ENDOMORPHISMS OF POWER SERIES FIELDS AND RESIDUE FIELDS OF FARGUES-FONTAINE CURVES

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Abstract. We show that for k a perfect field of characteristic p, there exist endomorphisms of the completed algebraic closure of $k((t))$ which are not bijective. As a corollary, we resolve a question of Fargues and Fontaine by showing that for p a prime and $\mathbb{C}_p$ a completed algebraic closure of $\mathbb{Q}_p$, there exist closed points of the Fargues-Fontaine curve associated to $\mathbb{C}_p$ whose residue fields are not (even abstractly) isomorphic to $\mathbb{C}_p$ as topological fields.

1. Introduction

In this short note, we address the following question. By an *analytic field*, we will always mean a field complete with respect to a nonarchimedean multiplicative absolute value (assumed to be real-valued and written multiplicatively); by default, we always allow the trivial absolute value.

Question 1.1. *Let K be an analytic field. Let k be a trivially valued subfield of K. Is every continuous k-linear homomorphism from K to itself which induces automorphisms of residue fields and value groups necessarily surjective (and hence an automorphism)?*

We will view Question 1.1 as a collection of distinct cases indexed by the choice of K, k. For example, one has affirmative answers in the following cases:

- when K is trivially valued, discretely valued, or more generally spherically complete (Proposition 3.1);
- when $\mathrm{char}(k) = 0$ and K is the completed algebraic closure of a power series field over k (Remark 3.3);

whereas one has negative answers in the following cases:

- in certain cases in characteristic 0 (Example 3.2);
- when $\mathrm{char}(k) > 0$ and K is the completed perfect closure of a power series field over k (see [9]).

Hereafter, fix a prime number p. Our main result is a negative answer to Question 1.1 when $\mathrm{char}(k) = p$ and K is the completed algebraic closure of a power series field over k.

Theorem 1.2. *Let K be a completed algebraic closure of $k((t))$ for some field k of characteristic p. Then there exists a continuous k-linear homomorphism $\tau : K \to K$ which is not an isomorphism.*

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The proof depends on a calculation using completed modules of Kähler differentials of analytic fields, as recently studied by the second author [12]. We develop here the bare minimum of this subject needed for the proof of Theorem 1.2; a more detailed treatment of completed differentials between analytic fields will be given by the second author elsewhere.

Theorem 1.2 was prompted by an application to a foundational question of p-adic Hodge theory, specifically in the *perfectoid correspondence* (commonly known as *tilting*) between nonarchimedean fields in mixed and equal characteristics (generalizing the *field of norms correspondence* of Fontaine and Wintenberger). A nonarchimedean field K of residue characteristic p is *perfectoid* if it is not discretely valued and the Frobenius automorphism on $\mathfrak{o}_K/(p)$ is surjective. Given such a field, let $K^\flat$ be the inverse limit of K under the p-power map; one then shows that $K^\flat$ naturally carries the structure of a perfectoid (and hence perfect) nonarchimedean field of equal characteristic p and that there is a canonical isomorphism between the absolute Galois groups of K and $K^\flat$ [8, 10, 11]. The functor $K \mapsto K^\flat$ is not fully faithful, even on fields of characteristic 0; for instance, one can construct many algebraic extensions of $\mathbb{Q}_p$ whose completions K map to the completed perfect closure of a power series field over $\mathbb{F}_p$ (e.g., the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$ and the Kummer extension $\mathbb{Q}_p(p^{1/p^\infty})$). However, Fargues and Fontaine have asked [5, Remark 2.24] (see also [4]) whether this can happen for a completed algebraic closure of $\mathbb{Q}_p$, and using Theorem 1.2 we are able to answer this question.

Theorem 1.3. *Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$. Then there exists a perfectoid field K which is not isomorphic to $\mathbb{C}_p$ as a topological field, but for which there exists an isomorphism $K^\flat \cong \mathbb{C}_p^\flat$.*

This result admits the following geometric interpretation. For each perfectoid field K, Fargues and Fontaine define an associated scheme X_K which is a "complete curve" (i.e., a regular one-dimensional noetherian scheme equipped with a surjection of its Picard group onto $\mathbb{Z}$) in terms of which p-adic Hodge theory over K can be simply formulated. Theorem 1.3 implies that for $K = \mathbb{C}_p$, there exists a closed point of X_K whose residue field is not isomorphic to $\mathbb{C}_p$.

2. Analytic fields and completed differentials

As a technical input into the proof of Theorem 1.2, we review some basic properties of analytic fields and completed differentials.

Definition 2.1. By an *analytic field*, we will mean a field equipped with a multiplicative nonarchimedean absolute value with respect to which the field is complete. By default, we allow the trivial absolute value. When we consider an *extension* L/K of analytic fields, we require that the absolute value on L restricts to the absolute value on K.

