Non-archimedean analytic spaces

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Introduction

The theory of non-archimedean analytic spaces was developed by Berkovich in two works in 1990 and 1993. Oversimplifying, its contribution to rigid non-archimedean geometry was in saturating Tate's rigid spaces with many "generic" points that made analytic spaces to honest topological spaces. One of major achievements of the theory was in constructing a theory of étale cohomology of such spaces.
Beyond generalizing the classical rigid geometry, this theory also makes sense over trivially valued and even archimedean fields. Moreover, geometric objects including both archimedean and non-archimedean worlds (e.g. $\mathbf{A}^n_{(\mathbf{Z},|\ |_\infty)}$) can be defined and studied.

Plan

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Semi-norms

- A **seminorm** on a ring A is a map $|\ |: A \to \mathbf{R}_+$ such that $|a+b| \leq |a| + |b|$, $|0| = 0$, $|-a| = |a|$ and $|ab| \leq |a||b|$.
- $|\ |$ is a **semivaluation** if $|ab| = |a||b|$ and $|1| = 1$.
- $|\ |$ is a **norm** (resp. a **valuation**) if its kernel is trivial.
- $|\ |$ is **non-archimedean** if $|a+b| \leq \max(|a|, |b|)$.
- A homomorphism $f : A \to B$ of normed rings is **bounded** if $|f(a)|_B \leq C|a|_A$.
- f is **admissible** if the residue seminorm on $f(A)$ is **equivalent** to the norm induced from B (i.e. both are bounded with respect to each other).

Analogous definitions make sense for normed modules over a normed ring.

Banach rings

Definition

- A **Banach ring** is a normed ring $(\mathcal{A}, |\ |)$ such that $\mathcal{A}$ is complete in the metric $d(a, b) = |a - b|$. Morphisms are bounded homomorphisms.
- The spectral semi-norm on $\mathcal{A}$ is the maximal bounded power-multiplicative seminorm ρ. Actually, $\rho(f) = \lim_{n\to\infty} |f^n|^{1/n}$.
- A Banach $\mathcal{A}$-module M is a complete **normed A-module** (i.e. $\|am\| \leq C|a|\|m\|$ for fixed C and all $a \in \mathcal{A}$, $m \in M$).
- **Complete tensor product** $M\widehat{\otimes}_{\mathcal{A}}N$ is the **separated completion** of $M \otimes_{\mathcal{A}} N$ with respect to the maximal seminorm such that $\|\sum_{i=1}^{l} m_i \otimes n_i\| \leq \sum_{i=1}^{l} \|m_i\|\|n_i\|$.

Examples

Here are few basic examples of Banach rings.

- $\mathbf{R}$ and $\mathbf{C}$ are the only archimedean Banach fields. The only multiplicative norms on them are $|\ |_\infty^r$ for $0 < r \le 1$.
- Some non-archimedean valued fields are: fields k with the trivial valuation $|\ |_0$ (i.e. $|x|_0 = 1$ for $x \neq 0$), $k(t)$ and $k((t))$ with the t-adic norm $|\ |_{t,s}$ normalized so that $|t|_{t,s} = s$, $s \in (0, 1)$, $\mathbf{Q}$ and $\mathbf{Q}_p$ with the p-adic norm $|\ |_{p,r}$ normalized so that $|p|_{p,r} = p^{-r}$, $r > 0$.
- The ring $\mathbf{Z}$ with the trivial norm $|\ |_0$ or an archimedean norm $|\ |_\infty^r$, $r \in (0, 1]$.
- Et cetera.

The definition

Definition

- For a Banach algebra $\mathcal{A}$ its **spectrum** $\mathcal{M}(\mathcal{A})$ is the set of all bounded semivaluations $|\ |_x : \mathcal{A} \to \mathbf{R}_+$. We denote such point x and write $|f(x)| = |f|_x$.
- $\mathcal{M}(\mathcal{A})$ is provided with the weakest topology such that the map $x \mapsto |f(x)|$ is continuous for any $f \in \mathcal{A}$.
- The **completed residue field** $\mathcal{H}(x)$ is the completion of $\mathrm{Frac}(\mathcal{A}/\mathrm{Ker}(|\ |_x))$. (Can view $f(x)$ as its element.) $\mathcal{H}(x)$ is an **analytic field**, i.e. a valued Banach field.

