Wild coverings of Berkovich curves

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The goals

- Fix a complete real-valued algebraically closed field k.
- A finite morphism $f\colon Y \to X$ of smooth k-analytic Berkovich curves can be pretty complicated in the non-tame case and our goal is to provide a satisfying description. In particular, we will describe the topological ramification locus of f.
- Main tools will be the following two invariants of f:
  - The different function $\delta_f\colon Y \to [0,1]$ given by $\delta_f(y) = \delta_{\mathcal{H}(y)/\mathcal{H}(f(y))}$.
  - A finer profile function $\phi_f\colon Y \to \mathcal{P}_{[0,1]}$ whose values are piecewise monomial bijections of $[0,1]$ onto itself. It is related to the higher ramification theory.

The sources

- The different function is studied in a work [CTT14] of Cohen-Temkin-Trushin.
- The profile function is studied in a work [Tem14].
- There is a short overview of the two works at [Tem15].
- These slides are available at www.math.huji.ac.il/∼temkin/lectures/wild_covers.pdf.

Plan

1. Basic results on Berkovich curves
2. Basic results on morphisms of curves
3. The different function and the genus formula
4. The profile function and the topological ramification locus

Conventiones

- If not assumed otherwise, a valued field K is a real-valued one, and the valuation $|\ |\colon K \to \mathbf{R}_{\geq 0}$ is written multiplicatively. Also, K° denotes the ring of integers and $\widetilde{K}$ denotes the residue field.
- We fix a complete real-valued algebraically closed ground field k.
- A k-analytic curve X will be called nice if it is smooth, proper and connected. (For simplicity, we do not consider rig-smooth curves with boundaries in this talk.) In particular, $X = \mathcal{X}^{\mathrm{an}}$ is the analytification of an algebraic k-curve $\mathcal{X}$.
- $f\colon Y \to X$ denotes a finite morphism of nice k-analytic curves.

Analytification

- In general, points of k-analytic spaces correspond to real valuations of k-affinoid algebras, and to any point x one assigns a completed residue field $\mathcal{H}(x)$.
- For any k-variety $\mathcal{X}$ Berkovich functorially defines an analytification $X = \mathcal{X}^{\mathrm{an}}$ and a surjective map $\pi \colon X \to \mathcal{X}$. The fiber $\pi^{-1}(z)$ consists of all real valuations $|\ |_x$ on $k(z)$ that extend $|\ |_k$, and $\mathcal{H}(x)$ is the completion of $k(z)$ with respect to $|\ |_x$.
- For any closed point $z \in \mathcal{X}$ we have that $k(z) = k$ and hence $\pi^{-1}(z) = \{x\}$. Such an x is called a rigid point (or a classical point).
- Thus, if $\mathcal{X}$ is an algebraic integral k-curve then set-theoretically X consists of the closed points of $\mathcal{X}$, called points of type 1, and the points given by valuations on $k(\mathcal{X})$.

Points of k-analytic curves

Points on k-analytic curves are divided to four types:

(1) Rigid points: $\mathcal{H}(x) = k$.

(2) $\widetilde{k} \subsetneq \widetilde{\mathcal{H}(x)}$. In this case, $|\mathcal{H}(x)^\times| = |k^\times|$ and $\widetilde{\mathcal{H}(x)}$ is the function field of a $\widetilde{k}$-curve that we denote C_x.

(3) $|k^\times| \subsetneq |\mathcal{H}(x)^\times|$. In this case, $|\mathcal{H}(x)^\times|/|k^\times| = \mathbf{Z}$ and $\widetilde{k} = \widetilde{\mathcal{H}(x)}$.

(4) $\mathcal{H}(x)/k$ is a non-trivial immediate extension.

Points of type 4 are the main obstacle for proving the semistable reduction theorem, but they are not essential for this work, so we will mainly ignore them.

The affine line

Let $X = \mathbf{A}^1_k$ with a fixed coordinate t, then

- For any x of type 1, 2 or 3 there exist $a \in k$ and $r \geq 0$ such that
$$\left|\sum_i c_i(t-a)^i\right|_x = \max_i |c_i| r^i.$$
- x is of type 1 iff $r = 0$, and so it is given by $t = a$.
- Otherwise x is the maximal point $p_{a,r}$ of the disc $E(a,r)$ given by $|t-a| \leq r$, and x is of type 2 iff $r \in |k^\times|$.
- $p_{a,r} = p_{b,s}$ iff $|a-b| \leq r = s$. Thus, X is a graph that can be visualized pretty well.
- (Points of type 4 correspond to intersections of sequences of discs $E_1 \supsetneq E_2 \supsetneq \dots$ without common rigid points.)

