RELATIVE RIEMANN-ZARISKI SPACES AND NAGATA COMPACTIFICATION

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Plan

Setup

- In the sequel, RZ means Riemann-Zariski and qcqs means "quasi-compact and quasi-separated".
- Unless said to the contrary, $f: Y \to X$ is a separated morphism between qcqs schemes.

Definition 0.1. (i) A Y-modification of X is a proper morphism $X' \to X$ with a schematically dominant X-morphism $Y \to X'$.

(ii) The relative Riemann-Zariski space attached to f is $\mathrm{RZ}_Y(X) := \operatorname{proj\,lim}(X', \mathcal{O}_{X'})$, where the limit is taken in the category of locally ringed spaces over all Y-modifications of X.

Remark 0.2. (i) The projective family of all Y modifications of X is filtered with final object corresponding to the schematic image of f.

(ii) $\mathfrak{X} = \mathrm{RZ}_Y(X)$ is a nice locally ringed space, which is usually not a scheme.

(iii) Motivation for introducing $\mathfrak{X}$, its description and applications will follow.

1. Classical RZ spaces

- We start with the semi-classical case when $Y = \mathrm{Spec}(K)$ is a point and f is schematically dominant.
- For example, if $K = k(X)$ (resp. $K = k(X)^a$), then K-modification of X is a modification (resp. alteration).

1.1. **Valuative description of** $\mathrm{RZ}_K(X)$.

Definition 1.1. (i) Let $\mathrm{Val}_K(X)$ be the set of pairs $(\mathcal{O}, \phi)$, where $\mathcal{O}$ is a valuation ring *of* K (i.e. $\mathrm{Frac}(\mathcal{O}) = K$), and $\phi : \mathrm{Spec}(\mathcal{O}) \to X$ is a morphism compatible with f.

(ii) One can easily provide $\mathrm{Val}_K(X)$ with a natural structure of a locally ringed space such that the local ring at $(\mathcal{O}, \phi)$ is $\mathcal{O}$.

(iii) For example, the sets $\mathrm{Val}_K(\mathbf{Z}[f_1, \ldots, f_n])$ generate the topology of $\mathrm{Val}_K(\mathbf{Z})$ and $\mathcal{O}(\mathrm{Val}_K(\mathbf{Z}[f_1, \ldots, f_n]))$ is the integral closure of $\mathbf{Z}[f_1, \ldots, f_n]$ in K.

By the valuative criterion of properness, the projection $\mathrm{Val}_K(X) \to X$ (given by taking the image of the closed point under ϕ) lifts to any K-modification X' of X. So, a map $\lambda : \mathrm{Val}_K(X) \to \mathrm{RZ}_K(X)$ arises.

Theorem 1.2. *λ is an isomorphism of locally ringed spaces.*

Remark 1.3. (i) The proof follows from the following "Chow lemma": for any $h \in K$ there exists a K-modification $X' \to X$ such that h induces a regular function $X' \to \mathbf{P}^1_{\mathbf{Z}}$. Hence for compatible sequence of points $x_i \in X_i$ on K-modifications, $\cup_i \mathcal{O}_{X_i, x_i}$ contains either h or h^{-1} and hence is a valuation ring of K. This gives an inverse of λ.

(ii) To prove the "Chow lemma" we build a "K-blow up" $X' \to X$ by taking X' to be the schematic image of $(f, h) : \mathrm{Spec}(K) \to X \times \mathbf{P}^1_{\mathbf{Z}}$.

1.2. **Historical remarks.**

- In 1930ies Zariski associated to a finitely generated field extension K/k a Riemann space $\mathrm{RZ}_K(k)$ and noticed that it is the projective limit of all proper k-models of K.
- Zariski used these spaces in his desingularization works. In particular, he used it for desingularization of threefolds of characteristic zero.
- Nagata used the Riemann-Zariski (or Zariski-Riemann) spaces to establish the following compactification theorem. His proof was very difficult for reading by other mathematicians. For this reason, one often uses the notion of compactifiable morphisms, though Nagata gives complete characterization of such morphisms.

Theorem 1.2.1 (Nagata compactification theorem)**.** *Let $f : Y \to X$ be a finite type morphism between qcqs schemes. Then f is separated iff it is compactifiable in the sense that it factors into a composition of a schematically dense open immersion $Y \hookrightarrow \overline{Y}$ with a proper morphism $\overline{Y} \to X$.*

1.3. **A categorical interpretation of** Val**.**

Definition 1.4. Fix a field k, and let K denote a varying k-field.