Definition 2.2. We say that an extension L/K of analytic fields is *primitive* if there exists $t \in L^\times$ such that $K(t)$ is dense in L; we will write $L = \widehat{K(t)}$ if we need to indicate the choice of t.

With t given, the extension $\widehat{K(t)}/K$ corresponds to a point in the projective line over K in the category of Berkovich nonarchimedean analytic spaces [2]. Without t given, the points associated to L/K are all of the same type 1–4 in Berkovich's classification [2, (1.4.4)]; we thus classify L/K accordingly. Write

$$E_{L/K} = \dim_{\mathbb{Q}}(|L^\times|/|K^\times|) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad F_{L/K} = \operatorname{trdeg}_{\kappa(K)} \kappa(L),$$

where $\kappa(*)$ denotes the residue field of $*$; these are determined by the type of L/K as follows.

Type of L/K	$E_{L/K}$	$F_{L/K}$
1	0	0
2	0	1
3	1	0
4	0	0

In all cases we have $E_{L/K} + F_{L/K} \leq 1$, as per Abhyankar's inequality. However, types 1 and 4 cannot be distinguished using $E_{L/K}$ and $F_{L/K}$ alone: one must instead observe that L/K is of type 1 if and only if the algebraic closure of K in L is dense.

In order to better distinguish between primitive extensions of types 1 and 4, we will use completed modules of differentials.

Definition 2.3. Let L/K be an extension of analytic fields. As described in [12, §4], the module $\Omega_{L/K}$ admits a maximal seminorm $\|\bullet\|$ (the *Kähler seminorm*) with respect to which $d_{L/K} : L \to \Omega_{L/K}$ is nonexpanding. Let $\widehat{\Omega}_{L/K}$ denote the completion of $\Omega_{L/K}$ with respect to $\|\bullet\|$; it receives an induced derivation $\widehat{d}_{L/K} : L \to \widehat{\Omega}_{L/K}$.

Lemma 2.4. *Let L/K be a primitive extension, and choose $t \in L$ such that $K(t)$ is dense in L.*

(a) *The module $\widehat{\Omega}_{L/K}$ is generated over L by the single element $\widehat{d}_{L/K}(t)$.*

(b) *The equality $\widehat{\Omega}_{L/K} = 0$ holds if and only if the separable closure of K in L is dense. (Note that this condition implies that L/K is of type 1, and conversely whenever $\operatorname{char}(K) = 0$.)*

Proof. Since $\Omega_{K(t)/K}$ is generated by $d_{L/K}(t)$, (a) is obvious.

Let l be the separable integral closure of K in L. If l is dense in L (which forces L/K to be of type 1), then $\Omega_{l/K} = 0$ and so $\widehat{\Omega}_{L/K} = 0$. This proves the inverse implication in (b).

Suppose that L/K is not of type 1. Let K' be a completed algebraic closure of K and put $L' = l\widehat{\otimes}_K K'$; then the natural map $\Omega_{L/K}\widehat{\otimes}_L L' \to \Omega_{L'/K'}$ sends $\widehat{d}_{L/K}(t) \otimes 1$ to $\widehat{d}_{L'/K'}(t)$. The latter is nonzero by [3, Theorem 2.3.2(i)], so $\widehat{d}_{L/K}(t) \neq 0$.

It remains to consider the case when L/K is of type 1 but l is not dense in L. (Note that this last step is not needed for the proof of Theorem 1.2, so the uninterested reader can skip it.) It suffices to show that $\widehat{d}_{L/\widehat{l}}(t) \neq 0$, so replacing K by $\widehat{l}$ we can assume that $K = l$, and then the Ax-Sen theorem [1] implies that $p = \operatorname{char} K > 0$ and $t \in \widehat{K^{1/p^\infty}} \setminus K$. Choose $a_0 = 0, a_1, \ldots \in K$ such that the sequence $r_n = |t - a_n^{1/p^n}|$ converges to zero. Then L is the completion of its subalgebra $\bigcup_n k\{r_0^{-1}t, r_n^{-p^n}(t^{p^n} - a_n)\}$; in particular, $k[t]$ is dense in L. Consider the Banach ring $\mathcal{A} := L\widehat{\otimes}_K L$ provided with the tensor product norm $\|\bullet\|$ and note that the ideal $J = \operatorname{Ker}(\mathcal{A} \to L)$ is generated by $T := 1 \otimes t - t \otimes 1$.

