Contents

1. Moduli spaces of stable n-pointed curves

1.1. **Reminds on regularity, smoothness and etaleness.**

smdef

Definition 1.1.1. (i) A scheme X is *regular* if all its local rings are noetherian and regular.

(ii) A morphism $f : Y \to X$ is *smooth* if it is finitely presented, flat and has geometrically regular fibers.

(iii) A morphism f is *etale* if it is smooth and of relative dimension zero.

(iv) A morphism $f : Y \to X$ is *unramified* if it is of finite presentation and for any $y \in Y$ with $x = f(y)$ one has that $m_y = m_x\mathcal{O}_y$ and $k(y)/k(x)$ is finite and separable.

presrem

Remark 1.1.2. In the first part of the course, one can assume that all schemes are noetherian. Then finite presentation is the same as finite type.

etflunr

Lemma 1.1.3. *A morphism $f : Y \to X$ is etale locally at a point $y \in Y$ if and only if it is flat and unramified at y.*

The proof is simple. The only slightly subtle point is that one should check (in both directions) that y is discrete in its fiber over X.

stretdef

Definition 1.1.4. A morphism $f : Y \to X$ is *strictly etale* at y if it is etale at y and induces an isomorphism of the residue fields.

etformprop

Proposition 1.1.5. *Let $f : Y \to X$ be a morphism of finite type between noetherian schemes, $y \in Y$ and $x = f(y)$. Then f is strictly etale at y if and only if it induces an isomorphism $\widehat{f}_y : \widehat{\mathcal{O}}_y \widetilde{\to} \widehat{\mathcal{O}}_x$ of completed local rings.*

Proof. To prove the direct implication we shrink Y so that y is the only preimage of x. Since $m_y = m_x\mathcal{O}_y$, the base change induces a strictly etale homomorphism $A = \mathcal{O}_x/m_x^n \to \mathcal{O}_y/m_y^n = B$ of Artin local rings. The latter is automatically an isomorphism: it is finite by the valuative criterion, hence B is A-finite, and then $A^n \widetilde{\to} B$ as A-modules by flatness of B. Since $A/m_A \widetilde{\to} B/m_B \widetilde{\to} B/m_A B$ by our assumption, $n = 1$ and $A \widetilde{\to} B$.

Conversely, if $\widehat{f_y}$ is an isomorphism, then f is, obviously, unramified at y and its flatness at y follows from the following algebraic fact for which we refer to texts on commutative algebra by Bourbaki or Matsumura.

faltcompl

Lemma 1.1.6. *If A is a noetherian ring, I is an ideal and $\widehat{A} = \operatorname{proj\,lim}_n A/I^n$ is the I-adic completion of A, then the completion homomorphism $\phi : A \to \widehat{A}$ is flat. In particular, if A is local and I is the maximal ideal, then ϕ is faithfully flat.*

□

1.2. **Simple nodes.** Nodal points (also called simple nodes) are simplest curve singularities. Intuitively such a point P is a transversal self-intersection or a curve which locally looks as the singularity O of the cross $X = \operatorname{Spec}(k[x, y]/(xy))$, but one should use étale locality rather than the Zariski one.

nodedef

Definition 1.2.1. A simple node on a curve C over an algebraically closed field k is a point P such that there exists a curve C' with a point P' and étale morphisms $C' \to C$ and $C' \to X$ taking P' to P and O, respectively.

Exercise 1.2.2. Show that in the above definition one can take $C' \hookrightarrow C$ to be an open immersion (i.e. work Zariski locally) if and only if C is reducible at P. In particular, show that $C = \operatorname{Spec}(k[x, y]/(x^2 - y^2 - y^3))$ such a neighborhood does not exist (Hint: send x and y to the local coordinates on the Zariski branches.)

Remark 1.2.3. In general, Zariski topology is not fine enough to separate the branches. Sometimes one says that C has normal crossings (nc) at a nodal point, and simple normal crossings (snc) if it is also reducible at the point.

Nodal points possess various equivalent characterizations in terms of normalizations or in terms of formal (or analytic when $k = \mathbb{C}$) localization. Before formulating them we give a few definitions.

semilocaldef

Definition 1.2.4. Let C be a separated k-curve with a finite set $P = \{P_1, \ldots, P_n\}$ of closed points. The *semi-local ring* $\mathcal{O}_P$ of P is defined as follows: take any affine neighborhood U of P (we use that any separated curve is quasi-projective) and define $\mathcal{O}_P$ as the localization of $\mathcal{O}(U)$ by all functions invertible at the points of P. The maximal ideals of $\mathcal{O}_P$ are the preimages of m_{P_i}, and we define m_P as the intersection of all maximal ideals of $\mathcal{O}_P$.

This definition will be useful because the preimage $\widetilde{P}$ of a point $P \in C$ under the normalization $\pi : \widetilde{C} \to C$ may be a finite set. Note normalization is compatible with localizations, hence $\mathcal{O}_{\widetilde{P}}$ is the normalization of $\mathcal{O}_P$. Now, we can define the following most basic invariant of a curve singularity:

Definition 1.2.5. The delta invariant of P is $\delta_P = \dim_k(\mathcal{O}_{\widetilde{P}}/\mathcal{O}_P)$.

pushcurlem **Lemma 1.2.6.** *If C is a reduced separated k-curve and $P \in C$ a finite set of closed points, whose preimage under the normalization $\pi\colon \widetilde{C} \to C$ is $\widetilde{P}$, then $m_{\widetilde{P}}$ contains an ideal I such that $I \subseteq \mathcal{O}_P$.*

Proof. For varieties over k normalization is a finite morphism, hence $\mathcal{O}_{\widetilde{P}}$ is a finite $\mathcal{O}_P$-module, the quotient $\mathcal{O}_{\widetilde{P}}/\mathcal{O}_P$ is a finite torsion module and hence is killed by some m_P^n. In particular, $I = m_P^n\mathcal{O}_{\widetilde{P}}$ is contained in $\mathcal{O}_P$. □

Note that one can take $I = m_{\widetilde{P}}^n$ for $n \gg 0$. As a corollary we obtain that δ_P is an analytic invariant in the following sense:

deltacor **Corollary 1.2.7.** *The delta invariant of P only depends on the formal completion ring $\widehat{\mathcal{O}}_P$.*

Proof. Note that δ_P is the dimension of the k-vector space

$$\mathcal{O}_{\widetilde{P}}/\mathcal{O}_P = (\mathcal{O}_{\widetilde{P}}/I)/(\mathcal{O}_P/I) = (\widehat{\mathcal{O}}_{\widetilde{P}}/\widehat{I})/(\widehat{\mathcal{O}}_P/\widehat{I}) = \widehat{\mathcal{O}}_{\widetilde{P}}/\widehat{\mathcal{O}}_P$$

and the righthand side depends only on the ring $\widehat{\mathcal{O}}_P$ because $\widehat{\mathcal{O}}_{\widetilde{P}} = \oplus_{i=1}^n k[[t_i]]$ is normal and hence is the normalization of $\widehat{\mathcal{O}}_P$. □

In addition, the lemma implies that C can be obtained from $\widetilde{C}$ by a special pushout operation called pinching, when one replaces a closed subscheme by a smaller one.

Corollary 1.2.8. *Keep the above notation and consider the closed subschemes $Z = \mathrm{Spec}(\mathcal{O}_P/I) \hookrightarrow C$ and $\widetilde{Z} = \mathrm{Spec}(\mathcal{O}_{\widetilde{P}}/I) \hookrightarrow \widetilde{C}$, then $\widetilde{C} \coprod_Z \widetilde{Z} = C$, that is, C is the pinching of $\widetilde{C}$ with respect to the morphism $\widetilde{Z} \to Z$.*

Proof. We should prove that a morphism $\widetilde{C} \to X$ with a given factorization of $\widetilde{Z} \to X$ through Z factors through C uniquely. The problem is local on X, hence we can assume that it is affine. Also, the question is local around P, hence we can assume that C and $\widetilde{C}$ are affine. Then the claim reduces to the observation that $\mathcal{O}_C$ is the preimage of $\mathcal{O}_C/I$ under $\mathcal{O}_{\widetilde{C}} \twoheadrightarrow \mathcal{O}_{\widetilde{C}}/I$, and hence $\mathcal{O}_C = \mathcal{O}_{\widetilde{C}} \times_{\mathcal{O}_{\widetilde{C}}/I} \mathcal{O}_C/I$. □

Example 1.2.9. (i) Let $\widetilde{C}$ be a smooth curve with points $\widetilde{P}_1, \dots, \widetilde{P}_n$ identifying them via the map $\coprod_{i=1}^n \mathrm{Spec}(k) \to P = \mathrm{Spec}(k)$ one obtains a curve C with an ordinary n-fold point P, which étale-locally looks as the union of coordinate axes in $\mathbb{A}^n$. On the level of function, one considers the subsheaf of functions attaining the same level at each P_i.

(ii) The ordinary cusp $Q \in C = \mathrm{Spec}(k[t^2, t^3])$ is obtained by pinching $\mathrm{Spec}(k([t]))$ along $\mathrm{Spec}(k[t]/(t^2)) \to Q = \mathrm{Spec}(k)$.

pushoutrem **Remark 1.2.10.** (i) In general, a pinching of a scheme $\widetilde{X}$ along a schematically dominant finite morphism $\widetilde{Z} \to Z$ with a closed subscheme $\widetilde{Z} \hookrightarrow \widetilde{X}$ always exists when $\widetilde{X}\mathrm{Spec}(\widetilde{A})$ is affine. Moreover, the pushout can be already computed in the affine category, that is, $Z = \mathrm{Spec}(\widetilde{A} \times_{\widetilde{B}} B)$ where $B \hookrightarrow \widetilde{B}$ is the homomorphism $\Gamma(\mathcal{O}_Z) \to \Gamma(\mathcal{O}_{\widetilde{Z}})$. For example, one can always past two disjoint closed subschemes

Y_1, Y_2 of X along any isomorphism $\phi : Y_1 \widetilde{\to} Y_2$. On the level of functions, we just restrict to the functions $f \in A$ which coincide on Y_1 and Y_2 (w.r.t. ϕ).

(ii) For non-affine schemes, it can be impossible to even past two disjoint isomorphic closed subschemes. For example, one cannot past two k-points without common affine neighborhood (on a non-separated curve or a non-projective proper surface). Also, one cannot past two isomorphic curves with different self-intersection numbers on a projective surface.

(iii) However, we will see that these "missing" pushouts are defined perfectly well in a larger category of algebraic spaces. In fact, one can also solve there some other problems, such as constructing certain blow downs.

After these generalities let us return to the case we are interested at:

nodeprop **Proposition 1.2.11.** *Let C be a reduced k-curve with a point P and let O be the origin in $X = \mathrm{Spec}(k[x,y]/(xy))$. Then the following conditions are equivalent:*

(i) Étale equivalence: P is a simple node.

(ii) Formal equivalence: $\widehat{\mathcal{O}}_P \widetilde{\to} \widehat{\mathcal{O}}_O$.

(iii) $\widetilde{P} = \{P_1, P_2\}$ and $\delta_P = 1$.

(iv) $\widetilde{P} = \{P_1, P_2\}$ and C is the pinching of $\widetilde{C}$ along $\widetilde{P} \to P$.

Proof. Étale morphisms of k-varieties are strictly étale over closed points, hence are formal isomorphisms. Therefore (i) implies (ii). Since δ_P is an analytic invariant by Corollary 1.2.7, (ii) implies (iii). Let P' be the image of $\widetilde{P}$ in $C' := \widetilde{C} \coprod_{\widetilde{P}} P$. The normalization morphism $\widetilde{C} \to C$ factors through C' and the morphism $\widetilde{C}' \to C$ is an isomorphism if and only if the embedding $\mathcal{O}_P \hookrightarrow \mathcal{O}'_P$ is an isomorphism. The latter happens if and only if the inequality $\delta_P \geq \delta_{P'} = |\widetilde{P}| - 1$ is an equality, hence (iii) is equivalent to (iv). Finally, if (iv) holds then taking a copy $\widetilde{C}'$ of C and pinching $\widetilde{C}'' = \widetilde{C} \coprod \widetilde{C}'$ by gluing $\widetilde{P}_1$ to $\widetilde{P}'_2$ and $\widetilde{P}_2$ to $\widetilde{P}'_1$ one obtains a reducible nodal curve C'' with a finite two-fold étale cover $C'' \to C$. Each node of C'' has a neighborhood which possesses an étale morphism to X. □

Exercise 1.2.12. Generalize the proposition to the case of ordinary n-fold curves.

There are two important invariants of points: tangent space $T_P = (m_P/m_P^2)'$, which is the $k(P)$-dual of the cotangent space $T'_P = m_P/m_P^2$, and a finer invariant called the tangent cone $C_P = \mathrm{Spec}(\oplus_{n=0}^{\infty} m_P^n/m_P^{n+1})$. Note that C_P is naturally embedded into the tangent space $\mathrm{Spec}(k(P)[C_P])$.

Exercise 1.2.13. (i) Let P be a node. Show that T_P is two-dimensional and C_P is a cross $V(xy)$ in T_P, where $x, y \in m_P$ are such that $xy \in m_P^3$.

(ii) Compute T_P and C_P for an ordinary n-fold point.

(iii) Compute T_P and C_P for a simple cusp $x^2 = y^3$.

embexer **Exercise 1.2.14.** (i) Let X be a variety over k and $P \in X$ a closed point. The number $e_P = \dim_{k(P)}(T_P)$ is an important invariant of P called the *embedding dimension*. Prove that e_P is the minimal dimension of a smooth k-variety M such that a neighborhood of P can be embedded into M as a closed subscheme. In particular, $e_P \geq \dim_P(X)$ and the equality holds if and only if X is smooth at P.

(ii)* A finer invariant of P and its local ring $\mathcal{O}_P$ is the whole *Hilbert-Samuel function* $f_P(n) = \dim_{k(P)}(\mathcal{O}_P/m_P^n)$, so $e_P = f_P(2) - 1$. Show that f_P is a polynomial of degree $d = \dim_P(X)$ for $n \gg 0$. It is called the Hilbert polynomial. Show that $f_P(n) \geq \binom{d+n-1}{d}$ and P is a smooth point if and only if the equality holds for any n. Show that the leading coefficient of the Hilbert polynomial is of the form $\frac{m}{d!}$ with $m \in \mathbb{N}$. The number m is called the *multiplicity* of P. In particular, check that the multiplicity of simple nodes and cusps is 2, and the multiplicity of an ordinary m-fold point is m.

1.3. **Stable n-pointed curves over an algebraically closed field.**

semistabledef **Definition 1.3.1.** (i) A curve C over an algebraically closed field k is *nodal* if all its singularities are simple nodes.

(ii) A *semistable* curve over k is a connected proper nodal curve.

(iii) An *n-pointed nodal* curve is a pair (C, D), where C is a nodal curve and $D = (D_1, \dots, D_n)$ is a an ordered set of n distinct smooth closed points of C.

semistablegen **Lemma 1.3.2.** *If C is semistable and connected, then $p_a(C) = h^1(C) = \sum g(\widetilde{C}_i) + h^1(\Gamma)$, where we sum the genera of all irreducible components of the normalization and $h^1(\Gamma)$ is the first Betti number of the incidence graph Γ of C (one vertex per irreducible component and one edge (perhaps a loop) per node).*

The lemma is deduced from a more general one.

curgen **Lemma 1.3.3.** *If C is a proper reduced connected curve over k with normalization $\pi : \widetilde{C} \to C$, then $p_a(C) = h^1(C) = \sum(g(\widetilde{C}_i) - 1) + \sum_P \delta_P - 1$.*

Proof. Compute dimensions in the long cohomological sequence corresponding to the short exact sequence

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{\widetilde{C}} \to \pi_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C \to 0$$

and use that $H^i(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \widetilde{\to} H^i(C, \pi_*\mathcal{O}_{\widetilde{C}})$ because π is affine. □

lcilem **Lemma 1.3.4.** *Any nodal curve C is lci (a locally complete intersection) at any its point P.*

Proof. If P is smooth then there is nothing to prove, so let P be a node, say, $\widehat{\mathcal{O}}_P \widetilde{\to} k[[x, y]]/(xy)$. We can assume that $i : C \to Z = \mathrm{Spec}(A)$ is a closed immersion, $C = \mathrm{Spec}(A/I)$, $z = i(P)$ and Z is smooth of dimension d at z. Then $m_z/m_z^2 = \oplus_{i=1}^d z_i k$ for a regular family of parameters $z_1, \dots, z_d \in \mathcal{O}_z$, and the cotangent map $\phi_2 : m_z/m_z^2 \to m_P/m_P^2 = xk \oplus yk$ is onto. So, we can find $f_1, \dots, f_{d-2} \in I$ which generate the kernel of ϕ_2. Replacing $Z = \mathrm{Spec}(A)$ with the closed subscheme $Z' = \mathrm{Spec}(A/(f_1, \dots, f_{d-2}))$, which is a smooth surface at z, we achieve that $d = 2$. Then ϕ_2 becomes an isomorphism, and we can assume that $\phi_2(z_1) = x$, $\phi_2(z_2) = y$.

Now, let us dig deeper: the map $\phi_3 : m_z^2/m_z^3 \to m_P^2/m_P^3$ sends z_1z_2 to $xy = 0$. Therefore, there exists $f \in I$ such that $f \in z_1z_2 + m_z^3$. A simple induction on $n \geq 3$ shows that translating the coordinates z_i by elements from m_z^{n-1} one can achieve that $f = z_1z_2 + m_z^{n+1}$, hence in the limit we obtain a coordinate change such that $f = z_1z_2$. In particular, $k[[z_1, z_2]]/(f)$ is a reduced two-dimensional ring and hence the surjection $k[[z_1, z_2]]/(f) \twoheadrightarrow k[[x, y]]/(xy)$ is an isomorphism. By Proposition 1.1.5 this implies that the closed immersion $C \to \mathrm{Spec}(A/(f))$ is étale at P. In particular, i is a flat closed immersion, hence a local isomorphism at P. Thus, locally at P we can describe C as a closed subscheme cutoff by the single element f from the smooth surface Z. In particular, C is l.c.i. at P. □

Remark 1.3.5. In fact, the above argument was as follows: first, starting from an embedding of C into a smooth variety M one uses a tangent space information to cut the dimension of M to $2 = e_P$. This is a general argument that solves Exercise 1.2.14(i). Then one uses the special form of the tangent cone via a concrete computation.

Next, we are going to develop a theory of dualizing sheaves on nodal curves C, but this will be done in a very ad hoc way. All what we really need as an outcome is a natural construction of a very ample sheaf on C. First, recall the most standard use of the Riemann-Roch theorem:

RRexer **Lemma 1.3.6.** *Let C be a smooth proper connected curve over k of genus g, let $D = \sum_{i=1}^n l_i P_i$ be an effective divisor of degree d and let $\mathcal{L}$ be an invertible sheaf such that $\deg(\mathcal{L}) > 2g-2+d$. Then the restriction map $\phi\colon H^0(\mathcal{L}) \to k_D = \oplus_{i=1}^n \mathcal{O}_C/m_{P_i}^{l_i}$ is surjective.*

Proof. By definition, $\mathrm{Ker}(\phi) = H^0(\mathcal{L}(-D))$, hence it suffices to prove that $d = \dim(k_D) = h^0(\mathcal{L}) - h^0(\mathcal{L}(-D))$. Both h^0 terms are computed by the Riemann-Roch, and the h^1 terms vanish as $\deg(K_C + D - \mathcal{L}) < 0$. Therefore, the difference of the h^0 terms is precisely the difference of the degrees, which is d. $\square$

This result can be used to produce sections on pinched curves.

projcur **Lemma 1.3.7.** *Let C be a semistable curve with normalization $\pi : \widetilde{C} = \coprod_{i=1}^n \widetilde{C}_i \to C$ and let $\mathcal{L}$ be an invertible sheaf on C. Then $\mathcal{L}$ is ample if and only if $\widetilde{\mathcal{L}} = \pi^*(L)$ is ample, and the latter happens if and only each restriction $\widetilde{\mathcal{L}}_i = \widetilde{\mathcal{L}}|_{\widetilde{C}_i}$ has positive degree.*

Proof. If $\mathcal{L}$ is ample, then for $n \gg 0$ the sheaf $\mathcal{L}^n$ has enough sections to embed C into $\mathbf{P}^N$. Pulling them back we obtain sections of $\widetilde{\mathcal{L}}^n$ non-trivial on all components of $\widetilde{C}$, hence the degrees are positive.

Conversely, assume that $\deg(\widetilde{\mathcal{L}}_i) > 0$. Let $C_{\mathrm{sing}} = \{P_1, \dots, P_m\}$ be the set of nodal points of C, let $\pi^{-1}(P_j) = \{\widetilde{P}_j, \widetilde{P}'_j\}$ and let m_i be the size of $\widetilde{C}_i \cap \pi^{-1}(C_{\mathrm{sing}})$. Choose n such that $\deg(\widetilde{\mathcal{L}}_i^n) > 2g_{C_i} - 2 + m_i + 2$, and note that $V = \Gamma(L^n)$ can be identified with the subspace $V \subset \Gamma(\widetilde{\mathcal{L}}^n)$ of sections s such that $s(\widetilde{P}_j) = s(\widetilde{P}'_j)$ for any j, where we identify the fibers $\widetilde{\mathcal{L}} \otimes k(\widetilde{P}_j) = \mathcal{L} \otimes k(P_j) = \widetilde{\mathcal{L}} \otimes k(\widetilde{P}'_j)$.

It easily follows from Lemma 1.3.6 that for any $Q, R \in \widetilde{C}$ not mapped to the same point in C there exists $s_1, s_2 \in V$ with $s_1(P) = 0$, $s_2(Q) \neq 0$. Similarly, for any $Q \in C$ and $t \in m_Q\mathcal{L}/m_Q^2\mathcal{L}$ there exists $s \in V$ mapped onto t. Indeed, only the case when $Q = P_j$ is a node needs an explanation. Then $m_Q\mathcal{L}/m_Q^2\mathcal{L}$ is two-dimensional and we should find $s, s' \in V$ with linearly independent images in $m_Q\mathcal{L}/m_Q^2\mathcal{L}$. For example, one can take s such that $s \in m_{\widetilde{P}_j}^2$, $s \notin m_{\widetilde{P}_j}^2$ and s' such that $s' \notin m_{\widetilde{P}_j}^2$, $s' \in m_{\widetilde{P}_j}^2$ (their images are the two directions corresponding to the tangent cone). Once we showed that $\Gamma(\mathcal{L})$ distinguishes points and tangent vectors of C, we obtain that $\mathcal{L}$ is very ample by [Har, Proposition II.7.3]. $\square$

Exercise 1.3.8. Extend the above result and argument to arbitrary pinchings: any reduced proper curve C over k is projective and an invertible sheaf on C is ample if and only if each its pullback to a component of the normalization has positive degree.

We will need the following simple observation:

Exercise 1.3.9. We observed that for a nodal curve C and an invertible sheaf $\mathcal{L}$, the pullback $\widetilde{\mathcal{L}}$ comes equipped with isomorphisms of the fibers $\widetilde{\mathcal{L}}\otimes k(\widetilde{P}_j)=\widetilde{\mathcal{L}}\otimes k(\widetilde{P}'_j)$. Show that, conversely, any invertible sheaf $\widetilde{\mathcal{L}}$ on $\widetilde{C}$ with such isomorphisms comes from an invertible sheaf on C.

The trivial example of the above is obtained for the structure sheaves $\mathcal{O}_C$ and $\mathcal{O}_{\widetilde{C}}$: the identifications are clear because each fiber of $\mathcal{O}_{\widetilde{C}}$ is canonically isomorphic to k. The situation with the sheaves of differentials is more interesting. First, we note that there is no natural identification of the fibers of $\Omega_{\widetilde{C}}$: clearly $\Omega_{\widetilde{C}}\otimes k(Q)$ is spanned by dt_Q, where t_Q is a uniformizer at $Q\in\widetilde{C}$, but its choice is non-canonical, for example it can be multiplied by an element of $k^\times$. However, the class of the logarithmic differential $\frac{dt_Q}{t_Q}$ in $m_Q^{-1}\Omega_{\widetilde{C}}/\Omega_{\widetilde{C}}$ is canonical because for any $u\in\mathcal{O}_Q^\times$ the difference $\frac{dt_Q}{t_Q}-\frac{d(ut_Q)}{ut_Q}=\frac{du}{u}$ lies in $\Omega_{\widetilde{C},Q}$. In fact, sending this element to 1 induces the residue isomorphism $\mathrm{res}_Q\colon m_Q^{-1}\Omega_{\widetilde{C}}/\Omega_{\widetilde{C}}\widetilde{\to}k$. Using the above exercise we obtain:

omegalem

Lemma 1.3.10. *Let C be a nodal curve with normalization $\pi\colon\widetilde{C}\to C$ and $D=\pi^{-1}(C_{\mathrm{sing}})$. Then there exists a unique invertible sheaf ω_C such that $\pi^*(\omega_C)=\Omega_{\widetilde{C}}(D)$ and the identifications of the fibers $k(\widetilde{P}_j)=k(\widetilde{P}'_j)$ are via the compositions $-\mathrm{res}_{\widetilde{P}'_j}\circ\mathrm{res}_{\widetilde{P}_j}$.*

Remark 1.3.11. (i) The minus sign is important. In descent of $\mathcal{O}_{\widetilde{C}}$ we identify functions with equal values at the pairs of points over the nodes, while descending $\Omega_{\widetilde{C}}(D)$ we identify forms with opposite residues at the pairs. In particular, this guarantees that sums of residues are zero.

(ii) In fact, the sheaf ω_C is the dualizing sheaf of C via the theory of Grothendieck-Serre. Since $\Omega_{\widetilde{C}}=\omega_{\widetilde{C}}$ is the dualizing sheaf of $\widetilde{C}$ we see that the dualizing sheaves are related in the most natural way $\omega_{\widetilde{C}}=\pi^*\omega_C$. See Exercise 1.3.13 below for comparison with the sheaves of differentials.

Now, let us relate the sheaves ω_C to the sheaves of usual differentials, obtaining another characterization of ω_C. Note that at any generic point $\eta\in C$ we have the canonical isomorphism of stalks $\omega_{C,\eta}=\Omega_{\widetilde{C},\eta}=\Omega_{C,\eta}$.

omega2lem

Lemma 1.3.12. *For any nodal curve C the isomorphism of the generic fibers extends to an embedding $\Omega_C\hookrightarrow\omega_C$ which is an isomorphism at any regular point and satisfies $\Omega_{C,P}=m_P\omega_{C,P}$ at any node. In particular, ω_C is the smallest invertible sheaf containing Ω_C.*

Proof. It suffices to study the situation at a node P. Moreover, it suffices to show that $\Omega_P\subset\omega_P$ and the quotient is isomorphic to k. This can be done using an explicit embedding of C into a smooth surface via Lemma 1.3.4 (see below), or by a formal computation with completed differentials, but we prefer an étale-local argument. It is easy to see that for an étale morphism $f\colon C'\to C$ we have that $f^*\Omega_C=\Omega_{C'}$, in addition, the morphism of normalizations is étale, hence $f^*\omega_C=\omega_{C'}$. Finally, f is flat, hence the claim for C' at P' implies the claim for C at $P=f(P')$, and vice versa.

Recall that C and the model nodal curve $X=\mathrm{Spec}(k[x,y])/(xy)$ are étale-locally isomorphic, hence it suffices to do the computations in this case. Let $A=\mathcal{O}_{X,O}$, then $\Omega_{A/k} = (Adx + Ady)/A(xdy + ydx)$ and $\omega_{A/k}$ is a free module with the generator $\frac{dx}{x} = -\frac{dy}{y}$. Clearly, $\Omega_{A/k} = m_A\omega_{A/k}$. □

Omegaexer **Exercise 1.3.13.** Show that $\pi^*(\Omega_C) = \Omega_{\widetilde{C}} \oplus \mathcal{O}_{\widetilde{C}}/m_D$, where $D = \pi^{-1}(C_{\rm sing})$. So not only the pullback is not locally free, it even has a non-trivial torsion.

Next, let us indicate how one can prove that ω_C is the dualizing sheaf. Recall that if $\widetilde{C}$ is smooth, then the main part of the duality is that sum of residues induces an isomorphism $t\colon H^1(\Omega_{\widetilde{C}})\widetilde{\to}k$. In terms of the Godement resolution

$$\Omega_{\widetilde{C}} \to \Omega_\eta \to \oplus_{Q\in\widetilde{C}}\Omega_\eta/\Omega_Q$$

the latter means that a finite tuple of meromorphic parts $(\phi_{Q_i}) \in \oplus_Q\Omega_\eta/\Omega_{Q_i}$ lifts to a meromorphic form $\phi \in \Omega_\eta$ if and only if $\sum_Q \mathrm{res}_Q(\phi_Q) = 0$.

Exercise 1.3.14. (i) For a nodal curve C use the above fact for the components of $\widetilde{C}$ to deduce that sum of the residues at the smooth points induces an isomorphism $t\colon H^1(\omega_C) = \mathrm{Coker}(\Omega_\eta \to \oplus_{Q\in C}\Omega_\eta/\omega_Q)\widetilde{\to}k$.

(ii) Deduce that ω_C is the dualizing sheaf: for any invertible sheaf $\mathcal{L}$ the spaces $H^i(C,\mathcal{L})$ and $H^{1-i}(C,\omega_C\otimes\mathcal{L}^{-1})$ are naturally dual.