(i) Var_k is the category of integral k-varieties with a fixed dominant point $\mathrm{Spec}(K) \to X$.

(ii) bir_k is the category of local homeomorphisms $\mathfrak{X} \to \mathrm{Val}_K(k)$ with a qcqs topological space $\mathfrak{X}$.

Theorem 1.5. *The functor* Val *induces an equivalence of the category* Var_k *localized by* K*-modifications onto the category* bir_k*.*

Remark 1.6. (i) Again, the proof is easy and uses the "Chow lemma".

(ii) The theorem is an easy analog of Raynaud's theory. In particular, Val is an analog of the generic fiber in Raynaud's theory.

(iii) The theorem was applied in [T1] to study local structure of Berkovich analytic spaces and was used to prove that properness in rigid geometry is stable under compositions.

(iv) There should be a similar theory for general qcqs X's, but this was not checked.

2. Raynaud's theory

- Let k be a complete valued field of height 1, and k° be its ring of integers.
- Recall that to a formal finitely presented $\mathrm{Spf}(k^\circ)$-scheme $\mathfrak{X}$ one associates generic fiber $\mathfrak{X}_\eta^{\mathrm{rig}}$ (resp. $\mathfrak{X}_\eta^{\mathrm{an}}$, resp. $\mathfrak{X}_\eta^{\mathrm{ad}}$) which is a rigid (resp. analytic, resp. adic) space over k.

Theorem 2.1 (Raynaud)**.** *The generic fiber functor establishes an equivalence of the category of $\mathfrak{X}$'s localized by blow ups along open ideals onto an appropriate category of generic fibers (e.g. qcqs rigid spaces or compact strictly analytic spaces).*

Remark 2.2. (i) The main part of the proof (as I see it) is a strong Chow lemma which asserts that any η-modification $\mathfrak{X}' \to \mathfrak{X}$ (i.e. a proper morphism inducing an isomorphism $\mathfrak{X}'_\eta \to \mathfrak{X}_\eta$ of generic fibers) is dominated by a formal blow up $\mathfrak{X}'' \to \mathfrak{X}$.

(ii) The theorem indicates that there is strong analogy between formal schemes, non-Archimedean spaces and functor η and schemes, RZ spaces and functor Val. Moreover, $\mathfrak{X}_\eta^{\mathrm{ad}} \widetilde{\to} \operatorname{proj\,lim}_{\mathfrak{X}_i \to \mathfrak{X}} \mathfrak{X}_i$ and adic points are actually valuations (while analytic points are valuations of height one and rigid points are Zariski closed points in $\mathfrak{X}^{\mathrm{ad}}$).

(iii) The main property of blow ups which is used in the theory is that given an open immersion $U \hookrightarrow X$ any blow up $U' \to U$ can be extended easily to a blow up $X' \to X$ because one can extend the defining ideal by EGA I. I can generalize this statement to all modifications only using Nagata's theorem.

3. P-modification theorems

3.1. Projective limits of schemes.

Theorem 3.1 (EGA IV, §8)**.** *(i) any filtered projective family* $(X_\alpha, f_{\alpha,\beta})$ *of qcqs schemes with affine transition morphisms possesses a projective limit* X*, which respects the locally ringed spaces, i.e.* $|X|\widetilde{\to}\operatorname{proj\,lim}|X_\alpha|$ *and* $\mathcal{O}_X\widetilde{\to}\operatorname{inj\,lim}\operatorname{pr}_\alpha^{-1}(\mathcal{O}_{X_\alpha})$*.*

(ii) $\operatorname{inj\,lim}_\alpha(\text{f.p. stuff}/X_\alpha)\widetilde{\to}(\text{f.p. stuff}/X)$ *for* $\mathcal{O}_X$*-sheaves,* X*-schemes and* X*-morphisms.*

Theorem 3.2 (Thomason and Trobaugh, "EGA V")**.** *A scheme* X *is qcqs iff* $X\widetilde{\to}\operatorname{proj\,lim}X_\alpha$ *for* X_α*'s of finite type over* $\mathbf{Z}$*.*

Remark 3.1.1. (i) Since any quasi-coherent sheaf is a direct limit of finitely presented sheaves by EGA I, this theorem is easily equivalent to an a priori weaker claim that X is affine over a scheme X_0 of finite type over $\mathbf{Z}$.

(ii) Thomason gently noted that probably this theorem should have appeared in EGA V.

