{C}_{i,j}$ provided by the Tate acyclicity theorem for rational coverings, where $\mathcal{C}_i$ and $C_{i,j}$ are the K-affinoid algebras of the preimages of Y_i and $Y_i \cap Y_j$ in Z, respectively. ∎

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