DESINGULARIZATION OF QUASI-EXCELLENT SCHEMES OVER Q

MICHAEL TEMKIN

Plan

Notation

- All schemes are locally noetherian.
- k is our base; it can be a field, a ring or a scheme.
- $\mathrm{Bl}_{\mathcal{I}}(X) \to X$ is the blow up of X along an ideal $\mathcal{I} \subset \mathcal{O}_X$.
- Usually, X is a noetherian scheme with a closed subscheme Z, $X' \to X$ is a blow up and $Z' = Z \times_X X'$.
- Assume $\mathcal{O}_{X,x}$ is regular, then Z is *strictly monomial* at x if locally it is given by an equation $x_1^{n_1} \dots x_m^{n_m} = 0$ where $n_i \geq 0$ and $x_1, x_2, \dots$ is a regular sequence of parameters of $\mathcal{O}_{X,x}$, and Z is *monomial* if it is etale-locally strictly monomial, i.e. x_i's exist in $\mathcal{O}^{\mathrm{sh}}_{X,x}$. For example, $Z = \emptyset$ is strictly monomial.
- $(X,Z)_{\mathrm{sing}}$ (resp. $(X,Z)_{\mathrm{ssing}}$) is the locus of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not regular or Z is not a monomial divisor (resp. strictly monomial divisor) at x.
- $(X,Z)_{\mathrm{reg}} = X \setminus (X,Z)_{\mathrm{sing}}$ and $X_{\mathrm{reg}} = (X,\emptyset)_{\mathrm{reg}}$.

1. Successive desingularization

Theorem 1.1 (Hironaka, 64)**.** *If k is a local ring containing $\mathbf{Q}$ whose completion homomorphism $\phi : k \to \widehat{k}$ is regular (i.e. the geometric fibers of ϕ are regular) and X is integral of finite type over k and with a closed subscheme Z then:*

Successive desingularization: there exists a sequence of blow ups $X' = X_n \to \cdots \to X_1 \to X_0 = X$ with regular X' and such that the centers are regular and lie over X_{sing}.

Successive embedded desingularization: if X is regular then there exists a sequence of blow ups $X' = X_n \to \cdots \to X_1 \to X_0 = X$ with monomial $Z' = Z \times_X X'$ and such that the centers are regular and lie over Z_{sing}, in particular $(X', Z')_{\mathrm{reg}} = X'$.

Remark 1.2. (i) The proof goes by successive improving of singularities.

(ii) The main problem is in pasting local solutions (say, over open subschemes).

(iii) An involved induction argument is used to bypass the above difficulty, it leads to a non-constructive proof.

(iv) It is important for the proof to allow k which are not fields.

"Recent" simplifications 1.3 (A very incomplete list is [BM97], [Vil89], [Wł05], [Kol07]). *If k is a field of characteristic zero, then there exists a canonical (or functorial with respect to smooth morphisms) successive embedded desingularization.*

Remark 1.4. (i) Canonicity ensures an easy pasting and leads to simplified and constructive proofs.

(ii) One must be careful about regularity of the centers in the not embedded case (actually, some proofs involve non-regular centers in this case).

(iii) An easy argument by Bierstone-Milman shows that one can choose the centers of blow ups over $(X, Z)_{\mathrm{sing}}$ (which can be strictly smaller than $X_{\mathrm{sing}} \cup Z_{\mathrm{sing}}$).

2. Quasi-excellence

Definition 2.1 (Grothendieck, [EGA $\mathrm{IV_{II}}$], 1965)**.** A *Quasi-excellent* scheme is a scheme pasted from spectra of *quasi-excellent* rings, which in their turn are noetherian rings satisfying two conditions:

(G) For any prime $p \subset k$, the completion homomorphism $k_p \to \widehat{k}_p$ is regular.

(N) For any integral finitely generated k-ring A the set $\mathrm{Spec}(A)_{\mathrm{sing}}$ is closed.

Remark 2.2. (i) Grothendieck introduced excellent schemes by imposing an additional condition of being universally catenary. The word "quasi-excellent" was introduced later.