Theorem 3.3 ([T3], "EGA VI")**.** *Any (separated) morphism* $Y \to X$ *of qcqs schemes factors into a composition of an affine morphism* $Y \to X'$ *and a finite type (separated) morphism* $X' \to X$*.*

Remark 3.4. (i) This theorem can be proved similarly to Thomason's theorem.

(ii) Both proofs can be shorten if one imitates a certain patching argument from Raynaud's theory.

3.2. **Three P-modification theorems.**

- There is an interesting application of semi-classical RZ spaces to **P**-modification theorems. The latter are results of the following general form.
- Let **P** be a property of a morphism (e.g. having geometrically reduced fibers), and $X \to S$ be a morphism with integral S and which is **P** over the generic point $\eta \hookrightarrow S$. Then there exists a K-modification $S' \to S$ for some fixed $K \supseteq k(\eta)$ and a proper morphism $\psi : X' \to X \times_S S'$ subject to certain restrictions (usually, a modification) such that $X' \to S'$ is **P**.
- We illustrate this too uncertain formulation by three theorems.

Theorem 3.5 (Flattening Theorem by Raynaud-Gruson)**.** **P** *is flatness,* $K = k(S)$ *and* ψ *is the proper transform (i.e.* X' *is the schematic closure of* $X_\eta = X \times_S \eta$*).*

Theorem 3.6 (Reduced fiber Theorem by Bosch-Lütkebohmert-Raynaud)**.** **P** *is having geometrically reduced fibers,* $K = k(S)^s$ *and* ψ *is a finite modification.*

Theorem 3.7 (Semi-stable modification Theorem by de Jong)**.** **P** *is being a relative semi-stable curve,* $K = k(S)^s$ *and* ψ *is a modification.*

Remark 3.8. (i) Although these three theorems were originally proved differently, they can be attacked by a similar method as follows:

(1) Solve **canonically** when S is the spectrum of a valuation ring (this stage is different for each problem).

(2) For any $\mathcal{O} \in \mathrm{RZ}_K(S)$, use EGA IV, §8, to find a finitely presented over S approximation for the solution over $\mathrm{Spec}(\mathcal{O})$.

(3) Glue these solutions together by canonicity and more EGA IV, §8.

(ii) The new proof of 3.5 is due to Fujiwara-Kato, and the new proofs of 3.6 and 3.7 are in [T2]. Moreover, the obtained curve in 3.7 is canonical (or *stable*), so one improves the original theorem by de Jong.

3.3. Refined P-modification.

- Assume that $U \hookrightarrow S$ is open and $X \to S$ is $\mathbf{P}$ over U, then it is a natural wish to preserve the U-fibers.
- Can one solve the $\mathbf{P}$-modification problem so that $S' \to S$ is an isomorphism over U or is at least etale over U? Note that flattening and reduced fiber theorems were originally proved in such stronger form.
- Idea ([T3], Fujiwara-Kato): to deal with the U-case use $\mathrm{RZ}_U(S)$ instead of $\mathrm{RZ}_K(S)$.
- To implement this idea one needs the following explicit description of $\mathfrak{X} = \mathrm{RZ}_U(S)$.

Theorem 3.9. *Identify U with a subset of $\mathfrak{X}$ via the obvious embedding $U \hookrightarrow \mathfrak{X}$, then any point $\mathfrak{x} \in \mathfrak{X}$ possesses a unique minimal generalization $u \in U$, $m_u \subseteq \mathcal{O}_{\mathfrak{X},\mathfrak{x}} \subseteq \mathcal{O}_{U,u}$ and $R_\mathfrak{x} = \mathcal{O}_{\mathfrak{X},\mathfrak{x}}/m_u$ is a valuation ring of $k(u)$.*

Remark 3.10. (i) One could expect that the theorem implies that $\mathfrak{X}$ is a family of classical RZ spaces with generic points $u \in U$. However, we will see that the situation is more subtle.

(ii) I call such $\mathcal{O}_{\mathfrak{X},\mathfrak{x}}$ a *semi-valuation ring with semi-fraction ring* $\mathcal{O}_{U,u}$. It is given by a local ring $\mathcal{O}_{U,u}$ with a valuation ring $R_\mathfrak{x}$ of $\mathcal{O}_{U,u}/m_u$, or by a valuation $|\ | : \mathcal{O}_{U,u} \to \Gamma \cup \{0\}$ with kernel m_u.