Skeletons

Definition

By a subgraph $\Gamma \subset X$ we mean a finite connected subgraph whose vertexes $v \in \Gamma^0$ are of types 1,2. Such a Γ is a skeleton if $X \setminus \Gamma^0$ is a disjoint union of open discs D_i and (semi-)annuli $A_1, \dots, A_n$ and the edges of Γ are the central chords (or skeletons) of $A_1, \dots, A_n$.

Any skeleton gives a good combinatorial approximation of X:

Fact

(i) $X \setminus \Gamma$ is a disjoint union of open discs, so a retraction $q_\Gamma : X \to \Gamma$ arises.
(ii) Any larger subgraph $\Gamma' \supseteq \Gamma$ is a skeleton.
(iii) $g(X) = \sum_{x \in X} g(x) + h^1(X) = \sum_{v \in \Gamma^0} g(v) + h^1(\Gamma)$, where $g(x) = g(C_x)$ for x of type 2 and $g(x) = 0$ otherwise.

Semistable reduction

Theorem

Any nice curve possesses a skeleton.

- This is a, so-called, skeletal formulation of the famous semistable reduction theorem.
- The classical formulation, which is equivalent to the skeletal one, is that X possesses a semistable formal model $\mathfrak{X}$ over $\mathrm{Spf}(k^\circ)$ (or even over $\mathrm{Spec}(k^\circ)$, since we assume that X is proper).
- The relation between the two formulations is as follows: if $\pi\colon X \to \mathfrak{X}_s$ is the reduction (or specialization) map then $\pi^{-1}(x)$ is:
    - a single point if x is generic,
    - an open annulus if x is a node,
    - an open disc if x is a smooth closed point.
- So, the incidence graph $\Gamma = \Gamma(\mathfrak{X}_s)$ is a skeleton with $\Gamma^0 = \pi^{-1}(\mathfrak{X}_s^{\mathrm{gen}})$.

The local structure

The following local description of a nice curve Y is (mainly) a consequence of the semistable reduction. On the other hand, it is not difficult to deduce the semistable reduction from it. (I used this method to give an analytic proof of the semistable reduction.)

- Y is a (huge) graph.
- Any $y \in Y$ of type 1 or 4 lies in an open disc $D \subset Y$.
- Any $y \in Y$ of type 3 lies in an open annulus $A \subset Y$.
- A point $y \in Y$ of type 2 is locally embeddable in $\mathbf{A}^1_k$ iff $g(x) = 0$. In general, the branches v at y are parameterized by the closed points of C_y.

The metric

Fact

Any curve Y possesses a canonical minimal metric d_Y such that each function $\log|f|$ is pl with integral slopes.

- We will work with the exponential (or radius) metric $r_Y([a,b]) = \exp^{d_Y([a,b])}$. Then each $|f|$ is pm (piecewise monomial) of integral degrees on intervals $I \subset Y$ with radius parametrization.
- On $Y = \mathbf{A}^1_k$ it is given by the usual radius: $r_Y([p_{a,s}, p_{a,t}]) = \frac{s}{t}$.
- Given a skeleton $\Gamma \subset Y$ we denote by $r_\Gamma : Y \to [0,1]$ the inverse exponential distance from Γ. In fact, if D is a connected component of $Y \setminus \Gamma$ and $y \in D$ then $r_\Gamma(y)$ is the radius of y in D, where the open disc D is normalized to be of radius 1.

The multiplicity function of f

- The multiplicity function $n_f \colon Y \to \mathbf{N}$ of f is given by $n_f(y) = [\mathcal{H}(y) : \mathcal{H}(f(y))]$ for types 2,3,4 and $n_f(y)$ equals to the ramification index e_y for rigid points.
- Fact: f is a local isomorphism at y iff $n_f(y) = 1$.
- Definition: f is topologically tame at y if $n_f(y) \in \widetilde{k}^\times$.
- One of our aims is to describe the multiplicity loci $N_{f,\geq d} = \{y \in Y |\ n_f(y) \geq d\}$, including the top. ramification locus $N_{f,\geq 2}$. This controls the metric properties of f due to the following

Fact

For any interval $I \subset Y$ the set $f(I)$ is a graph, the map $f|_I \colon I \to f(I)$ is pm and $|\deg(f|_I)| = n_f$ on I (with upper semicontinuity at the corners).