(ii) Grothendieck proved that it suffices to consider in (G) only completions $k_m \to \widehat{k}_m$ with a maximal m.

(iii) Grothendieck proved that quasi-excellence of X is inherited by schemes of locally finite type over X. The proof treats (G) and (N) conditions separately.

(iv) Grothendieck proved that for local rings it suffices to check the G-property only. Thus, Hironaka's base is a quasi-excellent local ring.

(v) A fundamental question asked by Grothendieck: does quasi-excellence survives completions? It suffices to prove that if k is quasi-excellent then $k[[T]]$ is. One of main problems is that the analog for the G-property (without (N)) is false.

(vi) In particular, it is difficult to define a reasonable notion of quasi-excellent formal schemes without knowing (v).

(vii) I was informed that Gabber established an affirmative answer to (v) (a proof is yet to be written).

Definition 2.3. By a *weak desingularization* of a scheme X we mean a proper birational morphism $f : X' \to X$ with a regular source.

Remark 2.4. Usually in the definition of a desingularization one requires also that f does not modify the regular locus of X.

Theorem 2.5 (Grothendieck, loc.cit.)**.** *If k is a locally noetherian scheme and any integral scheme of finite type over k admits a weak desingularization, then k is quasi-excellent.*

Conjecture 2.6 (Grothendieck, loc.cit.)**.** *The converse is probably true, i.e. any integral quasi-excellent scheme admits a desingularization.*

Remark 2.7. (i) Grothendieck stated without proof that Hironaka's proof applies to any noetherian quasi-excellent.

(ii) It was never checked in the published literature. Nevertheless, this statement has already been applied, e.g. for desingularization of affinoid algebras.

3. Desingularization of pairs and the main result

Definition 3.1. (i) We say that there is *resolution of singularities of pairs over* a scheme k if for any integral X of finite type over k and $Z \hookrightarrow X$, there exists a blow up $f : X' \to X$ with center supported on $T = X_{\mathrm{sing}} \cup Z_{\mathrm{sing}}$ and such that X' is regular and $Z' = Z \times_X X'$ is monomial.

(ii) A desingularization of pairs is *strict* (resp. *semi-strict*) if one can, furthermore, take $T = (X, Z)_{\mathrm{sing}}$ (resp. $T = (X, Z)_{\mathrm{ssing}}$).

Remark 3.2. (i) The resolution is simultaneous rather than successive, the center of f is usually very bad.

(ii) By [BM97], there is strict resolution of singularities of pairs over fields of characteristic zero.

(iii) If there is resolution of singularities of pairs over k, then it is quasi-excellent.

Main result 3.3. *In characteristic zero, resolution of singularities of pairs over fields implies resolution of singularities of pairs over quasi-excellent schemes.*

Remark 3.4. (i) The assertion of the theorem holds also for semi-strict resolution of singularities of pairs.

(ii) The strict case should hold true as well, but it is not proved so far.

In the sequel, we will mainly consider the particular case of $Z = \emptyset$ for the sake of simplicity.

4. Localization and reduction to Hironaka

Proposition 4.1. *There is resolution of singularities over k if and only if for any local integral scheme Y of essentially finite type over k and its blow up $f : Y' \to Y$ with $Y'_{\mathrm{sing}} \subset f^{-1}(y)$ for the closed point $y \in Y$, there is a desingularization $Y'' \to Y'$.*

Sketch of proof. The direct implication is easy. To prove the opposite one we would like to desingularize an integral X of finite type over k. We will build a resolving blow up $f : X' \to X$ by decreasing noetherian induction on the closed set $S = f(X'_{\mathrm{sing}})$. Choose a generic point $x \in S$ and set $X_x = \mathrm{Spec}(\mathcal{O}_x)$ and $X'_x = X_x \times_X X'$. Then $(X'_x)_{\mathrm{sing}}$ sits over x, so we can find a desingularization $X''_x = \mathrm{Bl}_{\mathcal{I}'_x}(X') \to X'_x$, where $\mathcal{I}'_x$ is supported on the preimage of x. Our aim is to extend the blow up $g_x : X''_x \to X'_x$ to a blow up $g : X'' \to X'$

$$\begin{tikzcd} X''_x \arrow[r, "g_x"] \arrow[d, hook, dashed] & X'_x \arrow[r, "f_x"] \arrow[d, hook] & X_x \arrow[d, hook] \\ X'' \arrow[r, dashed, "g"] & X' \arrow[r, "f"] & X \end{tikzcd}$$