For the sake of completeness we show how ω_C can be constructed from the general theory of dualizing sheaves as developed in [Har, Chapter III]. The idea is that ω_C is obtained from an embedding $C \hookrightarrow \mathbf{P}^n$ by (correct) restricting a dualizing sheaf on $\mathbf{P}^n$, and this can be computed explicitly since C is lci. Recall that any reduced k-curve C is Cohen-Maucauley, hence possesses a dualizing sheaf ω_C such that $\mathrm{Hom}(\mathcal{F},\omega_C)$ is dual to $\mathrm{Ext}^1(\mathcal{O}_C,\mathcal{F})\widetilde{\to}H^1(C,\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on C. Moreover, if C is l.c.i., then ω_C is locally free and for any closed immersion $i: C \to \mathbf{P} = \mathbf{P}^n_k$ the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_C$-sheaf of rank $n-1$ (here $\mathcal{I}$ is the sheaf of ideals on $\mathbf{P}$ which defines the closed subscheme C), and $\omega_C\widetilde{\to}i^*\omega_{\mathbf{P}}\otimes\det(\mathcal{I}/\mathcal{I}^2)'$, where as usual $\omega_{\mathbf{P}} = \det(\Omega^1_{\mathbf{P}})$ and $\mathcal{L}^*$ is the dual of $\mathcal{L}$.

dualprop **Proposition 1.3.15.** *Let C be a semistable curve with the set of nodes $C_{\rm sing}$ and normalization $\pi : \widetilde{C} \to C$. Then $\pi^*\omega_C\widetilde{\to}\omega_{\widetilde{C}}(E)$ where $\omega_{\widetilde{C}} = \Omega^1_{\widetilde{C}}$ is the dualizing sheaf of $\widetilde{C}$ and $E = \pi^{-1}(C_{\rm sing})$ as a reduced subscheme.*

Proof. By Lemma 1.3.7, there exists a closed immersion $i : C \to \mathbf{P} = \mathbf{P}^n_k$. The second fundamental sequence for i is

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega^1_{\mathbf{P}} \to \Omega^1_C \to 0$$

and pulling it back to $\widetilde{C}$ we get an exact sequence

$$\pi^*(\mathcal{I}/\mathcal{I}^2) \xrightarrow{\widetilde{d}} \widetilde{i}^*\Omega^1_{\mathbf{P}} \to \pi^*\Omega^1_C \to 0$$

where $\widetilde{i} = i\circ\pi$. We claim that $\widetilde{d}$ is injective and $\pi^*\Omega^1_C\widetilde{\to}\Omega_{\widetilde{C}}\oplus\mathcal{O}_E/m_E$. It suffices to check this claim on stalks at a point $Q \in \widetilde{C}$, and only the case of $Q \in E$ requires an explanation. Set $P = \pi(Q)$, then from Lemma 1.3.4 and the argument from its proof, we know that there exist local parameters $t_1, \ldots, t_{n-2}, x, y \in \mathcal{O}_{\mathbf{P},P}$ such that $\mathcal{I}$ is generated locally around P by elements $t_i, g = xy + \ldots$ ($\ldots$ means everything involving t_i's and cubic expressions in x and y, and we leave it as an exercise to worry for $\ldots$ in the sequel). Also, $\{dt_i, dx, dy\}$ is a basis of $(i^*\Omega^1_{\mathbf{P}})_P$, and d_P (i.e. the map induced by d on P-stalks) takes t_i to dt_i and $dg = xdy + ydx + \ldots$. We can assume that Q lies on a branch of $\widetilde{C}$ where x is a uniformizer and y vanishes to at least second order, then $\widetilde{d}_Q$ maps a free $\mathcal{O}_Q$-module with a basis t_i, g to a free $\mathcal{O}_Q$-module with a basis dt_i, dx, dy, hence the map d is injective and its cokernel is of the form

$$(dx\mathcal{O}_Q \oplus dy\mathcal{O}_Q)/(xdy + ydx + \dots)\mathcal{O}_Q \widetilde{\to} (dx\mathcal{O}_Q \oplus dy\mathcal{O}_Q)/x(dy + \frac{y}{x}dx + \dots)\mathcal{O}_Q \widetilde{\to} dx\mathcal{O}_Q \oplus \mathcal{O}_Q/m_Q \widetilde{\to} \Omega^1_{\widetilde{C},Q} \oplus \mathcal{O}_Q/m_Q$$

Thus, we obtain an exact sequence

$$0 \to \pi^*(\mathcal{I}/\mathcal{I}^2) \xrightarrow{\widetilde{d}} \widetilde{i}^*\Omega^1_{\mathbf{P}} \to \omega_{\widetilde{C}} \oplus \mathcal{O}_E/m_E \to 0$$

and taking the determinants we conclude that $\widetilde{i}^*\omega_{\mathbf{P}} \widetilde{\to} \det(\pi^*\mathcal{I}/\mathcal{I}^2)(E) \otimes \omega_{\widetilde{C}}$. (This is clear when $E = 0$ because the sheaves are locally free, check that this is ok in general.) Multiplying by the dual of $\det(\pi^*\mathcal{I}/\mathcal{I}^2)$ we obtain that $\omega_{\widetilde{C}}(E) \widetilde{\to} \widetilde{i}^*\omega_{\mathbf{P}} \otimes \det(\pi^*\mathcal{I}/\mathcal{I}^2)^*$, and the lemma now follows by applying π^* to the isomorphism $\omega_C \widetilde{\to} i^*\omega_{\mathbf{P}} \otimes \det(\mathcal{I}/\mathcal{I}^2)^*$ from [Har, III.7.11]. □

stabdef **Definition 1.3.16.** For an n-pointed nodal curve (C, D) define the dualizing sheaf to be $\omega_{C,D} = \omega_C(D)$. An n-pointed semistable curve is is *stable* if $\omega_{C,D}$ is ample. By the (arithmetic) *genus* of a stable curve C we mean $p_a = h^1(C) = 1 - \chi(C)$.

stablem **Lemma 1.3.17.** *A semistable n-pointed curve (C, D) is stable if and only if any rational component of the normalization $\widetilde{C}$ contains at least three points from $\pi^{-1}(D \cup C_{\text{sing}})$, and any component with $g(\widetilde{C}_i) = 1$ contains at least one such point.*

Proof. By Lemma 1.3.7 we should check that $\pi^*\omega_C(D) = \omega_{\widetilde{C}}(\pi^{-1}(D \cup C_{\text{sing}}))$ has positive degree on any component $\widetilde{C}_i$ of $\widetilde{C}$. It remains to use that the degree of $\omega_{\widetilde{C}}$ on $\widetilde{C}_i$ is $2g(\widetilde{C}_i) - 2$. □

stabrem **Remark 1.3.18.** (i) Another way to reformulate the stability condition in the lemma is to say that $2g(\widetilde{C}_i) - 2 + n(\widetilde{C}_i) > 0$ for any component $\widetilde{C}_i \hookrightarrow \widetilde{C}$, where $g(\widetilde{C}_i)$ is the genus and $n(\widetilde{C}_i)$ is the number of *marked points* (i.e. points from $\pi^{-1}(D \cup C_{\text{sing}})$).

(ii) One can show that a semistable (C, D) is stable if and only if its group of automorphisms is finite. The inverse implication is almost obvious, but one has to work to establish the direct one (in particular, one has to show that a smooth projective connected curve of genus $g \geq 2$ has finitely many automorphisms.)

The following is [DM, Theorem 1.2] (for $D = 0$).

ampleth **Theorem 1.3.19.** *Let (C, D) be a stable curve. Then $\omega_{(C,D)}^{\otimes m}$ is very ample for $m \geq 3$ and $h^1(\omega_{(C,D)}^{\otimes m}) = 0$ for $m \geq 2$.*

Proof. By duality, $h^1(\omega_{(C,D)}^{\otimes m}) = h^0(\omega_C \otimes \omega_{(C,D)}^{\otimes -m}) = h^0(\omega_C^{1-m}(-mD))$. It follows from Proposition 1.3.15 that the degree of the pullback of $\omega_C^{1-m}(-mD)$ to any component of $\widetilde{C}$ is negative as soon as $m > 1$. This proves the second claim.

By [Har, Proposition II.7.3], to prove that $\mathcal{L}_m := \omega_{(C,D)}^{\otimes m}$ is very ample we should establish surjectivity of $H^0(\mathcal{L}_m) \to H^0(\mathcal{L}/m_P m_Q \mathcal{L}_m)$ for any pair of points $P, Q \in C$. The latter would follow if $h^1(m_P m_Q \mathcal{L}_m) = 0$, hence by duality it suffices to prove that $\mathrm{Hom}(m_P m_Q \mathcal{L}_m, \omega_C) = 0$. Since $\pi^*\mathcal{L}_1$ is of positive degree on all components of $\widetilde{C}$, we have that $\deg(\pi^*\mathcal{L}_2) > \deg(\pi^*\omega_C)$ on each component of $\widetilde{C}$. But the length of the skyscraper $\mathcal{L}/m_P m_Q\mathcal{L}$ cannot exceed 3 (3 is obtained when $P = Q$ is a node), hence $\deg(\pi^*(m_P m_Q \mathcal{L}_5)) > \deg(\pi^*\omega_C)$. Thus the first assertion holds for $m \geq 5$, and we refer to [DM], for a more pedantic check that $m \geq 3$ is also ok. □

amplerem **Remark 1.3.20.** In principle, it is only important that the theorem holds for $m \geq m_0$ with a uniform m_0 not depending on a curve (or depending only on its genus). So, we did not tried to prove to the sharpest result $m_0 = 3$.

amplecor **Corollary 1.3.21.** *Any stable n-pointed curve (C, D) admits a tri-canonical embedding into $\mathbf{P}_k^{N_{g,n}}$, where $N_{g,n} = 5g - 6 + 3n$. The Hilbert polynomial of this embedding is $P_{g,n}(m) = 3(2g - 2 + n)m + 1 - g$.*

Proof. Since $h^1(\omega_C(D)^{\otimes l}) = 0$, we obtain that

$$h^0(\omega_C(D)^{\otimes l}) = \deg(\omega_C(D)^{\otimes l}) + 1 - g = l(2g - 2 + n) + 1 - g$$

by the Riemann-Roch theorem, which follows from the duality precisely in the same way as in the classical smooth case:

$$\chi(\mathcal{L}) - \deg(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L})$$

is constant and equal to $1 - g = h^0(\mathcal{O}) - h^1(\mathcal{O})$. For $l = 3$ we obtain that h^0 is equal to $5g - 5 + 3n$, hence the first assertion is clear. The second one follows from the fact that

$$P(m) = h^0(\omega_C(D)^{\otimes 3m}) = 3(2g - 2 + n)m + 1 - g.$$

□

Note that the coefficient $2g-2+n$ appeared in remark 1.3.18(i), but the corollary explains why it is very common to express the stability condition by an "artificial" inequality $2g - 2 + n > 0$.

familyrem **Remark 1.3.22.** (i) The corollary implies that each stable n-pointed curve of genus g can be realized as a curve $C \hookrightarrow \mathbf{P}_k^{N_{g,n}}$ with divisor D such that the Hilbert polynomial of C is $P_{g,n}(m)$, $\mathcal{O}_C(1)\widetilde{\to}\omega_{(C,D)}$ and (C, D) is an n-pointed nodal curve. Moreover, any ambiguity is only up to the choice of a basis in $\Gamma(\mathcal{O}_C(1))$, and choosing another basis we obtain a $\mathrm{PGL}(N_{g,n} + 1)$-translate of the original C. So, factorizing the above sets of data by the action of PGL we obtain the set $\mathcal{M}_{g,n}(k)$ of all stable n-pointed curves of genus g over k.

(ii) From this description it is not so clear if this set can be naturally identified with the set of k-points on a scheme $\mathcal{M}_{g,n}$ (a moduli scheme). The basic idea in search for such a scheme (or a more refined creature, e.g. stack) is that though a scheme is not defined by the set of its points, it is defined by the set of its X*-points* for all schemes X. By Yoneda lemma it would suffice to know all sets $\mathrm{Hom}(X, \mathcal{M}_{g,n})$ (and one can easily see that affine X's would suffice). It is natural to expect that $\mathrm{Hom}(X, \mathcal{M}_{g,n})$ should be the set of isomorphism classes of families of stable n-pointed curves over X, so it is easy to define $\mathcal{M}_{g,n}$ as a functor. But in order to construct $\mathcal{M}_{g,n}$ as a scheme, i.e. in order to prove representability of the functor $\mathcal{M}_{g,n}$, we have to redo all steps of (i) scheme-theoretically and not just on the level of sets. In particular, we will have to work with flat families of stable pointed curves, and we will have to check that one can make constructions, impose restrictions and form the quotient scheme-theoretically.

We will need two more lemmas concerning stable curves over algebraically closed fields. By an *elementary contraction* of n-pointed nodal curve (C, D) we mean a map to another n-pointed nodal curve $(\overline{C}, \overline{D})$ which maps D onto $\overline{D}$ bijectively, contracts one $\mathbf{P}^1_k$ irreducible component of C and does not change anything else. The following lemma admits an easy combinatorial proof, which is left as an exercise. Check also that for $g = 0$ and $g = 1$, there is no canonical stable contraction.

contractlem

Lemma 1.3.23. *(i) Elementary contraction preserves genus and a rational component can be contracted if and only if its preimage in $\widetilde{C}$ contains at most two marked points;*

(ii) If C is semistable of genus $g \geq 2$, then any maximal sequence of successive elementary contractions leads to a canonical stable contraction of $(C_{\rm st}, D_{\rm st})$.

(iii) If C is nodal and we extend the set of marked point by including a non-empty set C_∞ of generic points (in other words, we forbid to contract some components), then any sequence of successive elementary contractions leads to a unique n-pointed nodal curve called the stable contraction with respect to C_∞. *In particular, this is the case when there are non-rational components.*

The last lemma concerns Fitting ideals of nodal curves. The definition and basic facts about Fitting ideals are recalled in §1.4.5 below.

Fitsinglem

Lemma 1.3.24. *For a nodal k-curve C, the set $C_{\rm sing}$ with the reduced scheme structure is the scheme-theoretic support of the first Fitting ideal* $\mathrm{Fitt}_1\Omega^1_C$ *of the differential sheaf.*

Proof. As in the proof of Lemma 1.3.12 both Ω and its fitting ideal are compatible with the étale morphisms, and this reduces the claim to a computation for the model curve $X = \mathrm{Spec}(k[x,y]/(xy))$. The differential sheaf if locally free of rank one at a smooth point $P \in X$, hence $(\mathrm{Fitt}_1\Omega^1_X)_P = \mathcal{O}_P$. If P is the node, then we have an exact sequence $M \stackrel{d}{\to} L \to \Omega^1_{C,P} \to 0$, where M is locally free with a basis xy, L is locally free with a basis dx, dy and $dg = xdy + ydx$. Then $(\mathrm{Fitt}_1\Omega^1_C)_P = (x,y) = m_P$. □

The importance of the lemma is that it gives an algebraic (or a scheme-theoretic a-la Remark 1.3.22) way to define $C_{\rm sing}$.

1.4. **Flattening.**

1.4.1. *Monomorphisms.* Recall that a morphism $Y \to X$ is a *monomorphism* if all induced maps $\mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, X)$ are injective.

Exercise 1.4.1. (i) Prove that $f : Y \to X$ is a monomorphism if and only if its diagonal is an isomorphism.

(ii) Show that any locally closed immersion is a monomorphism.

(iii) Show that a flat bijective monomorphism is an isomorphism.

There are plenty of examples of other monomorphisms of increasing nastiness. To the best of my knowledge there is no reasonable classification or a simple description of general monomorphisms.

Example 1.4.2. (i) The morphism $\mathbf{P}^n \coprod \mathbf{A}^{n+1} \to \mathbf{P}^{n+1}$ is a bijective monomorphism, which is not a locally closed immersion. It is a stratification, as defined below.

(ii) Take $X = \mathrm{Spec}(k[x,y]/(xy))$, $X_1 = \mathrm{Spec}(k[x])$ and $X_0 = \mathrm{Spec}(k[y,y^{-1}])$. Then $X_1 \coprod X_0 \to X$ is a bijective monomorphism, which is only a weak stratification in our sense.

(iii) Take the normalization $\widetilde{X} \to X$ and let $\widetilde{X}'$ be obtained by removing of the preimages of the node. Then $\widetilde{X}' \to X$ is a weak stratification only étale-locally on X, that is, there is an étale cover $Y \to X$, in fact any one separating the branches to different irreducible components, such that the base change $\widetilde{Y}' \to Y$ is a weak stratification.

1.4.2. *Stratifications.* It seems that the definitions of the notion of stratification vary in the literature, so we suggest the following:

stratdef **Definition 1.4.3.** (i) By a *stratification* of X we mean a finite disjoint union $\mathcal{X} = \sqcup X_j$ of locally closed subschemes which cover X on the level of sets. If, in addition, each closure $\overline{\mathcal{X}}_i$ is the union of the strata it contains, then we say that $\mathcal{X}$ is a *strong stratification* of X.

(ii) Often we will not distinguish between $\mathcal{X}$ and the bijective monomorphism $i_\mathcal{X} : \mathcal{X} \to X$, and say that $\mathcal{X}$ *refines* $\mathcal{X}'$ if $i_\mathcal{X}$ factors through $i_{\mathcal{X}'}$ (in particular, $\mathcal{X}$ is a stratification of $\mathcal{X}'$).

The following lemma is a simple exercise on noetherian induction, so we skip the argument.

refinlem **Lemma 1.4.4.** *The family of all stratifications is filtered. If X is noetherian, then any family of stratifications which admit a common refinement possesses a coarsest mutual refinement.*

1.4.3. *Flattening.*

Definition 1.4.5. Let $f\colon Y \to X$ be a morphism and let $\mathcal{F}$ an $\mathcal{O}_Y$-module. A morphism $i\colon X' \to X$ *flattens* $\mathcal{F}$ if the pullback of $\mathcal{F}$ to $Y' = Y \times_X X'$ is X-flat. We say that i *flattens* f or Y if it flattens $\mathcal{O}_Y$. If i is also a stratification, then we say that it is a *flattening stratification* for $\mathcal{F}$ or Y.

As often happens, even if one is mainly interested in flattening X-schemes, flattening modules provides an additional flexibility and makes it possible to argue in the category of modules. This is well illustrated by the following proof.

generalflattenth **Theorem 1.4.6.** *Let $f\colon Y \to X$ be a morphism of finite type between noetherian schemes. Then for any coherent module $\mathcal{F}$ on Y there exists a flattening stratification $\mathcal{X} \to X$.*

Proof. We will tacitly use that the stratifications form a filtered family (use the fiber products). In particular, throughout the proof we can replace X by its stratification and update Y and $\mathcal{F}$ accordingly. By this argument we can replace X by its reduction, making it reduced. Now, $\mathcal{F}$ is flat over a generic point $\eta \in X$, and it suffices to find a neighborhood U of η such that $\mathcal{F}$ is flat over U because the theorem will follow by noetherian induction. In particular, we can assume that $X = \mathrm{Spec}(A)$ is integral. Also, since the family of stratifications is filtered, we can deal separately with elements of an affine covering of Y, so assume that $Y = \mathrm{Spec}(B)$.

Any finite B-module $\mathcal{F}$ is composed of modules of the form B/p for a prime ideal $p \hookrightarrow B$, hence it suffices to shrink X so that all B/p are flat. Therefore, we can replace B with B/p achieving that it is a finitely generated integral A-algebra. By Noether normalization, $B \otimes_A K$ is finite over a subring $K[x_1, \dots, x_n]$, where $K = \mathrm{Frac}(A)$ and n is the dimension of Y_η, hence shrinking X (or localizing A) we can achieve that B is finite over its subring $C = A[x_1, \dots, x_n]$. Then $\mathcal{F}$ is finite over C and its composition series consists of C^d for $d = [\mathrm{Frac}(B) : \mathrm{Frac}(C)]$, and modules which are finite over X-schemes Y' with $\dim(Y'_\eta) < n$. Hence it remains to use induction on n. □

1.4.4. *Universal flattening.* The next natural question is if there exists a coarsest flattening stratification, such that any other flattening stratification factors through it. Similarly, one may wonder whether there exists a universal flattening morphism. In general, the answer is negative.

Example 1.4.7. Consider the stratification

$$f\colon \mathrm{Spec}(k[x]) \coprod \mathrm{Spec}(k[y, y^{-1}]) \to X = \mathrm{Spec}(k[x, y]/(xy)).$$

Any stratification which flattens it should contain a closed stratum whose reduction is the origin. Set $X_1 = \mathrm{Spec}(k[x, x^{-1}])$, $X_2 = \mathrm{Spec}(k[y, y^{-1}])$ and $Y_n = \mathrm{Spec}(k[x]/(x^n))$. Then each $X_1 \coprod X_2 \coprod Y_n$ is a flattening stratification for f and it is easy to see that there exists no coarsest flattening strong stratification.

In the above example there exists a universal flattening morphism; it is f itself. In particular, it is at least a stratification.

Exercise 1.4.8. (i) Construct a similar example using nastier monomorphisms to show that even a coarsest flattening stratification may not exist.

(ii) Even worse, show that if $X = \mathrm{Spec}(k[x, y]) = \mathbf{A}^2$ and $Y = \mathrm{Spec}(k[x, x^{-1}, y]/(y))$ is the x-axis punctured at the origin, then the embedding $Y \hookrightarrow X$ has no universal flattening morphism. (Hint: show that it is flattened by any $\mathrm{Spec}(k[x, y]/(xy, y^n))$ (the x-axis with an embedded component at the origin) and its complement $\mathrm{Spec}(k[x, y, y^{-1}])$.)

On the positive side, we will show that the universal flattening exists when X is projective. The first case to deal with is when $\mathcal{F}$ is finite over $\mathcal{O}_X$, for example f is finite. It will be solved using the theory of Fitting ideals.

fittsec

1.4.5. *Fitting ideals.* We only recall the definition and basic properties.

Fittdef **Definition 1.4.9.** Let R be a ring and M be a finitely presented module with a given presentation $F \overset{d}{\to} G \to M \to 0$, where F and G are free of finite ranks and G is of rank r. Then ideal $\mathrm{Fitt}_j(d)$ which is the image of $\wedge^{r-j}(F) \otimes \wedge^{r-j} G^* \to R$ is called the j-th Fitting ideal of d. (Then $\mathrm{Fitt}_r(M) = R$, and we agree that $\mathrm{Fitt}_j(M) = R$ for $j > r$.)

Fittth **Theorem 1.4.10.** *(i) The Fitting ideal does not depend on the choice of a presentation, so it is an invariant* $\mathrm{Fitt}_j(M)$ *of* M*.*

(ii) The formation of Fitting ideals commutes with any base change $R \to R'$*, i.e.* $\mathrm{Fitt}_j(M \otimes_R R') = \mathrm{Fitt}_j(M)R'$*.*

(iii) The sequence of Fitting ideals increases, and for a local R*, a module* M *can be generated by* r *elements if and only if* $\mathrm{Fitt}_r = R$*.*

(iv) A finitely presented module M *is projective of constant rank* r *if and only if* $\mathrm{Fitt}_r(M) = R$ *and* $\mathrm{Fitt}_{r-1}(M) = 0$*.*

Exercise 1.4.11. Prove the theorem or read [E, §20.2]. (Hint: observe that the $\mathrm{Fitt}_j(d)$ is generated by all $(r-j)\times(r-j)$ minors of the matrix representing d and deduce (i) by comparing two presentations by embedding them into a third one; (ii) follows from the right exactness of the tensor products; (iii) follows from the linear algebra over the residue field and Nakayama's lemma, and (iv) follows from (iii).)

In particular, the formation of Fitting ideals is compatible with localizations, hence for any scheme X with a coherent module $\mathcal{O}_X$-module M we can define the ideals $\mathrm{Fitt}_j(M)\hookrightarrow\mathcal{O}_X$.

flatcoherent **Proposition 1.4.12.** *For any coherent sheaf $\mathcal{F}$ on a noetherian scheme X there exists a universal flattening morphism $i\colon\mathcal{X}\to X$. Moreover, i is a stratification of X (hence the coarsest flattening stratification).*

Proof. Since $\mathcal{F}$ is coherent on X, flatness is the same as projectivity. So, the flattening should kill each Fitting ideal $I_j=\mathrm{Fitt}_j(M)$ by making it 0 on a union of few components of the stratification and 1 on the complement. Killing I_j has the obvious universal solution: $\mathcal{X}_j=X_j\sqcup X'_j$, where $X_j=\mathrm{Spec}(\mathcal{O}_X/I_j)$ is the vanishing locus of I_j and $X'_j=X\setminus X'_j$ is its complement. Hence the fiber product over X of all $\mathcal{X}_i$ yields the universal flattening morphism $i\colon\mathcal{X}\to X$. Clearly, each $\mathcal{X}_j\to X$ is a stratification, hence $\mathcal{X}\to X$ is a stratification too. □

Exercise 1.4.13. Show that the stratification $\mathcal{X}\to X$ does not have to be strong. (Hint: take $X=\mathrm{Spec}(k[x,y]/(xy))$ and $\mathcal{F}=\mathcal{O}_X/x\mathcal{O}_X$.)

1.4.6. *The main flattening theorem.* As we saw, this result cannot be extended to non-finite affine morphisms of finite type. However, projective morphisms are better controlled by coherent sheaves, so one might hope to repeat this argument after twisting $\mathcal{F}$ enough by a very ample sheaf. Throughout this subsection we fix a projective morphism $f\colon Y\to X$, a very ample with respect to f sheaf $\mathcal{O}(1)$ on Y, and a coherent $\mathcal{O}_Y$-module $\mathcal{F}$.

To simplify notation we will often assume that $X=\mathrm{Spec}(A)$ is affine. Then instead of $\mathcal{O}_X$-module $f_*\mathcal{F}$ one can work with the A-module $\Gamma(\mathcal{F})$. Recall that ([Har, Exercise II.5.9]) the category $\mathrm{Coh}(Y)$ of coherent $\mathcal{O}_Y$-modules is equivalent to the category $\mathrm{Coh}(f_*\mathcal{O}_Y(\bullet))$ of finitely generated graded $\oplus_n f_*\mathcal{O}_Y(n)$-modules modulo the equivalence $\sim$ which identifies $\oplus_n\mathcal{G}_n$ with $\oplus_n\mathcal{G}'_n$ if $\mathcal{G}_n=\mathcal{G}'_n$ for $n\gg 0$. For example, if X is affine, then one associates $\oplus_n\Gamma(\mathcal{F}(n))$ to $\mathcal{F}$ and reconstructs $\mathcal{F}$ as follows: Y is covered by affines Y_s for $s\in H^0(Y,\mathcal{O}(1))$ and

$$\Gamma(Y_s,\mathcal{F})=\mathrm{colim}_n\, s^{-n}H^0(Y,\mathcal{F}(n)).$$

We will need two results about the equivalence $\mathrm{Coh}(Y)\widetilde{\to}\mathrm{Coh}(f_*\mathcal{O}_Y(\bullet))/\sim$.

flatprojmod **Lemma 1.4.14.** *Keep the above notation, then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if each $\Gamma(\mathcal{F}(n))$ is $\mathcal{O}_X$-flat for $n\gg 0$. Moreover, in this case, the $\mathcal{O}_X$-modules $\Gamma(\mathcal{F}(n))$ are flat for each n such that $H^i(Y,\mathcal{F}(l))=0$ whenever $i>0$ and $l\geq n$.*

Proof. This is shown in the proof of [Har, 9.9], so we just outline the argument. The problem is local on X, so we can assume $X=\mathrm{Spec}(A)$. If $\Gamma(\mathcal{F}(n))$ are flat for $n\gg 0$, then using that flatness is preserved by colimits we obtain that the sections $\Gamma(Y_s,\mathcal{F})$ are flat, that is $\mathcal{F}$ is flat.

Assume now that $\mathcal{F}$ is A-flat and choose n as in the second assertion. Choose an affine covering $\mathcal{U}$ of Y and consider the associated Čech complex $C^{\cdot}_{\mathcal{U}}(\mathcal{F}(l))$ which computes the cohomology of $\mathcal{F}(l)$. Its terms are direct sums of sections of $\mathcal{F}$ on open subschemes, hence they are A-flat. For $l \geq m$ the higher cohomologies vanish hence this complex provides an A-flat resolution of $H^0(Y, \mathcal{F}(l))$. Therefore the latter A-module is flat. $\square$

The following result essentially states that the equivalence between $\mathcal{O}_Y$-modules and equivalence classes of graded modules is compatible with base changes.

basechangelem

Lemma 1.4.15. *Let $f\colon Y \to X$ and $\mathcal{F}$ be as above, let $g\colon X' \to X$ be a morphism and let $f'\colon Y' \to X'$ be the base change with the projection $g'\colon Y' \to Y$. Then natural morphisms $g^* f_*(\mathcal{F}(n)) \to f'_* g'^*(\mathcal{F}(n))$ are isomorphisms for $n \gg 0$.*

Proof. Since the schemes are quasi-compact and we can enlarge n as we wish, the question is local on X and X'. So, we can assume that $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ and should show that for a large n the map

$$H^0(\mathcal{F}(n)) \otimes_A A' \to H^0(\mathcal{F}(n) \otimes_A A')$$

induced, for example, by Čhech complexes is an isomorphism. Find a surjection

$$\phi\colon \mathcal{F}_0 = \oplus_{i=1}^m \mathcal{O}_Y(d_i) \to \mathcal{F}.$$

Let $\mathcal{G} = \mathrm{Ker}(\phi)$ and find a map $\psi\colon \mathcal{F}_1 = \oplus_{i=1}^n \mathcal{O}_Y(l_i) \to \mathcal{F}_0$, whose image is $\mathcal{G}$. Then $\mathcal{F} = \mathrm{Coker}(\mathcal{F}_1 \to \mathcal{F}_0)$. Exactness of sequences of sheaves is preserved by twists, so if $H^1(\mathcal{G}(n)) = 0$, then the sequence

$$\Gamma(\mathcal{G}(n)) \to \Gamma(\mathcal{F}_0(n)) \to \Gamma(\mathcal{F}(n)) \to 0$$

is exact, and if in addition, $H^1(\mathrm{Ker}(\psi)(n))) = 0$, then the map $\Gamma(\mathcal{F}_1(n)) \to \Gamma(\mathcal{G}(n))$ is surjective and hence the sequence

$$\Gamma(\mathcal{F}_1(n)) \to \Gamma(\mathcal{F}_0(n)) \to \Gamma(\mathcal{F}(n)) \to 0$$

is exact too.

Pullback to Y' is right exact, hence $\mathcal{F}' = \mathrm{Coker}(\mathcal{F}'_1 \to \mathcal{F}'_0)$, where $\mathcal{F}'_0 = \oplus_{i=1}^m \mathcal{O}_{Y'}(d_i)$ and similarly for $\mathcal{F}'_1$. As above, increasing n so that two first cohomologies on Y' vanish we achieve also that

$$\Gamma(\mathcal{F}'_1(n)) \to \Gamma(\mathcal{F}'_0(n)) \to \Gamma(\mathcal{F}'(n)) \to 0$$

is exact. By a direct inspection $\Gamma(\mathcal{F}_i(n)) \otimes_A A' \widetilde{\to} \Gamma(\mathcal{F}'_i(n))$ for $i = 0, 1$, and since the tensor product preserves cokernels (it is right exact), $\Gamma(\mathcal{F}(n)) \otimes_A A' \widetilde{\to} \Gamma(\mathcal{F}'(n))$. $\square$

Remark 1.4.16. In general, using Čech complex one obtains base change morphisms $g^* R^i f_*(\mathcal{F}) \to R^i f'_* g'^*(\mathcal{F})$ which are isomorphisms if $X' \to X$ is flat (in this case pulling back is exact and hence respects the cohomology of complexes). However, in general it is not an isomorphism even when $\mathcal{F}$ is flat. In this case, the modules in complexes are flat, but the base change does not respect the cohomology. This situation is studied in [Har, §III.12] and will be used.

The two lemmas imply that if $i\colon X' \to X$ flattens $\mathcal{F}$ then there exists n_0 such that $\Gamma(\mathcal{F}'(n)) = i^* \Gamma(\mathcal{F}(n))$ for $n \geq n_0$ and these sheaves are flat, that is, i flattens each $\Gamma(\mathcal{F}(n))$ for $n \geq n_0$. Moreover, once this happens we obtain from the first lemma that $\mathcal{F}'$ is X'-flat. So, the idea is very simple: choose n large enough and flatten all $\mathcal{O}_X$-modules $\Gamma(\mathcal{F}(n))$. The only issue is how to choose the bound n_0.

flattenth

Theorem 1.4.17. *Let X be a noetherian scheme and $\mathcal{F}$ be a coherent sheaf on an X-projective scheme Y. Then there exists a universal flattening morphism $i\colon X' \to X$ for $\mathcal{F}$ and i is a stratification.*

Proof. The problem is local on X, so we can assume that $X = \mathrm{Spec}(A)$. Let us also fix a very ample sheaf $\mathcal{O}(1)$. By Theorem 1.4.6 there exists a flattening stratification $i_0\colon X'' \to X$. Choosing n_0 for j as in Lemma 1.4.15 we obtain that all $\mathcal{F}(n)$ with $n \geq n_0$ can be flattened simultaneously, hence by Lemma 1.4.4 there exists a coarsest flattening stratification $\mathcal{X} \to X$ which flattens each $\mathcal{F}(n)$ with $n \geq n_0$ and applying the two lemmas to $\mathcal{X} \to X$ we obtain that it flattens $\mathcal{F}$. Unfortunately, this is not enough because an arbitrary flattening stratification $\mathcal{X}' \to X$ flattens all $\mathcal{F}(n)$ for $n \geq n_0'$, but this n_0' might be larger than n_0. So, it remains to show that if $\mathcal{X}_m \to X$ is the coarsest flattening of all $\mathcal{F}(n)$ with $n \geq m$ then the sequence of coarsenings $\mathcal{X}_{n_0} \hookrightarrow \mathcal{X}_{n_0+1} \hookrightarrow \dots$ stabilizes. In other words, the nilpotent structure is bounded unlike what can happen in the non-projective case.

Remark 1.4.18. Is there an elementary argument, which shows that for any flattening $\mathcal{X} \to X$ which is coarser than $X_0 \to X$ the same number n_0 works – if twisting by n_0 suffices for the base change to X_0 to be compatible with f_*, then it also suffices for the base change to $\mathcal{X}$? This seems plausible and I would not be surprised if the answer is positive, but the standard solution in the literature (after Grothendieck and then Mumford) is to invoke a relatively heavy artillery about higher direct images of flat modules.

In the sequel, for $x \in X$ let

$$\phi_{X,x}(l)\colon \Gamma(\mathcal{F}(l)) \otimes_A k(x) \to \Gamma(\mathcal{F}_x(l))$$

denote the fiber base change map, where $\mathcal{F}_x$ is the pullback of $\mathcal{F}$ to the fiber Y_x.

Claim: *There exists m such that for any $l \geq m$ and a point $x \in X$ the map $\phi_{X,x}(l)$ is surjective.*

First, let us deduce the theorem assuming the claim. Assume that $X' \to X$ is a flattening stratification and to simplify notation we will work locally on X' using the notation $X' = \mathrm{Spec}(A')$. Then $\phi_{X,x}(l)$ is the composition of the base change map $\phi_{X,X'}(l)\colon \Gamma(\mathcal{F}(l)) \otimes_A A' \to \Gamma(\mathcal{F}'(l))$ tensored with $k(x)$ and the base change map $\phi_{X',x}(l)\colon \Gamma(\mathcal{F}'(l)) \otimes_{A'} k(x) \to \Gamma(\mathcal{F}_x(l))$, in particular, the latter is surjective and hence an isomorphism by [Har, 12.11(a)] applied to $\mathcal{F}'(l)$, which is A'-flat by our assumption. By [Har, 12.11(b)] this implies that $\Gamma(\mathcal{F}'(l))$ is A'-flat. Thus any flattening stratification of $\mathcal{F}$ automatically flattens each $\mathcal{F}(l)$ with $l \geq m$, and the universal flattening is the coarsest stratification $X' \to X$ which does this.