Simultaneous semistable reduction

Definition

A skeleton of $f\colon Y \to X$ is a pair $\Gamma_f = (\Gamma_Y, \Gamma_X)$ of skeletons such that $\Gamma_Y = f^{-1}(\Gamma_X)$ and $\mathrm{Ram}(f) \subset \Gamma_Y^0$.

Theorem

Any finite morphism between nice curves possesses a skeleton.

- Since any enlarging of a skeleton is a skeleton, this theorem is not essentially stronger than the semistable reduction of curves.
- It is more or less equivalent to existence of a finite formal model $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with semistable $\mathfrak{Y}$ and $\mathfrak{X}$.
- $Y \setminus \Gamma_Y \to X \setminus \Gamma_X$ is a disjoint union of finite étale covers of discs by discs.

Description of tame morphisms

- Any top. tame étale cover of a disc by a disc splits, and any top. tame étale cover of an annulus by annulus is Kummer: $t \mapsto t^e$.
- Thus, if f is top. tame then $n_f = 1$ outside of Γ_Y and on any edge $e \subset \Gamma_Y$ we have that $n_f = n_e$ is constant.
- In particular, the map $\Gamma_f \colon \Gamma_Y \to \Gamma_X$ of graphs with multiplicities n_v, n_e is a good combinatorial approximation of f. It satisfies:
- Constancy of multiplicity: $\sum_{v \in f^{-1}(u)} n_v = \deg(f)$ for any $u \in \Gamma_X^0$.
- Local constancy of multiplicity: $n_v = \sum_{e \in f^{-1}(h) \cap \mathrm{Br}(v)} n_e$ for any vertex $v \in \Gamma_Y^0$ and an edge $h \in \mathrm{Br}(f(v))$ in Γ_X.
- Local Riemann-Hurwitz: $2g(v) - 2 - 2n_v(g(u) - 1) = \sum_{e \in \mathrm{Br}(v)}(n_e - 1)$.
- Proofs are by the usual reduction (or formal models) technique, e.g. we use the RH formula for the map $C_v \to C_u$ corresponding to the extension $\widetilde{\mathcal{H}(v)}/\widetilde{\mathcal{H}(u)}$.

Problems with the wild case

- An étale cover of a disc by a disc can be pretty complicated.
- The extension $\widetilde{\mathcal{H}(v)}/\widetilde{\mathcal{H}(u)}$ can be purely inseparable, making the reduction $C_v \to C_u$ rather non-informative.
- Even if $\widetilde{\mathcal{H}(v)}/\widetilde{\mathcal{H}(u)}$ is separable, the local term of e involves the different and exceeds $n_e - 1$ if $\mathrm{char}(\widetilde{k})|n_e$.
- The non-splitting set $N_{f,>1}$ can be huge, e.g. it is the metric neighborhood of $[0,\infty]$ of radius $|p|^{1/(p-1)}$ for the map $f_p\colon \mathbf{P}^1_{\mathbf{C}_p} \to \mathbf{P}^1_{\mathbf{C}_p}$ given by $t \mapsto t^p$.
- An even stranger picture is for the covering $f_\lambda\colon E \to \mathbf{P}^1_{\mathbf{C}_2}$ given by $y^2 = t(t-1)(t-\lambda)$ with $|\lambda - 1| = 1 \le |\lambda| < |2|^{-4}$ (this is the supersingular reduction case since $|j| = |2^4\lambda| < 1$ and so $\widetilde{j} = 0$).

The different

The point with local terms calls for the different. In addition, the different is the main invariant to measure wildness of the extension, so this is the most natural thing we can look at when $\widetilde{\mathcal{H}(v)}/\widetilde{\mathcal{H}(u)}$ is inseparable.

Definition

The different of a separable algebraic extension of valued fields L/K is $\delta_{L/K} = |\mathrm{Ann}(\Omega_{L^\circ/K^\circ})| \in [0,1]$.