Using [EGA I], we can extend $\mathcal{I}'_x$ to an ideal $\mathcal{I}' \subset \mathcal{O}_X$ sitting over the Zariski closure of x. Then $X'' = \mathrm{Bl}_{\mathcal{I}'}(X')$ is isomorphic to X' over $X \setminus S$ and desingularizes the preimage of x, hence the image of X''_{sing} in X lies in $S \setminus \{x\}$. Since the composition $h : X'' \to X' \to X$ is a blow up with center at X_{sing} by Raynaud-Gruson, we see that the induction works fine. $\square$

Corollary 4.2. *Hironaka's work implies desingularization of any noetherian quasi-excellent scheme over* $\mathbf{Q}$.

Corollary 4.3. *To prove the main result it suffices to desingularize schemes* X *such that* $Z = X_{\mathrm{sing}}$ *is an algebraic variety (i.e. of finite type over a field).*

Remark 4.4. (i) The proposition works in any characteristic.

(ii) A funny observation is that the proposition applies even when the dimension of X and X_{sing} is infinite. I do not expect any application for this case, but it indicates that we are on the right way: the proposition reduces the case of any X_{sing} (including the infinite dimensional ones) to the case of an algebraic variety.

(iii) The induction from the proof of the proposition does not simplify local structure of singularities, but it does simplify their global structure.

(iv) One of the advantages of general blow ups is that one can easily extend them. We loose successiveness of the desingularization when extending $\mathcal{I}'_x$.

(v) The proposition is partially motivated by Raynaud's theory of formal blow ups.

5. Reduction to desingularization of formal schemes

Idea: a quasi-excellent scheme X may have no good floor, but it has such a roof (or roofs) $\widehat{X}_T$, and it may share a roof with much better schemes, e.g. algebraic varieties. So, the idea is to pass to a completion and then to algebraize it in such a form that the initial desingularization problem reduces to the case of varieties. Quasi-excellence should be used to connect desingularizations of X and of its roof.

Example: if $\widehat{A}$ is a complete local ring such that $\widehat{X} = \mathrm{Spec}(\widehat{A})$ has an isolated singularity, then the latter is algebraizable due to Artin. It allows to resolve isolated singularities.

Remark 5.1. (i) In general, we will need a more general algebraization result due to Elkik.

(ii) We prefer to define roofs as formal schemes.

Definition 5.2. (i) We say that a formal scheme $\mathfrak{X}$ is *absolutely quasi-excellent* if for any open affine subscheme $\mathrm{Spf}(A)$ the ring A is quasi-excellent. We say that $\mathfrak{X}$ is *quasi-excellent* if it admits an open covering by absolutely quasi-excellent subschemes.

(ii) We say that $\mathfrak{X}$ is *special* if $\mathfrak{X}_s$ is a variety.

Theorem 5.3 (Valabrega, 1976)**.** *Special formal schemes are excellent.*

Remark 5.4. (i) I was informed that Gabber proved that an I-adic noetherian ring A is quasi-excellent iff A/I is so (a proof is yet to be written).

(ii) This result is an ultimate result on quasi-excellence of formal schemes. In particular, it eliminates any need to introduce absolute quasi-excellence.

Definition 5.5. (i) If $\mathfrak{X} = \mathrm{Spf}(A)$ is absolutely quasi-excellent, $X = \mathrm{Spec}(A)$ and $I \subset A$ defines the subscheme X_{sing}, then we set $\mathfrak{X}_{\mathrm{sing}} = \mathrm{Spf}(A/I)$. This definition globalizes for quasi-excellent formal schemes!