(iii) The theorem is proved similarly to Raynaud's theory but using U-modifications and U-admissible blow ups (i.e. blow ups along centers disjoint from U).

(iv) $\mathrm{RZ}_U(S)$ is an algebraic analog of rigid (analytic or adic) spaces.

(v) Using the theorem, one easily strengthens the stable modification theorem so that it involves only base change $S' \to S$ which is etale over U (though, then one has to allow non-proper covers).

4. Relative RZ spaces

- So far, we studied $\mathfrak{X} = \mathrm{RZ}_Y(X)$ in the cases when f is an open immersion or Y is a point (but f does not have to be a monomorphism). It seems natural to study the case of more general f's.
- Let $i : Y \to \mathfrak{X}$ be the natural map, then we introduce a sheaf $\mathcal{M}_{\mathfrak{X}} := i_*(\mathcal{O}_Y)$ and notice that it is naturally an $\mathcal{O}_{\mathfrak{X}}$-algebra.
- Intuitively, the sheaf $\mathcal{M}_{\mathfrak{X}}$ can be considered as the sheaf of meromorphic functions on $\mathfrak{X}$. This agrees with the intuition in the cases when Y is a point or f is a monomorphism.
- The pair $(\mathcal{M}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ is an analog of the pair $(\mathcal{O}_X, \mathcal{O}_X^{\circ})$ (resp. $(\mathcal{O}_X, \mathcal{O}_X^{+})$) in rigid (resp. adic) geometry.

4.1. X-valuations on Y.

Definition 4.1. (i) An *X-valuation on Y* consists of a point $y \in Y$, a valuation ring R of $k(y)$ and a morphism $\mathrm{Spec}(R) \to X$ compatible with the rest. In other words, it is a valuative diagram

$$\begin{array}{ccc} y & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

The set of all X-valuations on Y is denoted $\mathrm{Spa}(Y, X)$ (an adic space of R. Huber when f is affine).

(ii) An X-valuation is *minimal* if the morphism $y \to Y \times_X \mathrm{Spec}(R)$ is a closed immersion. The set of all minimal X-valuations is denoted as $\mathrm{Val}_Y(X)$.

- Each X-valuation on Y can be naturally "cut" to a uniquely defined minimal X-valuation on Y. Thus a set-theoretical retraction $r : \mathrm{Spa}(Y, X) \to \mathrm{Val}_Y(X)$ arises.
- The valuative criterion of properness produces a natural projection $\mathrm{Spa}(Y, X) \to \mathfrak{X}$.
- It was a surprise for me that this map is not an isomorphism, but factors through r.
- Perhaps, this becomes less surprising after one observes that minimal X-valuative diagrams suffice to test properness via the valuative criterion. (This is not obvious, and requires a proof.)

4.2. **Main results for relative RZ spaces.**

Theorem 4.2. *(i) r is continuous;*

(ii) $i : Y \hookrightarrow \mathfrak{X}$ and any point $\mathfrak{x} \in \mathfrak{X}$ has a unique minimal generalization $y \in i(Y)$;

(iii) $\mathcal{O}_{\mathfrak{X}} \hookrightarrow \mathcal{M}_{\mathfrak{X}}$, and for any $\mathfrak{x}$ and y as above $\mathcal{O}_{\mathfrak{X},\mathfrak{x}}$ is a semi-valuation ring with the semi-fraction ring $\mathcal{O}_{Y,y} = \mathcal{M}_{\mathfrak{X},\mathfrak{x}}$;

(iv) $\mathrm{Val}_Y(X) \widetilde{\to} \mathfrak{X}$.

Remark 4.3. (i) All the above claims are proved together, and none is much simpler than another one.

(ii) Should have an analog of Raynaud's theory for valuative spaces $\mathrm{Val}_Y(X)$, but this was not checked.

4.3. **Decomposable morphisms.**

- We say that f is decomposable if it is a composition of an affine morphism $Y \to X'$ followed by a proper morphism $X' \to X$.
- The analysis of spaces $\mathrm{RZ}_U(X)$ can be easily generalized to the case when $f : Y \to X$ is decomposable, but U-admissible blow ups of X should be replaced with Y-blow ups of X'.
- If f is affine, a Y-modification $\overline{X} \to X$ is called a *Y-blow up* if there exists an X-ample $\mathcal{L}$ on $\overline{X}$ and a *Y-trivialization* $\varepsilon : \mathcal{O}_{\overline{X}} \to \mathcal{L}$ inducing an isomorphism after pulling back to Y.