- We use the multiplicative language, while the usual different is $\delta^{\mathrm{add}}_{L/K} = -\log(\delta_{L/K})$.
- This definition is the "right" one only when Ω_{L°/K° is of “almost rank one" (i.e. almost isomorphic to a subquotient of L°). It is ok for $\mathcal{H}(v)/\mathcal{H}(u)$ on curves (our case) or for extensions of DVF's with perfect residue fields, but in general one should use the zeroth (almost) Fitting ideal of Ω_{L°/K° or something similar.

The log different

- One can similarly define the logarithmic different $\delta^{\log}_{L/K} = |\mathrm{Ann}(\Omega^{\log}_{L^\circ/K^\circ})|$.
- If K is discretely valued then $\delta^{\log}_{L/K} = \delta_{L/K}|\pi_L|/|\pi_K|$, and $\delta^{\log}_{L/K} = \delta_{L/K}$ otherwise.
- The classical RH formula for a morphism $f\colon Y \to X$ of algebraic curves of degree n is

$$2g(Y) - 2 - 2n(g(X) - 1) = \sum_{y \in Y} \delta^{\mathrm{add}}_{y/x} = \sum_{y \in Y} (\delta^{\log,\mathrm{add}}_{y/x} + n_y - 1).$$

The different function

- Let Y^{hyp} denote the set of points of Y of types 2, 3 (and 4).
- We assign to a generically étale $f\colon Y \to X$ the different function $\delta_f\colon Y^{\mathrm{hyp}} \to (0,1]$ by $\delta_f(y) = \delta_{\mathcal{H}(y)/\mathcal{H}(f(y))}$.
- Since $\mathcal{H}(y)$ is not discretely valued, there is no difference between δ_f and $\delta_f^{\log}$. But, in fact, few places in the sequel where discrete valuations show up (e.g. the limit behavior at type 1 points) indicate that δ_f is the log different function.
- δ_f measures the wildness of f. In particular, if f is top. tame at y then $\delta_f(y) = 1$.
- δ_f easily explains all phenomena we saw in the examples f_p and f_λ. For example, for f_p the different equals to $|p|$ along $[0,\infty]$ and it is monomially increasing of degree $p-1$ in all directions outside of $[0,\infty]$.

Main properties of δ_f

Theorem

- δ_f *is pm on intervals.*
- δ_f *extends to a pm function* $\delta_f \colon Y \to [0,1]$ *and for* y *of type 1* $\mathrm{slope}_y(\delta_f) = \delta^{\mathrm{log,add}}_{y/x}$. *In particular,* δ_f *is not constant near* y *iff* $\delta_f(y) = 0$ *iff* f *is wildly ramified at* y.
- *Balancing condition at* $y \in Y$ *of type 2 with* $x = f(y)$*:*

$$2g(y) - 2 - 2n_y(g(x) - 1) = \sum_{v \in C_y} (-\mathrm{slope}_v \delta_f + n_v - 1).$$

In particular, almost all slopes of δ_f *at* y *equal to* $n^i_y - 1$, *where* n^i_y *is the inseparability degree of* $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(f(y))}$.

Balancing condition: the method

Our proof of the balancing condition is simple and we roughly outline it here.

- Idea: δ_f is a family of differents, so sheafify the definition of $\delta_{L/K}$.
- The “lattice" $\Omega^\diamond_X = \mathcal{O}^\circ_X d(\mathcal{O}^\circ_X)$ of Ω_X is a version of $\Omega_{\mathcal{O}^\circ_X/k^\circ}$.
- $\Omega^\diamond_Y/f^*\Omega^\diamond_X$ is a torsion sheaf of k°-modules, its stalk at y is almost cyclic with absolute value of the annihilator equal to $\delta_f(y)$.
- Choose $a \in k^\circ$ with $|a| = \delta_f(y)$. Reductions of $\Omega^\diamond_Y$ and $a^{-1}f^*\Omega^\diamond_X$ at y induce a non-zero meromorphic map $\lambda\colon \widetilde{f}^*\Omega_{C_y} \to \Omega_{C_x}$, where $\widetilde{f}\colon C_y \to C_x$ is the map of $\widetilde{k}$-curves associated with $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$.
- The balancing condition boils down to computing the degree of $\Omega_{C_y} \otimes \widetilde{f}^*\Omega^{-1}_{C_x}$ via poles and zeros of λ.