Remark

$x \in \mathcal{M}(\mathcal{A}) \Leftrightarrow$ an equiv. class of bounded char. $\chi_x : \mathcal{A} \to K$, where K is generated by the image of $\mathcal{A}$ as an analytic field.

First properties

- $\mathcal{M}$ is a functor: a bounded homomorphism $f : \mathcal{A} \to \mathcal{B}$ induces a continuous map $\mathcal{M}(f) : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ by composing the semivaluations on $\mathcal{B}$ with f.
- Spectrum of an analytic field is a point (but spectrum of a Banach field can be more complicated).
- If k is an analytic field and $\mathcal{A}$ is a Banach k-algebra then $\mathcal{M}(\mathcal{A}) \widetilde{\to} \mathcal{M}(\mathcal{A}\widehat{\otimes}_k k^a)/\mathrm{Gal}(k^s/k)$.

Fact

Let $\mathcal{A}$ be a Banach ring and $X = \mathcal{M}(\mathcal{A})$ then

- *X is compact.*
- *$X = \emptyset$ if and only if $\mathcal{A} = 0$.*
- *Maximum modulus principle: $\rho(f) = \max_{x \in X} |f(x)|$.*

The spectrum of $\mathbf{Z}$

- Ostrowsky's theorem classifies valuations on $\mathbf{Z}$ and it gives a simple description of $X = \mathcal{M}((\mathbf{Z}, |\ |_\infty))$.
- X is a tree consisting of the "central point" $|\ |_0$ and intervals that connect it to the leaves $|\ |_{p,\infty}$ and $|\ |_\infty$, where $|x|_{p,\infty} = 0$ if $(x, p) = p$ and $|x|_{p,\infty} = 1$ if $(x, p) = 1$.
- The open intervals consist of p-adic valuations $|\ |_{p,r}$ with $0 < r < \infty$ or archimedean valuations $|\ |_\infty^r$ with $0 < r < 1$, respectively.
- $\mathcal{M}((\mathbf{Z}, |\ |_0))$ is the non-archimedean part of X, it is obtained from X by removing the archimedean semi-open interval.

The definition

Definition

- The n-**dimensional affine space** over a Banach ring $\mathcal{A}$ is the set $\mathbf{A}^n_{\mathcal{A}} = \mathcal{M}(\mathcal{A}[T_1, \dots, T_n])$ of all real semivaluations on $\mathcal{A}[T]$ that are bounded on $\mathcal{A}$. Topology on $X = \mathbf{A}^n_{\mathcal{A}}$ is defined by functions $|f| : X \to \mathbf{R}_+$, $f \in \mathcal{A}[T]$.
- X is provided with the **structure sheaf** $\mathcal{O}_X$ of **analytic functions**: $f \in \mathcal{O}_X(U)$ is a map $f : U \to \coprod_{x \in U} \mathcal{H}(x)$ which locally at each $x \in U$ is a limit of rational functions $\frac{g}{h}$ with $g, h \in \mathcal{A}$ and $g(x) \neq 0$.

Remark

The fiber of $\mathcal{A}^n_{\mathcal{A}}$ over $x \in \mathcal{M}(\mathcal{A})$ is $\mathbf{A}^n_{\mathcal{H}(x)} \widetilde{\to} \mathbf{A}^n_{\widehat{\mathcal{H}(x)^a}} / \mathrm{Gal}(\mathcal{H}(x))$ (so $\mathbf{A}^0_{\mathcal{A}} \widetilde{\to} \mathcal{M}(\mathcal{A})$).