Lemma 4.4. *Each Y-blow up $\overline{X} \to X$ is given by finitely generated $\mathcal{O}_X$-submodules $\mathcal{E}$ of $f_*(\mathcal{O}_Y)$ that contain* 1. *Namely, $\overline{X} \widetilde{\to} \mathrm{Proj}(\oplus_{n=0}^{\infty} \mathcal{E}^n)$, where $\mathcal{E}^n$ is the submodule of $f_*(\mathcal{O}_Y)$.*

- The unit section $1 : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ gives rise to ε.
- Y-blow ups are as convenient to work with as usual blow ups. In particular, one can extend them from an open subscheme, and prove for them a version of Chow lemma.
- Using Y-blow ups one can imitate Raynaud's theory.

4.4. **Decomposition theorem and Nagata compactification.** It turns out that the description of RZ spaces in the decomposable case is actually the general description of RZ spaces because of the following decomposition theorem.

Theorem 4.5. *Let $f : Y \to X$ be a morphism between qcqs schemes. Then f is separated iff it is a composition of an affine morphism $Y \to X'$ with a proper morphism $X' \to X$.*

- Although the formulation of the decomposition theorem seems to be new, it is equivalent to the union of Nagata compactification and "EGA VI" (the separated case).
- Thus, one can deduce the theorem from known results, and as a consequence obtain a description of a general RZ space.
- However, the interesting direction is the opposite one. Namely, one can describe RZ spaces directly, and deduce the decomposition theorem.
- I hope that the second approach can lead to a proof of the decomposition theorem for algebraic spaces.

Remark 4.6. It is very natural to look for counterexamples to the decomposition theorem. Here are two failing candidates:

(i) The scheme $\mathrm{Proj}(\mathbf{Z}[x_0, x_1, \dots, x_n, \dots])$ is not qcqs because it is covered by infinitely many affine charts.

(ii) The qcqs locally ringed space $(\mathbf{P}_{\mathbf{Z}})^{\mathbf{N}}$ is not a scheme because its points do not have affine neighborhoods.

4.5. **Main stages in direct description of RZ spaces.**

- If $Y' \hookrightarrow Y$ is an open immersion, $X' \to X$ is separated of finite type, and $f' : Y' \to X'$ is an X-morphism, then we have an embedding $i : \mathrm{Spa}(Y', X') \hookrightarrow \mathrm{Spa}(Y, X)$ by the valuative criterion of separatedness.
- The situation with Val's is more subtle since the minimality condition can be destroyed by i.
- It turns out that i induces an embedding $\mathrm{Val}_{Y'}(X') \hookrightarrow \mathrm{Val}_Y(X)$ iff $Y' \to Y \times_X X'$ is a closed immersion.
- Deligne following Nagata says that in such case Y' is a quasi-domination of Y over X', and this notion plays an important role in his proof of Nagata's theorem.
- One describes an RZ space $\mathfrak{X} = \mathrm{RZ}_Y(X)$ in two stages.
- Local stage.
  - At this stage for each point $\mathfrak{x} \in \mathfrak{X}$ one finds a neighborhood of the form $\mathfrak{X}_i = \mathrm{RZ}_{Y_i}(X_i)$ where Y_i is a quasi-domination of Y over X_i and $Y_i \to X_i$ is affine. Then using a quasi-compactness argument one is left with finitely many such quasi-dominations.
  - This stage is done by using EGA IV, §8 to approximate a valuative diagram with schemes of finite type.
  - This stage has no analog in Raynaud's theory, since each rigid space is automatically covered by finitely many affinoid ones (and the latter have affine formal models).
- Patching stage.
  - This stage is an imitation of Raynaud's theory.
  - We use that the decomposable case (including each $\mathfrak{X}_i$) is described very concretely using Y-blow ups.

Remark 4.7. (i) Although RZ spaces and their analogy with the Raynaud's theory helped a lot to find the above proof of the decomposition theorem, one can easily eliminate them from all formulations and arguments.

(ii) In such a proof, one simply approximates minimal valuative diagrams with affine quasi-dominations, and then patches them together after modifying them with appropriate Y-blow ups.

(iii) I hope that, up to small changes, the same proof should apply to algebraic spaces.

(iv) It also seems probable that there is an analog of relative RZ spaces for a morphism $\mathcal{Y} \to \mathcal{X}$ of algebraic spaces, which is the quotient of a relative RZ space by an etale equivalence relation.

References

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