Minimal skeletons

Definition

A branch v at a point y of type 2 is δ_f-trivial if $\mathrm{slope}_v\delta_f = n_v - 1$.

Theorem

Let Γ_X be a skeleton of X and $\Gamma_Y = f^{-1}(\Gamma_X)$. Then (Γ_Y, Γ_X) is a skeleton of f if and only if $\mathrm{Ram}(f) \subseteq \Gamma_Y^0$ and for any point $y \in \Gamma_Y$ all branches at y pointing outside of Γ_Y are δ_f-trivial.

- Thus, the different function controls the minimal skeleton of f containing a fixed skeleton Γ of X and allows to construct it algorithmically.
- Also δ_f controls the set $N_{f,p}$ when $\deg(f) = p$, but it does not control the sets $N_{f,p}$, N_{f,p^2}, etc., in general.

Radial sets

Definition

A closed subset $S \subseteq X$ is called Γ-radial of radius r, where $r\colon \Gamma \to \mathbf{R}$, if S consists of all points $x \in X$ satisfying $r_\Gamma(x) \geq r(q_\Gamma(x))$.

Theorem

There exists a skeleton of f such that Γ_Y radializes the sets $N_{f,\geq d}$ and then any larger skeleton does so. Moreover, any skeleton of f radializes these sets in each of the following cases: (1) f is a normal covering (e.g. Galois), (2) f is tame, (3) f is of degree p.

Example

If f is of degree p then $N_{f,p}$ is Γ-radial of radius $\delta_f^{1/(p-1)}|_\Gamma$ for any skeleton (Γ, Γ_X) of f.

The splitting method

The following method is often used to prove results about extensions of valued fields:

- Prove the result for tame extensions and wild extensions of degree p. Often this is simpler and can be done by hands.
- Extend the result to compositions, obtaining the case of Galois extensions.
- Use some form of descent to deduce the non-normal case.

The radialization is proved by running this method locally over X. This works since the category of étale covers of a germ (X, x) is equivalent to the category of étale covers of $\mathrm{Spec}(\mathcal{H}(x))$ by a theorem of Berkovich.

The profile function

- Choose Γ that radializes all sets $N_{f,\geq d}$. Then Γ_Y contains each $N_{f,d}$ with $d \notin p^{\mathbf{N}}$ and our last goal is to express the radius $r_n \colon \Gamma_Y \to \mathbf{R}$ of $N_{f,\geq p^n}$ in classical terms.
- For $y \in \Gamma_Y$ of type 2 the list $r_f(y) = (r_1(y), r_2(y), \dots)$ is a bad invariant because it is hard to compute compositions $r_{f\circ g}$.
- This motivates the following definition: choose any interval $I = [c, y]$ with a rigid c and $I \cap \Gamma_Y = \{c\}$, and identify I and $f(I)$ with $[0, 1]$ via the radius parametrization. Then $f|_I$ induces an element $\phi_f(y) \in \mathcal{P}_{[0,1]}$ (a pm bijection $[0, 1] \to [0, 1]$) independent of c that we call the profile function of f at y. Obviously, $\phi_{f\circ g} = \phi_f \circ \phi_g$.

Relation to the higher ramification

Theorem

If $f\colon Y \to X$ is generically étale then for any point $y \in Y$ of type 2 with $x = f(y)$ the profile function ϕ_y coincides with the Herbrand function $\phi_{\mathcal{H}(y)/\mathcal{H}(x)}$ of the extension $\mathcal{H}(y)/\mathcal{H}(x)$.

- Even to formulate the theorem, one has to extend the higher ramification theory to non-discrete setting and almost monogeneous extensions.
- Once this is done, the proof is, again, by a simple use of the splitting method.

Theorem

The family $\{\phi_y\}$ extends uniquely to a pm function $\phi\colon Y \to P_{[0,1]}$.

Happy Birthday Arthur!

- Adina Cohen, Michael Temkin, and Dmitri Trushin, Morphisms of Berkovich curves and the different function, ArXiv e-prints (2014), http://arxiv.org/abs/1408.2949.
- Michael Temkin, Metric uniformization of morphisms of Berkovich curves, ArXiv e-prints (2014), http://arxiv.org/abs/1410.6892.
- ______, Wild coverings of Berkovich curves, ArXiv e-prints (2015), http://arxiv.org/abs/1509.06063.