It remains to prove the claim, and to do so we will use a flattening $X'' \to X$ which exists by Lemma1.4.15. Again just to simplify notation we work locally on $X'' = \mathrm{Spec}(A'')$. By Lemma1.4.15 there exists m such that for $l \geq m$ the base change maps $\phi_{X,X''}(l)$ are isomorphisms: $\Gamma(\mathcal{F}''(l)) = \Gamma(\mathcal{F}(l)) \otimes_A A''$. Since $\phi_{X,x}(l)$ is composed from $\phi_{X,X''}(l) \otimes_{A''} k(x)$ and the fiber base change map $\phi_{X'',x}(l)$, it remains to show that the latter are isomorphisms for large enough l.

By [Har, Theorem III.8.8] there exists m such that $R^i f_* \mathcal{F}''(l) = H^i(\mathcal{F}''(l)) = 0$ whenever $i > 0$ and $l \geq m$. Fix an affine cover of Y and consider the associated Čech complex $\mathcal{C}^\cdot$ of $\mathcal{F}(l)$. Its cohomologies vanish for $i > 0$, hence this complex is an A''-flat resolution of $\Gamma(\mathcal{F}''(l))$, in particular, the latter is A''-flat. Acyclic complexes of flat modules are preserved by any base changes, hence the complex

$$0 \to \Gamma(\mathcal{F}''(l)) \otimes_{A''} k(x) \to \mathcal{C}^{\cdot} \otimes_{A''} k(x)$$

is acyclic. Without the first term this is nothing else but the Čech complex of $\mathcal{F}_x$, hence higher cohomologies of $\mathcal{F}_x$ vanish and the zero cohomology $\Gamma(\mathcal{F}_x)$ is precisely the base change $\Gamma(\mathcal{F}''(l)) \otimes_{A''} k(x)$, which means that $\phi_{X'',x}(l)$ is an isomorphism. □

Remark 1.4.19. In the classical works one uses heavier tools – uses existence of X'' to show that only finally many Hilbert polynomials show up in the fibers and proves a uniform bound on the twist, depending only on the Hilbert polynomial, which suffices to kill all $H^1(\mathcal{F}_x)$ at once. We will prove these bounds later, but managed to avoid using them here.

Higher direct images were essentially used in the proof of the flattening theorem and we conclude the section with a couple of remarks about them.

Remark 1.4.20. (i) The Hilbert polynomial $P(l) = \sum_i (-1)^i h^i(\mathcal{F}(l))$ is locally constant in flat projective families because the Euler-Poincare characteristic is locally constant. However, separate cohomology can jump on the fibers and controlling this phenomenon to some extent is the main tool of the theory of higher images.

(ii) There is one remarkable case, when it is easier to control jumping of h^i. If X is normal and $Y \to X$ is flat projective of relative dimension one, then $h^i(\mathcal{O}_Y)$ can jump only on the fibers with embedded components, as follows from Stein factorization and constancy of Euler characteristic of $\mathcal{F}_x = \mathcal{F} \otimes k(x)$ for $x \in X$.

Example 1.4.21. (i) Here is an example of a flat family of curves, where jumping happens. Let $S = \mathrm{Spec}(k[\pi])$ and let C be a family of curves in $\mathbf{P}^3_S$ given parametrically by $(\pi t, t^2, t^3)$. For $a \neq 0$ the curve C_a is a rational space curve and $h^0(C_a) = 1$, $h^1(C_a) = 0$. At $a = 0$ it degenerates to a cuspidal plane curve with an embedded component pointing outside the plane (so the resulting curve is also non-plane). It has $h^0(C_0) = 0$ (the global sections are constants and nilpotents at 0) and $h^1(C_0) = 1$ because the delta invariant of the cusp is 1. Note that the Euler characteristic is 1 everywhere.

(ii) A careful study of this example is very instructive as it illustrates all basic pathologies that can happen if the conditions of [Har, Theorem III.12.11] are not satisfied. The first base change $R^1 f_* \mathcal{O}_C \otimes k \to H^1(\mathcal{O}_{C_0})$ is surjective (the highest one is always surjective), hence $R^1 f_* \mathcal{O}_C$ is not locally free. In fact, it is a skyscraper at 0. The base change $f_* \mathcal{O}_C \otimes k \to \Gamma(\mathcal{O}_{C_0})$ is not surjective as there are no nilpotents in $\mathcal{O}_C$.

1.5. **Hilbert schemes.** Thanks to the flattening results one can build a universal family of subschemes of $\mathbf{P}^n$ with a fixed Hilbert polynomial. More, specifically, we will prove representability (compare to Remark 1.3.22(ii)) of a functor $Hilb_{X/S,P}$, where P is a polynomial and $X \to S$ is a projective morphism with a fixed relatively ample sheaf $\mathcal{O}(1)$ on X.

Hilbdef **Definition 1.5.1.** The functor $Hilb_{X/S,P}$ on the category of S-schemes is defined by setting $Hilb_{X/S,P}(T)$ equal to the set of subschemes $Z_T \hookrightarrow X_T = X \times_S T$ which are T-flat and have Hilbert polynomial equal to P on each T-fiber.

The main goal of this section is to represent the Hilbert functor by a scheme.

repdef **Definition 1.5.2.** A functor $F : \mathcal{C} \to Sets$ is *represented* by an object X if there is an isomorphism of functors $\varepsilon : F \to \mathrm{Hom}(\cdot, X)$. Since any choice of ε is uniquely determined by the object $f = \varepsilon^{-1}(\mathrm{Id}_X) \in F(X)$, we say that a pair $(X, f \in F(X))$ represents F.

The idea how to represent $Hilb$ is very simple: if $Y = \mathbf{P}^n_X$ and $\mathcal{I}$ is the ideal defining Z_T in Y_T, then for large enough m the sheaf $\mathcal{I}(m)$ is generated by its pushforward to T (the relative the global sections), which is a T-family of vector subspaces of $\mathcal{O}_{Z_T}(m)$ of fixed dimension. Therefore $Hilb_{\mathbf{P}^n_X/X,P}$ is represented by a subscheme of the corresponding Grassmannian in this case, and in general, $\mathrm{Hilb}_{Y/X}$ is a closed subscheme of $\mathrm{Hilb}_{\mathbf{P}^n_X/X}$. The main obstacle to working out this program is the cohomology groups, and we have to control them uniformly for all closed subschemes with a given Hilbert polynomial P – by cohomology base change properties it suffices to control the fibers over the closed points.

1.6. **Mumford's regularity.** The following notion is introduced by Mumford, it turns out to be very convenient for inductive proofs of vanishing results for cohomologies.

mregdef **Definition 1.6.1.** Let k be a field. A coherent $\mathcal{O}_{\mathbf{P}^n_k}$-sheaf $\mathcal{F}$ is called (Castelnuovo-Mumford) m-regular if $H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) = 0$ for any $i > 0$.

The logic of the definition is clear: since higher cohomologies are "responsible for obstructions of higher order", we want them to die earlier. Here is an indication that the notion is working well:

mregprop **Proposition 1.6.2.** *If $\mathcal{F}$ is m-regular then for any $l \geq m$:*
(i) $\mathcal{F}$ is l regular;
(ii) $\mathcal{F}(l)$ is generated by global sections;
(iii) the map

$$\phi : H^0(\mathbf{P}^n_k, \mathcal{F}(l)) \otimes H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbf{P}^n_k, \mathcal{F}(l+1))$$

is onto.

Proof. Since cohomologies commute with flat base change, we can extend k and assume that it is infinite. Then the key idea is to consider a *general* hyperplane H with a restriction $\mathcal{F}_H$ of $\mathcal{F}$ and to apply the induction hypothesis to $\mathcal{F}_H$.

genex **Exercise 1.6.3.** (i) Show that for a general choice of H and any i the following sequence is exact

$$0 \to \mathcal{F}(i) \to \mathcal{F}(i+1) \to \mathcal{F}_H(i+1) \to 0.$$

(Hint: The genericity is needed for the injectivity of $\mathcal{F}(i) \to \mathcal{F}(i+1)$. If H is locally given by the vanishing of s, this amounts to s being a non-zero divisor in $\mathcal{F}$. Show that this happens if and only if H does not contain associated points of $\mathcal{F}$ (locally these are points corresponding to ideals p such that the composition series of an A-module F contains A/p). Since $\mathrm{Ass}(\mathcal{F})$ is a finite set and k is infinite, such an H exists.)

(ii) Use the corresponding long exact sequence for an appropriate i to check that $\mathcal{F}_H$ is m-regular, and apply the induction assumption for $\mathcal{F}_H$ to deduce that $\mathcal{F}$ is l-regular for any $l \geq m$.

To show surjectivity of ϕ we consider the commutative diagram whose bottom line is a part of the long exact sequence.

$$\begin{array}{ccccc}
 & & H^0(\mathbf{P}^n_k, \mathcal{F}(l)) \otimes H^0(\mathbf{P}^n_k, \mathcal{O}(1)) & \xrightarrow{f} & H^0(H, \mathcal{F}_H(l)) \otimes H^0(H, \mathcal{O}_H(1)) \\
 & & \downarrow \phi & & \downarrow \phi_H \\
H^0(\mathbf{P}^n_k, \mathcal{F}(l)) & \xrightarrow{h} & H^0(\mathbf{P}^n_k, \mathcal{F}(l+1)) & \xrightarrow{g} & H^0(H, \mathcal{F}_H(l+1))
\end{array}$$

Note that f is surjective because $H^1(\mathcal{F}(l)) = 0$ by the l-regularity and ϕ_H is onto by the induction assumption on n. Hence the image of ϕ is mapped onto $H^0(H, \mathcal{F}_H(l+1))$, and by exactness in the middle term, the images of ϕ and h span $H^0(\mathcal{F}(l+1))$. Since the first image obviously contains the second one, ϕ is surjective, as claimed.

Consider the submodule $\mathcal{E} \hookrightarrow \mathcal{F}(l)$ generated by the global sections of $\mathcal{F}(l)$. The surjectivity of ϕ implies that the embedding $H^0(\mathcal{E}(1)) \hookrightarrow H^0(\mathcal{F}(l+1))$ is an equality. Since any $\mathcal{F}(m)$ is l-regular for $m \geq 0$, twisting by m we obtain that $\Gamma(\mathcal{E}(m)) = \Gamma(\mathcal{F}(l+m))$ and hence $\mathcal{E} = \mathcal{F}(l)$ by the equivalence of the categories of modules. Thus, $\mathcal{F}(l)$ is generated by global sections. □

Definition 1.6.4. Let S be a noetherian schemes and $\mathcal{F}$ be an $\mathcal{O}_S$-flat $\mathcal{O}_{\mathbf{P}^n_S}$-coherent module. We say that $\mathcal{F}$ is m-regular if its restrictions to S-fibers are so.

The following result follows from Lemma 1.6.2 by applying the theorem on direct images [Har, III.12.11].

mregcor **Corollary 1.6.5.** *Keep the above notation, and let $f : \mathbf{P}^n_S \to S$ denote the projection and $\mathcal{F}$ be an m-regular $\mathcal{O}_X$-coherent sheaf, then for any $l \geq m$:*
(i) $R^i f_* \mathcal{F}(l-i) = 0$ for $i > 0$;
(ii) $f^* f_* \mathcal{F}(l) \to \mathcal{F}(l)$ is onto;
(iii) the map

$$f_* \mathcal{F}(l) \otimes f_* \mathcal{O}(1) \to f_* \mathcal{F}(l+1)$$

is onto;
(iv) the sheaf $f_* \mathcal{F}(l)$ is locally free and taking this direct image commutes with any base change: given a morphism $g : S' \to S$, the product $X' = X \times_S S'$ and the projections $f' : X' \to S'$ and $g' : X' \to X$ the base change morphism $g^* f_* \mathcal{F}(l) \to f'_* g'^* \mathcal{F}(l)$ is an isomorphism.

Finally, we prove that there exists a uniform m-regularity bound depending only on the Hilbert polynomial.

mregth **Theorem 1.6.6.** *Let $\mathcal{F} \hookrightarrow \mathcal{O}_{\mathbf{P}^n_k}$ be a sheaf of ideals on $\mathbf{P}^n_k$ and let P be its Hilbert polynomial. Then there exists a number m_P depending only on P such that $\mathcal{F}$ is m_P-regular.*

Proof. We use the same construction with induction on n as earlier, with the case of $n = 0$ being trivial. Choose a general hyperplane H and let $\mathcal{F}_H$ be the induced ideal sheaf on H and P_H be its Hilbert polynomial. Then $P_H(m) = P(m) - P(m-1)$, hence by induction assumption $\mathcal{F}_H$ is m'-regular for a number m' depending only on P. From the long exact sequence for each $i > 1$ we have that $H^i(\mathcal{F}(l)) \xrightarrow{\sim} H^i(\mathcal{F}(l+1))$ for any $l \geq m'$, hence all these groups vanish by the Serre's vanishing. For $i = 1$, we still have a map $H^1(\mathcal{F}(l)) \to H^1(\mathcal{F}(l+1))$, but this time its kernel, which equals to the image of $H^0(\mathcal{F}_H(l+1))$, can be non-trivial. Moreover, the kernel is trivial if and only if the map $H^0(\mathcal{F}(l+1)) \to H^0(\mathcal{F}_H(l+1))$ is onto, and then all maps $H^0(\mathcal{F}(L)) \to H^0(\mathcal{F}_H(L))$ would be onto for $L > l$ because the maps

$$H^0(\mathcal{F}_H(L)) \otimes H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{F}_H(L+1))$$

are onto by the m'-regularity of $\mathcal{F}_H$. In particular, it would follow that all maps $H^1(\mathcal{F}(l)) \widetilde{\to} H^1(\mathcal{F}(l+1))$ have no kernels, and hence are isomorphisms. Since the latter would contradict Serre's vanishing, each map $H^1(\mathcal{F}(l)) \to H^1(\mathcal{F}(l+1))$ with $l \geq m'$ has a non-trivial kernel, unless its source is zero. In particular, $H^1(\mathcal{F}(m)) = 0$ for any $m \geq m' + h^1(\mathcal{F}(m'))$. So, it only remains to bound $h^1(\mathcal{F}(m'))$. But

$$h^1(\mathcal{F}(m')) = h^0(\mathcal{F}(m')) - P(m') \leq h^0(\mathcal{O}_{\mathbf{P}^n}(m')) - P(m') = \binom{m'+n}{n}$$

is bounded by a function of P, $n = \deg(P)$ and $m' = m'(P)$. □

Now we are prepared to prove the main result of this section.

Hilbth **Theorem 1.6.7.** *Let $f : X \to S$ be a projective morphism of Noetherian schemes with a relatively ample sheaf $\mathcal{O}(1)$ and P be a fixed polynomial. Then the functor $Hilb_{X/S,P}$ is represented by a projective S-scheme* $\mathrm{Hilb}(X/S,P)$ *of finite type with a universal subscheme* $\mathrm{Univ}(X/S,P) \hookrightarrow X \times_S \mathrm{Hilb}(X/S,P)$.

Proof. Step 1. *Embedding into Grassmannian functor.* Embed X into $\mathbf{P} = \mathbf{P}^n_S$ using $\mathcal{O}(1)$ and let $g : \mathbf{P} \to S$ be the projection. The main task will be to represent the functor $Hilb_{\mathbf{P}/S,P}$. Let S' be any S-scheme with the projection $g' : \mathbf{P}' = S' \times_S \mathbf{P} \to S'$ and a closed subscheme $Z' \hookrightarrow \mathbf{P}'$ such that the Hilbert polynomial of the fibers is P. Consider the ideal sheaf $\mathcal{I}' \hookrightarrow \mathcal{O}_{\mathbf{P}'}$ which defines Z', then we have a short exact sequence $0 \to \mathcal{I}' \to \mathcal{O}_{\mathbf{P}'} \to \mathcal{O}_{Z'} \to 0$ hence the Hilbert polynomials of $\mathcal{I}'$ on the S'-fibers are all equal to $Q(m) = \binom{m+n}{n} - P(m)$. By Theorem 1.6.6, there exists $m = m_Q$ such that $\mathcal{I}'$ is m-regular. Replacing m with $\max(m, 0)$ we achieve that $\mathcal{O}_{\mathbf{P}'}$ is m-regular (see [Har, III.5.1]), and then $\mathcal{O}_{Z'}$ also has to be m-regular, as can be seen from the long cohomological sequence.

Twisting the short exact sequence by m we kill all higher direct images by Corollary 1.6.5 and hence obtain the exact sequence

$$0 \to g'_*\mathcal{I}'(m) \to g'_*\mathcal{O}_{\mathbf{P}'}(m) \to g'_*\mathcal{O}_Z(m) \to 0.$$

Moreover, by 1.6.5(iv) the sheaves in the sequence are locally free and by 1.6.5(ii) the submodule $\mathcal{I}'(m) \hookrightarrow \mathcal{O}_{\mathbf{P}'}(m)$ is generated by relative global sections in the sense that $g'^*g'_*\mathcal{I}'(m) \to \mathcal{I}'(m)$ is onto. In particular, to any Z' we can attach the locally free subsheaf $\mathcal{L}' = g'_*\mathcal{I}'(m)$ of $g'_*\mathcal{O}_{\mathbf{P}'}(m)$ which defines Z' uniquely (Exercise 1.6.8 below) and such that $g'_*\mathcal{O}_{\mathbf{P}'}(m)/\mathcal{L}'$ is locally free. In particular, we have embedded the functor $Hilb_{\mathbf{P}/S,P}$ into the Grasmann functor $Grass_{S,g_*\mathcal{O}_{\mathbf{P}}(m),Q(m)}$ recalled below.

embedex **Exercise 1.6.8.** Check that $Z' = \mathbf{Proj}(\mathrm{Coker}(g'^*\mathcal{L}'(-m) \to \mathcal{O}_{\mathbf{P}'}))$.

Step 2. *Representation of the Grassmannian.*

grassdef **Definition 1.6.9.** Given a scheme S with a locally free sheaf $\mathcal{E}$ of rank n and a number $0 \leq m \leq n$, the Grassmannian functor $Grass_{S,\mathcal{E},m}$ is defined as follows. For a morphism $f : S' \to S$ the set $Grass_{S,\mathcal{E},m}(S')$ consists of all rank m locally free subsheaves $\mathcal{L}' \hookrightarrow f^*\mathcal{E}$ with a locally free quotient $f^*\mathcal{E}/\mathcal{L}'$.

Remark 1.6.10. (i) Our definition is dual to the definition from [EGA I', 6.9].

(ii) The sheaves are finitely presented, hence local freeness is equivalent to flatness. The subtlety here is in requiring that the quotients also form a flat family.

Exercise 1.6.11. (i) Let $E = \mathbf{Spec}(\mathcal{O}_S[\mathcal{E}])$ be the vector bundle attached to $\mathcal{E}$, and $E^* = \mathbf{Spec}(\mathcal{O}_S[\mathcal{E}^*])$ be its dual. Identify $Grass_{S,\mathcal{E},m}(S')$ with the set of vector factor-bundles in $E \times_S S'$ of rank m, and with the set of vector subbundles of $E^* \times_S S'$ of rank m.

(ii) Show that $Grass_{S,\wedge^m\mathcal{E},1}$ is represented by the projective fiber $\mathbf{P}(\wedge^m\mathcal{E}^*)$.

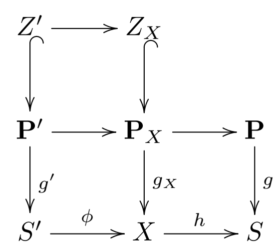
(iii) Represent $Grass_{S,\mathcal{E},m}$ by a closed subscheme $\mathrm{Grass}(S,\mathcal{E},m) \hookrightarrow \mathbf{P}(\wedge^m\mathcal{E}^*)$.

(iv) Show that $\mathrm{Grass}(S,\mathcal{E},m)$ is equipped with a universal rank m locally free subsheaf $\mathcal{L}_{S,\mathcal{E},m} \hookrightarrow f^*\mathcal{E}$, where $f : \mathrm{Grass}(S,\mathcal{E},m) \to S$ is the structure map.

(v) Check that if $S = \mathrm{Spec}(A)$ and $\mathcal{E} \approx \mathcal{O}_S^n$ is free, then $\mathrm{Grass}(S,\mathcal{E},m)$ is given by quadratic equations in $\mathbf{P}(\wedge^m\mathcal{E}^*)$.

Step 3. *Hilbert schemes of* $\mathbf{P}/S$*.*

To simplify the notation set $X = \mathrm{Grass}(S, g_*\mathcal{O}_{\mathbf{P}}(m), Q(m))$ and $\mathbf{P}_X = \mathbf{P}^n_X$. Let $h : X \to S$ and $g_X : \mathbf{P}_X \to X$ be the natural projections, and let $\mathcal{L}_X \hookrightarrow h^*g_*\mathcal{O}_{\mathbf{P}}(m) \xrightarrow{\sim} (g_X)_*\mathcal{O}_{\mathbf{P}_X}(m)$ be the universal subsheaf of rank $Q(m)$. The coherent subsheaf of $\mathcal{O}_{\mathbf{P}_X}(m)$ generated by $\mathcal{L}_X$ is the image of the map $g_X^*\mathcal{L}_X \to g_X^*(g_X)_*\mathcal{O}_{\mathbf{P}_X}(m) \to \mathcal{O}_{\mathbf{P}_X}(m)$, hence the closed subscheme $Z_X \hookrightarrow \mathbf{P}_X$ defined by vanishing of $\mathcal{L}_X$ can be described as $\mathbf{Proj}(\mathrm{Coker}(g_X^*\mathcal{L}_X(-m) \to \mathcal{O}_{\mathbf{P}_X}))$. So far, we have the right part of the following diagram:



Let now S' and a closed subscheme $Z' \hookrightarrow \mathbf{P}' = \mathbf{P}^n_{S'}$ be as in Step 1. By the universal property of the Grassmannian, the locally free sheaf $\mathcal{L}' = g'_*\mathcal{I}'(m)$ is induced from $\mathcal{L}_X$ via a uniquely defined map $\phi : S' \to X$. Then it follows from the formula in Exercise 1.6.8 and the fact that the direct image $(g_X)_*\mathcal{I}_X(m)$ is compatible with the base change with respect to ϕ by Corollary 1.6.5(iv) that Z' is induced from Z_X, i.e. $Z' = Z_X \times_X S'$ and all squares in the diagram are Cartesian.

Let $\mathcal{X} \to X$ be the flattening stratification for the X-scheme Z_X (or its structure sheaf) whose existence is guaranteed by Theorem 1.4.17. Let, furthermore, $\mathrm{Hilb}(\mathbf{P}/S, P)$ be the union of strata of $\mathcal{X}$ where the Hilbert polynomial of Z_X is equal to P. Then just by the definition of the flattening stratification, $\phi : S' \to X$ factors through $\mathrm{Hilb}(\mathbf{P}/S, P)$, and Z' is induced from the $\mathrm{Hilb}(\mathbf{P}/S, P)$-flat subscheme $\mathrm{Univ}(X/S, P) := \mathrm{Hilb}(\mathbf{P}/S, P) \times_X Z_X$ of $\mathrm{Hilb}(\mathbf{P}/S, P) \times_S \mathbf{P}$. It proves that the embedding $Hilb_{\mathbf{P}/S,P} \hookrightarrow Grass_{S,g_*\mathcal{O}_{\mathbf{P}}(m),Q(m)}$ of functors is actually induced from a locally closed immersion $i : \mathrm{Hilb}(\mathbf{P}/S, P) \hookrightarrow \mathrm{Grass}(S, g_*\mathcal{O}_{\mathbf{P}}(m), Q(m))$ of S-schemes.

Step 4. *The immersion* i *is a closed immersion, in particular,* $\mathrm{Hilb}(\mathbf{P}/S, P)$ *is* S*-projective.* It suffices to show that $\mathrm{Hilb}(\mathbf{P}/S, P)$ is proper, and this can be done by the valuative criterion (since the schemes are noetherian, it even suffices to consider only discrete valuations, but we do not need this observation). By the functorial definition of $\mathrm{Hilb}(\mathbf{P}/S,P)$ the valuative criterion reduces to the following: if R is a valuation ring with fraction field K, then any closed subscheme $Z_K \hookrightarrow \mathbf{P}^n_K$ extends to an R-flat closed subscheme of $Z \hookrightarrow \mathbf{P}^n_R$.

Exercise 1.6.12. Prove this statement using that an R-module is flat if and only if it has no non-zero π-torsion for any non-zero $\pi \in m_R$.

Step 5. *Hilbert schemes of X/S.*

Exercise 1.6.13. Check that the natural embedding of functors $Hilb_{X/S,P} \hookrightarrow Hilb_{\mathbf{P}/S,P}$ is induced from a closed immersion of schemes $\mathrm{Hilb}(X/S,P) \hookrightarrow \mathrm{Hilb}(\mathbf{P}/S,P)$

□

Remark 1.6.14. It is an important feature that the flattening stratification, and hence Hilbert schemes, can be non-reduced. A famous example by Mumford shows that the Hilbert schemes of curves of degree 14 and genus 24 in $\mathbf{P}^3$ is non-reduced at the generic point of one of its irreducible component. Actually, it is a typical situation that Hilbert schemes are very singular schemes with many irreducible components of various dimensions. In fact, there is a theorem of Vakil that Murphy's law holds for Hilbert schemes – any isomorphism class of a singularity on varieties over $\mathbf{Q}$ occurs on some Hilbert schemes.

Exercise 1.6.15. (i) If X is a smooth projective scheme of a field k, then the Hilbert scheme of n-points $\mathrm{Hilb}(X/k,n)$ admits a natural map $\phi(X,n)$ to the symmetric power $X^{(n)} = X^n/S_n$.

(ii) If X is a curve, then $\phi(X,n)$ is an isomorphism, but in larger dimensions the fibers over the diagonal of $X^{(n)}$ are not zero-dimensional.

Here is an important corollary of the representability of Hilbert functors.

Exercise 1.6.16. Use graph of a morphism to prove that if X and Y are S-projective and X is S-flat, then the functor $\mathrm{Hom}_S(X,Y)$ is represented by a scheme of locally finite type over S.

Finally, we note that although we have defined the Hilbert scheme over any scheme, if S is equicharacteristic, then the Hilbert scheme of $\mathbf{P}$ is induced from the prime field and hence forms a constant family over S. The situation in the mixed characteristic is different, as there might be components of the moduli spaces (e.g. supersingular $K3$ surfaces) which are special for the characteristic. In particular, the Hilbert schemes over $\mathbb{Z}$ does not have to be even flat.

Exercise 1.6.17. (i) Show that for any scheme S and polynomial P one has that $\mathrm{Hilb}(\mathbf{P}^n_S/S,P) = \mathrm{Hilb}(\mathbf{P}^n_{\mathbf{Z}}/\mathbf{Z},P) \times_{\mathbf{Z}} S$ and deduce that the Hilbert schemes of projective spaces are S-flat if S is equicharacteristic.

(ii)* Give an example of a Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^n_{\mathbf{Z}}/\mathbf{Z},P)$ which is not $\mathbf{Z}$-flat.

1.7. **Relative stable curves.**

1.7.1. *An explicit description.*

relstabdefin

Definition 1.7.1. A *relative curve* is a flat finitely presented morphism $f : C \to S$ of relative dimension one. An *n-pointed* S-curve is an S-curve C provided with an ordered n-tuple of closed subschemes $D_1, \dots, D_n$ which are mapped isomorphically onto C. An n-pointed curve is called *nodal*, *proper of genus g*, *semistable*, or *stable* if all its geometric fibers $(C_{\overline{s}}, D_{\overline{s}})$ are so.

Note that it suffices to consider geometric points of the form $\mathrm{Spec}(\overline{k(s)}) \to S$, where $s \in S$ is a usual point. In particular, a nodal curve C over a field is a curve such that $C \otimes_k \overline{k}$ is nodal. Each its singularity P gives rise to $[k(P) : k]$ geometric nodes (which are taken into account in the definition of stability), but even if $k(P) = k$ it can happen that the tangents at P are not defined over k.

nodex **Exercise 1.7.2.** (i) Let P be as above and assume that $k(P) = k$. Prove that one of the following possibilities hold:

(a) $\widehat{\mathcal{O}}_P \widetilde{\to} k[[x,y]]/(xy)$, and then P is obtained by pasting two different k-points in the normalization, i.e. C is the pushout of $\widetilde{C}$ which contracts its closed subscheme as $\widetilde{P} = \mathrm{Spec}(k \oplus k) \to \mathrm{Spec}(k) = P$. The tangent cone consists of two affine k-lines in this case.

(b) $\widehat{\mathcal{O}}_P \widetilde{\to} k[[x,y]]/(Q(x,y))$ where $Q(x,y)$ is a quadratic form over k whose roots lie in a quadratic extension l/k, and then C is the pushout of $\widetilde{C}$ which contracts its closed subscheme as $\widetilde{P} = \mathrm{Spec}(l) \to \mathrm{Spec}(k) = P$. The tangent cone is an affine l-line with k-rational origin (e.g. $\mathrm{Spec}(\mathbf{R}[x,y]/(x^2+y^2))$).

(ii) Prove that each node contributes $k(\widetilde{P})^\times/k(P)^\times$ to $\mathrm{Pic}(C)$. In particular, the contribution is $k^\times$ or $l^\times/k^\times$, depending on the type of the node.

Now, let us consider an arbitrary relative nodal curve $f : C \to S$. It follows from our definition of smoothness that on the level of sets the singular locus $(C/S)_{\mathrm{sing}}$ of f coincides with the set of nodes in the fibers.

nodsinglem **Lemma 1.7.3.** *The zero locus of the first Fitting ideal* $\mathrm{Fitt}_1(\Omega^1_{C/S})$ *is unramified over* S *and coincides with* $(C/S)_{\mathrm{sing}}$ *set-theoretically. In particular we obtain a natural (may be non-reduced) scheme-structure on* $(C/S)_{\mathrm{sing}}$.

Proof. Since the formations of differentials and Fitting ideals are compatible with any base changes, it suffices to check this claim on the geometric fibers. The latter has already been done in Lemma 1.3.24. □

Now, we can describe the local structure of relative nodal curves. Only the situation at a point of the singular locus requires such a description. The following proposition generalizes Exercise 1.7.2, though we will use the exercise in the proof.

nodsingprop **Proposition 1.7.4.** *Let* $f : C \to S$ *be a nodal curve with noetherian base,* $s \in S$ *be a point and* x *be a nodal point in the* s*-fiber and such that* $k(x) = k(s)$*. If* $B = \widehat{\mathcal{O}}_x$ *and* $A = \widehat{\mathcal{O}}_s$ *denote the completed local rings, then* B *is* A*-isomorphic to a ring* $A[[u,v]]/(Q(u,v) - h_0)$*, where* $h_0 \in m_A$ *and* $Q(u,v) = au^2 + buv + cv^2$ *is a quadratic form over* A *with invertible discriminant.*

Proof. The local structure of the s-fiber of f at x was described in Exercise 1.7.2: $B/m_AB \widetilde{\to} \widehat{\mathcal{O}}_{C_s,x} \widetilde{\to} k(s)[[u,v]]/(q(u,v))$. Lift q to a quadratic form $Q(u,v)$ with coefficients in A and lift the homomorphism $k(s)[[u,v]] \to B/m_AB$ to a homomorphism $\phi : A[[u,v]] \to B$, then ϕ is onto and we obtain an exact sequence $0 \to I \to A[[u,v]] \to B \to 0$. By flatness of B tensoring this sequence with A/m_A we obtain an exact sequence $0 \to I/m_AI \to k(s)[[u,v]] \to B/m_AB \to 0$. So, I/m_AI is a free $k(s)[[u,v]]$-module generated by $q(u,v)$, hence $I/(m_AI + uI + vI)$ is of rank one and the Nakayama lemma implies that I is generated by an element z. We can take z to be a lifting of q and then $z \in Q + m_AA[[u,v]]$. So, $A[[u,v]]/(Q-h) \widetilde{\to} B$ for some $h = h_0 + h_uu + h_vv + \dots$ with $h_u \in m_A$, and a direct computation shows that replacing the coordinates u and v we can achieve that the element $Q - h$ rewrites as $Q(u', v') - h'_0$ with $h'_0 \in m_A$. In particular, $A[[u', v']]/(Q(u', v') - h'_0) \widetilde{\to} B$. $\square$

nodeexer **Exercise 1.7.5.** (i) Keep the notation of the proposition. Show that locally at the node x the scheme $(C/S)_{\rm sing}$ with the Fitting scheme structure is S-isomorphic to a closed subscheme of S which is the zero locus of h_0.