Affine lines over algebraically closed analytic fields

- $\mathbf{A}^n_{\mathbf{C}} \widetilde{\to} \mathbf{C}^n$ by the classification of archimedean fields.
- If $k = k^a$ is non-archimedean then $X = \mathbf{A}^1_k$ consists of the classical points parameterized by $a \in k$ (these are semivaluations on $k[T]$ with kernels $(T - a)$), and "generic" points with trivial kernels. To give the latter is the same as to extend the valuation on k to $k(T)$. For example, to each disc with center at a of radius r we associate its maximal point $p_{a,r}$ given by $|\sum a_i(T - a)^i| = \max |a_i| r^i$.
- X is a (sort of) infinite tree and classical points are its leaves. They are connected by maximal points of discs. If k is not spherically complete (i.e. there are nested sequences of discs without common classical points) then there are non-classical leaves too.

Analytic spaces over $\mathcal{A}$

Definition

A chart of an analytic space over $\mathcal{A}$ is the closed subspace of an open subspace $U \subseteq M = \mathbf{A}^n_{\mathcal{A}}$ given by vanishing of analytic functions $f_1, \dots, f_n \in \mathcal{O}_M(U)$. A general analytic space over $\mathcal{A}$ is a locally ringed space $(X, \mathcal{O}_X)$ obtained by gluing charts.

Remark

- The construction depends only on the **Banach reduction** of $\mathcal{A}$, in which we replace $|\ |$ with ρ and then replace $\mathcal{A}$ with its separated completion. In particular, $\mathbf{A}^0_{\mathcal{A}}$ is reduced, though $\mathcal{O}_X$ may have nilpotents.
- Only partial results are known. The structure sheaf of $\mathbf{A}^1_{\mathbf{Z}}$ has noetherian stalks (Poineau); the case of $\mathbf{A}^2_{\mathbf{Z}}$ is open.

The definition

From now on, we will discuss analytic geometry "of finite type" over a non-archimedean analytic field k. Building blocks will be spectra of Banach k-algebras of "finite type".

- $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} := \{\sum a_i T^i | \; a_i \in k, |a_i|r^i \to 0\}$ is the algebra of power series converging on a polydisc of radii $r_1, \dots, r_n$; it is Banach algebra with multiplicative (spectral) norm $\|\sum a_i T^i\|_r = \max |a_i| r^i$
- A k-affinoid algebra is an admissible quotient of $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$; it is strictly k-affinoid if one can take $r_i \in \sqrt{|k^\times|}$ (and then can even take $r_i = 1$).

Basic properties

Here are basic properties of affinoid algebras.

- Any affinoid algebra $\mathcal{A}$ is noetherian, excellent and all its ideals are closed.
- If $f \in \mathcal{A}$ is not nilpotent then there exists $C > 0$ such that $\|f^n\| \leq C\rho(f)^n$ for all $n \geq 1$ (so, $\|f^n\|^{1/n}$ converges to $\rho(f)$ very fast).
- If $\mathcal{A}$ is reduced then the Banach norm on $\mathcal{A}$ is equivalent to the spectral norm.
- The categories of finite $\mathcal{A}$-modules and finite Banach $\mathcal{A}$-modules are equivalent.

Strict and non-strict algebras

Here are two important results that hold only for strictly k-affinoid algebras $\mathcal{A}$.

- Hilbert Nullstelensatz: $\mathcal{A}$ possesses a k^a-point.
- Noether normalization: $\mathcal{A}$ is finite over a subalgebra isomorphic to $k\{T_1, \ldots, T_m\}$.

This is false for non-strict algebras because if $r \notin \sqrt{|k^\times|}$ then $K_r = k\{r^{-1}T, rT^{-1}\}$ is a field. by successive tensoring with K_r's, and this often allows to reduce proofs of various results (e.g. the results from the previous page) to the strict case.

Category of k-affinoid spaces

Definition

If $\mathcal{A}$ is a (strictly) k-affinoid algebra then $\mathcal{M}(\mathcal{A})$ is called a **(strictly) k-affinoid space**. A morphism $\mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is a map of the form $\mathcal{M}(\phi)$ for a bounded k-homomorphism $\phi : \mathcal{B} \to \mathcal{A}$. The category of k-affinoid spaces is denoted k-*Aff*.