We claim that $\|T^{p^n}\| \leq r_n^{p^n}$. Indeed, since $|t^{p^n} - a_n| = r_n^{p^n}$, we have that $\|1 \otimes t^{p^n} - a_n\| \leq r_n^{p^n}$ and $\|t^{p^n} \otimes 1 - a_n\| \leq r_n^{p^n}$. (Note, for the sake of completeness, that T is quasi-nilpotent, i.e. its spectral norm vanishes, and hence L is the uniform completion of $\mathcal{A}$, i.e. the completion with respect to the spectral seminorm. This is a topological extension of the classical fact that T is nilpotent and L is the reduction of $\mathcal{A}$ when L/K is finite and purely inseparable.)

By [12, Remark 4.3.4(ii)], there is an isomorphism $J/J^2 \xrightarrow{\sim} \widehat{\Omega}_{L/K}$ that takes T to $\widehat{d}_{L/K}(t)$. Thus, we should only show that $T \neq aT^2$ in $\mathcal{A}$. Assume, to the contrary, that $T = aT^2$ and set $s = \|a\|$. Then, $\|T\| = \|a^{p^n-1}T^{p^n}\| \leq s^{p^n-1}r_n^{p^n}$ for any n and hence $\|T\| = 0$. Thus $L\widehat{\otimes}_K L = L$ and since $L \otimes_K L$ embeds into $L\widehat{\otimes}_K L$ by [6, 3.2.1(4)], we obtain a contradiction. □

3. Proofs and examples

We now settle the questions raised in the introduction.

Proposition 3.1. *Question 1.1 admits an affirmative answer if K is spherically complete.*

Proof. Let $\tau : K \to K$ be a homomorphism as in Question 1.1. Suppose by way of contradiction that there exists $x \in K$ with $x \notin \tau(K)$. Since K is spherically complete, the set of possible valuations of $x - \tau(y)$ for $y \in K$ has a least element. If y realizes this valuation, then by the matching of value groups, we can find $y' \in K$ such that $\tau(y')$ and $x - \tau(y)$ have the same valuation; by the matching of residue fields, we can further choose y' such that $(x - \tau(y))/\tau(y')$ maps to 1 in k. But then $x - \tau(y + y')$ has smaller valuation than $x - \tau(y)$, a contradiction. □

Example 3.2. Let k be an analytic field whose absolute value is nondiscrete, and choose a sequence $x_1, x_2, \ldots \in k^\times$ such that $|x_i| < 1$ and $\lim_n |x_1 \cdots x_n| > 0$. (For a more concrete example, take k to be a completed algebraic closure of $\mathbb{C}((t))$ and take $x_n = t^{2^{-n}}$.) Let K be the completion of $k(t_1, t_2, \ldots)$ for the Gauss valuation (i.e., the valuation of a nonzero polynomial is the maximum valuation of its coefficients); then K admits a unique valuation-preserving endomorphism τ fixing k and taking t_n to $t_n - x_n t_{n+1}$ for each n. We will show that the image of τ does not contain t_1, and hence τ is not an isomorphism.

Suppose to the contrary that there exists $y \in K$ with $\tau(y) = t_1$. By hypothesis, there exists some $\lambda \in k$ such that $|\lambda| < |x_1 \cdots x_n|$ for all n. We may then choose $y' \in K_0(t_1, \ldots, t_n)$ for some positive integer n in such a way that $|y - y'| < |\lambda|$. Put $y'' = t_1 + x_1 t_2 + \cdots + x_1 \cdots x_n t_{n+1}$; then $\tau(y'') = t_1 - x_1 \cdots x_{n+1} t_{n+2}$, so $|y'' - y| = |\tau(y'' - y)| = |x_1 \cdots x_{n+1}| > |\lambda|$. Hence $|y'' - y'| = |x_1 \cdots x_{n+1}|$, but $y'' - y'$ equals $x_1 \cdots x_n t_{n+1}$ plus an element of $k(t_1, \ldots, t_n)$ and so cannot have valuation less than $|x_1 \cdots x_n|$. This yields the desired contradiction.

Remark 3.3. Let k be a field of characteristic 0. For each positive integer n, the derivation $\frac{d}{dt}$ on $k((t))$ extends to the derivation $\partial_n = n^{-1} t^{1/n-1} \frac{d}{dt^{1/n}}$ on $k((t^{1/n}))$ satisfying $|\partial_n f| \leq |t|^{-1} |f|$ for any $f \in k((t^{1/n}))$. Let K be a completed algebraic closure of $k((t))$; by Puiseux's theorem, K is the completion of $\bigcup_{n=1}^\infty \overline{k}((t^{1/n}))$ for $\overline{k}$ the algebraic closure of k in K, so the derivation $\frac{d}{dt}$ extends uniquely to a continuous derivation on K. Consequently, $\widehat{\Omega}_{K/k}$ is generated by $\widehat{d}_{K/k}(x)$ for any $x \in K - \overline{k}$. For any k-linear automorphism τ of K, let L be the completion of $\tau(K)(t)$ within K; taking $x = \tau(t)$ in the previous discussion shows that $\widehat{\Omega}_{L/\tau(K)} = 0$. By Lemma 2.4(b) we conclude that $L/\tau(K)$ is of type 1. Since $\tau(K)$ is algebraically closed, it follows that $L = \tau(K)$ and hence τ is an isomorphism.