(ii) Let R be a ring. Show that an R-scheme C is a nodal relative curve if and only if étale-locally it is equivalent to R-schemes of the form $\mathrm{Spec}(R[x, y]/(xy - h))$ for $h \in R$. (Hint: split the quadratic form by an étale cover, also achieve that the branches in the fiber belong to two different irreducible components locally given by the vanishing of x and y, e.g. lift an appropriate étale cover of the fiber.)

Remark 1.7.6. Actually the lemma proves that the universal deformation of a node is smooth and one-dimensional: if k is a field and $k[[u, v]]/(uv)$ is a completed local ring of a k-node, then the universal deformation of the node is given by the homomorphism $k[[h]] \to k[[h]][[u, v]]/(uv - h)$.

Let us also briefly discuss possible generalizations to higher dimensions. This will not be used in the sequel.

Remark 1.7.7. (i) If S is a curve, then the notion of semistability naturally extends to higher dimensions: a morphism $f : X \to S$ is semistable if étale-locally it is of the form $\mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_n - h)) \to \mathrm{Spec}(R)$. It is not difficult to see, that this is equivalent to requiring that f is flat, its fibers are normal crossings divisors, and each irreducible component of the fiber is a Cartier divisor. The latter condition is automatic in the relative curve case, but cannot be omitted in general. One explanation for this phenomenon is that the singularities are not isolated anymore, so the deformations are of infinite dimensions.

(ii) If S is not a curve, then the above definition is not general enough and a better version is of *polystable* S-schemes which are locally a fiber product of S-semistable ones: such a morphism $X \to S$ is étale-locally isomorphic to

$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(x_1 \cdots x_{n_1} - h_1, x_{n_1+1} \cdots x_{n_2} - h_2, \ldots)) \to \mathrm{Spec}(R).$$

In particular, a series of very recent results on resolution of morphisms shows that any dominant morphism between varieties of characteristic zero can be modified to a polystable one by blowing up the base and the source. On the other hand, already the polystable morphism $\mathrm{Spec}(k[x, y, z, t]/(xy - u, zt - v)) \to \mathrm{Spec}(k[u, v])$ cannot be modified to a semistable one.

1.7.2. *The dualizing sheaf.* Next we want to generalize sheaves ω_C to the relative situation. In [DM] a dualizing sheaf $\omega_{C/S}$ is defined using general Grothendieck's duality. We will use a bit more elementary approach.

rellcilem **Lemma 1.7.8.** *Any nodal curve $f : C \to S$ is locally a complete intersection: for any closed immersion $C \to \mathbf{A}^n_S$ (which always exist locally on C and S), the ideal $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbf{A}^n_S}$ which defines C is locally generated by $n - 1$ elements and the $\mathcal{O}_S$-ideal $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $n - 1$.*

Proof. Proposition 1.7.4 provides a local description of C/S. Using it one can describe the generators of $\mathcal{I}$ as in the proof of Lemma 1.3.4. $\square$

relprojcur **Lemma 1.7.9.** *Any proper nodal curve $C \to S$ is projective locally over S, i.e for any point $s \in S$ it is projective over a neighborhood of s.*

Proof. Choose points $P_1, \dots, P_n$ on each component in the smooth locus of the fiber C_s and shrink S so that the closed immersions $P_i \to C_s$ extend to closed subschemes $D_i \hookrightarrow C$ (i.e. $D_i \times_C \mathrm{Spec}(s) \widetilde{\to} \mathrm{Spec}(k(P_i))$). Shrinking S, we can achieve that each D_i is a Cartier divisor and D is contained in the smooth locus of C/S and hits all irreducible components of the S-fibers. Then the restriction of $\mathcal{L} = \mathcal{O}_C(D)$ to each fiber is ample by Lemma 1.3.7. Hence $\mathcal{L}$ itself is ample by the criterion of ampleness [EGA, III.4.7]. □

Exercise 1.7.10. Read the proof of [EGA, III.4.7] (straightforward but computational), or deduce this result using (much more advanced) results of [Har, III.12]. (Hint: using semi-continuity find n such that $h^1(\mathcal{L}_s^{\otimes n})$ vanishes for each $s \in S$ (here $\mathcal{L}_s^{\otimes n} = \mathcal{L}^{\otimes n} \otimes k(s)$ is the restriction to the s-fiber), and deduce that $f_*(\mathcal{L}^{\otimes})$ commutes with any base change $S' \to S$; enlarge n so that each $\mathcal{L}_s^{\otimes n}$ is very ample and observe that one obtains a morphism $i : C \to \mathbf{P}(f_*(\mathcal{L}^{\otimes}))$ whose restrictions to the S-fibers are closed immersions; deduce that i is a proper monomorphism, hence a closed immersion.)

Let (C, D) be a semistable n-pointed S-curve. If C embeds into $\mathbf{P} = \mathbf{P}_S^n$ and $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbf{P}}$ is its ideal, then we set $\omega_{C/S} = (\omega_{\mathbf{P}/S} \otimes \mathcal{O}_C) \otimes (\wedge^{n-1} \mathcal{I}/\mathcal{I}^2)^*$ and $\omega_{(C,D)/S} = \omega_{C/S}(D)$. Clearly, these sheaves are locally free.

Exercise 1.7.11. (i) Check that this definition is independent of the projective embedding (Hint: dominate two embeddings by a third one).

(ii) Check that the so-defined $\omega_{(C,D)/S}$ (with projective $C \to S$) commutes with base changes.

Using the exercise we can extend the definition to an arbitrary (C, D): by Lemma 1.7.9 $C \to S$ is projective locally on the base, so we can define $\omega_{(C,D)/S}$ over small pieces of S, but these sheaves are canonically isomorphic, so we can glue them to a global sheaf. Clearly the formation of the sheaves $\omega_{(C,D)/S}$ is compatible with any base change.

relampleth **Theorem 1.7.12.** *Let (C, D) be a stable S-curve. Then $\omega_{(C,D)/S}^{\otimes m}$ is relatively very ample for $m \geq 3$ and $R^1 f_*(\omega_{(C,D)/S}^{\otimes m}) = 0$ for $m \geq 2$, where $f : C \to S$ is the structure morphism.*

Proof. By Theorem 1.3.19 $h^1(\omega_{(C,D)/S}^{\otimes m} \otimes k(s)) = 0$ for $m > 1$ and any point $s \in S$. It follows from the theorem on cohomology base change that $R^1 f_*(\omega_{(C,D)/S}^{\otimes m}) = 0$ and $f_*(\omega_{(C,D)/S}^{\otimes m})$ is locally free for $m \geq 2$. So, for $m \geq 3$ we have that (i) the sheaf $\omega_{(C,D)/S}^{\otimes m} \otimes k(s)$ is very ample, (ii) its push-forward with respect to f commutes with any base change $S' \to S$, (iii) the restrictions of this sheaf to the fibers are very ample. It follows that $i : C \to \mathbf{P}(f_*(\omega_{(C,D)/S}^{\otimes m}))$ is a proper morphism which induces closed immersions on S-fibers. Hence i is a closed immersion and $\omega_{(C,D)/S}^{\otimes m}$ is relatively very ample. □

1.7.3. *Rigidified stable curves.* The theorem implies that any stable curve admits a tri-canonical embedding $C \to \mathbf{P}_S(f_*(\omega_{(C,D)/S}^{\otimes 3}))$, where $\mathbf{P}_S(V) = \mathrm{Proj}_S(\mathrm{Sym}(V))$, is the projective S-bundle defined by $\mathcal{L}$. In general, this bundle depends on (C, D) via $f_*(\omega_{(C,D)/S}^{\otimes 3})$, but locally it is just isomorphic to $\mathbf{P}_S^{N_{g,n}}$, where $N_{g,n} = 5g - 6 + 3n$ by Corollary 1.3.21. So, working locally on S we can fix a *trivialization* $\mathbf{P}_S(f_*(\omega_{(C,D)/S}^{\otimes 3})) \widetilde{\to} \mathbf{P}_S^{N_{g,n}}$. In a sense we just fix the coordinates on the trivial projective bundle. Equivalently, one can rigidify the vector bundle: by a *twisted trivialization* of the vector bundle $f_*(\omega^{\otimes 3}_{(C,D)/S})$ we mean an isomorphism $f_*(\omega^{\otimes 3}_{(C,D)/S}) \widetilde{\to} \mathcal{L}^{N_{g,n}+1}$, where $\mathcal{L}$ is a line bundle on S. Note that $\mathcal{L}$ does not have to be trivial.

Exercise 1.7.13. Construct a natural bijection between trivializations of a vector bundle V on S and twisted trivializations of the associated projective bundle $\mathbf{P}_S(V)$.

Once a trivialization of $\mathbf{P}_S(f_*(\omega^{\otimes 3}_{(C,D)/S}))$ is fixed we obtain a closed immersion $C \to \mathbf{P}_S^{N_{g,n}}$. Since C is S-flat and the Hilbert polynomial on the fibers is $P_{g,n}$ by Corollary 1.3.21, the latter immersion gives rise to a morphism $S \to \mathrm{Hilb}(\mathbf{P}_{\mathbf{Z}}^{N_{g,n}}/\mathbf{Z}, P_{g,n}(m))$, whose target will be denoted Hilb for simplicity. We can also formulate this observation in the language of functors: there is a natural morphism of functors $\mathcal{H}_{g,n} \to \mathrm{Hilb}_{g,n}$, where we identify the Hilbert scheme with the corresponding functor and the source functor is defined below.

Hdef **Definition 1.7.14.** Define $\mathcal{H}_{g,n}$ to be a functor which assigns to a scheme S isomorphism classes of n-pointed S-schemes (C, D) of genus g with trivialized $f_*(\omega^{\otimes 3}_{(C,D)/S})$ (where isomorphisms should respect the trivializations).

Let $Z \hookrightarrow \mathbf{P}^N \times \mathrm{Hilb}_{g,n}$ be the universal subscheme and let Z_{sm} be the smooth locus of the projection $Z \to \mathrm{Hilb}_{g,n}$. The scheme $\mathrm{Hilb}^n_{g,n}$ obtained from the n-th fiber power of Z_{sm} over $\mathrm{Hilb}_{g,n}$ by removing the diagonal represents the functor which assigns to S a subscheme $Z_S \to \mathbf{P}^N_S$ with Hilbert polynomial $P_{g,n}$ and n disjoint sections $S \to Z_S$ whose image lies in the smooth locus of the projection $Z_S \to S$. In particular, the morphism $\mathcal{H}_{g,n} \to \mathrm{Hilb}_{g,n}$ lifts to $\phi_{g,n} : \mathcal{H}_{g,n} \to \mathrm{Hilb}^n_{g,n}$ and the latter morphism is already injective. Thus, we embedded $\mathcal{H}_{g,n}$ into a representable functor and our aim now is to show that this embedding identifies $\mathcal{H}_{g,n}$ with the functor attached to a closed subscheme of $\mathrm{Hilb}^n_{g,n}$. Note that the image of $\phi_{g,n}$ is a subfunctor of $\mathrm{Hilb}^n_{g,n}$ which assigns to each S the set of subschemes $C \subset \mathbf{P}_S^{N_{g,n}}$ with Hilbert polynomial $P_{g,n}(m)$ and n sections $S \widetilde{\to} D_i \hookrightarrow C$ such that the following three conditions are satisfied:

1. $(C, D) \to S$ is a stable n-pointed curve,
2. $\mathcal{O}_C(1) \otimes f^*\mathcal{L} \widetilde{\to} \omega^{\otimes 3}_{(C,D)/S}$ for an invertible $\mathcal{O}_S$-sheaf $\mathcal{L}$, where $\mathcal{O}_C(1)$ is induced from the projective embedding and $f : C \to S$ is the structure map,
3. C spans $\mathbf{P}^N_S$ in the sense that the homomorphism $f_*\mathcal{O}_C(1) \to pr_*\mathcal{O}_{\mathbf{P}^N}(1)$ is onto for the projection $pr : \mathbf{P}^N_S \to S$ (it is equivalent to requiring that C is not contained in a smaller linear subspace $\mathbf{P}^m_S$, or that C is embedded by a complete linear system of $\mathcal{O}_C(1)$).

Theorem 1.7.15. *The functor $\mathcal{H}_{g,n}$ is represented by a locally closed subscheme $H_{g,n}$ of $\mathrm{Hilb}^n_{g,n}$.*

Proof. We should check that the above three conditions define a locally closed subscheme in $\mathrm{Hilb}^n_{g,n}$. Let (C, D) be the universal n-pointed curve over $\mathrm{Hilb}^n_{g,n}$. The first condition defines an open subscheme, because the set of points $x \in \mathrm{Hilb}^n_{g,n}$ with stable fiber (C_x, D_x) is open (use Proposition 1.7.4). On the open locus defined by (1) both $f_*\mathcal{O}_C(1)$ and $pr_*\mathcal{O}_{\mathbf{P}^N}(1)$ are of the same rank, hence the condition is that the map between is an isomorphism. This is an open condition, so (1) and (3) just define an open subscheme $X \subset \mathrm{Hilb}^n_{g,n}$.

The last condition is more subtle, because it defines a closed subscheme in X (hence one has to worry about the scheme structure). One way to do this is to note that the invertible sheaves $\mathcal{O}_C(1)$ and $\omega^{\otimes 3}_{(C,D)/S}$ define maps $h_1, h_2 : S \to \mathrm{Pic}_{C/S}$, and by the universal property of Picard schemes, condition (2) defines the maximal closed subscheme $H_{g,n} \hookrightarrow X$ such that $h_1 = h_2$ over $H_{g,n}$ (check that it exists!). An alternative argument which avoids the use of Picard schemes is based on the observation that $H_{g,n}$ is the maximal closed subscheme such that $f_*(\mathcal{O}_C(1) \otimes \omega^{\otimes -3}_{(C,D)/S})$ and $f_*(\mathcal{O}_C(-1) \otimes \omega^{\otimes 3}_{(C,D)/S})$ are locally free of rank one on $H_{g,n}$. □

Remark 1.7.16. (i) The scheme $H_{g,n}$ classifies stable n-pointed curves of genus g with a trivialization of the cohomology of the tri-canonical sheaf. Unlike the moduli stack $\mathcal{M}_{g,n}$, it is obtained rather straightforwardly from Hilbert schemes, that is a great advantage. Its disadvantages with respect to the moduli stack are as follows: not any stable curve $(C, D) \to S$ is induced from $H_{g,n}$, but only those for which the sheaf $f_*(\omega^{\otimes 3}_{(C,D)/S})$ is of the form $\mathcal{L}^n$; and if the sheaf is free then there are many possible trivializations (corresponding to the action of $\mathrm{PGL}(N+1)$ on $\mathbf{P}^N$), and some of them can lead to a compactification of the initial stable family, while others can admit no compactification. All these problems will be solved in the moduli space $\mathcal{M}_{g,n}$: the space is proper and each stable curve is induced from the universal stable curve over $\mathcal{M}_{g,n}$ in a unique way.

(ii) Actually, it is a standard situation that a problem of classifying certain objects admits a "covering" when one classifies an object with an extra-structure, which is very often a trivialization of some data canonically assigned to an object. Then the moduli space $\mathcal{M}$ which classifies the initial problem is covered by a larger moduli space $\mathcal{M}'$ which classifies the extended problem, and the group Aut of automorphisms of the trivialized data acts on $\mathcal{M}'$ so that $\mathcal{M}'/\mathrm{Aut} \widetilde{\to} \mathcal{M}$. Very often one trivializes etale cohomology groups (or Jacobian torsion subgroups) obtaining finite etale covers of $\mathcal{M}$ – these are standard level structures. In our case, we trivialized a coherent cohomology group (canonically attached to a stable curve). To build the moduli space $\mathcal{M}_{g,n}$ we should now just forget about the trivialization. Namely, we will build $\mathcal{M}_{g,n}$ as $H_{g,n}/\mathrm{PGL}(N+1)$.

Since we have built a moduli space $H_{g,n}$ which classifies stable curves with trivialized cohomology of the tri-canonical sheaf, it seems we are just one step from the construction of the moduli space $\mathcal{M}_{g,n}$ of stable n-pointed curves of genus g. However this a huge step because we have to leave the category of schemes. On the level of functors the definition is simple, but malfunctioning:

Mdef **Definition 1.7.17.** We define a set-valued functor $\mathcal{M}_{g,n}$ on the category of schemes by requiring that $\mathcal{M}_{g,n}(S)$ is the set of isomorphism classes of n-pointed stable S-curves of genus g.

nonrepex **Exercise 1.7.18.** (i) Give an example of a non-constant hyperelliptic family $C \to S$ which becomes trivial after an etale base change of $S' \to S$ of rank 2. (Hint: take $S = \mathrm{Spec}(k[t, t^{-1}])$ and C given by $tx^2 = (y-a)(y-b)\dots$.)

(ii) Give an example of a discrete valuation ring R with fraction field K and a stable and smooth K-curve which does not extend to a stable R-curve, but does extend to a stable smooth curve over a discrete valuation ring R' finite over R. (Hint: extend the above example to the punched origin of S.)

(iii) Deduce from either (i) or (ii) that $\mathcal{M}_{g,n}$ is not representable.

nonreprem **Remark 1.7.19.** (i) The fact that $\mathcal{M}_{g,n}$ is not representable was a striking disappointing discovering in 60ies. Historically, a partial salvation from this conundrum was an observation that there exists a scheme $M_{g,n}$ called the *coarse moduli space* which approximates the functor in a best possible way, in the sense that any morphism from $\mathcal{M}_{g,n}$ to a scheme (we mean the natural transformation of functors) factors through $M_{g,n}$. Moreover, the geometric points of $M_{g,n}$ are exactly what we want (the isomorphism classes of stable n-pointed curves), so it can be considered as a reasonably good moduli space. Unfortunately, $M_{g,n}$ does not admit a universal family (just by definition), so it is much less useful than the fine moduli space would be. In particular, it is useless for our application to de Jong's theorems. A better approach would be to extend the category of schemes so that certain good functors, including $\mathcal{M}_{g,n}$, become representable in the extended setting.

(ii) Another way to introduce $\mathcal{M}_{g,n}$ is to define it as a quotient $H_{g,n}/\mathrm{PGL}(N_{g,n}+1)$. However, we should decide what is a quotient of a scheme X by a group scheme G with respect to an action $G \times X \to X$ (i.e. a group object in the category of schemes and a categorical action). One can give at least two definitions of quotient schemes: categorical quotient and geometric quotient. The first states that X/G is the coequalizer of the diagram $G \times X \rightrightarrows X$, where one map is the projection and another map is the action. The second definition is more geometric and we refer to [GIT, Ch. 0] for the definition (it requires that X/G is the set of orbits on the level of geometric points, $X \to X/G$ is the topological quotient, and few more things). In addition, both notions can be not stable under base changes so there are universal analogs of these notions. Using these approaches one should be able (up to many technical difficulties) define $M_{g,n}$. Instead of trying to stick to one of these definitions, we will invent one more definition. It is the most natural one, but its drawback is that we have to leave the category of schemes.

(iii) In the context of moduli spaces, a natural attempt to define the quotient $H_{g,n}/\mathrm{PGL}(N_{g,n}+1)$ is just to take the quotient of functors. However, this does not works out well because we do not get $\mathcal{M}_{g,n}$: for example, some stable curves over a base S have non-free $f_*\omega^{\otimes 3}$, so they do not admit a trivialization, and cannot appear in the quotient functor. From this description it is clear that the difficulty is that $\mathcal{M}_{g,n}$ is a sheaf in the Zariski topology, while the quotient is not. Moreover, taking the sheafification of the functor $H_{g,n}/\mathrm{PGL}(N_{g,n}+1)$ we get $\mathcal{M}_{g,n}$.

In the next chapter we will study sheafified quotients as described in (iii). Moreover, the sheafification should be done with respect to étale covers, as becomes clear from the above bad example.

2. Stacks

2.1. **Descent and Hironaka's example.** We start with a very important example by Hironaka. Let X be a smooth proper threefold over a field k with two smooth curves C, C' meeting transversally at points P and P'. Locally over P we blow up C and the blow up the strict transform of C', locally over P' we reverse the order, and at each other point we just blow them up simultaneously. The result is a threefold Y with over P consisting of two projective lines l and m, where l appears after the first blow up, and the fiber over P' is a union of l' and m', where l' appeared first.

Hirexer **Exercise 2.1.1.** (i) Show that $l'+m$ is numerically equivalent to zero and deduce that Y is not projective.

(ii) Use this to construct an example of a semistable curve $C \to S$ with S a surface and such that $C \to S$ is not projective (note that $C \to S$ is projective locally over S by lemma 1.7.9).

(iii) Show that no affine subscheme of Y intersects both l' and m.

(iv) Find an example of Y as above and with an action of $G = \mathbf{Z}/2$ (i.e. $G = \mathrm{Spec}(k \oplus k)$ as a k-scheme) which is free and switches l' and m, and deduce that the geometric quotient Y/G does not exist.

The problem in the above example is that Zariski topology is not fine enough, in particular, it can happen that two points do not have a common affine neighborhood. This problem is resolved by passing to etale (or finer) topology on the *category* of schemes, so in the sequel we will work with etale topology on the category of schemes. The idea is to replace open immersions with a wider class of morphisms (e.g. etale, flat and finitely presented, flat and quasi-compact; note that fpqc topology is mainly considered in the case when non-Noetherian schemes are allowed) and to define what are the covering sets of morphisms (in all the above cases a covering is just the set of morphisms $U_i \to X$ whose images cover X). Few natural compatibility conditions are required (e.g. transitivity of coverings and stability under base changes). One of the main motivations for introducing Grothendieck topology on categories was the fact that one can easily define sheaves on such "spaces" (in particular, one can introduce cohomology). A presheaf is just a contravariant functor (to sets, groups, rings, etc.) and a sheaf is a presheaf $\mathcal{F}$ such that for any covering $U \to X$ (we assume that the category possesses disjoint unions, otherwise an obvious modification must be done) the sequence $\mathcal{F}(X) \to \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U)$ is exact (i.e. $\mathcal{F}(X)$ is the equalizer of the double arrow).

Exercise 2.1.2. Read about Grothendieck topologies, sheaves on them and the sheafification (you can take any book on etale cohomology). In particular, find out the definitions of the topologies Zar, Et, fppf and fpqc corresponding to the above examples.

Since we have decided to switch to etale (or finer) topology, the first natural question is if any representable functor is an etale sheaf. The answer is yes, but it is not so easy to see this: this fact is proved in the framework of the theory of descent. We refer to [Ful, §A] for a detailed exposition of the theory, and give here only a very brief exposition.

Let us first discuss the usual Zariski topology. Any scheme is pasted from affine ones, so it can be described by a covering $X = \cap U_i$. Then X is defined by U_i's and their intersections, and the best way to encode this information is as follows: a covering map $U = \sqcup U_i \to X$ and gluing datum $p_1, p_2 : U \times_X U \rightrightarrows U$ (first and second projections). Then X is the coequalizer of the gluing datum. Note that any gluing datum must satisfy a natural compatibility condition (the cocycle condition usually written as $p_{13}^* i = p_{23}^* i \circ p_{12}^* i$ for the involution $i : U \times_X U \to U \times_X U$ and the projections $p_{ij} : U \times_X U \times_X U \Rrightarrow U \times_X U$).

pastrem **Remark 2.1.3.** (i) Note that the category of all schemes is obtained from the category of affine schemes (opposite to the category of rings) by adding certain pushouts: coequalizers of gluing datum. Beware however, that they the same pushout problem can have different solution in the category of affine schemes, so one has to use locally ringed spaces to define the pushouts.

(ii) Actually, it usually happens that one can easily work with projective limits (including fibred products) in the category of schemes, but injective limits (including pushouts and coequalizers) are much more subtle. For example, one can define a union of closed subschemes (using intersections of ideals), but it is not stable under base changes and where is no unique way to decompose a subscheme as a union of irreducible ones.

(iii) In many cases, it is a very subtle question if a certain pushout exists (e.g. contraction of a closed subscheme or a scheme pasted from other schemes along closed subschemes). However, some pushout operations become possible when we enlarge the category of schemes to algebraic spaces. See next section for such examples.

Almost any local construction on a scheme X is done through an atlas corresponding to a gluing datum $R = U \times_X U \rightrightarrows U \to X$. For example, a sheaf $\mathcal{F}$ on X is defined as a sheaf $\mathcal{F}_U$ (i.e. collections of sheaves $\mathcal{F}_i$'s on U_i's) with a gluing isomorphism $p_1^*\mathcal{F}\widetilde{\to}p_2^*\mathcal{F}$ subject to a natural cocycle condition.

Now, let us switch to other topologies: we will be mainly interested in Et and fppf. The above definition of locality makes sense for any topology because we have not use any specific property of open immersions in the above discussion. For example, a local construction in a flat topology is done by specifying a faithfully flat covering $U \to X$ (we automatically assume that it is either fppf or fpqc) and performing a construction on U and $R = U \times_X U$ such that natural compatibility (gluing isomorphism and cocycle condition) are satisfied. In general one calls statements about such constructions *descent statements*, because they state that certain constructions/properties descent from an atlas $R \rightrightarrows U$ to a scheme X. One says that a descent datum (for a sheave or scheme) is *effective* if it instead leads descents to an object on X. Obviously, this makes sense for sheaves (in the corresponding topology), but the descent theory does much more: any quasi-coherent Zariski sheaf is also an fpqc sheaf, any descent datum for a Zariski quasi-coherent sheaf is effective and descent datum is effective for morphisms of quasi-coherent sheaves. In other words, the category of quasi-coherent sheaves $Coh(X)$ is equivalent to the category of descent datum $Coh(R \rightrightarrows U)$.

Exercise 2.1.4. Prove this, or read a proof in the literature (e.g. [Ful]).

Since the descent works perfectly for quasi-coherent modules, it works perfectly for X-affine schemes (i.e. affine morphisms $Y \to X$) and, therefore, for quasi-affine morphisms to X. A very important property is that for any fppf or fpqc covering $U \to X$ with $R = U \times_X U$ one has that X is the coequalizer of $R \rightrightarrows U$ (in the category of sets this claim is true for any surjection $U \to X$, so flat covers behave as surjections). In particular, any representable functor $h_Z = \mathrm{Hom}(\cdot, Z)$ is a sheaf in the fppf and fpqc (hence etale) topologies. It follows that descent works perfectly for morphisms of X-affine schemes, and therefore for morphisms between any pair of X-schemes. However, using Hironaka's example it is easy to construct a non-effective descent datum for an X-proper scheme, i.e. proper morphisms $Y_U \to U$ with a gluing datum $Y_U \times_{U,p_1} R\widetilde{\to}Y_U \times_{U,p_2} R$ subject to a cocycle condition. It turns out nevertheless that one can descent X-projective schemes with a fixed ample sheaf.

Exercise 2.1.5. Prove this, or read a proof in the literature (e.g. [Ful]).

Remark 2.1.6. Note that the whole theory of descent is based on descent of modules. That is why we can descent X-affine schemes and X-projective ones with fixed ample sheaves.

2.2. **Algebraic spaces.** The classical reference for the theory of algebraic spaces is the book [Knu] of Knutson. The category of algebraic spaces extends the category of schemes with respect to etale-local constructions in the same way as the category of schemes extends the category of affine schemes with respect to Zariski-local constructions. In the category of algebraic spaces there is enough room to perform some operations impossible in the category of schemes: any descent datum is effective (even fppf and fpqc) and quotients by free actions of finite groups always exist (e.g. Hironaka's quotient from 2.1.1(iv) is an algebraic space). A general algebraic space $\mathcal{X}$ should be the coequalizer of an etale gluing datum $p_1, p_2 : R \rightrightarrows U$ (it will help to your intuition to think that $R = U \times_{\mathcal{X}} U$) with etale p_i's and which satisfies certain compatibility conditions described below. An equivalent way is to talk about equivalence relations.

Exercise 2.2.1. (i) Check that $p : R \rightrightarrows U$ is an equivalence relation of sets if and only if $R\widetilde{\to}U \times_X U$ for the coequalizer X if p. Show that it happens if and only if $R \to U \times U$ is an injection, R is symmetric, i.e. there exists an involution $i : R \to R$ which switches p_i's, and R is transitive (a cocycle condition on $R \times_{p_1,U,p_2} R = U \times_X U \times_X U$).

(ii) Define a *categorical equivalence relation* $R \rightrightarrows U$ by the condition that all maps $\mathrm{Hom}(Z, R) \rightrightarrows \mathrm{Hom}(Z, U)$ are equivalence relations. Show that $R \to U \times U$ is a monomorphism. Give an equivalent definition of categorical equivalence relation using a symmetrization $i : R \to R$ and a cocycle condition.

We can now define algebraic spaces as quotients (or coequalizers) of etale equivalence relations, but we have also to decide what are the morphisms in the new category: we saw in 2.1.3(i) that the passage from affine schemes to schemes required to use locally ringed spaces. In our case the trick is somewhat similar: we know that the category of schemes embeds faithfully into the category of etale sheaves, and coequalizers exist in that larger category (they are sheafifications of presheaf coequalizers). So, we have just one possibility to define morphisms between algebraic spaces so that they form a full subcategory in the category of etale sheaves. If $p : R \rightrightarrows U$ is an etale equivalence relation and $\mathcal{X}$ is the quotient algebraic space, then we say that p is an *atlas* for $\mathcal{X}$ and write $\mathcal{X} = U/R$. The parts (i) and (ii) of the following exercise show that it actually suffices to work with sheaves on affine schemes and with affine atlases.

algspexer **Exercise 2.2.2.** (i) Check that the category of algebraic spaces embeds faithfully into the category of etale sheaves on the category of affine schemes.

(ii) Check that any algebraic space admits an affine atlas in the sense that U is affine (R can be not affine already when U/R is a non-separated scheme).

(iii) Prove that if $\mathcal{X} = U/R$ for an etale equivalence relation $R \rightrightarrows U$, then $R\widetilde{\to}U \times_{\mathcal{X}} U$.

(iv) Prove that if $U' \to U$ is an etale covering and $p : R' = U' \times_{U,p_1} R \times_{p_2,U} U'$ (often one simply writes $R' = R \times_{U\times U} (U' \times U')$), then $p' : R' \rightrightarrows U'$ is an equivalence relation with $U'/R'\widetilde{\to}U/R$. One says that the atlas p' for $\mathcal{X}$ is a refinement of p.

(v) If X is an S-scheme and G is an etale S-group with a categorically free S-action $\mu : X \times_S G \to X$ (i.e. $X \times_S G \to X \times_S X$ is a monomorphism), then $\mathrm{pr}, \mu : X \times_S G \to X$ is an etale equivalence relation. In particular, the quotient X/G (we will use this notation instead of $X/(X \times G)$) is defined as an algebraic space. For example, the Hironaka's quotient is an algebraic space.

artinrem **Remark 2.2.3.** (i) One could make similar construction for other topologies, e.g. the fppf topology. Surprisingly, it would lead to equivalent category. However, the only published proof of that is based on Artin's theorem about stacks. It is an interesting question, if one can prove it directly with reasonable efforts.

(ii) Usually, one also requires that the diagonal $R \to U \times U$ is quasi-compact (if $X = U/R$ is a scheme, then this means that X is quasi-separated). Since we work with noetherian schemes in this part of the course, we do not care about this point.

There are few equivalent conditions which describe algebraic spaces among all etale sheaves. Usually, one of those conditions is taken for the definition of algebraic spaces (instead of our definition with an atlas). Actually, almost all local constructions work with atlases (similarly to schemes with the Zariski topology).

atlaslem **Lemma 2.2.4.** *Any morphism of algebraic spaces $\mathcal{X}' \to \mathcal{X}$ admits an affine atlas $(R' \rightrightarrows U') \to (R \rightrightarrows U)$.*

Note that the map of atlases defines the map of quotients, and the map $(U' \to \mathcal{X}') \to (U \to \mathcal{X})$ defines a map on relations (so that everything fits into one commutative diagram). Using the language of functors one just checks this on the level of sets.