(ii) We say that $\mathfrak{X}$ is *regular* (resp. *rig-regular*) if $\mathfrak{X}_{\mathrm{sing}} = \emptyset$ (resp. is defined by an open ideal).

(iii) By a *desingularization* of a rig-regular formal scheme $\mathfrak{X}$ we mean a formal blow up $\mathfrak{X}' \to \mathfrak{X}$ along an open ideal such that $\mathfrak{X}'$ is regular.

Proposition 5.6. *If X is quasi-excellent and $Z = X_{\mathrm{sing}}$ is a variety, then $\mathfrak{X} = \widehat{X}_Z$ is a special rig-regular formal scheme and any desingularization $\mathfrak{X}' \to \mathfrak{X}$ leads to a desingularization of X.*

Proof. Any open ideal on $\mathfrak{X}$ algebraizes to an ideal on X, and blow ups are taken by completions to formal blow ups. $\square$

6. Reduction to varieties

Theorem 6.1. *In characteristic zero, desingularization of rig-regular special formal schemes follows from desingularization of varieties.*

Main points of the proof are as follows.

- Blowing up an ideal of definition we can make it locally principal.
- A decreasing noetherian induction argument similar to an argument used earlier allows to localize the problem. In particular, it suffices to desingularize an affine rig-regular special formal scheme $\mathfrak{X}$ with a principal ideal of definition.
- A special affine formal scheme with a principal ideal of definition is of finite type over $k[[\pi]]$ for appropriate choice of a field k.
- In characteristic zero, a rig-regular $\mathfrak{X}$ is actually rig-smooth over $k[[\pi]]$.
- By Elkik's theorem, if affine $\mathfrak{X}$ is rig-smooth then it is algebraizable (e.g. is isomorphic to a completion of a scheme X of finite type over $k[\pi]$).
- Finally, any desingularization of X gives rise to a desingularization of $\mathfrak{X}$.

7. Complements. A: some details on desingularization of pairs

Remark 7.1. Elkik's theorem does not treat algebraization of rig-monomial divisors (though, probably, some results in that direction can be proven by similar methods). Therefore, the previous proof does not carry over straightforwardly to the case of pairs. The following proposition which monomializes strict transform of a closed subscheme allows to treat the general case as well.

Proposition 7.2. *If there is resolution of singularities of pairs over k up to dimension d, X is of finite type over k of dimension d and Z is a closed subscheme, then there exists a blow up $X' \to X$ with center in $(X,Z)_{\mathrm{ssing}}$ and such that $Z \times_X X'$ is strictly monomial along the strict transform of Z.*

Remark 7.3. (i) The proposition is valid without any restriction on the characteristic.

(ii) The proposition allows to obtain semi-strict desingularization of pairs from the desingularization results of [Vil89], [Wł05], [Kol07], and other works (where semi-strictness is not studied).

(iii) It seems certain that one can replace $(X,Z)_{\mathrm{ssing}}$ and strict monomiality with $(X,Z)_{\mathrm{sing}}$ and monomiality in the proposition. In this case one would also deduce strict desingularization of pairs of quasi-excellent schemes of characteristic zero.

8. Complements. B: Questions for further study

- Can one use a similar method to reduce desingularization of quasi-excellent schemes to the case of schemes of finite type over a quasi-excellent DVR? (The hope is yes.)
- Can one prove successive desingularization of quasi-excellent schemes by a similar method? (Yes.)
- Can one desingularize not rig-regular formal scheme by a similar method? (The main difficulty is that non-open ideals on an open formal subscheme may not admit an extension to the whole formal scheme. Most probably, one has to use functorial desingularization.)
- Can one prove functorial desingularization of quasi-excellent schemes by a similar method? (Work in progress. Most probably the answer is positive. An analog of localizing proposition has been proved. Algebraization of special formal schemes requires a more delicate treatment since it is absolutely not canonical. One has to include in the argument some knowledge about the desingularization algorithms rather than use them as a black box (as opposed to our method in this lecture). Functoriality proved in [BM97],[Vil89],[Wł05] or [Kol07] does not suffice for a straightforward argument.)

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