Proof of Theorem 1.2. Choose a sequence $\{d_i\}_{i=1}^\infty$ of positive integers in such a way that:

(a) d_i is not divisible by p;

(b) $\lim_{i\to\infty}(d_{i+1} - pd_i) = \infty$; and

(c) the sequence $\{p^{-i}d_i\}_{i=1}^\infty$ is strictly increasing (for large i, this follows from (b)) and bounded.

For a concrete example, take

$$d_i := 1 + pi + p^2(i-1) + p^3(i-2) + \cdots + p^i.$$

Choose a sequence $\{c_i\}_{n=1}^\infty$ of elements of k such that each field $\mathbb{F}_p(c_i)$ is finite, but the field $\mathbb{F}_p(c_1, c_2, \dots)$ is infinite (in fact any $c_i \neq 0$ will do, but this assumption shortens the argument). Set

$$\alpha_n := \sum_{i=0}^{n} c_i t^{p^{-i}d_i} \in K$$

and

$$r_n := |\alpha_{n+1} - \alpha_n| = |t|^{p^{-n-1}d_{n+1}};$$

by construction, $\{r_n\}_{n=1}^\infty$ is a strictly decreasing sequence with nonzero limit. Let E_n be the closed disc of radius r_n centered at α_n; the intersection of the E_n does not contain any element of any finite extension of $k((t))$ (e.g., by [7, Theorem 4.1.3]), so it consists of a single point of type 4. This point corresponds to a primitive extension L of $k((t))$ topologically generated by an element x for which $\left|x^{p^n} - \alpha_n^{p^n}\right| = r_n^{p^n}$ for each n. Since $\widehat{d}_{L/k}$ is nonexpanding, we have $\left\|\widehat{d}_{L/k}(\alpha_n^{p^n})\right\| \leq r_n^{p^n}$. Furthermore, in the expression $\alpha_n^{p^n} = \sum_{i=0}^n t^{p^{n-i}d_i}$ only the term t^{d_n} is not a p-th power, so

$$\widehat{d}_{L/k}(\alpha_n^{p^n}) = \widehat{d}_{L/k}(t^{d_n}) = d_n t^{d_n - 1}\widehat{d}_{L/k}(t)$$

and hence (since d_n is not divisible by p)

$$\left\|\widehat{d}_{L/k}(t)\right\| \leq r_n^{p^n}|t|^{1-d_n} = |t|^{p^{-1}d_{n+1} - d_n + 1}.$$

Since this holds for all n, we conclude that $\left\|\widehat{d}_{L/k}(t)\right\| = 0$.

By the previous paragraph, $\widehat{d}_{L/k}(t) = 0$ and hence $\widehat{d}_{L/k((x))}(t) = 0$. By Lemma 2.4, $L/k((x))$ is a primitive extension of type 1; the inclusion $k((x)) \to L$ thus induces an isomorphism of completed algebraic closures. That is, t belongs to the completed algebraic closure of $k((x))$, but x does not belong to the completed algebraic closure of $k((t))$. If we write K' for a completed algebraic closure of $k((x))$, we then have a strict inclusion $K \to K'$; composing this with an identification $K' \cong K$ yields the desired endomorphism. □

Proof of Theorem 1.3. We use [8, Theorem 1.5.6] as our blanket reference concerning the perfectoid correspondence. By [8, Example 1.3.5], there is an algebraic extension of $\mathbb{Q}_p$ whose completion is perfectoid with tilt isomorphic to the completed perfect closure of $\mathbb{F}_p((t))$; hence $\mathbb{C}_p$ is perfectoid and $\mathbb{C}_p^\flat$ is isomorphic to the completed algebraic closure of $\mathbb{F}_p((t))$. By Theorem 1.2, there exists an endomorphism $\tau : \mathbb{C}_p^\flat \to \mathbb{C}_p^\flat$ which is not surjective; this corresponds to a morphism $\mathbb{C}_p \to K$ of perfectoid fields which is not surjective either. In particular, the integral closure of $\mathbb{Q}_p$ in K is not dense, so K cannot admit any isomorphism to $\mathbb{C}_p$ in the category of topological fields. □

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