Proof. Accordingly to the intuition coming from the experience with schemes and Zariski topology, we start with arbitrary atlases, and we will see that it suffices to refine U'. If $U' \to \mathcal{X}$ factors through U then the map of atlases exists even without refining because the map $(U' \to \mathcal{X}') \to (U \to \mathcal{X})$ defines the map of R's. In general, we note that $U \to \mathcal{X}$ is a surjection of etale sheaves. So, even if the map $U(U') \to \mathcal{X}(U')$ is not surjective, there exists an etale covering $U'' \to U'$ such that the map $U(U'') \to \mathcal{X}(U'')$ is surjective. Replacing $R' \rightrightarrows U'$ with $R' \times_{U' \times U'} U'' \times U'' \rightrightarrows U''$ we obtain the required refinement because $U'' \to \mathcal{X}$ factors through U. $\square$

atlasex **Exercise 2.2.5.** Construct fibred products of algebraic spaces using atlases (or read [Knu, II.1.5]).

algspprop **Proposition 2.2.6.** *The following conditions on an etale sheaf $\mathcal{F}$ are equivalent (we identify schemes and algebraic spaces with their functors):*

(i) $\mathcal{F}$ is an algebraic space;

(ii) there exists a scheme U and an etale covering $U \to \mathcal{F}$ *which is* schematic *or* representable *in the sense that for any morphism from a scheme V to $\mathcal{F}$ the base change morphism $U \times_{\mathcal{F}} V \to U$ is a surjective etale morphism of schemes;*

(iii) the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is schematic (i.e. its base change to any scheme is a morphism of schemes) and there exists an etale sheaf-theoretic epimorphism from a scheme $f : U \to \mathcal{F}$ (f is automatically schematic and etaleness means that any its base change to a scheme is an etale morphism);

(iv) any map $V \to \mathcal{F}$ from a scheme is schematic and there exists an etale sheaf-theoretic epimorphism from a scheme $f : U \to \mathcal{F}$.

Proof. Equivalence of (iii) and (iv) follows from the fact that any fibred product over $\mathcal{F}$ is obtained from the direct product by base change with respect to the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$, i.e. $U \times_{\mathcal{F}} V \widetilde{\to} U \times V \times_{\mathcal{F}\times\mathcal{F}} \mathcal{F}$ (thus Δ is schematic iff any morphism from a scheme to $\mathcal{F}$ is schematic). To deduce (ii) from (iii) we should only prove that the base changes of $U \to \mathcal{F}$ are surjective, but the latter follows from the fact that an etale map $f : U \to V$ is surjective if and only if it induces surjection of sheaves (use that can one split f by an etale base change so that it admits a section).

Remark 2.2.7. This surjectivity is of crucial importance. It explains why one has to consider etale sheaves instead of Zariski sheaves to be able to define algebraic spaces. Similarly, if one wants to define "flat" algebraic spaces by use of flat equivalence relations, see remark 2.2.3(i), then one must work with fppf sheaves (though the resulting category will be the same).

Also, (i) follows from (ii) because $R = U \times_{\mathcal{F}} U$ is a scheme and $p : R \rightrightarrows U$ is an equivalence relation of sheaves, hence p is an equivalence relation of schemes and it is etale by etaleness of f. It remains to deduce (iv) from (i). So, we assume that $p : R \rightrightarrows U$ is an etale equivalence relation with $U/R \widetilde{\to} \mathcal{F}$ and we have to prove that if V_1 and V_2 are two schemes with a morphism to $\mathcal{F}$ then $V_1 \times_{\mathcal{F}} V_2$ is a scheme. First, let us assume that the morphisms $V_i \to \mathcal{F}$ factor through U. Then $V_1 \times_{\mathcal{F}} V_2 \widetilde{\to} V_1 \times_U U \times_{\mathcal{F}} U \times_U V_2 \widetilde{\to} V_1 \times_{U,p_1} R \times_{p_2,U} V_2$ is a scheme. In general, we apply lemma 2.2.4 to find etale covers $W_i \to V_i$ such that the morphisms $W_i \to \mathcal{F}$ factor through U. In particular, we obtain morphisms of the quotient sequences $(W_i \times_{V_i} W_i \rightrightarrows W_i \to V_i) \to (R \rightrightarrows U \to \mathcal{F})$ (it is a triple of morphisms compatible with all the rest). Set $T_i = W_i \times_{V_i} W_i$ for simplicity, then by the above particular case $W_1 \times_{\mathcal{F}} W_2$ and $T_1 \times_{\mathcal{F}} T_2$ are schemes and, in particular, the monomorphisms $W_1 \times_{\mathcal{F}} W_2 \to W_1 \times W_2$ and $T_1 \times_{\mathcal{F}} T_2 \to T_1 \times T_2$ are quasi-affine morphisms of schemes. It remains to apply descent of quasi-affine morphisms to obtain that $V_1 \times_{\mathcal{F}} V_2$ is a scheme quasi-affine over $V_1 \times V_2$. □

Remark 2.2.8. (i) Note that the implication (i)⇒(iv) is rather subtle (we used descent in the proof). Its analog for affine atlases does not hold because a product of affine schemes over a non-separated scheme does not have to be affine.

(ii) Representability of the diagonal and existence of an etale covering (as in parts (iii) or (iv) of 2.2.6) are often taken for the definition of an algebraic space, though Knutson uses 2.2.6(ii) as his definition.

Now we will briefly outline a further development of the theory of algebraic spaces. We will not make too much use of it, but it will serve as a simple analog of the theory of stacks we will discuss later. Also, we will discuss few examples of concrete algebraic spaces, so that the reader can built some intuition of working with such objects.

bfpdef **Definition 2.2.9.** Let **P** be any property of a morphism which is stable under base changes and is etale-local on the base (i.e. f satisfies **P** iff its base change with respect to a surjective etale map does so). Actually, **P** can be almost everything except (quasi-) projectivity, see [Ful, 5.5] for a list of 30 properties including monomorphism, (locally) closed immersion, etale, radiciel, affine, etc. We say that a schematic morphism $\mathcal{Y} \to \mathcal{X}$ of algebraic spaces satisfies **P** if it is so etale-locally on the base (i.e. the base change with respect to an etale covering of $\mathcal{X}$ by a scheme, which is automatically a morphism of schemes, satisfies **P**).

Using this definition we can talk about (surjective) etale and flat morphisms and open immersions, hence the category of algebraic spaces is naturally provided with the topologies Zar, Et and fppf. The most natural topology for algebraic spaces is the etale topology (an analog of the Zariski topology for schemes). Each representable functor is an etale sheaf in the category of algebraic spaces. A structure sheaf $\mathcal{O}_{\mathcal{X}}$ is defined as the etale sheaf satisfying $\mathcal{O}_{\mathcal{X}}(U) = \mathcal{O}_U(U)$ for any morphism $U \to \mathcal{X}$ from a scheme. A quasi-coherent sheaf is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules which is etale-locally quasi-coherent. The topology Zar can be convenient for some applications, but one should remember that it does not describe $\mathcal{X}$ completely (see exercise 2.2.12 below). Namely, a *point* p of $\mathcal{X}$ is a monomorphism $p \to \mathcal{X}$ with p being the spectrum of a field. The set of points $|\mathcal{X}|$ is a topological space which adequately describes the Zariski topology of $\mathcal{X}$ (i.e. there is a one-to-one correspondence between open subspaces of $|\mathcal{X}|$ and the isomorphism classes of open immersions into $\mathcal{X}$). The structure sheaf $\mathcal{O}_{\mathcal{X}}$ restricts to a sheaf of rings on $|\mathcal{X}|$ making it to a locally ringed space. Using Zariski topology, one can define the notions of closed and universally closed morphisms. Since we want to be able to talk about separatedness and properness in the case of algebraic spaces which are not necessarily schemes, we have to give the following definition (instead of 2.2.9).

bfp2def **Definition 2.2.10.** (i) A morphism $\mathcal{Y} \to \mathcal{X}$ is called *(locally) separated* if its diagonal $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ (which is always a monomorphism) is a (locally) closed immersion.

(ii) A morphism is *proper* if it is of finite type and universally closed.

(iii) A morphism is *(quasi-) projective* if it is a composition of a (locally) closed immersion $\mathcal{Y} \to \mathbf{P}_{\mathcal{X}}$ with the projection $\mathbf{P}_{\mathcal{X}} \to \mathcal{X}$.

Remark 2.2.11. One can generalize to non-schematic morphisms the notions of etaleness, smoothness, flatness, and any other property $\mathbf{P}$ which is etale-local on the source (i.e. $Y \to X$ satisfies $\mathbf{P}$ if and only if $Y' \to X$ satisfies $\mathbf{P}$ for an etale covering $Y' \to Y$). For example, the Hironaka's quotient is a smooth proper three-dimensional algebraic space which is not a scheme.

algspex **Exercise 2.2.12.** (i) Let X be an affine scheme with a closed subscheme Y and a complement U. Define a non-separated scheme $\overline{X}$ obtained from X by doubling Y via the gluing datum $X_{11} \sqcup X_{22} \sqcup U_{12} \sqcup U_{21} \rightrightarrows X_1 \sqcup X_2$.

(ii) Assume, more generally, that X admits a non-trivial finite etale covering $X' \to X$ such that the preimage of Y is Y'. Imitate the above construction to build a non-separated algebraic space $\overline{\mathcal{X}}$ obtained from X by inserting Y' instead of Y.

(iii) Show that one can always "multiple" a closed subscheme Y in X with respect to a finite etale covering $Y' \to Y$ obtaining an algebraic space $\overline{\mathcal{X}}$ (you can assume that the etale covering $Y' \to Y$ locally extends to X).

(iv) Assume that $Y = \mathrm{Spec}(k)$ is a closed point and $Y' = \mathrm{Spec}(k')$ is a larger point. Show that $\mathcal{X}$ and X have equal Zariski locally ringed spaces (we feel non-separatedness only etale locally), hence the functor $\mathcal{X} \to (|\mathcal{X}|, \mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$ is not full. Show that if k'/k is Galois, then $\mathcal{X}$ has a non-trivial automorphism over X, hence the functor is not faithful.

(v) If X is a surface over $\mathbf{C}$ and $Y' \to Y$ is a non-split etale covering of a curve, then $\overline{\mathcal{X}}$ from the above exercise is not a scheme, but its analytification can be defined as a (non-Hausdorff) analytic space because $Y' \to Y$ is a local isomorphism in analytic topology.

Despite (iv) above, Zariski topology can be useful for some applications. The following exercise provides such an example. An etale equivalence relation $p : R \rightrightarrows U$ is called *effective* if U/R is a scheme.

Exercise 2.2.13. Prove that if $U = \cup U_i$ is an open covering and $p_i : R_i = U_i \times_{U,p_1} R \times_{p_2,U} U_i \rightrightarrows U_i$ are the induced equivalence relations, then the algebraic spaces U_i/R_i form a Zariski covering of U/R. In particular, effectivity of an equivalence relation is Zariski local on U, in the sense that p is effective iff all p_i's are.

There is one important case when effectivity is granted.

genscheme **Exercise 2.2.14.** (i) Show that any affine finite etale equivalence relation $\mathrm{Spec}(B) \rightrightarrows \mathrm{Spec}(A)$ is effective and the quotient equals to $\mathrm{Spec}(\mathrm{Ker}(A \rightrightarrows B))$. (Hint: refine the atlas so that it splits as $\mathrm{Spec}(A) \times G \to \mathrm{Spec}(A)$ for a discrete finite group G, then the quotient will be $\mathrm{Spec}(A^G)$).

(ii)* Deduce that any algebraic space is generically a scheme, i.e. contains a Zariski dense open subscheme. (Hint: starting with an atlas $R \rightrightarrows U$ one should find an open affine $U' \hookrightarrow U$ such that the induced equivalence relation $R' \rightrightarrows U'$ is finite. Though this is obviously so over the generic points of U, it is not so easy to find U'.)

Remark 2.2.15. (i) Chow lemma holds for separated morphisms of algebraic spaces, in particular any $\mathcal{X}$ of separated and of finite type over a scheme S can be modified to a scheme X' quasi-projective over S. The proof of this is not easy, see [Knu, IV.3.1].

(ii) An important theorem by Artin states that any contraction of a subvariety can be performed in the category of algebraic spaces if and only if it can be contracted formally (in other words, formal blow downs can be algebraized in the category of algebraic spaces). For example, a curve on a proper surface can be contracted (using algebraic spaces!) if and only if its self-intersection matrix is totally negative.

(iii) Artin's theorem plays an important role in the proof that any Moishezon analytic space (i.e. proper n-dimensional space with n algebraically independent meromorphic functions) algebraizes by an algebraic space over $\mathbf{C}$. A non-archimedean analog of this was proved very recently by B. Conrad.

We finish this section with few more examples of algebraic spaces.

Exercise 2.2.16. (i) Let X be a scheme with a closed subscheme Y and $Y \to Y'$ be a finite morphism. Construct the pushout X' of $X \leftarrow Y \to Y'$ in the category of algebraic spaces. Give an example when X' is not a scheme (e.g. when we glue curves on a surface with different self-intersections).

(ii) Let X be an affine line with doubled origin (the most typical example of a non-separated scheme), Y be the doubled origin (just two points) and Y' be one point (the origin). Show that X' (i.e. the scheme obtained by gluing the twin points together) is not locally separated. Show that this gives an equivalent description of the example of a not locally separated space in the book of Knutson (Example 1 in chapter I). Intuitively this example looks like affine line with doubled tangent direction at the origin.

(iii)* Show that an algebraic space over $\mathbf{C}$ admits an analytification if and only if it is locally separated. (Hint: you should prove that if an etale equivalence relation on analytic spaces has a locally closed diagonal then it is effective, and that the analytification depends only on the algebraic space, i.e. does not depend on the atlas).

It was recently proved by B. Conrad and me that any separated algebraic space of finite type over a non-archimedean field admits an analytification (as a Berkovich analytic space or as a rigid space).

Exercise 2.2.17. (i)* Give an example of a non-analytifiable locally separated algebraic space over $\mathbf{Q}_p$. (Hint: use that $\mathbf{Q}_p$ is not algebraically closed).

(ii)** Construct an analogous example over $\mathbf{C}_p$. (Hint: it is much more difficult to invent such example in the framework of rigid geometry (using admissible covers); switch to the analytic language and use that on Berkovich analytic spaces there exist points defined over larger fields.)

2.3. **Groupoids.** Algebraic spaces are a first step towards constructing group quotients in the category of schemes. They work well for categorically free finite group actions (including a Galois action on a field), but cannot treat the non-free case because $X\times G\rightrightarrows X$ is not an equivalence relation then. Actually, the problem can be tracked already on the level of sets: for a non-free group action $X\times G\to X$ the map $\phi: X\times G\to X\times_{X/G}X$ is not an isomorphism (let us say that the quotient is fine if ϕ is an isomorphism). Since we would like to define fine quotients in the category of schemes, it is natural first to extend the category of sets so that fine quotients exist in a larger category $\mathcal{C}$, and then embed the category of set-valued functors (where algebraic spaces form a full subcategory) into the category of $\mathcal{C}$-valued functors or, more specifically, into the category of $\mathcal{C}$-valued sheaves. This plan works out fine with one serious complication: a natural definition of $\mathcal{C}$ makes it into a 2-category rather than a usual category.

A natural generalization of an equivalence relation $R\rightrightarrows U$ is a *groupoid* $s,t: X_1\rightrightarrows X_0$ which consists of two maps s,t, a multiplication $m: X_2=X_1\times_{s,X_0,t}X_1\to X_1$, an inverse $i: X_1\to X_1$ which switches s and t and an identity $e: X_0\to X_1$. An equivalent way to give groupoid is to give a category $\mathcal{X}$ where all morphisms are isomorphism (often groupoid means category, and we will use both interpretations): $\mathrm{Ob}(\mathcal{X})=X_0$, $\mathrm{Mor}(\mathcal{X})=X_1$, s and t are the source and the target of morphisms, m is the composition law and i is the inverse of morphisms (all morphisms are invertible by our assumption).

Exercise 2.3.1. (i) Using the categorical interpretation of groupoid write down all axioms s,t,i,m,e should satisfy.

(ii) Show that an equivalence relation is a groupoid.

(iii) Attach to any group action a groupoid $X\times G\rightrightarrows X$.

In particular, for $G=e$ we obtain a banal groupoid $\mathrm{Id}_X: X\rightrightarrows X$ which will be freely identified with X. So, by saying that a groupoid is a set we mean that it is strictly isomorphic to such a groupoid. Next we have to define the category of groupoids. It is clear how to define morphisms: a map $\phi:(X_1\rightrightarrows X_0)\to(Y_1\rightrightarrows Y_0)$ is a pair of maps $\phi_0: X_0\to Y_0$, $\phi_1: X_1\to Y_1$ which satisfies natural compatibility, and such a map is called a *strict isomorphism* if ϕ_i's are isomorphisms. The latter notion is too restrictive, because we would like also to define isomorphism (or equivalence) of groupoids which corresponds to an isomorphism of the quotients at least when the quotients are sets. The natural definition of equivalence is based on the interpretation of groupoid as a category: there is a rather useless notion of isomorphism of categories and there is a notion of equivalence of categories. Let us switch therefore to the categorical interpretation of groupoids.

2catdef **Definition 2.3.2.** Groupoids (and, more generally, all categories) form a *2-category* Grp (resp. Cat). Its objects are groupoids, but Hom's are categories denoted $\mathrm{HOM}(\mathcal{X},\mathcal{Y})$. The objects of $\mathrm{HOM}(\mathcal{X},\mathcal{Y})$ are functors $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$ and they are called 1-morphisms; for two such functors $\mathcal{F}$ and $\mathcal{G}$ the natural transformations form a set $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and they are called 2-morphisms. Since we consider only groupoids in Grp, any natural transformation is invertible, i.e. any 2-morphism is an isomorphism in our case. A 1-morphism $\mathcal{F}$ is an isomorphism (or equivalence) if there exists an inverse $\mathcal{G}$ (going in opposite direction) such that the compositions $\mathcal{G}\circ\mathcal{F}$ and $\mathcal{F}\circ\mathcal{G}$ are 2-isomorphic to identities (note that these 2-isomorphisms can be non-unique). For 2-morphisms two types of compositions are defined: a so-called vertical composition $g\circ g:\mathcal{F}\to\mathcal{H}$ for $f:\mathcal{F}\to\mathcal{G}$ and $f:\mathcal{G}\to\mathcal{H}$ and a so-called horizontal composition $f'*f:\mathcal{F}\circ\mathcal{F}'\to\mathcal{G}\circ\mathcal{G}'$ for $f:\mathcal{F}\to\mathcal{G}$ and $f':\mathcal{F}'\to\mathcal{G}'$. A diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{t} & T \end{array}$$

is called *2-commutative* (resp. *strictly 2-commutative*) if $\phi:t\circ g\to f\circ h$ is a 2-isomorphism (resp. an identity). Sometimes, one can omit ϕ to ease the language, but such 2-isomorphisms must be taken into account in almost any diagram chasing (e.g. in order to establish two commutativity of other cycles in complicated diagrams one has to compose 2-morphisms using vertical or horizontal composition).

grex **Exercise 2.3.3.** (i) Reformulate everything in the non-categorical language. For example, for two 1-morphisms $\mathcal{F},\mathcal{G}:\mathcal{X}\to\mathcal{Y}$ a 2-morphism $\mathcal{F}\to\mathcal{G}$ is given by a map $X_0\to Y_1$ which satisfies certain compatibility conditions.

(ii) For any group G let BG denote the groupoid $\mathrm{pt}\times G\rightrightarrows\mathrm{pt}$. Show that any groupoid is equivalent to a uniquely defined groupoid $\sqcup_i BG_i$, and the groupoid is a set iff all groups are trivial. (Hint: take the skeleton of the groupoid.)

(iii) Given two groups G and H describe the category $\mathrm{HOM}(BG,BH)$. What are the isomorphism classes of objects of this category?

Note that the above interpretation of 2-morphisms provides a very close analogy between 2-morphisms and homotopies (in some sense a 2-morphism is a homotopy between 1-morphisms).

Definition 2.3.4. Let $\phi:\mathcal{Y}\to\mathcal{X}$ and $\psi:\mathcal{Z}\to\mathcal{X}$ be two 1-morphisms, then the fibred product groupoid $\mathcal{T}=\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z}$ is defined as follows: T_0 is the set of triples $t=(y\in Y_0,z\in Z_0,f\in\mathrm{Mor}(\phi(y),\psi(z))$ and $\mathrm{Mor}(t,t')$ (i.e. the preimage of (t,t') under the map $T_1\to T_0\times T_0$) is the set of pairs $a:y\to y',b:z\to z'$ such that $f'\circ\phi(a)=\psi(b)\circ f$. Note that the map $\mathcal{T}_0\to\mathcal{X}_1$ which takes (y,z,f) to f induces a 2-isomorphism of the composed maps $\mathcal{T}\to\mathcal{Y}\to\mathcal{X}$ and $\mathcal{T}\to\mathcal{Z}\to\mathcal{X}$. A *strictly 2-cartesian* diagram is the 2-commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{Y} \\ \downarrow & \overset{f}{\Rightarrow} & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{X} \end{array}$$

Note that if Y and Z are sets, then $Y \times_{\mathcal{X}} Z$ is always a set.

fibex1 **Exercise 2.3.5.** (i) Show that the fibred product satisfies the following strict universal property: for any pair of morphism $a : \mathcal{T}' \to \mathcal{Y}$ and $b : \mathcal{T}' \to \mathcal{Z}$ with a 2-isomorphism f' between the compositions $\mathcal{T}' \to \mathcal{Y} \to \mathcal{X}$ and $\mathcal{T}' \to \mathcal{Z} \to \mathcal{X}$, there is a unique morphism $(a, b) : \mathcal{T}' \to \mathcal{T}$ such that a and b factor through it and f' is induced from f in the sense that $f' = \mathrm{Id}_{(a,b)} * f$.

(ii) Check that $\mathrm{pt} \times_{BG} \mathrm{pt} \widetilde{\to} G$. In particular, one cannot just take product of atlases to obtain product of groupoids (unlike the case when the groupoids are equivalence relations).

(iii) More generally, consider the covering of a groupoid $\mathcal{X}$ by the set X_0 (i.e. a morphism $(e, Id_{X_0}) : (X_0 \rightrightarrows X_0) \to (X_1 \rightrightarrows X_0))$. Show that $X_0 \times_{\mathcal{X}} X_0 \widetilde{\to} X_1$ as a set (i.e. category with on. (Yes!)

(iv) Even more generally, show that for morphisms $\mathcal{X}' \to \mathcal{X}$ and $\mathcal{X}'' \to \mathcal{X}$ of groupoids $\mathcal{X} = (R \rightrightarrows U)$, $\mathcal{X}' = (R' \rightrightarrows U')$ and $\mathcal{X}'' = (R'' \rightrightarrows U')$, the fibred product $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}''$ is the groupoid $R' \times_R R'' \rightrightarrows U' \times_{U,s} R \times_{t,U} U''$.

(v) Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{T}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of groupoids. Show that the natural map $\mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{T}} \mathcal{Z}) \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Z}$ is an isomorphism which does not have to be a strict isomorphism.

The last part of the exercise indicates that it is much more useful to weaken the strict cartesianity condition as follows.

Definition 2.3.6. A 2-commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{b} & \mathcal{Y} \\ {\scriptstyle a}\downarrow & \overset{f}{\Rightarrow} & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{X} \end{array}$$

is called *2-cartesian* if the morphism $(a, b) : \mathcal{T} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ is an isomorphism.

fibex2 **Exercise 2.3.7.** (i) Show that $\mathcal{T} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ possesses a natural universal property or read [Ful, §C.5.4]; this is a good example of a 2-commutative diagram chasing with vertical and horizontal compositions of 2-morphisms. (Hint: loosely speaking, the property states that for any pair of morphism $f : \mathcal{T}' \to \mathcal{Y}$ and $g : \mathcal{T}' \to \mathcal{Z}$ with a 2-isomorphism between the compositions $\mathcal{T}' \to \mathcal{Y} \to \mathcal{X}$ and $\mathcal{T}' \to \mathcal{Z} \to \mathcal{X}$, both f and g factor through a morphism $f \times_{\mathcal{X}} g : \mathcal{T}' \to \mathcal{T}$ up to a 2-isomorphism and $f \times_{\mathcal{X}} g$ is unique up to a unique 2-isomorphism.)

(ii) Show that $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ depends only on isomorphism classes of $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ up to a 2-isomorphism.

(iii) Establish 1-isomorphisms $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \widetilde{\to} (\mathcal{Y} \times \mathcal{Z}) \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ and $\mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{T}} \mathcal{Z}) \widetilde{\to} \mathcal{X} \times_{\mathcal{T}} \mathcal{Z}$.

Definition 2.3.8. A morphism $f : \mathcal{Y} \to \mathcal{X}$ is a *(strict) monomorphism* if it is (injective and) fully faithful, and it is a *(strict) epimorphism* if it is essentially surjective (resp. surjective).

It follows immediately that f is a (strict) isomorphism iff it is a (strict) monomorphism and a (strict) epimorphism.

fibex3 **Exercise 2.3.9.** Prove that the following conditions for a morphism $f : \mathcal{Y} \to \mathcal{X}$ are equivalent:

(i) f is faithful;
(ii) the diagonal $\Delta_{\mathcal{Y}/\mathcal{X}} : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is a monomorphism;
(iii) f is *representable* in the sense that for any morphism $X \to \mathcal{X}$ with X isomorphic to a set the base change $X \times_{\mathcal{X}} \mathcal{Y}$ is isomorphic to a set.

(Hint: use exercise 2.3.3(ii) and for any homomorphism of groups $G \to H$ and induced morphism $BG \to BH$ compute the map $\Delta_{BG/BH} : BG \to BG \times_{BH} BG$, in particular, deduce from exercise 2.3.5(iv) or compute directly that $BG \times_{BH} BG \widetilde{\to} G \setminus H/G$ is isomorphic to the factor of H by the left-right action of $G \times G$, i.e. it is (even strictly) isomorphic to the groupoid $G \times G \rightrightarrows H$.)

Remark 2.3.10. We see that the behavior of a morphism f between groupoids is very different from the behavior of a usual map of sets if and only if f is not faithful, i.e. f kills automorphism.

2.4. **CFGs.** Since we cooked up the replacement for the category of sets, we can try to generalize etale sheaves and algebraic spaces by stacks and algebraic (Artin or Deligne-Mumford) stacks, respectively. The idea is clear: stack is an etale sheaf of groupoids and algebraic stack should be defined either by a groupoid atlas or by some conditions analogous to proposition 2.2.6. Naturally, we should start with presheaves because many constructions on sheaves use presheaves. At this point we are going to feel the difference between sheaves of sets and sheaves of groupoids, namely, it is possible to give a direct definition of such a presheaf, but it should involve 2-morphisms: a presheaf $\mathcal{F}$ consists of a groupoid $\mathcal{F}(U)$ for any scheme U, a 1-morphism $f_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any morphism $V \to U$ and a 2-isomorphism $\gamma_{U,V,W} : f_{V,W} \circ f_{U,V} \to f_{U,W}$ for any tower $W \to V \to U$. The usual transitivity condition (whose role is played here by γ) must be replaced with a generalized transitivity condition on γ's.

Exercise 2.4.1. Write down the compatibility conditions γ's should satisfy for any tower $T \to W \to V \to U$, or see [Ful, p. 39].

There is, however, a standard way to reformulate this definition in essentially equivalent way, and it leads to a notion of categories fibred in groupoids. Since moduli problems data can be interpreted in the language of CFG's, one usually prefers to work in that language.

Definition 2.4.2. Let $\mathcal{S}$ be a category (normally, it will be the category of S-schemes). A *category fibred in groupoids* or *CFG* over $\mathcal{S}$ is a functor $p : \mathcal{X} \to \mathcal{S}$ such that

(i) if $f : Y \to X$ is a morphism and $X = p(x)$, then there exists a morphism $\overline{f} : y \to x$ with $p(\overline{f}) = f$ (in particular, $p(y) = Y$);
(ii) if $\overline{f} : y \to x$ and $\overline{g} : z \to x$ are two morphisms, then any morphism $h : p(y) \to p(z)$ with $p(\overline{g}) \circ h = p(\overline{f})$ admits a unique lifting $\overline{h} : y \to z$ with $\overline{g} \circ \overline{h} = \overline{f}$.

A 1-morphism of CFGs is a functor $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ such that $p_{\mathcal{X}} \circ \mathcal{F} = p_{\mathcal{Y}}$, and a 2-morphism of CFGs is a natural transformation of functors. A 1-morphism is a 1-isomorphism if it is an equivalence of categories.

Clearly, $\mathcal{F}$ is a 1-isomorphism iff it has an inverse $\mathcal{G}$ up to 2-isomorphisms, i.e. such that both $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are 2-isomorphic to identity functors.

CFGexam **Example 2.4.3.** In moduli problems CFG's appear as follows. If we want to classify certain objects over S-schemes (e.g. stable n-pointed curves), then we can define a moduli CFG $\mathcal{X} \to \mathcal{S}$ as follows: x with $p(x) = X$ corresponds to an object over X and $\overline{f} : y \to x$ with $p(\overline{f}) = f : Y \to X$ corresponds to an isomorphism $\overline{f} : y \to f^*x$ where f^*x is the pullback of x with respect to f.

The above example provides a good source of intuition for working with CFG's. Also, it explains some notation we are going to introduce. For any object U in $\mathcal{S}$ let $\mathcal{X}_U$ be the category whose objects live over U and morphisms live over Id_U.

Exercise 2.4.4. Prove that $\mathcal{X}_U$ is a groupoid.

Naturally, one can think about $\mathcal{X}_U$ as the value of $\mathcal{X}$ on U. To make $\mathcal{X}$ to a presheaf we have to define the restriction functors $\gamma_{U,V}$ for a morphism $f : V \to U$. It is done as follows: for any u over U let us fix a lifting $\overline{f} : f^*u \to u$ of f (use condition (i)). Though f^*u is not defined uniquely, it is defined uniquely up to a unique isomorphism (use condition (ii)). Moreover, thanks to (ii) any morphism $u_1 \to u_2$ lifts uniquely to a morphism $f^*u_1 \to f^*u_2$, hence we obtain a change of base functor $\gamma_{U,V} = f^* : \mathcal{X}_U \to \mathcal{X}_V$ (which depends on our choice, but is unique up to a unique isomorphism regardless to the choices). It is natural to interpret $\mathcal{X}_T$ as S-points of $\mathcal{X}$, however it is a category unlike the case of algebraic spaces.

Exercise 2.4.5. Let T be an object in $\mathcal{S}$ and $\mathcal{T}$ be the category of T-objects. Then $\mathcal{T}$ has a natural structure of a CFG over $\mathcal{C}$ and the evaluation on (T, Id_T) induces a functor $\mathrm{HOM}(\mathcal{T}, \mathcal{X}) \to \mathcal{X}_T$. Check that it is surjective and fully faithful, in particular, it is an equivalence of categories.

Few more constructions are done exactly as in the case of groupoids, so we omit the details. (The informal reason is that a CFG behaves as a relative groupoid.) The reader can consult [Ful, §§2.4-2.5] for a very detailed exposition of this subject. Given two 1-morphisms $\mathcal{Y} \to \mathcal{X}$ and $\mathcal{Z} \to \mathcal{X}$ of CFGs over $\mathcal{S}$ one defines a fibred product $\mathcal{T} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ with 1-projections to the factors and a 2-isomorphism between $\mathcal{T} \to \mathcal{Y} \to \mathcal{X}$ and $\mathcal{T} \to \mathcal{Z} \to \mathcal{X}$. This data possesses the same strict universal property as the fibred product of groupoids. A diagram

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{Y} \\ \downarrow & \overset{f}{\Rightarrow} & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{X} \end{array}$$

is called (strictly) 2-cartesian if the induced map $\mathcal{T} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ is a (strict) 1-isomorphism. The diagram is called (strictly) 2-commutative if f is a 2-isomorphism (resp. identity), and 2-commutativity of triangular diagrams is defined similarly. The universal property of the fibred product gives rise to the diagonal morphisms $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ and $\Delta_{\mathcal{Y}/\mathcal{X}} : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$.

Exercise 2.4.6. Formulate and prove the analogs of the properties from exercises 2.3.5 and 2.3.7.

Note that to each presheaf of sets F on $\mathcal{S}$ (i.e. a set-valued contravariant functor) one can attach a CFG $\mathcal{F}$ by setting $\mathcal{F}_T = F(T)$ and defining the morphisms as follows: for a morphism $f : V \to U$ and $u \in F(U), v \in F(V)$ the set $\mathrm{Mor}(u, v)$ contains one element if u is mapped to v by $F(f)$, and is empty otherwise. Conversely, if $\mathcal{F}$ is a CFG such that each groupoid $\mathcal{F}_T$ is a set then we can define the presheaf of sets with $F(U) = \mathcal{F}_U$ and $F(U) \to F(V)$ coming from the change of base functor. So, by slight abuse of language we will say that $\mathcal{F}$ is a presheaf of sets.

Exercise 2.4.7. (i) Show that a CFG $\mathcal{X}$ is 1-isomorphic to a presheaf of sets X if and only if its objects have no non-trivial automorphisms. (Hint: take X to be the set of isomorphism classes of objects from $\mathcal{X}$.)

(ii) Formulate and prove the analog of the claim of exercise 2.3.9

In particular, to each algebraic space (including schemes) we can associate a CFG. As usually, an interesting property is that a CFG is 1-isomorphic to an algebraic space or scheme rather than strictly isomorphic.

stackPdef **Definition 2.4.8.** (i) A morphism $\mathcal{Y} \to \mathcal{X}$ is *representable* (resp. *schematic*) if for any algebraic space (resp. scheme) X and a morphism $X \to \mathcal{X}$ the CFG $X \times_{\mathcal{X}} \mathcal{Y}$ is isomorphic to an algebraic space (reps. scheme).

(ii) Let $\mathbf{P}$ be a property of morphisms of algebraic spaces (resp. schemes) which is stable under base changes and etale-local on the base (see definition 2.2.9). Then we say that a representable (resp. schematic) morphism $\mathcal{Y} \to \mathcal{X}$ possesses $\mathbf{P}$ if any its base change $X \times_{\mathcal{X}} \mathcal{Y} \to X$ with X being an algebraic space (resp. a scheme) possesses $\mathbf{P}$.

2.5. **Stacks.** As we discussed earlier, a stack should be defined as a CFG which is a sheaf. In particular, if we want to be able to talk about stacks, then the category $\mathcal{S}$ should be provided with a Grothendieck topology. For the sake of simplicity we assume in the sequel that $\mathcal{S}$ is the category of schemes over S provided with the etale topology. Since we are working with presheaves of groupoids, the sheaf condition should hold both for the objects and for the morphisms, and this is reflected in the following definition. Note that for any $T \in \mathcal{S}$ and $t, t' \in \mathcal{X}_T$ the presheaf $\mathrm{Isom}(t,t') = \mathrm{Mor}(t,t')$ on the category of T-schemes can be defined as $\mathrm{Isom}(t,t')(Z) = \mathrm{Isom}(f^*t, f^*t')$ for any morphism $f : Z \to T$.

stackdef **Definition 2.5.1.** (i) A CFG $\mathcal{X}$ is a *prestack* if each presheaf $\mathrm{Isom}(t,t')$ is a sheaf.

(ii) A prestack $\mathcal{X}$ is a *stack* if any descent datum on its objects is effective: if $p : T' \to T$ is an etale covering with $p_{1,2} : T'' = T' \times_T T' \rightrightarrows T'$ and $(t' \in \mathcal{X}_{T'}, \phi : p_1^* t' \widetilde{\to} p_2^* t')$ is a descent datum (satisfying the cocycle condition $p_{23}^*\phi \circ p_{12}^*\phi^* = p_{13}^*\phi$), then there exists $t \in \mathcal{X}_T$ with an isomorphism $\psi : p^*t \widetilde{\to} t'$ such that $p_2^*\psi = \phi \circ p_1^*\psi$.

As we will later see many naturally arising CFGs are actually prestacks, while the second condition is more subtle and is less "automatic". It is instructive to see how these conditions apply in the particular case of presheaves of sets.

Exercise 2.5.2. Let $\mathcal{F}$ be a CFG isomorphic a presheaf of sets. prove that $\mathcal{F}$ is a prestack (resp. a stack) if and only if for each etale covering $T' \to T$ with $T'' = T' \times_T T'$ the map $\mathcal{F}(T) \to \mathrm{Ker}(\mathcal{F}(T') \rightrightarrows \mathcal{F}(T''))$ is injective (resp. bijective) (i.e. $\mathcal{F}$ is a separated presheaf (resp. a sheaf) of sets).

A very important example of prestacks is obtained from quotients by group actions or, more generally, from groupoid schemes.

Definition 2.5.3. A *groupoid scheme* is a groupoid object in the category of schemes. It consists of a pair of morphisms $s, t : X_1 \rightrightarrows X_0$, an involution $i : X_1 \to X_1$ which switches s and t, an identity $e : X_0 \to X_1$ and a multiplication $m : X_1 \times_{s,X_0,t} X_1 \to X_1$ which satisfy natural compatibility conditions. A groupoid scheme is etale or flat if s and t are so.

Exercise 2.5.4. (i) Formulate the compatibility conditions or read [Ful, §3.1].
(ii) Check that any group action induces a groupoid scheme $X \times G \rightrightarrows X$.

To any groupoid scheme $X_1 \rightrightarrows X_0$ we can associate a CFG $\mathcal{X} = [X_1 \rightrightarrows X_0]^{pre}$ by evaluating on S-schemes, i.e. objects of $\mathcal{X}_U$ are elements of $X_0(U)$ and its morphisms are elements of $X_1(U)$. More generally, if $f : V \to U$ is a morphism, $u \in X_0(U)$ and $v \in X_0(V)$, then a morphism $\overline{f} : v \to u$ is an element of $X_1(V)$ with $s(\overline{f}) = v$ and $t(\overline{f}) = f^*u$ (we have a canonical map $f^* : X_0(U) \to X_0(V)$). The composition of the so-defined morphisms is defined using the groupoid multiplication (see the diagram in definition [Ful, 3.11]).

Exercise 2.5.5. Check that $\mathcal{X}$ is a prestack.

Usually, the prestack $\mathcal{X}$ as above is not a stack. It is clear already in the case of atlases for algebraic spaces or even a finite free group action. The reason is that in the definition of the etale sheaf quotient U/R we first define a presheaf $(U/R)^{pre}$ and then apply the sheafification. Similarly, we saw in remark 1.7.19(iii) that the moduli functor $\mathcal{M}_{g,n}$ can be obtained from the Hilbert functor by taking the presheaf quotient and applying sheafification. Let us consider one more example of the same kind.

Example 2.5.6. Let G be a group scheme over S acting trivially on the base scheme S. Then the prestack BG^{pre} is defined by the atlas $G \rightrightarrows S$ with s and t being the structure morphism and the multiplication given by the multiplication $m : G \times_S G \to G$. Each category BG^{pre}_T contains exactly one object and the group of automorphisms is $G(T)$.

Remark 2.5.7. We can identify the object of BG^{pre}_T with the trivial G-bundle $G \times_S T \to T$ (its automorphisms are as required). The prestack BG^{pre} is not a stack because the descent datum is not effective in general: indeed, any G-torsor, which is not trivial but can be trivialized by an etale covering, can arise from such a datum. The stack BG which classifies G-torsors can be obtained from the prestack BG^{pre} via a stackification operation (i.e. we add "solutions" for any descent datum on BG^{pre}).

Definition 2.5.8. Let $f : \mathcal{X}_0 \to \mathcal{X}$ be a morphism of CFGs with $\mathcal{X}$ being a stack. Then f is called the *stackification* of $\mathcal{X}_0$ is for any stack $\mathcal{Y}$ the functor $\mathrm{HOM}(\mathcal{X}, \mathcal{Y}) \to \mathrm{HOM}(\mathcal{X}_0, \mathcal{Y})$ is an equivalence.

Stackification can always be constructed by first "correcting" the morphisms in order to satisfy the first condition and second "correcting" the objects by adding the solutions for descent data. Since interesting CFGs are usually prestacks, the second stage is more useful (actually, we will not need the first stage). Using the stackification we can attach to any groupoid scheme $X_1 \rightrightarrows X_0$ a stack $[X_1 \rightrightarrows X_0]$ by stackifying the prestack $[X_1 \rightrightarrows X_0]^{pre}$.

Exercise 2.5.9. (i) Prove that any prestack $\widetilde{\mathcal{X}}$ admits a stackification $\mathcal{X}$ and then the functors $\widetilde{\mathcal{X}}_U \to \mathcal{X}_U$ are fully faithfully (actually, $\mathcal{X}$ is obtained from its full subcategory $\widetilde{\mathcal{X}}_U$ by adding solutions to non-effective descent data).
(ii) Check that BG is the stackification of BG^{pre}.

Note also that any stack $[R \rightrightarrows U]$ with a quasi-affine relative diagonal $(s, t) : R \to U \times U$ can be described as a classifying stack of $(R \rightrightarrows U)$-torsors, see [Ful, §4.3].

2.6. **Algebraic stacks.** Starting with stacks one defines algebraic stacks in the same way as algebraic spaces are defined starting with etale sheaves (or, as one sometimes says, *spaces*).

algstackdef **Definition 2.6.1.** A stack $\mathcal{X}$ is called *algebraic* or Artin stack (resp. Deligne-Mumford or *DM* stack) if it satisfies the following two conditions:

(i) Presentation: there exists a surjective smooth (resp. etale) morphism $U \to \mathcal{X}$ with U being an algebraic space.

(ii) Representability of the diagonal: the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is separated, representable and quasi-compact.

Remark 2.6.2. Sometimes (e.g. in [Ful]) one defines algebraic stacks in a more restrictive way by requiring that the diagonal is schematic.

Since our base category $\mathcal{S}$ is fixed we can use the notation $\mathcal{X} \times \mathcal{X}$, but if the dependence on $\mathcal{S}$ (or the scheme S) must be emphasized, then one uses the notation $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ (or $\mathcal{X} \times_S \mathcal{X}$). As usually, we do not have to care for quasi-compactness of the diagonal until we leave the noetherian world. The 1-morphisms between algebraic stacks are just the morphisms of stacks (i.e. the morphisms of CFGs), and the 2-morphisms are defined similarly.

stacktospace **Exercise 2.6.3.** Prove that the following conditions on an algebraic stack $\mathcal{X}$ are equivalent:

(i)* $\mathcal{X}$ is isomorphic to an algebraic space (the only difficult claim is that $\mathcal{X}$ admits an etale presentation; you can try to prove it or use theorem 2.7.1 from the next section);

(ii) $\mathcal{X}$ is isomorphic to a sheaf of sets;

(iii) the diagonal $\Delta_{\mathcal{X}}$ is a monomorphism;

(iv) for any $x \in \mathcal{X}_T$ the group $\mathrm{Aut}(x)$ (the stabilizer) is trivial.

Similarly to the case of algebraic spaces (parts (iii) and (iv) of proposition 2.2.6) there are few reformulations of the diagonal property which are easily seen to be equivalent.

algstex **Exercise 2.6.4.** Prove that the following properties of a stack $\mathcal{X}$ are equivalent:

(i) $\Delta_{\mathcal{X}}$ is representable (resp. schematic);

(ii) each fiber product $Y \times_{\mathcal{X}} Z$ with Y and Z isomorphic to algebraic spaces (resp. schemes) is isomorphic to an algebraic space (resp. scheme);

(iii) for any scheme T in $\mathcal{S}$ and objects $x, y \in \mathcal{S}_T$ the sheaf $\mathrm{Isom}(x, y)$ is isomorphic to an algebraic space (resp. scheme).

(Hint: the isomorphism

$$\mathrm{Isom}(x,y) \widetilde{\to} T \times_{x,\mathcal{X},y} T \widetilde{\to} T \times_{(x,y),\mathcal{X}\times\mathcal{X},\Delta_{\mathcal{X}}} \mathcal{X}$$

where $\mathrm{Isom}(x, y)$ is considered as a stack.)

The essential implication in proposition 2.2.6 (which required descent) was (i)⇒(iii). In the same way, a more involved (and usually more useful, at least for explicit constructions) characterization of algebraic stacks involves atlases via groupoid schemes.

atstackprop **Proposition 2.6.5.** *Let $s, t : X_1 \rightrightarrows X_0$ be a smooth (resp. etale) groupoid scheme with quasi-compact separated diagonal $X_1 \to X_0 \times X_0$, then the stack $\mathcal{X} = [X_1 \rightrightarrows X_0]$ is algebraic (resp. DM) and $X_1 \widetilde{\to} X_0 \times_{\mathcal{X}} X_0$.*

Proof. By exercise 2.6.4(iii), it suffices to prove representability of $\mathrm{Isom}_{\mathcal{X}}(x, y)$ for a scheme T and $x, y \in \mathcal{X}_T$. Recall that $\mathcal{X}$ is the stackification of the prestack $\widetilde{\mathcal{X}} = [X_1 \rightrightarrows X_0]^{pre}$. If $x, y \in \widetilde{\mathcal{X}}_T \hookrightarrow \mathcal{X}_T$ then we can interpret them as elements of $X_0(T)$, but in general we have to refine T: there exists an etale covering $f : T' \to T$ such that $f^*x, f^*y \in \widetilde{\mathcal{X}}_{T'}$, hence $f^*x, f^*y \in X_0(T')$. Note that $\mathrm{Isom}_{\mathcal{X}}(f^*x, f^*y) \widetilde{\to} \mathrm{Isom}_{\widetilde{\mathcal{X}}}(f^*x, f^*y)$ because $\widetilde{\mathcal{X}}$ is a prestack. The righthand sheaf is represented by (or isomorphic to) an algebraic T'-space $X_1 \times_{(s,t), X_0 \times X_0, (f^*x, f^*y)} T'$, hence the isomorphism sheaf for the pullbacks with respect to the morphism $T' \times_T T' \to T$ is also represented by an algebraic space. Using that the descent datum for an algebraic space over T' is effective, we obtain that $\mathrm{Isom}_{\mathcal{X}}(x, y)$ is represented by an algebraic T-space. □

atstackex **Exercise 2.6.6.** In the situation of proposition 2.6.5, prove that if the the diagonal of $X_1 \rightrightarrows X_0$ is quasi-affine then the stack $\mathcal{X}$ has schematic diagonal. (Hint: use descent for quasi-affine T'-schemes.)

The immediate corollary from the proposition is that we can now define fine group quotients for non-finite groups and non-free actions. Recall, that this feature was our main motivation for introducing stacks.

Corollary 2.6.7. *If a smooth (resp. etale) S-group scheme G acts on an S-scheme X, then the quotient stack $X/G = [X \times G \rightrightarrows X]$ is an Artin (resp. DM) stack and $X \times G \widetilde{\to} X \times_{X/G} X$.*

stackexam **Example 2.6.8.** (i) For any smooth S-group G the stack $BG = S/G$ is algebraic.

(ii) For each g, n with $2g + n \geq 3$, the ($\mathbf{Z}$-) stack $\mathcal{M}_{g,n} = H_{g,n}/\mathrm{PGL}(N_{g,n} + 1)$ from remark 1.7.19 is algebraic.

(iii) Note that considering $\mathcal{M}_{g,n}$ as a stack we enrich the defined earlier etale sheaf structure on $\mathcal{M}_{g,n}$ to a stack structure as follows: objects of $(\mathcal{M}_{g,n})_T$ are stable n-pointed T-curves of genus g (instead of the isomorphism classes) and the morphisms are isomorphisms. The latter definition makes sense for all g and n, though we have yet to check if $\mathcal{M}_{g,n}$ is algebraic for $2g + n < 3$.

Exercise 2.6.9. In this exercise we work over $S = \mathrm{Spec}(\mathbf{Z})$.

(i) For any $n \leq 3$ let H_n be the subgroup of $G = \mathrm{PGL}(2, \mathbf{Z})$ which fixes n points (i.e. $H_0 = G$, H_1 is a parabolic subgroup (e.g. the upper triangular subgroup), H_2 is a torus (e.g. the diagonal subgroup) and $H_3 = e$). Prove that $\mathcal{M}_{0,n} \widetilde{\to} H/G$, in particular, $\mathcal{M}_{0,0} \widetilde{\to} BG$ and $\mathcal{M}_{0,3} \widetilde{\to} S$.

(ii) Show that for any $n \geq 3$ the stack $\mathcal{M}_{0,n}$ is representable (i.e. isomorphic to an algebraic space). (Hint: show that the stabilizers are trivial.)

(iii) Show that the stacks $\mathcal{M}_{g,n}$ with $g > 0$ are not representable. (Hint: find non-trivial stabilizers in the geometric fibers.)

(iv) Show that $\mathcal{M}_{g,n}$ has Zariski open substack which classifies smooth n-pointed curves of genus g. Prove that this substack is a scheme for $n \gg g$.

stackPdef2 **Definition 2.6.10.** If a property $\mathbf{P}$ from definition 2.4.8 is also a smooth-local (resp. etale-local) on the source, then we extend it to non-representable morphisms of Artin (resp. DM) stacks as follows: $\mathcal{Y} \to \mathcal{X}$ satisfies $\mathbf{P}$ if for any algebraic space X with a morphism $X \to \mathcal{X}$ where exists a smooth (resp. etale) covering $Y \to X \times_{\mathcal{X}} \mathcal{Y}$ such that the composition $Y \to X$ satisfies $\mathbf{P}$.

In particular, the definition allows to talk about non-representable smooth (resp. etale) morphisms of Artin (resp. DM) stacks. A *Zariski point* of a stack $\mathcal{X}$ is an isomorphism class of morphisms $i : \mathrm{Spec}(k) \to \mathcal{X}$ such that k is a field and i does not factor through $\mathrm{Spec}(k')$ with $k' \subsetneq k$. As in the case of algebraic spaces, we obtain a topological space $|\mathcal{X}|$ of Zariski points, where the topology comes from open immersions (check that any monomorphism of stacks induces an embedding of underlying topological spaces). Caution: the morphism i does not have to be a monomorphism (e.g. $S = \mathrm{Spec}(k)$ and $i : S \to BG$). The Zariski topology is used to define properness and separatedness.

propsepdef **Definition 2.6.11.** Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of Artin stacks, then
(i) f is *separated* if its diagonal $\Delta_{\mathcal{Y}/\mathcal{X}}$ is universally closed.
(ii) f is proper if it is separated, of finite type and universally closed.

2.7. **Two theorems about DM stacks.** Our aim in this section is to find a finite scheme covering of the stacks $\mathcal{M}_{g,n}$ with $2g + n > 2$. It will be done in two stages: first we will show that they are DM, though the presentation with Hilbert scheme is smooth but not etale! Then we will prove that any DM stack admits such a covering. We saw in exercise 2.6.3 that algebraic spaces are characterized by "small" diagonal (i.e. the diagonal is a monomorphism). It turns out that the DM stacks for a next level in this hierarchy: their diagonal can be larger than a monomorphism, but is unramified (in particular, it is quasi-finite). Intuitively, it means that the automorphisms groups (or stabilizers of points) are small in DM stacks.

DMth **Theorem 2.7.1.** *Let $\mathcal{X}$ be an Artin stack. Then the following conditions are equivalent:*
(i) $\mathcal{X}$ is a DM stack;
(ii) the diagonal $\Delta_{\mathcal{X}}$ is unramified;
(iii) for any scheme T with $x, y \in \mathcal{X}_T$ the scheme $\mathrm{Isom}(x, y)$ *is unramified over T;*
(iv) the any scheme T with $x \in \mathcal{X}_T$ the scheme $\mathrm{Aut}(x)$ *is unramified;*
(v) for any geometric point $T \to \mathcal{X}$ with $x \in \mathcal{X}_T$ the scheme $\mathrm{Aut}(x)$ *is isomorphic to a disjoint union of finitely many copies of T.*

Proof. We leave as an exercise to prove that the last three claims are equivalent. Equivalence of (ii) and (iii) follows from the fact that each isomorphism T-scheme as above can be obtained as the base change of the diagonal with respect to the morphism $(x, y) : T \to \mathcal{X} \times \mathcal{X}$. To prove the implication (i)⇒(ii) assume that there exists an etale presentation $p : U \to \mathcal{X}$. Then $\Delta_{\mathcal{X}}$ is unramified if and only if its base change with respect to the etale covering $U \times U \to \mathcal{X} \times \mathcal{X}$ is unramified. But the latter is the relative diagonal $R = U \times_{\mathcal{X}} U \to U \times U$ which is unramified because $p_{1,2} : R \rightrightarrows U$ is etale.

The deep implication of the theorem is that (iii) or (ii) implies (i). Naturally, we can start with any smooth presentation $p : U \to \mathcal{X}$ with affine U. Intuitively, it should suffice to replace U with its "generical" closed subscheme U', i.e. to find an etale quasi-section $\mathcal{X} \dashrightarrow U'$ of p (we refer to [Ful, 5.3.1] for such approach). We prefer a similar but faster method from [LMB, §8]. Set $R = U \times_{\mathcal{X}} U$ and note that the relative diagonal $(s, t) : R \to U \times U$ is non-ramified because it is a base change of $\Delta_{\mathcal{X}}$. Therefore the morphism $s^*\Omega^1_U \oplus t^*\Omega^1_U \to \Omega^1_R$ is surjective and there exists a natural morphism $t^*\Omega^1_U \to \Omega^1_s$ (the target is $\Omega^1_{R/U}$ with respect to s). Consider the following diagram with a 2-cartesian right square and where the left square is cartesian with respect to either top or bottom morphisms

$$\begin{array}{ccccc} R\times_U R & \overset{\mu}{\underset{\mathrm{pr}_1}{\rightrightarrows}} & R & \xrightarrow{t} & U \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle s} & \Rightarrow & \downarrow{\scriptstyle p} \\ R & \overset{s}{\underset{t}{\rightrightarrows}} & U & \xrightarrow{p} & \mathcal{X} \end{array}$$

Probably the best way to understand the diagram is to recall that $R\times_{s,U,t}R\widetilde{\to}U\times_{\mathcal{X}} U\times_{\mathcal{X}} U$ and the maps $\mathrm{pr}_1, \mathrm{pr}_2, \mu$ (where μ is the action map) are the three projections p_{12}, p_{23}, p_{13}. If $\mathcal{X}$ is a scheme then we set $\Omega^1_p = \Omega^1_{U/\mathcal{X}}$ and using the base change with respect to p we obtain that $t^*\Omega^1_p\widetilde{\to}\Omega^1_s$. In addition, applying the base changes corresponding to the left square we also obtain that $\mu^*\Omega^1_s\widetilde{\to}\Omega^1_{\mathrm{pr}_2}\widetilde{\to}\mathrm{pr}_1^*\Omega^1_s$. The latter isomorphism gives a descent datum which allows to define Ω^1_p. Moreover, even if $\mathcal{X}$ is not a scheme we still have a locally free sheaf Ω^1_s with a descent datum $\mu^*\Omega^1_s\widetilde{\to}\mathrm{pr}_1^*\Omega^1_s$ (check the cocycle condition: it requires to extend the diagram by one more cartesian square). Therefore, for any Artin stack $\mathcal{X}$ with a smooth presentation p we have a locally free sheaf Ω^1_p on U. The crucial property of DM stacks is that for them the natural map $\psi : \Omega^1_U \to \Omega^1_p$ (defined using descent) is onto because we have already seen that the map $t^*\Omega^1_U \to \Omega^1_s\widetilde{\to}t^*\Omega^1_p$ is onto (so the surjectivity follows from the descent along t).

From now on we will work Zariski locally on $\mathcal{X}$, so let us fix a point $y : \mathrm{Spec}(k) \to \mathcal{X}$ with a lifting $x : \mathrm{Spec}(k') \to U$. By surjectivity of ψ we can find functions $f_1, \dots, f_n$ on X such that $df(x)$ is the basis of $\Omega^1_p(x)$ (in particular, n is the rank of Ω^1_p at x). Then f_i's induce a morphism $f : U \to \mathbf{A}_{\mathcal{X}} := \mathcal{X}\times\mathbf{A}^n$ (by our convention, the latter means $\mathcal{X}\times_S\mathbf{A}^n_S$). We claim that f is etale at x as can be verified smooth-locally after the base change with respect to $\mathbf{A}^n_U \to \mathbf{A}^n_{\mathcal{X}}$. So, replacing U with a neighborhood U' of x and replacing $\mathcal{X}$ with the image of U' we can assume that f is etale.

Exercise 2.7.2. (i) Define the image of U' as a Zariski open substack of $\mathcal{X}$. (Hint: use atlases.).

(ii) Show that if U_i's cover U then their images cover $\mathcal{X}$. (Hint: they form a smooth covering.)

We claim that there exists an S-etale subscheme $V \hookrightarrow \mathbf{A}^n_S$ such that the subscheme $V\times\mathcal{X} \hookrightarrow \mathbf{A}^n_{\mathcal{X}}$ contains a preimage of y (more precisely, its underlying Zariski space contains a preimage of y). To prove this set $U_y = U\times_{p,\mathcal{X},y}\mathrm{Spec}(k)$. Then we have a sequence of morphisms $U_y \to \mathbf{A}^n_k \to \mathbf{A}^n_{k(s)}$, where the first one is etale and s is the image of x in S. Clearly, there exists a point $z\in\mathbf{A}^n_{k(s)}$ with finite separable extension $k(z)/k(s)$ and extending it to a subscheme of $\mathbf{A}^n_S$ we can find V as required. Now, let W be the preimage of V, i.e. $W = U\times_{\mathbf{A}^n_{\mathcal{X}}} V\times\mathcal{X}$. Then W is a scheme (immersion of a subscheme is a schematic morphism), W is etale over $\mathcal{X}$ (morphisms $W\to V\times\mathcal{X}\to\mathcal{X}$ are etale) and its image contains y (a direct computation shows that $W_y\neq\emptyset$). $\square$

We will need one more theorem about DM stacks.

DMcov **Theorem 2.7.3.** *Let $\mathcal{X}$ be a noetherian DM stack. Then there exists a scheme X and a morphism $f : X\to\mathcal{X}$ which is finite and surjective. If $\mathcal{X}$ is reduced then f can be chosen generically etale.*

Note that the reducedness assumption is redundant, see [LMB, 16.6], but it allows us to give a simpler proof.

Proof. We can replace $\mathcal{X}$ with the disjoint union of its irreducible components. So, it suffices to prove the theorem for an irreducible $\mathcal{X}$. Furthermore, we replace $\mathcal{X}$ with its reduction, achieving that it is reduced (this stage can produce a morphism which is not generically etale). Let $U \to \mathcal{X}$ be an etale presentation of $\mathcal{X}$ and set $R = U \times_{\mathcal{X}} U$ with $p : R \rightrightarrows U$. Clearly, we can take U to be a scheme and let $\operatorname{Spec}(K) \to U$ be a generic point of U. The idea of finding f is very simple: take X to be the normalization of $\mathcal{X}$ in L for a sufficiently large separable extension L/K. A good way to think about this is to think about $\mathcal{X}$ as a scheme given by the atlas $R \rightrightarrows U$ and to construct $\mathcal{N}r_L(\mathcal{X})$ in terms of R and U.

Let R^0 and U^0 be the sets of the generic points of R and U, then $R^0 \rightrightarrows U^0$ is a groupoid scheme whose stack quotient $\mathcal{X}^0$ should be thought off as the generic point of $\mathcal{X}$: it is the intersection of all dense Zariski open substacks (even without the irreducibility assumption).

Exercise 2.7.4. (i) Prove that $\mathcal{X}^0$ is isomorphic to a stack $\operatorname{Spec}(k)/G$ where k is a field and G is a finite group acting trivially. (Hint: start with an atlas $R^0 \rightrightarrows U^0$ and refine it so that $U^0 = \operatorname{Spec}(L)$ is a point (it is possible because $\mathcal{X}$ is irreducible), then refine the atlas further by extending L so that $R^0 \widetilde{\to} \sqcup_{g\in G} R^0_g$ becomes a disjoint union of copies of U^0. Show that the last groupoid scheme $R^0 \rightrightarrows U^0$ reduces to a homomorphism $\phi : G \to \operatorname{Aut}(L)$ with a finite group G via the action $U^0 \stackrel{t^{-1}}{\to} R^0_g \stackrel{s}{\to} U^0$. Then $\operatorname{Spec}(L^G)/H \widetilde{\to} \mathcal{X}^0$ for $H = \operatorname{Ker}(\phi)$.

(ii)* Deduce the following generalization of exercise 2.2.14(ii): an integral DM stack $\mathcal{X}$ contains an open dense substack of the form V/G where V is a scheme and a finite group G acts trivially. (Hint: approximate the generic points of the atlas from (i) with a finite etale groupoid $R' \rightrightarrows U'$ where R' and U' are $\mathcal{X}$-etale.)

Let k be as in the first part of the exercise (we will not use part (ii)). Any point of U^0 is of the form $\operatorname{Spec}(K_i)$ for a finite extension K_i/k. Let L/k be a Galois extension which contains all K_i's and set $X_0 = \operatorname{Spec}(L)$, $U_0 = X_0 \times_{\mathcal{X}} U$ and $R_0 = X_0 \times_{\mathcal{X}} R$, then it follows that $R_0 \widetilde{\to} U_0 \times_{U,p_1} R \widetilde{\to} U_0 \times_{U,p_2} R$. Note that L contains any field $k(r)$ for $r \in R_0$, hence U_0 and R_0 are disjoint unions of copies of X_0 and it follows that the atlas $R_0 \rightrightarrows U_0$ for X_0 is trivial, i.e. $U_0 \widetilde{\to} X_0 \times G$ and $R_0 \widetilde{\to} X_0 \times G^2$ for a finite discrete set G. Let $\overline{U} := \mathcal{N}r_{U_0}(U)$ be the normalization of U in U_0 in the following sense. We have that $U_0 = \sqcup_{g\in G} u_g$ with $u_g \widetilde{\to} \operatorname{Spec}(L)$. Consider the Zariski closure of the image of u_g in U, and let $\overline{U}_g$ be its normalization in L. Then we set $\overline{U} := \sqcup_{g\in G} \overline{U}_g$ and define $\overline{R} = \mathcal{N}r_{R_0}(R) = \sqcup_{h\in G^2} \overline{R}_h$ similarly. The maps $R_0 \rightrightarrows U_0$ then extend to the maps $\overline{p} : \overline{R} \to \overline{U}$.

Exercise 2.7.5. (i) Prove that if $Y \to X$ is an etale morphism between reduced schemes and $\mathcal{N}r(X)$ is the normalization of X, then $\mathcal{N}r(Y) \widetilde{\to} \mathcal{N}r(X) \times_X Y$.

(ii) Deduce that $\overline{p}_1$ is the base change of p_1, i.e. $\overline{R} \widetilde{\to} \overline{U} \times_{U,p_1} R$ and similarly for $\overline{p}_2$.

Let X be the stack with the atlas $\overline{p}$, then the map of atlases $\overline{p} \to p$ consists of integral morphisms $\overline{R} \to R$ and $\overline{U} \to U$. For the sake of simplicity we assume that these morphisms are finite since this is always the case for "non-pathologic" noetherian schemes, e.g. excellent schemes. (The general case can be done using the fact that the normalization is always the projective limit of finite modifications. It is a typical application of projective limits, and we will study this technic later in the course.) Since the map of atlases is finite, it induces a finite morphism $f : X \to \mathcal{X}$ (prove the finiteness!). Also, the maps $\overline{p}_i$ are etale and generically isomorphisms, hence they are open immersions. Moreover, the groupoid $R_0 \rightrightarrows U_0$ is trivial, i.e. components of the equivalence relation can induce only trivial automorphisms on the components of U_0, hence $R_0 \rightrightarrows U_0$ is actually a usual Zariski gluing data. The latter implies that X is a scheme, as required. (Actually, $X = \mathcal{N}r_L(\mathcal{X})$.) □

As a corollary we can prove valuative criteria for noetherian DM stacks. (There are much more general valuative criteria whose proofs are more involved however.)

Corollary 2.7.6. *A noetherian DM S-stack $\mathcal{X}$ is separated (resp. proper) over S if and only if for any field K with a DVR $R \hookrightarrow K$ and compatible morphisms $\mathrm{Spec}(K) \to \mathcal{X}$ and $\mathrm{Spec}(R) \to S$ the following condition is satisfied: for any finite extension R'/R of DVR's the morphism $\mathrm{Spec}(R') \to S$ admits at most one (up to an isomorphism) lifting $\mathrm{Spec}(R') \to \mathcal{X}$ compatible with the morphism $\mathrm{Spec}(\mathrm{Frac}(R')) \to \mathcal{X}$ (resp. and there is such a lifting for some choice of R').*

Exercise 2.7.7. (i) Prove that $\mathcal{X}$ is separated/proper over S iff a scheme X from theorem 2.7.3 is so.

(ii) Deduce the corollary.

2.8. Application to moduli spaces of stable n-pointed curves.

Theorem 2.8.1. *The moduli stacks $\mathcal{M}_{g,n}$ for $2g + n \geq 3$ are DM stacks.*

Proof. By theorem 2.7.1, it suffices to check that the groups $\mathrm{Aut}_T(x)$ are T-unramified, and the latter can be checked on geometric points. So we can assume that $T = \mathrm{Spec}(k)$ for an algebraically closed field k and x corresponds to a stable n-pointed curve of genus g, and our aim is to prove that the scheme $\mathrm{Aut}_k(C)$ is a disjoint union of copies of T. Let $z \in \mathrm{Aut}_k(C)$ be a k-point corresponding to the identity, then it suffices to prove that the tangent space to z is zero-dimensional because it would imply that the point is discrete and reduced, and the local structure at all points is the same. The tangent space can be identified with R-points which agree with z, where $R = \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$. Therefore it can be identified with a ring automorphism of $\mathcal{O}_C \oplus \varepsilon\mathcal{O}_C$ which is trivial on $\mathcal{O}_C$, i.e. acts as $(a, b) \mapsto (a, \partial(a)\varepsilon + b)$, and the latter automorphism is completely defined by a derivation $\partial : \mathcal{O}_C \to \mathcal{O}_C$. To give a derivation is the same as to give a homomorphism $\Omega^1_{C/T} \to \mathcal{O}_C$, or to give a vector field on C (given by a dual map). The vector field must vanish at all singular and marked points (because the automorphism must be trivial above marked points), hence to give the vector field is the same as to give a vector field $\Omega^1_{\widetilde{C}/T} \to \mathcal{O}_{\widetilde{C}}$ which vanishes on all marked points $\widetilde{D}$ (preimages of D and C_{sing}), i.e. to give a homomorphism $\Omega^1_{\widetilde{C}/T}(\widetilde{D}) \to \mathcal{O}_{\widetilde{C}}$. A direct computation shows that the degree of the source is positive on each irreducible component of $\widetilde{C}$, hence the latter homomorphism must vanish. □

Remark 2.8.2. (i) An important corollary of the theorem is that any stack $\mathcal{M}_{g,n}$ for $2g + n \geq 3$ admits a finite covering $M \to \mathcal{M}_{g,n}$ by a scheme. There is a much more canonical way to construct M, but it involves etale cohomology (or, at least, relative Picard's schemes). Namely, it turns out that a good way to kill all automorphisms of a non-rational smooth proper curve over an algebraically closed field is to trivialize the cohomology group $H^1_{\rm et}(C,\mathbf{Z}/l\mathbf{Z})$ for $l\geq 3$ and prime to the characteristic. (Compare to the construction of the schemes $H_{g,n}$, where coherent cohomology was trivialized.) So, consider a functor $\mathcal{M}^{{\rm sm},l}_{g,n}$ which classifies smooth stable n-pointed S-curves (C,D) with an isomorphism $R^1_{\rm et}f_*(\mathbf{Z}/l\mathbf{Z})\widetilde{\to}(\mathbf{Z}/l\mathbf{Z})^{2g}$ of etale S-sheaves, where $f:C\to S$ is the structure morphism and l is invertible on S. Then $\mathcal{M}^{{\rm sm},l}_{g,n}$ is representable by a proper $\mathbf{Z}[l^{-1}]$-scheme which is an etale covering of $\mathcal{M}^{\rm sm}_{g,n}\otimes_{\mathbf{Z}}\mathbf{Z}[l^{-1}]$. (Note also, that one usually considers a smaller etale covering of $\mathcal{M}^{\rm sm}_{g,n}$ which classifies trivializations compatible with the symplectic structure on $H^1_{\rm et}(C,\mathbf{Z}/l\mathbf{Z})$ coming from the cup product.

(ii) It turns out (see [AO, §14] that $\mathcal{M}^{{\rm sm},l}_{g,n}$ extends to a finite covering $\mathcal{M}^l_{g,n}$ of the whole $\mathcal{M}_{g,n}\otimes_{\mathbf{Z}}\mathbf{Z}[l^{-1}]$ which is a scheme but does not have to be smooth (just normalize $\mathcal{M}_{g,n}$ in the generic point of $\mathcal{M}^{{\rm sm},l}_{g,n}$). In particular, $\mathcal{M}^l_{g,n}$ does not have a nice moduli description. Moreover, using two such schemes with different l's one can construct a (non-etale) finite covering of $\mathcal{M}_{g,n}$ (take its normalization in the composite of the fields of rational functions of two $\mathcal{M}^l_{g,n}$'s).

Exercise 2.8.3 (*)**.** Construct the scheme $\mathcal{M}^{{\rm sm},l}_{g,n}$. (Hint: first construct the corresponding etale cover $H^{{\rm sm},l}_{g,n}\to H^{\rm sm}_{g,n}$ of the Hilbert scheme $H^{\rm sm}_{g,n}\hookrightarrow H_{g,n}$ parametrizing smooth stable n-pointed curves (use proper base change theorem instead of the results on coherent direct images), then construct $\mathcal{M}^{{\rm sm},l}_{g,n}$ as the quotient $H^{{\rm sm},l}_{g,n}/{\rm PGL}(N_{g,n}+1)$ and check that this stack is a scheme by showing that all automorphism schemes are trivial. Note that the level structure is trivial when $n=0$, but $\mathcal{M}_{0,n}$ is already a scheme for $n\geq 3$.)

Here is another very important result about the stacks $\mathcal{M}_{g,n}$.

properM **Theorem 2.8.4.** *The moduli stacks $\mathcal{M}_{g,n}$ for $2g+n\geq 3$ are proper.*

Proof. We already know that the stacks are DM and of finite type over $\mathbf{Z}$. By corollary 2.7.1, it suffices to check the valuative criterion. Since we will not prove it here, we leave it to the reader to formulate the corresponding statement. Currently, we only note that the uniqueness part (the separatedness) is much easier though requires some work, while the existence part is the famous stable reduction theorem of Deligne-Mumford (which loosely speaking states that any stable curve over a fraction field of a DVR extends to a stable curve over a finite extension of the DVR). For expository reasons we postpone any further discussion on the stable reduction theorem until the next chapter. □

Finally, we have all ingredients to prove the main theorem of this chapter. The following definition is due to de Jong (though the theorem after the definition was well known much earlier).

Definition 2.8.5. Let S be an integral scheme with a generic point $\eta={\rm Spec}(K)$. Then an alteration $S'\to S$ is a proper dominant generically finite morphism with integral source.

extth **Theorem 2.8.6** (Stable extension theorem)**.** *Let S be an integral scheme with a generic point $\eta={\rm Spec}(K)$ and (C_η,D_η) be a stable n-pointed curve over η. Then there exists a generically etale alteration $S'\to S$ with $\eta'={\rm Spec}(K')$ the generic point of S' such that the stable curve $(C_\eta\times_\eta\eta',D_\eta\times_\eta\eta')$ extends to a stable n-pointed curve over S'.*

Proof. The curve (C_η, D_η) induces a morphism $\eta \to \mathcal{M}_{g,n}$ (where g is the genus of C_η) which factors through the reduction $\mathcal{M}$ of $\mathcal{M}_{g,n}$ (the latter stack is actually **Z**-smooth but we did not prove it). By theorems 2.7.3 and 2.8.4, there exists a proper scheme M' with a finite surjective generically etale morphism $M' \to \mathcal{M}$. Hence we obtain a finite separable K-scheme $\eta \times_{\mathcal{M}} M'$ and there exists a finite separable extension K'/K such that the morphism $\eta' := \mathrm{Spec}(K') \to \eta$ factors through $\eta \times_{\mathcal{M}} M'$. In particular, the n-pointed curve $(C'_\eta, D'_\eta) := (C_\eta \times_\eta \eta', D_\eta \times_\eta \eta')$ is induced from the n-pointed curve (C'_M, D'_M) corresponding to the map $M' \to \mathcal{M}_{g,n}$.

Exercise 2.8.7. Show that there exists an alteration $S' \to S$ such that η' is the generic point of S' and the map $\eta' \to M'$ extends to a map $\phi : S' \to M'$. (Hint: first take $S' = \mathcal{N}r_{K'}(S)$ so that ϕ is defined as a rational map, then refine S' by replacing it with the Zariski closure of the diagonal image of η' in $S' \times_{\mathrm{Spec}(\mathbf{Z})} M'$.)

Taking S' and $\phi : S' \to M'$ as in the exercise we can simply take (C', D') to be the n-pointed curve induced via ϕ from (C'_M, D'_M). □

Remark 2.8.8. (i) The stable reduction theorem is just the particular case of the stable extension theorem obtained when S is a *trait*, i.e. the spectrum of a DVR. (Strictly speaking the classical stable reduction theorem does not requires that $S' \to S$ is finite, but it is easily achieved.) In particular, the whole machinery of Hilbert schemes and stacks is used in the de Jong's proof only in order to show that theorem 2.8.6 follows from its particular case (which should be proved by a different method).

(ii) We will see that (using Riemann-Zariski spaces) one can easily deduce the stable extension (and more general stable modification) theorem from a slightly more general variant of the stable reduction theorem when S can be the spectrum of an arbitrary valuation.

3. RZ spaces and stable modification theorem

In this chapter we will introduce Riemann-Zariski spaces of valuations and apply them to prove the stable modification and reduction theorems. However, we will discuss first another approaches which do not involve the RZ spaces.

3.1. **On semistable modification theorem.** The following theorem (in a larger generality) is the main theorem of [dJ2].

Theorem 3.1.1 (Semi-stable modification theorem)**.** *Let $C \to S$ be a dominant proper morphism between integral noetherian schemes whose generic fiber $C_\eta \to \eta$ is a curve and $D \hookrightarrow C$ be a reduced closed subscheme which is the Zariski closure of a smooth zero-dimensional η-scheme $D_\eta \hookrightarrow C_\eta$. Assume that (C_η, D_η) is a semistable n-pointed η-curve. Then there exists an alteration $f : S' \to S$ with a semistable n-pointed S'-curve (C', D') which admit an alteration $(C' \to C, D' \to D)$ compatible with f and such that $(C'_{\eta'}, D'_{\eta'}) \widetilde{\to} (C, D) \times_\eta \eta'$ where η' is the generic point of S'.*

Later we will prove a more general stable modification theorem using only stable reduction theorem and RZ spaces. The original de Jong's proof uses a very nice three point lemma trick (which, as I think, becomes more clear when one works with RZ or analytic non-Archimedean spaces). In the following exercise we outline this proof in the particular case when all irreducible components of the S-fibers of C have at least three smooth points. See [AO, §§4.8-4.9] for details.

Exercise 3.1.2. During the proof we can replace S and (C,D) with their alterations until (C,D) becomes semistable.

(i)* Use flattening theorem to prove that after a modification of S we can achieve that C and D are S-flat.

(ii) Replace D with a larger divisor $\overline{D}$ which hits any irreducible component C_i in any fiber C_s over $s \in S$ in at least three smooth points. (Actually, it suffices to hit only rational components. Note that we achieve, in particular, that (C_η, D_η) becomes stable.)

(iii) Apply the stable extension theorem to S and $(C_\eta, \overline{D}_\eta)$ to get an alteration $f : S' \to S$ with a stable n'-pointed S'-curve $(C'\overline{D}')$ with $(C,D) \times_\eta \eta' \widetilde{\to} (C'_{\eta'}, \overline{D}_{\eta'})$.

(iv)* The point! (or the three point lemma). Prove that (C', D') is an alteration of (C,D) compatible with f.

Remark 3.1.3. (i) The suggested proof is due to de Jong. The stable extension theorem with the redundant assumption on three smooth points appeared in [dJ1]. Already this particular case sufficed for proving the famous theorem of de Jong that using alterations one can desingularize any integral algebraic variety.

(ii) In [dJ2], de Jong gets rid of this assumption. The trick is to first replace C with its Galois covering $\overline{C}$ such that the irreducible components of the S-fibers of $\overline{C}$ are of genus at least 2 (then no extension of D is required) and to find a semistable modification $\overline{C}_{\rm st} \to \overline{C}$ (after a sufficiently large alteration of S) which is equivariant with respect to the action of the Galois group $G = \mathrm{Gal}_{\overline{C}/C}$ on $\overline{C}$. Then one can show that the quotient $\overline{C}_{\rm st}/G$ is a semistable modification of the original C.

Here we give material which was covered/reviewed on lectures 17-20.

Lecture 17. The desingularization theory for surfaces, including factorization of birational morphisms between smooth surfaces, minimal model and minimal desingularization theorems, local desingularization and its equivalence to local uniformization, and reduction of desingularization to local uniformization.

Lecture 18. Riemann-Zariski spaces $\mathrm{RZ}_K(X)$ attached to a scheme X with a dominant point $\mathrm{Spec}(K) \to X$. Equivalence of valuative and projective limit descriptions, and quasi-compactness. (A reference is [Tem, §3.1].)

Lecture 19. Approximation theory of [EGA, IV §8]. The main results are existence of filtered projective limits $S = \mathrm{proj\,lim}_\alpha S_\alpha$ in the case when the transition morphisms $S_\alpha \to S_\beta$ are affine, and its realization by the projective limit in the category of locally ringed spaces; and equivalence of the categories of finitely presented S-schemes (resp. quasi-coherent $\mathcal{O}_S$-modules) to the projective limit of their S_α-analogs.

Lecture 20. Valuation theory. Definitions of valuations and valuation rings, height, invariants of algebraic and transcendental extensions of valued fields. Criteria for unramifiedness of extensions (completion criterion for $h = 1$ and composition criterion for $h > 1$, see [Tem], 2.4.1 and 2.4.2). The problem of uniformization of valued fields, its connection to local uniformization and their equivalence for algebraic surfaces.

We saw that desingularization of algebraic surfaces reduces to local uniformization, and the latter reduces to uniformization of valued fields of transcendence degree 2. I do not know a reasonable direct proof of the latter results when the characteristic is positive (surely, one can deduce it from the global desingularization, which is known), but at least the proof in the zero characteristic case is easy.

I present it here because it is similar to the argument we will use to prove the stable reduction theorem. For simplicity, we work over an algebraically closed ground field.

surfunifth

Theorem 3.1.4. *Let k be an algebraically closed field of characteristic zero and provided with the trivial valuation, and let K/k be a finitely generated extension with* $\text{tr.deg.}_k(K) = 2$. *Then K is uniformizable over k, i.e. K is unramified over a subfield $k(x, y)$.*

Proof. If K/k is Abhyankar, i.e. there $E_{K/k} + F_{K/k} = 2$ and there is no transcendental defect, then $|K^\times| \widetilde{\to} \mathbf{Z}^F$. In particular, we can find a transcendence basis x, y whose image generates $|K^\times|$. Then $K/k(x, y)$ is a finite unramified extension, as required. It remains to treat the case when $D_{K/k} = 1$. Since, we have that $F \geq 1$ for a non-trivial valuation on K, we must have $E = 0, F = D = 1$.

Exercise 3.1.5. Construct examples of valuations on the field of rational functions of $\mathbf{P}^2_k$ with $D = F = 1$. So, $|K^\times|$ is a subgroup of $\mathbf{Q}$. Give an example when $|K^\times|$ coincides with $\mathbf{Q}$, in particular it is not discrete.

The exercise illustrates the obvious intuition that the defect is a bad guy. In particular, we cannot give a nice receipt how to choose an unramified transcendence basis x, y. Fortunately, the ramification theory is easy in our case because there is only tame ramification (unlike the positive characteristic case, where one often has to struggle with algebraic defect). Note that $h \leq 1$ and where is nothing to prove when $h = 0$, so we can assume that $h = 1$. Choose any transcendence basis $x, y \in K$ with $|x| \neq 1$. Since K is finite over $L = k(x, y)$, $\widetilde{K} = k$ is algebraically closed and the extension is tame (by our assumption on the characteristic), $f_{L/K} = [K : L]$. We will need the following well known result

Exercise 3.1.6. Show that if F is a complete valued field of height one and such that $\widetilde{F}$ is algebraically closed and of characteristic zero, then $\pi \in F$ is an n-th power if and only if $|\pi|$ is an n-th power in $|F^\times|$. In particular, if $\sqrt{|F^\times|}/|F^\times| \subseteq \mathbf{Q}/\mathbf{Z}$ (e.g. $|F^\times| \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$), then any finite extension of F is of the form $F(\pi^{1/n})$.

The exercise implies that the completion $\widehat{K}$ is generated over $\widehat{k(x, y)}$ by an element $x^{1/n}$, i.e. $\widehat{K} = \widehat{k(x^{1/n}, y)}$ is topologically generated by two elements. It remains to approximate the topological generators of $\widehat{K}$ with algebraic generators of K because $x^{1/n}$ does not have to be in K. There are two possible solutions of this problem which we outline in exercises.

Exercise 3.1.7. (i) Show that $L = K(x^{1/n})$ is unramified over both $k(x^{1/n}, y)$ and K. In particular, L is uniformizable.

(ii)* Deduce that K is uniformizable. (Hint: extend the etale morphism $\text{Spec}(L^\circ) \to \text{Spec}(K^\circ)$ to an etale morphism of local schemes of essentially finite type over k, and use that L° is locally uniformizable to prove that the source can be taken essentially smooth over k.)

The direct algebraization method has to separate the cases accordingly to the topological dimension of $\widehat{K}$.

Exercise 3.1.8. (i)* Show that if x is not in the completion of $k(y)^a$ (i.e. $\widehat{K}$ is two-dimensional), then $r = \inf_{c \in k(y)^a} |x^{1/n} - c|$ is positive and for any $x' \in \widehat{K}$ with

$|x' - x^{1/n}| < r$ one has that $\widehat{K} = \widehat{k(x^{1/n}, y)} = \widehat{k(x', y)}$. Deduce that taking such x' in the dense subfield K, one obtains a required transcendence basis x', y of K.

(ii)* Establish the case when x lies in the completion of $k(y)^a$, i.e. $\widehat{K}$ is one-dimensional (currently I do not see a way to do it directly without any use of analytic geometry over $k((y))$, but I expect this to be possible).

□

3.2. **Stable modification theorem: reduction to uniformization of one-dimensional extensions of valued fields.** Let us recall the formulation of the stable modification theorem (though it was mentioned in lecture 17). We start with few definitions.

Definition 3.2.1. (i) A *multipointed curve* (C, D) over a base scheme S is an S-*curve* C, i.e. a flat finitely presented S-scheme, with an S-*divisor* D, i.e. a closed subscheme that is flat and finitely presented over S.

(ii) A *semistable modification* is a multipointed curve (C', D') with a modification $(C', D') \to (C, D)$, $C' \to C$ and $D' \to D$ are modifications, and such that the S-fibers of (C', D') are semistable.

(iii) If S is integral with generic point η, when an η-*modification* is a modification which does not changes η-fibers.

(iv) A semistable modification is *stable* if the fibers C'_s for $s \in S$ have no *exceptional* irreducible components, i.e. rational curves with at most two marked points (points from $D_s \cup (C_s)_{\rm sing}$) and contracted in C to a point.

Theorem 3.2.2 (Stable modification theorem)**.** *If S is integral and qcqs and (C, D) is a multipointed S-curve with semistable η-fiber, then there exists a separable alteration $S' \to S$ such that $(C', D') = (C, D) \times_S S'$ admits a stable η-modification $(C'_{\rm st}, D'_{\rm st})$. If S' is normal, then the stable η-modification is the minimal semistable modification of (C', D'), in particular it is unique up to a unique isomorphism and is compatible with automorphisms of S and (C, D).*

Remark 3.2.3. (i) Unlike the stable extension theorem, one allows non-proper and even non-separated curves.

(ii) Unlike semistable modification theorem, the theorem states that there exists a canonical modification. This canonicity is heavily used in the process of proof. As a matter of fact, it happens often that canonicity (or functoriality) makes a desingularization proof easier, even though one proves a stronger statement.

Exercise 3.2.4. (i) Deduce the theorem from its particular case when C is normal. (Hint: show that to find a semistable modification of (C, D) is equivalent to find a semistable modification of the multipointed curve (C', D') where $C' = \mathrm{Nor}(C)$ and D' is the union of the preimage of D and the modification locus of $C' \to C$).

(ii) Deduce the theorem from its particular case when S is of finite type over $\mathbf{Z}$. (Obvious hint: use approximation.)

Though it is not necessary, it will be convenient to work with irreducible C's in the sequel. So, we will always assume that C is irreducible. As was explained earlier, the theorem is an analog of the minimal desingularization of surfaces, and our proof will be similar. First we localize the problem by reducing it to a kind of local uniformization problem. However, unlike the case of surfaces we can localize both on C and on S. The latter is much easier and is accomplished in the following proposition.

3.2.1. *Localization on the base.*

loconbase **Proposition 3.2.5.** *The stable modification theorem follows from its particular case when the base scheme is Spec*(R) *and* R *is a valuation ring with a separably closed fraction field* K*.*

Proof. First we will establish minimality. So, let us assume that S is normal, $(C', D') \to (C, D)$ is a semistable modification and $(C_{\rm st}, D_{\rm st}) \to (C, D)$ is a stable one. We know that if R is a valuation ring of the separable closure K^s of the field of fractions $K = k(S)$ of S, and $T := \mathrm{Spec}(R) \to S$ extends $\mathrm{Spec}(K^s) \to S$, then the semistable modification $C' \times_S T \to C \times_S T$ refines the stable modification $C_{\rm st} \times_S T$. Now, the minimality follows from the following lemma.

refinlem2 **Lemma 3.2.6.** *Assume that* S *is normal, and let* X *be an integral* S*-curve with normal modifications* X' *and* X''*. If for any valuation ring* $R \in \mathrm{RZ}_{K^s}(S)$ *we have that* $X' \times_S \mathrm{Spec}(R)$ *is a refinement of* $X'' \times_S \mathrm{Spec}(R)$*, then* X' *refines* X''*.*

Proof. Let us assume on the contrary that X' does not refines X''. Note that the minimal refinement $\overline{X}$ of X' and X'' is constructed as follows: let ε be the generic point of X, then $\overline{X}$ is the Zariski closure of the diagonal image of ε in $X' \times_X X''$. By our assumption, $\overline{X}$ is a non-trivial modification of a normal scheme X', hence the map $\overline{X} \to X'$ has a non-discrete fiber over a closed point $x' \in X'$. Let $\overline{x}$ be a generic point of a non-zero dimensional component over x' and x'' be its image in $\overline{x}$. We claim that x'' is not closed in its S-fiber, because otherwise the fiber over (x', x'') in $X' \times_X X''$ were discrete in its S-fiber. So, x'' is the generic point of an irreducible component in the fiber X''_s over a point $s \in S$. Now the following exercise provides a contradiction to our assumption, thus proving the lemma.

Exercise 3.2.7. For a valuation ring $R \in \mathrm{RZ}_{K^s}(S)$ centered on s set $T = \mathrm{Spec}(R)$, $X'_T = X' \times_S T$, $X''_T = X'' \times_S T$, and let $\overline{X}_T$ be the minimal refinement the T-curves of X'_T and X''_T.

(i) Show that $\overline{X}_T$ is the Zariski closure of ε in $\overline{X} \times_S T$.

(ii) Show that for an appropriate choice of R, there exists points $x'_T \in X'_T$, $x''_T \in X''_T$ and $\overline{x} \in \overline{X}_T$ sitting over x', x'' and $\overline{x}$, respectively.

(iii) Show that x'_T is closed in the closed T-fiber, x''_T is not closed in its T-fiber, and $\overline{x}_T$ is mapped to x'_T and x''_T.

(iv) Deduce that X'_T cannot be a modification of X''_T oppositely to our assumptions.

□

Minimality of stable modification will be very important in tuning RZ-local (or valuation-local) solutions. For any valuation ring $R \in \mathrm{RZ}_{K^s}(S)$ with $T = \mathrm{Spec}(R)$, the multipointed curve $(C_T, D_T) = (C, D) \times_S T$ admits a stable modification $(C_{T,\rm st}, D_{T,\rm st})$ by our assumption. Let S_α be the family of all separable alterations of S and s_α be the centers of R on S_α's, and set $\overline{S}_\alpha = \mathrm{Spec}(\mathcal{O}_{S_\alpha, s_\alpha})$. Since we know that the $R = \cup \mathcal{O}_{S_\alpha, s_\alpha}$, the approximation implies that the stable modification over T is induced from a modification (C'_α, D'_α) of $(C_\alpha, D_\alpha) = (C, D) \times_S \overline{S}_\alpha$.

Exercise 3.2.8. (i) Show that for large enough $\beta \geq \alpha$ we have that $(C'_\alpha, D'_\alpha) \times_{\overline{S}_\alpha} \overline{S}_\beta)$ is a stable modification of $(C, D) \times_S \overline{S}_\beta$.

(ii) Deduce that the stable modification $(C, D) \times_S \overline{S}_\beta$ extends to a stable modification of $(C, D) \times_S \widetilde{S}_\beta$, where $\widetilde{S}_\beta$ is an open neighborhood of s_β.

(iii) Use quasi-compactness of $\mathrm{RZ}_{K^s}(S)$ to show that there exists finitely many S_{β_i}'s with open subschemes $\widetilde{S}_{\beta_i}$ such that $(C,D)\times_S \widetilde{S}_{\beta_i})$ admit stable modifications and the preimages of all $\widetilde{S}_{\beta_i}$ cover $\mathrm{RZ}_{K^s}(S)$.

(iv) Enlarge β_i's so that they all are equal to some β and S_β is normal, observe that in this case $\widetilde{S}_{\beta_i}$ form an open covering of S_β, and show that the stable modifications of $(C,D)\times_S \widetilde{S}_{\beta_i})$'s agree over the intersections $S_{ij} = S_{\beta_i}\cap S_{\beta_j}$ by uniqueness of the stable modification over a normal base S_{ij}. Deduce, that there exists a stable modification of $(C,D)\times_S S_\beta$.

□

3.2.2. *Reduction to semistable modification.* In previous section we reduced the proof of the stable modification theorem to the case when $S = \mathrm{Spec}(R)$, where R is a valuation ring of finite height and with separably closed fraction field K. Also, we reduced to the case of normal C_η.

Exercise 3.2.9. Show that it is enough to establish the case when the valuation of R is finite. (Hint: use approximation.)

blowdownprop

Proposition 3.2.10. *Let $S = \mathrm{Spec}(R)$ be as above. If (C,D) is a semistable modification of an S-curve (C_0,D_0), then there exists a unique way to blow (C,D) down to a stable modification of (C_0,D_0).*

Proof. By our assumption S is a finite chain of points $s_0 = \eta \succ s_1 \succ \cdots \succ s_h$. Let X_i be any proper irreducible component in $C_i := C_{s_i}$ and X be its Zariski closure, and assume that X is S-proper. We claim that the arithmetic genus of the non-empty S-fibers of X is constant. Indeed, it is known for noetherian bases, but C (and hence X) is induced from a curve over a noetherian base. Similarly, the fact that X_i is geometrically connected implies that the non-empty fibers are connected. So, $h^1(X_j) = h^1(X_i)$ for any fiber X_j with $j \ge i$.

Exercise 3.2.11. (i) Deduce that if X_i is an exceptional component, then each X_j is an "exceptional tree" (i.e. a tree of $\mathbf{P}^1$'s with at most two outer marked points coming from the intersections with D or outer irreducible components). In particular, each X_j contains an exceptional component.

(ii) Deduce that (C',D') is not stable if and only if it contains an exceptional component X which is closed. (Hint: if X' is exceptional, then it is contracted to a point in C; hence any X'_i is contracted in C, in particular, X'_i is proper.)

Now, our proof runs as follows: we must show that any closed exceptional component can be blown down, and we must show that it leads to a semistable modification again. Then uniqueness of the final blow down will follow from an easy combinatorics with exceptional trees.

Let X be a closed exceptional component. Since the question is local on the images $x_0 \in C_0$ and $s \in S$ of X, we can shrink them both achieving that s is closed and C_0 is affine. Then it will be convenient to compactify C_0, so we can assume that C_0 and C are S-proper. Find Cartier divisors P and Z which are disjoint, do not contains components of C_s and hit (both P and Z) all components of C_s except X. The following exercise is similar to the computations on curves we have done in the beginning of the course.

Exercise 3.2.12. Show that for sufficiently large natural number m the invertible sheaf $\mathcal{L} = \mathcal{O}_C(mZ - P)$ satisfies $h^1(\mathcal{L}_s) = 0$ and there is a non-zero section $f_s \in H^0(C_s, \mathcal{L}_s)$.

Choose $\mathcal{L}$ as in the exercise. By semi-continuity $h^1(\mathcal{L}_\eta) = 0$, and by the theorem on direct images, the map $H^0(C, \mathcal{L}) \to H^0(C_s, \mathcal{L}_s)$ is onto, in particular we can lift f_s to $f \in H^0(\mathcal{L})$. We claim that f provides a function on C which is constant on X and is not constant on other components. So, if we take a neighborhood C_1' of X where f has no poles, we obtain a morphism $C_1' \to C_1 \hookrightarrow C_0 \times \mathbf{A}^1_S$ which contracts X. Clearly, we can extend it outside by an isomorphism obtaining a contraction $C' \to C$ which is an isomorphism over $C_s \setminus x$ where x is the image of X. It remains to check that C is semistable at x because if C is semi-stale along C_s, then it is semistable. Since $h^1(C_s) = h^1(C_\eta) = h^1(C_s')$, the formula for h^1 of curves implies that $\delta_x = 1$, i.e. C_s is semistable at x. □

3.2.3. *Reduction to the height one case.* Let h be the height of R. Our aim is to establish induction on h assuming that the main case of $h = 1$ is known. We know that if $h > 1$, then R is composed from valuation rings A and B of positive height, where A is a localization of R and B is a valuation ring of $k = A/m_A$. In particular, $m_A \subset R$, $B = R/m_A$ and R is the preimage of B under $A \to k$. Then $\operatorname{Spec}(R)$ is pasted from $U = \operatorname{Spec}(A)$ and $T = \operatorname{Spec}(B)$ along $\varepsilon = \operatorname{Spec}(k)$, in particular

Exercise 3.2.13. Prove that the following "gluing" diagram is bi-Cartesian

$$\begin{array}{ccc} \varepsilon & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

Moreover, we will see that U-admissible S-schemes (resp. quasi-coherent $\mathcal{O}_S$-modules) can be glued from T-schemes and U-schemes (resp. modules). Given a quasi-coherent $\mathcal{O}_S$-module M, which we identify with an $\mathcal{O}$-module, set $M_U = M \otimes_R A$, $M_T = M \otimes_R B = M/m_A M$ and $M_\varepsilon = M \otimes_R k$. We say that M is U-admissible if the localization homomorphism $M \to M_U$ is injective. Note that any $\mathcal{O}$-module M defines a descent datum consisting of M_U, M_T and an isomorphism $\phi_M : M_U \otimes_A k \widetilde{\to} M_T \otimes_B k$, and a similar claim holds for S-schemes. The corresponding categories of descent data are defined in an obvious way, and, naturally, we have the following gluing lemma.

gluelem **Lemma 3.2.14.** *Keep the above notation.*

(i) The natural functor from the category of U-admissible quasi-coherent $\mathcal{O}_S$-modules M to the category of descents data (M_U, M_T, ϕ_M) with quasi-coherent M_U and quasi-coherent ε-admissible M_T is an equivalence of categories.

(ii) The natural functor from the category of qcqs U-admissible S-schemes X to the category of descents data (X_U, X_T, ϕ_X) with qcqs X_U and qcqs ε-admissible X_T is an equivalence of categories.

(iii) a qcqs U-admissible S-scheme X is of finite type (resp. finite presentation) if and only if X_U and X_T are so.

Proof. The assertion (iii) of the lemma is exactly Step 1 from the proof of [Tem, 2.4.3]. To prove (i) we note $m_A M_U = m_A M$, hence $M_T = M/m_A M$ embeds into $M_\varepsilon = M_U/m_A M_U$. So, M_T is ε-admissible and the embedding $M \hookrightarrow M_U$ identifies

M with the preimage of M_T under the projection $M_U \to M_\varepsilon$. In particular, an exact sequence $0 \to M \to M_U \oplus M_T \to M_\varepsilon \to 0$ arises. Conversely, given a descent datum as in (i), we can define an $\mathcal{O}$-module $M = \mathrm{Ker}(M_U \oplus M_T \to M_\varepsilon)$, and one easily sees that M_U and M_T are the base changes of this M. We constructed maps from $\mathcal{O}_S$-modules to descent data and vice versa, and one immediately sees that these maps extend to functors. Then it is obvious from the above that these functors are actually equivalence of categories which are essentially inverse one to another.

We proved (i) and similarly to the classical case one deduces quasi-affine descent rather automatically. Indeed, it follows obviously that the category of affine U-admissible S-schemes is equivalent to the category of affine descent data. In order to extend this equivalence to the categories of all qcqs schemes, the only non-obvious claim is effectivity of descent. So, let assume that $\varepsilon \times_T X_T \widetilde{\to} X_\varepsilon \widetilde{\to} \varepsilon \times_U X_U$ is a descent data as in the assertion of (ii). We know that the descent holds in the affine case, and the case of quasi-affine descent data follows because X_T (resp. X_U) is an open subscheme of its affine hull $\overline{X}_T = \mathrm{Spec}(\Gamma(\mathcal{O}_{X_T}))$ (resp. $\overline{X}_U$) and one easily checks that $\varepsilon \times_T \overline{X}_T \widetilde{\to} \varepsilon \times_U \overline{X}_U$. Hence affine hulls define an affine descent data which gives rise to an S-scheme $\overline{X}$, and the desired scheme X is realized as an open subscheme in $\overline{X}$. Finally, in our case the general descent follows from the quasi-affine one because one can easily construct open quasi-affine coverings $X_T = \cup_{i=1}^n X_{T,i}$ and $X_U = \cup_{i=1}^n X_{U,i}$ with $\varepsilon \times_T X_{T,i} \widetilde{\to} X_\varepsilon \widetilde{\to} \varepsilon \times_U X_{U,i}$ for each i (use that open subschemes of X_ε extend to open subschemes in X_U and X_T). □

Exercise 3.2.15. Deduce that if semistable modification holds over T and U, then it holds over S. In particular, it suffices to prove the semistable modification when $h = 1$.

Remark 3.2.16. Note that in the above reduction we used the case when the generic fiber is not normal, because even if the generic fiber of C_U is normal, the closed fiber of its stable modification can be non-normal, and then we have to apply the theorem to a a curve C'_T with not normal generic fiber.

In the sequel, we assume that $h = 1$, so S has two points: the generic point η and the closed point s. We also use the following notation: $L = k(C)$ (recall that C_η is smooth, so it is harmless to assume that C is integral), $S = \mathrm{Spec}(R)$ and $K = \mathrm{Frac}(R)$ is a valued field with $K^\circ = R$.

redlusec

3.2.4. *Reduction to local uniformization.* Throughout §3.2.4 we will be concerned with the case when $D = \emptyset$, and only in the very end we will show how to treat the divisor. Let $\mathfrak{C} = \mathrm{RZ}_L(C)$ be the Riemann-Zariski space of C. Note that $\mathfrak{C}$ admits a natural projection to S (which is the Riemann-Zariski space of itself) and the generic fiber $\mathfrak{C}_\eta$ is just C_η because the latter is a smooth curve (so it does not admit non-trivial modifications and its local rings are already valuation rings). The interesting part of $\mathfrak{C}$ is its closed fiber $\mathfrak{C}_s$.

Exercise 3.2.17. (i) Prove that any generic point $x \in C_s$ possesses finitely many preimages in $\mathfrak{C}$.

(ii) Prove that if C is normal then the local ring $\mathcal{O}_{C_s,x}$ is a valuation ring, we can identify x with its preimage in $\mathfrak{C}$.

(iii)* Generalize the above claims to any flat finitely presented morphism $X \to S$.

Remark 3.2.18. One should be careful with normalizations because a priori it is not clear why normalization of C is a curve, i.e. is of finite presentation. We will see later that normalization is finite in our situation (and, more generally, over any valuation ring with a separably closed fraction field).

Exercise 3.2.19. Give an example of a valuation ring R and a curve C over R with non-finite normalization. (Hint: use that already normalization of R in its finite extensions does not have to be finite.)

Let $\Gamma(C) \subset \mathfrak{C}$ be the preimage of the set of generic points of C_s. Note that any element $\mathbf{x} \in \Gamma(C)$ is a valuation $\mathcal{O}_{\mathbf{x}}$ on $k(C)$ which extends R (i.e. $\mathcal{O}_{\mathbf{x}} \cap K = R$) and such that if L is provided with the valuation corresponding to $\mathcal{O}_{\mathbf{x}}$, then $F_{L/K} = 1$. Actually, $\mathcal{O}_{\mathbf{x}}$ is a direct analog of divisorial valuations on the field of rational functions of a surface.

Exercise 3.2.20. Show that for two normal modifications C' and C'' of C, one has that C' is a refinement of C'' if and only if $\Gamma(C'') \subset \Gamma(C')$. (Hint: use the argument from lemma 3.2.6, i.e. find the minimal mutual refinement of C' and C'' and show that its Γ coincides with the union $\Gamma(C'') \cup \Gamma(C')$.)

We know from exercise 1.7.5 that C at x is etale-locally isomorphic to an S-curve $\mathrm{Spec}(R[x,y]/(xy-\pi))$ where $\pi \in R$. But it will be convenient to consider a special class of nodal curves as follows.

Definition 3.2.21. An S-curve C is a called *strictly nodal* at a point x if a Zariski neighborhood of x admits an etale morphism to some $\mathrm{Spec}(R[u,v]/(uv-\pi))$.

Exercise 3.2.22. (i) Prove that any strictly nodal S-curve is normal.
(ii) Prove that any nodal S-curve is normal.

nodmodprop **Proposition 3.2.23.** *Assume that C is strictly nodal. Then the family of strictly nodal modifications of C is cofinal in the family of all its modifications.*

Proof. It suffices to show that if $\mathbf{x}$ is an element of $\mathfrak{C}$ such that $\mathcal{O}_{\mathbf{x}}/m_{\mathbf{x}}$ is transcendent over R/m_R, then there exists a modification $C' \to C$ such that C' is nodal and $\mathbf{x}$ is centered on a generic point of C'_s. Let $x \in C$ be the center of $\mathbf{x}$. We will show that a required modification can be chosen so that it does not modify $C \setminus \{x\}$. This stronger problem is local at x, so we can localize C. In particular, we can assume that C is etale over $\overline{C} = \mathrm{Spec}(R[u,v]/(uv-\pi))$ and then it suffices to solve the problem for $\overline{C}$ with the image $\overline{x} \in \overline{C}$ of x. So, we can assume that $C = \overline{C}$. Provide L with the valuation corresponding to $\mathbf{x}$. Assume first that x is the origin.

blowex **Exercise 3.2.24.** Find $\omega \in R$ with $|u| = |\omega|$ and show that after blowing up the ideal (ω, x) we obtain a strictly nodal modification of C such that $\mathbf{x}$ is centered on its smooth point. (Hint: $|K^\times|$ is divisible, hence $|K^\times| = |L^\times|$.)

Now we can assume that x is smooth, and using etale map to $\mathrm{Spec}(R[u])$ we can now assume that $C = \mathrm{Spec}(R[u])$ and x is the origin. Then the following exercise finishes the proof.

Exercise 3.2.25. Show that $\widetilde{L}$ is generated by the residue of an element $(T-a)/\omega$ with $a, \omega \in R$, and then blowing up the ideal (a, ω) we obtain a strictly nodal modification C' of C such that $\mathbf{x} \in \Gamma(C')$. (Hint: use that any polynomial in $R[u]$ factors as a product of p^n-th powers of linear terms $(T-a)^{p^n}$, and moving it slightly (w.r.t. the valuation of L) we can achieve that all factors are linear.)

□

In the following proposition we deduce semistable modification from that local uniformization for the R-curve C (with empty divisor).

locunifprop

Proposition 3.2.26. *Assume that for any valuation ring $\mathbf{x} \in \mathfrak{C}$ there exists a semistable R-curve $C'_{\mathbf{x}}$ with a separated morphism $C'_{\mathbf{x}} \to C$ which induces an open immersion $(C'_{\mathbf{x}})_\eta \hookrightarrow C_\eta$ and such that $\mathbf{x}$ is centered on $C'_{\mathbf{x}}$ (i.e. the morphism $\mathrm{Spec}(\mathcal{O}_{\mathbf{x}}) \to C$ factors through $C'_{\mathbf{x}}$). Then C admits a semistable modification C'.*

Proof. By quasi-compactness of $\mathfrak{C}$, we need only finitely many points $\mathbf{x}_i$ such that the corresponding curves, let us denote them C'_i, induce a covering of $\mathfrak{C}$. Then a standard argument about RZ spaces implies that there exists modifications $C''_i \to C'_i$ such that all C''_i's glue to a single scheme C'', which is automatically a modification of C. The only our problem is that we can loose semi-stability of C'_i's. By proposition 3.2.23, each C''_i admits a semistable modification, hence it admits a stable modification $\overline{C}_i$ by proposition 3.2.10. Moreover, all stable modifications of C''_i are isomorphic because they are dominated by a larger semistable modification (by using 3.2.23 again). It follows that the stable modifications $\overline{C}_i \to C''_i$ agree over intersections $C''_i \cap C''_j$, hence we obtain a stable modification $\overline{C} \to C''$ and a semistable modification $\overline{C} \to C$. □

It remains to treat the divisors, and it is done in the following exercise.

Exercise 3.2.27. Let C be a semistable R-curve with smooth C_η and a divisor D whose generic fiber D_η is K-smooth (i.e. the generic point of D are $\mathrm{Spec}(K)$).

(i) Show that there exists a semistable modification C' of C which separates the irreducible components of D. (Hint: first find any modification which separates the components, and then refine it.)

(ii) Show that the strict transform $D' \hookrightarrow C'$ of D is isomorphic to the disjoint union of copies of $\mathrm{Spec}(K)$ and $\mathrm{Spec}(R)$ ($\mathrm{Spec}(K)$ can appear only if C is not proper).

(iii) Show that if a component $D_i = \mathrm{Spec}(R)$ hits the singular locus of C_s at a point x, then by an additional blow up at x one can achieve that C is still semistable and D_i lies in the smooth locus of C/S. (Hint: use the same blow up as in exercise 3.2.24.)

Combining this exercise with the above propositions, we see that to prove semistable modification theorem for (C, D) it suffices to prove that any valuation ring $\mathcal{O}_{\mathbf{x}} \in \mathfrak{C}$ can be uniformized by a strictly semistable R-curve C' with $C'_\eta \hookrightarrow C_\eta$. Clearly, it suffices to prove this for points $\mathbf{x} \in \mathfrak{C}_s$.

3.2.5. *Reduction to uniformization of one-dimensional valued fields over K.* Any point $\mathbf{x} \in \mathfrak{C}_s$ provides L with a structure of valued K-field, which is also finitely generated and of transcendence degree one. We say that such an L is *uniformizable* if it is unramified over its subfield of the form $K(x)$.

Proposition 3.2.28. *Let $\mathbf{x} \in \mathfrak{C}_s$ be such that the field L provided with the valuation corresponding to $\mathbf{x}$ is uniformizable. Then there exists a strictly semistable curve C' and a separated morphism $C' \to C$ which induces an open immersion on generic fibers and such that $\mathbf{x}$ is centered on C'.*

Proof. We have an etale homomorphism $\phi : K(t)^\circ \to L^\circ = \mathcal{O}_{\mathbf{x}}$. Let $\overline{C}_i$ be the family of all strictly semistable affine models of $K(t)$ such that $K(t)^\circ$ is centered on a point $\overline{x}_i \in \overline{C}_i$. This family is obviously non-empty, and then it follows from proposition 3.2.23 that $K(t)^\circ = \cup_i \overline{\mathcal{O}}_i$ for $\overline{\mathcal{O}}_i = \mathcal{O}_{\overline{C}_i, \overline{x}_i}$. By approximation, for $i \geq i_0$ we have etale homomorphisms $\phi_i : \overline{\mathcal{O}}_i \to \mathcal{O}_i$ which induce ϕ, in particular, $\cup \mathcal{O}_i = \mathcal{O}_{\mathbf{x}}$. Let x be the center of $\mathbf{x}$ on C, and find $f_1, \dots, f_n$ such that $\mathcal{O}_{C,x}$ is a localization of its subring $R[f_1, \dots, f_n]$. If we take i large enough then $f_1, \dots, f_n \in \mathcal{O}_i$, and therefore $\mathcal{O}_{C,x} \subset \mathcal{O}_i$.

Exercise 3.2.29. Deduce that for sufficiently small neighborhood $\overline{C}'_i$ of x_i, ϕ_i extends to an etale morphism $C' \to \overline{C}'_i$ where C' is a model of L such that $\mathbf{x}$ is centered on C' and the isomorphism of the generic points extends to a morphism $C' \to C$ with $C'_\eta \hookrightarrow C_\eta$.

Since $\overline{C}'_i$ is strictly semistable, C' is strictly semistable, and shrinking C' around the center of $\mathbf{x}$ we can achieve that the morphism $C' \to C$ is separated. ☐

The proposition reduces the local uniformization of C to uniformization of one-dimensional valued extensions of K. So, we actually reduced the stable modification theorem to a purely valuation theoretic problem.

3.3. **Uniformization of one-dimensional valued fields.**

3.3.1. *Reduction to uniformization of analytic fields.* Note that the completion $\widehat{K}$ is separably closed, but any complete separably closed field is algebraically closed: indeed any inseparable polynomial becomes separable after an arbitrary small perturbation. So, in the sequel $k = \widehat{K}$ denotes an algebraically closed analytic field. By a one-dimensional analytic field over k we mean any analytic k-field l finite over a subfield $\widehat{k(x)}$. We say that l is *uniformizable* if one can achieve that $l/\widehat{k(x)}$ is unramified.

Proposition 3.3.1. *Let K be a separably closed valued field of height one and L/K be a finitely generated separable extension of valued fields of transcendence degree one. The valued K-field L is uniformizable if and only if the analytic $\widehat{K}$-field $\widehat{L}$ is uniformizable.*

Proof. The direct implication follows from the criterion of unramifiedness. The same criterion implies that to prove the opposite it is enough to find $x \in L$ such that l is unramified over $\widehat{k(x)}$ and such that L is separable over $k(x)$. By our assumption, l is unramified over $\widehat{k(x)}$ and x can approximated with any precision by elements $y \in L$ and such that $L/K(y)$ is separable. So, proving the following claim will finish the proof: for any $y \in l$ with $|y - x| < \inf_{a \in k} |a - x|$ the extension $l/\widehat{k(y)}$ is unramified. We will prove below even a more general lemma. ☐

Lemma 3.3.2. *Assume that l is one-dimensional over an algebraically closed analytic field k. Then for any $x \in l$ with $x \notin k$ the extension $l/\widehat{k(x)}$ is finite, and for any $y \in l$ with $|y - x| < \inf_{a \in k} |a - x|$ the fields $\widehat{k(x)}$ and $\widehat{k(y)}$ are isomorphic l is of the same degree over both these fields.*

We outline the proof in the exercise.

Exercise 3.3.3. Choose $r<1$ such that $|y-x|<r\inf_{a\in k}|a-x|$.

(i) Prove that for any rational function $f(T)$ one has that $|f(x)-f(y)|<r|f(x)|$, and deduce that there is a unique isomorphism $\phi:\widehat{k(x)}\widetilde{\to}\widehat{k(y)}$ taking x to y, and this ϕ satisfies $|z-\phi(z)|<r|z|$ for any $z\in\widehat{k(x)}$.

(ii) Find a basis $z_1,\dots,z_n$ of l over $\widehat{k(x)}$ which is orthogonal up to a factor of r' with $r<r'<1$, or r'-*orthogonal*, in the sense that $|\sum_{i=1}^n a_iz_i|\geq r'\max_{1\leq i\leq n}|a_iz_i|$.

(iii) Prove that z_i generates l over $\widehat{k(y)}$, hence $[l:\widehat{k(x)}]\geq[l:\widehat{k(y)}]$, and by symmetry this is actually an equality.

To explain the importance of r-orthogonality we give one more easy exercise.

Exercise 3.3.4. Show that a finite extension l/k of analytic fields is defectless (i.e. satisfies $ef=n$) if and only if it admits an orthogonal basis (with $r=1$).

The exercises imply that if there exists a non-trivial defect, then we cannot find an orthogonal basis (which describes the extension in the best possible way), but at least there exist arbitrary close approximations to such a basis. Sometimes, one says that l/k is Cartesian if it admits an orthogonal basis, and it is weakly Cartesian if it admits r-orthogonal basis for any $r<1$.

Exercise 3.3.5. Let L/K be an extension of valued fields of height one, and assume that the valuation of K admits exactly one extension to L (i.e. $L^\circ=\mathcal{N}r_L(K^\circ)$). Prove that L/K is weakly Cartesian if and only if $L\otimes_K\widehat{K}$ is a field (it is automatically local, but can be non-reduced if $\widehat{K}/K$ is not separable).

3.3.2. *Analytic uniformization: setup and stability theorem.* In the sequel k is an algebraically closed analytic field and l is a one-dimensional field we want to uniformize. There are three possibilities which we will call E, F and D cases accordingly to the non-zero invariant of l/k. Clearly, the first two cases should be somewhat easier, but we will see that not too much easier. We will see that all three cases are proved by controlling the defect, but the control will be different. We also remark that in E and D cases the uniformization actually means that $l=\widehat{k(x)}$ because $\widetilde{l}=\widetilde{k}$ is algebraically closed. In the E and F cases, the uniformization theorem is equivalent to the following stability theorem.

onedimstab **Theorem 3.3.6.** *If k is an algebraically closed analytic field and l is a one-dimensional field of type E or F, then l is stable.*

Indeed, if the stability theorem is known then to uniformize l we simply choose x such that $|x|$ generates $|l^\times|$ in the E-case and $\widetilde{x}$ is the separable transcendence basis of $\widetilde{l}$ in the F-case. The converse follows from the following exercise.

Exercise 3.3.7. Check by a direct computation that if l is uniformizable, $x\in l\setminus k$ and we are in E or F case, then the extension $l/\widehat{k(x)}$ is defectless.

The above argument shows that if $p=\mathrm{char}(\widetilde{k})$ is zero, then E and F cases are established. Also, in the D case there is nothing to prove because $\widehat{k(x)}$ is algebraically closed for any $x\in l$ (any its algebraic extension is immediate), hence automatically $l=\widehat{k(x)}$ for any $x\notin k$. In general, however, type D field are not algebraically closed.

Exercise 3.3.8. Construct examples of non-stable l's of type D both in mixed and positive characteristics.

3.3.3. *How to gain a certain control on immediate extensions.* Though the result and the method of this section are very natural, I consider them as one of two central results in the uniformization of one-dimensional analytic extensions. In this section we give a fine enough description of an immediate extension of analytic fields l/k of degree p. The idea is very simple: although l/k does not have an orthogonal basis, at least it has r-orthogonal ones, and taking r close enough to 1 we should obtain a good description of l.

Exercise 3.3.9. Let l/k be of degree p (maybe not immediate) and $\alpha \in l \setminus k$. Prove that by choosing $c \in k$ such that $|c-\alpha|$ is close enough to $r_\alpha := \inf_{c\in k} |c-\alpha|$ one achieves that $1, \alpha, \dots, \alpha^{p-1}$ is an r-orthogonal basis for any given $r<1$. Show that $r=1$ is achieved if and only if $|c-\alpha|$ is the infimum and the latter is possible if and only if l/k is defectless.

It turns out that the minimal polynomial $f_c(T)$ of $\alpha - c$ becomes of a rather special form as $|\alpha - c|$ approaches r_α. The following proposition generalizes the obvious case when l/k is defectless.

Proposition 3.3.10. *Let l/k and α be as above, and for $c \in k$ let $f_c(T)$ be of the form $T^p + \dots - aT + b$. Then for $c \in k$ with $|\alpha - c|$ close enough to r_α one of the following possibilities hold:*

(i) $|\alpha^p + a| < s = \inf_{c\in k} |c^p + a|$ and $|pa| < s$;

(ii) $|\alpha^p - b\alpha + a| < s = \inf_{c\in k} |c^p - bc + a|$, $|b| = s^{\frac{p-1}{p}}$ and $|pa| < s$.

Moreover, in the second case l contains a root of $T^p - bT + a$, so $l \xrightarrow{\sim} k[T]/(T^p - bT + a)$.

Remark 3.3.11. The condition on $|pa|$ is essential only in the mixed characteristic case. It requires that already taking $c=0$ we are close enough to the infimum (or, that can be shown to be equivalent, $|\alpha|$ is already nearly orthogonal to k, i.e. $|\alpha - c|$ cannot be too much smaller than $|\alpha|$). This condition is very important in the mixed characteristic case because it allows to work with p-th powers in the additive form: $(c_1+c_2)^p = c_1^p + c_2^p + p(\dots)$, and we can remove all terms involving p because the absolute value is does not exceed s.

The proposition is proved by a rather straightforward studying of the coefficients of $f_c(T)$ given by the binomial formula. We leave it as a difficult exercise to complete the details, or consult [Tem, 2.1.3] instead. Note that it was natural to expect that a field k admits an immediate extension is there exists a polynomial $f(T)$ such that $\inf_{c\in k} |f(T)|$ is not achieved, but we obtain that one can take $f(T)$ to be of a very special form.

holecor **Corollary 3.3.12.** *A field k admits an immediate extension of degree p if and only if one of the following is true:*

(i) there exists $a \in k$ such that $s = \inf_{c\in k} |c^p + a|$ is not achieved and $|pa| < s$.

(ii) there exists $a, b \in k$ such that $s = \inf_{c\in k} |c^p - bc + a|$ is not achieved, $|b| = s^{\frac{p-1}{p}}$ and $|pa| < s$.

Exercise 3.3.13. Assume we are in the situation of the second case.

(i) Show that it never happens that $|b| > s^{\frac{p-1}{p}}$. (Hint: assume it does and deduce that the polynomial $T^p - aT + b$ has a solution in k then.)

(ii) Show that if $|b| < s^{\frac{p-1}{p}}$, then one can simply remove the $b\alpha$ term, in the sense that already the infimum $|k^p + a|$ is not achieved.

Remark 3.3.14. The exercise explains the role of the condition on $|b|$. Actually, this is indeed a boundary case in the sense that one can show that the extension l/k has zero different (or is almost etale as defined by Faltings) iff the case (ii) holds.

3.3.4. *On the proof of stability theorem.* The idea in both E and F cases is very simple: we should show that any one-dimensional k-field l of type E or F does not have "holes" described in 3.3.12. In other words, we should prove that for any element a the infimum $\inf_{c\in l}|a+c^p|$ is achieved and similarly for infimum of the form $\inf_{c\in l}|a-bc+c^p|$.

Exercise 3.3.15. Show that if l contains $b^{1/(p-1)}$ (as in our case), then it suffices to consider the case when $b=1$. (Hint: replace T with $b^{1/(p-1)}T$.)

Theorem 3.3.16. *If l is of one-dimensional of E or F type over an algebraically closed analytic field k, then l is stable.*

Proof. It suffices to prove that a field $\widehat{k(x)}$ of E or F-type is stable. If it is not, then there exists a Cartesian extension $K/\widehat{k(x)}$ with an immediate extension L/K of degree p. Moreover, by the above exercise replacing K with a tame extension we can find $a\in K$ such that either $s=\inf_{c\in K}|a+c^p|$ or $1=s=\inf_{c\in K}|a+c+c^p|$ is not achieved for $c\in K$ and also $|pa|<s$. Now, the strategy of the proof is very simple: we start with any $a\in K$ and modify it by replacing it with elements $a+c^p$ or $a+c+c^p$ of smaller absolute value until the infimum is achieved: we may do that because c^p is additive up to terms of negligible magnitude (if $|a+c^p|<|a|$ then $|pc^p|=|pa|<s$). The E case is technically easier because $\widehat{k(x)}\widetilde{\to}k\{r^{-1}T,rT^{-1}\}$ for $r=|x|\notin|k^\times|$ (i.e. any element of this field is of the form $\sum_{i=-\infty}^{\infty}a_ix^i$ with $|a_ix^i|$ tending to zero in both directions). It is obvious that removing from $a=\sum a_ix^i$ all terms with $p|i$ we obtain an element that cannot be decreased by adding a power of p, and a similar argument shows that by adding some element of the form c^p-c we can get rid of all elements with $p|i$. Since the remainder is still of absolute value larger than 1, it cannot be reduced further by adding elements of the form c^p-c, and we obtain that the infimum is attained. The F case is outlined in the following exercise, or you may consult [Tem, 2.2.4].

Exercise 3.3.17. (i)* Assume that L is Cartesian over an F field of the form $\widehat{k(x)}$. Show that there exists a Schauder basis $B=\{1\}\sqcup U\sqcup U^p\sqcup U^{p^2}\dots$ of L over k (i.e. any subset of B is orthogonal over k and B topologically spans L over k) and such that U is orthogonal to L^p (i.e. $|u-c^p|\geq|u|$ for any $u\in U$ and $c\in L$).

(ii) Deduce that L has no immediate extensions similarly to the E case. (Note that in the E case we used the Schauder basis $T^{\mathbf{Z}}$ with $U=T^{\mathbf{Z}}\setminus p\mathbf{Z}$).

□

4. Desingularization by alterations

4.1. **The main theorem.** Using the stable modification theorem it is not difficult to prove that any integral algebraic variety can be desingularized by alteration. Moreover, we will work in a greater generality which which covers schemes over excellent curves such as $\mathrm{Spec}(Z)$ or $\mathrm{Spec}(\mathbf{Z}_p)$ (and surfaces if we know (or trust) that excellent surfaces admit desingularization). Let S be a noetherian base scheme and X be an integral S-scheme of finite type and with a closed subset $Z\subsetneq X$. We say that the pair (X,Z) is *desingularized* by an alteration $f:X'\to X$ if X' is regular and $Z' = f^{-1}(Z)$ is a normal crossing divisor. The following theorem (in a slightly different formulation) was proved by de Jong in [dJ2].

Theorem 4.1.1. *Assume that any S-pair (X, Z) as above and such that X is generically S-finite can be desingularized by an alteration (for example, X is excellent of dimension not exceeding 1). Then any S-pair (X, Z) as above can be desingularized by an alteration.*

Proof. Step 1. *Application of the stable modification theorem.* We can replace S with the Zariski closure of the image of X (with the reduces scheme structure), so assume that the structure morphism $\phi : X \to S$ is dominant. In particular, S is integral with generic point η. By Chow lemma we can modify X so that it becomes quasi-projective over S. In particular, ϕ factors into a composition of a dominant morphism $X \to S'$ and a dominant morphism $S' \to S$ such that S'_η is a curve. If we would know that S' satisfies the same assumption as S does, then the theorem would follow by induction on the dimension of the generic fiber X_η, because the generic fiber of X over S' has smaller dimension. Thus it suffices to prove the theorem for X's that are generically finite over S', and we can assume that X_η is a curve. Find a finite purely inseparable extension $K/k(S)$ such that $\mathrm{Nor}_K(X_\eta)$ is a smooth K-curve (the normalization of X_η is regular but can be non-smooth). Replace S and X with $\mathrm{Nor}_K(S)$ and $\mathrm{Nor}_{Kk(X)}(X)$, and update Z accordingly (the preimage of the original Z), then we achieve that X_η is normal but the new S satisfies the assumption of the theorem as well. By the same argument we can achieve that the divisor $Z_\eta = Z \cap X_\eta$ is η-smooth. Blowing up X along Z we achieve that Z is a divisor, and then we decompose it to a vertical component Z_v, which is the preimage of a divisor on S, and a horizontal component Z_h, which is the Zariski closure of Z_η. By the flattening theorem, replacing S with its modification and replacing X and Z_h with their strict transforms, we can achieve that (X, Z_h) is a multipointed S-curve, i.e. the structure morphisms become flat. Note also that we replace Z_v with its preimage, and then the new Z is the preimage of the old one. Since we worried to make the generic fiber (X_η, Z_η) nodal, the stable modification theorem implies that we can alter the base and then modify the multipointed curve so that (X, Z_h) becomes nodal over S.

Step 2. *Explicit resolution of few mild singularities.* Consider the vertical divisor Z_v and the singularity locus $\phi_{\mathrm{sing}} = (X/S)_{\mathrm{sing}}$ of ϕ, and let $D = \phi(Z_v \cup \phi_s ing)$. Enlarging Z_v we can assume that it is the preimage of D. By our assumption on S, there exists an alteration $S' \to S$ such that S' is regular, the preimage of D is a normal crossing divisor. Replacing S with S' and updating all the rest we achieve the following situation: S is regular, $(X, Z_h$ is S-nodal, $Z_v = \phi^{-1}(D)$ for a normal crossing divisor D and $\phi_{\mathrm{sing}} \subset Z_v$ is a normal crossing divisor. We will see that the singularities of the pair (X, Z) are very mild and can be resolved explicitly (actually, they are already very special toric singularities).

The singular locus of X is contained in the singular locus ϕ_{sing} which is unramified over S by lemma 1.7.3 and lives over D. So, X is regular in codimension one and Z. Since Z_h is disjoint with ϕ_{sing}, it is an S-etale scheme contained in the regular locus of X. Moreover, it follows that Z is a normal crossing divisor at each point of Z_h. Since, we will modify in the sequel only the singular locus ϕ_{sing}, we can forget about Z_h starting with this moment.

Exercise 4.1.2. Show that altering S one can achieve that

(i) ϕ_{sing} splits over S in the sense that any irreducible component of $\phi_{\text{sing}} \to S$ is mapped isomorphically onto a closed subscheme of S;

(ii) the self-intersection points in the S-fibers have rational tangent cone.

Let $x \in \phi_{\text{sing}}$ be a point and $s = \phi(x)$. Then by proposition 1.7.4 we have that $\widehat{\mathcal{O}}_{X,x} \widetilde{\to} \widehat{\mathcal{O}}_{S,s}[[u,v]]/(uv-\pi)$ for an element π which vanishes only along D. It follows that $\pi = a \prod \pi_i^{n_i}$ where π_i define the irreducible component of D locally at s and a is a unit. Replacing u with u/a we can get rid of a. We will first resolve singularities in codimension two; these are exactly the singularities a semistable curve over a DVR can have. This will be done by blowing up the components of D until $n_i = 1$. Choose a component D_1 of ϕ_{sing} and let X' be the blow up of X along D_1 taken with the reduced scheme structure. For any point $x \in D_1$ etale-locally the S-scheme C looks as $Y = \operatorname{Spec}(\mathcal{O}_{S,s})[u,v]/(uv - \pi_1^{n_1} \dots \pi_m^{n_m})$. Note that n_1 is an invariant of the singularity along D_1 because it is the same at x and at the generic point of D_1, moreover $n_1 > 1$ if and only if there is a singularity at the generic point ε of D_1, i.e. $D_1 \in X_{\text{sing}}$. (Note that $\mathcal{O}_{S,\phi(\varepsilon)}$ is a DVR with an uniformizer π_1 and the fiber over $\operatorname{Spec}(\mathcal{O}_{S,\phi(\varepsilon)})$ is a nodal curve with singularity at ε, so we are actually dealing with resolving singularities on nodal curves over DVR's; the singularity type at ε is actually A_ε). Since blow ups are compatible with blow up, the preimage of x in X' etale-locally looks as the blow up of Y along (u, v, π_1).

Exercise 4.1.3. Describe the blow up of Y by an explicit computation with charts of the blow up, and show that it is nodal over S, smooth over $S \setminus D$ and the invariant n_1 on the preimage of D_1 drops.

The exercise implies that by successive blowing up irreducible components of X_{sing} we achieve that there are no singularities in codimension 2, or, that is equivalent, the singularity at x is described etale-locally as $uv = t_1 \dots t_m$.

The remaining procedure will go by blowing up the ideals of the form (u, t_i). It turns out that the geometry of such blow ups is rather funny, so we prefer to work out the simplest example in the the following exercise.

Exercise 4.1.4. Let $X = \operatorname{Spec}(k[u,v,x,y])/(uv - xy)$ and p be its origin (i.e. $m_p = (u,v,x,y)$).

(i) Show that blowing up the maximal ideal of p one obtains a desingularization $X' \to X$ where the preimage of p is a surface.

(ii) Show that X has two smaller desingularizations X_1 and X_2 (usually called *small*) obtained by blowing up the Weil divisors (u,x) (or that is the same (v,y)) and (u,y), which are not Cartier divisors. Show that the preimage of p in X_i is a line, in particular, X_i cannot be obtained by a blow up of a subscheme sitting supported at p, even though X_i is mapped isomorphically on X outside of p.

(iii) Show that X' is the minimal mutual refinement of X_i's.

Note that X_1 and X_2 are absolutely symmetric, and the procedure of passing from X_1 to X_2 is a typical example of a flop (a birational transformation which is an isomorphism in codimension 1).

The desingularization of a general X is similar. Find a singularity x which is locally-etale of the form $uv = \pi_1 \dots \pi_m$ with maximal possible m. Let D_i be the component of D defined by vanishing of π_i, set $D' = \cap_{i=1}^m D_i$ and let E be the component of the preimage of D' in ϕ_{sing} that contains x. Then the singularity of X along E is etale-locally of the form $Y = \operatorname{Spec}(\mathcal{O}_{S,s})[u,v]/(uv - \pi_1 \dots \pi_m)$, in particular m is an invariant of the singularity.

Exercise 4.1.5. Describe the blow up of Y by an explicit computation with charts of the blow up, and show that it is nodal over S, smooth over $S\setminus D$ and the invariant m on the preimage of E drops.

Blowing up such E's successively we obtain in the end the situation with regular X. It remains to check that D_v is a normal crossing divisor, and we can check that $\phi^{-1}(D \cup D_v)$ is so. Locally D is given by $\pi_1 \ldots \pi_m = 0$ and etale-locally the pair X looks as $\mathrm{Spec}\mathcal{O}_{S,s})[u,v]/(uv - \pi_j)$ and Z_v is given by $\pi_1 \ldots \pi_m = uv\pi_1 \ldots \pi_{j-1}\pi_{j+1} \ldots \pi_m = 0$. Since u, v and π_i's excluding π_j form a regular family of parameters, Z_v is a normal crossing divisor, as claimed. □

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