We will later provide affinoid spaces X with a richer structure. In particular, we would like to introduce a structure sheaf $\mathcal{O}_X$. This requires to introduce a notion of affinoid subdomains $V \subseteq X$, which have to be compact and hence closed.

Affinoid domains

Definition

A subset V of $X = \mathcal{M}(\mathcal{A})$ is an **affinoid domain** if the category of *k-Aff* morphisms $f : Y \to X$ with $f(Y) \subseteq V$ has a universal object $i_V : \mathcal{M}(\mathcal{A}_V) \to X$ whose image is the whole V.

Fact

- *i_V is a bijection and $\mathcal{H}(i_V(z)) \widetilde{\to} \mathcal{H}(z)$, so we can identify V and $\mathcal{M}(\mathcal{A}_V)$.*
- *If $f_1, \dots, f_n, g \in \mathcal{A}$ have no common zeros and $r_i > 0$, then $X\{r^{-1}\frac{f}{g}\} = \{x \in X |\ |f_i(x)| \leq r_i|g(x)|\}$ is an affinoid domain called* **rational**, *and $\mathcal{A}_V = \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}/(f_i - gT_i)$.*
- *Gerritzen-Grauert: any affinoid domain is a finite union of rational ones.*

The structure sheaf

Definition

G**-topology** τ^c **of compact analytic domains** in $X = \mathcal{M}(\mathcal{A})$:
τ^c-open sets are finite unions of affinoid domains,
τ^c-admissible coverings are those that have a finite refinement.

Fact

- *Tate's acyclicity: the correspondence $V \mapsto \mathcal{A}_V$ uniquely extends to a τ^c-sheaf $\mathcal{O}_{X_G^c}$. For any finite Banach $\mathcal{A}$-module M, $\mathcal{O}_{X_G^c} \otimes_{\mathcal{A}} M$ is a sheaf of Banach $\mathcal{O}_{X_G^c}$-modules whose Čech complex is exact and admissible for finite (or τ^c-admissible) affinoid coverings $X = \cup_{i \in I} V_i$.*
- *Kiehl: the categories of finite Banach $\mathcal{A}$-modules and finite Banach $\mathcal{O}_{X_G^c}$-modules are equivalent.*

Nets

Definition

Let X be a topological space with a set of subsets T.

- T is a **quasi-net** if any point $x \in X$ has a neighborhood of the form $\cup_{i=1}^{n} V_i$ with $x \in V_i \in T$ for $1 \leq i \leq n$.
- A quasi-net T is a **net** if for any choice of $U, V \in T$ the restriction $T|_{U \cap V} = \{W \in T|\ W \subseteq U \cap V\}$ is a quasi-net.

Remark

The definition of nets axiomatizes properties an affinoid atlas should satisfy, and the definition of quasi-nets axiomatizes the properties of admissible coverings by analytic domains.

Analytic spaces with atlases

Definition

A **(strictly) k-analytic space** is a triple (X, A, τ) where X is a locally compact space, τ is a net of compact spaces and A is a **k-affinoid atlas on X with net τ**, which consists of a functor A from τ (viewed as a category with respect to embeddings) to (strictly) k-affinoid spaces and an isomorphism i of the topological realization functors $\tau \stackrel{A}{\to} k\text{-}Aff \to Top$ and $\tau \to Top$.

So, A associates a k-affinoid algebra $\mathcal{A}_U$ to $U \in \tau$ and i provides a homeomorphism $i_U : \mathcal{M}(\mathcal{A}_U) \widetilde{\to} U$ so that both A and i are compatible with embeddings $V \hookrightarrow U$ in τ.

Morphisms between k-analytic spaces

Definition

- A strong morphism $\phi : (X, A, \tau) \to (X', A', \tau')$ consists of a continuous map $X \to X'$, such that each $U \in \tau$ is mapped to some $V \in \tau'$, and of a family of compatible morphisms of k-affinoid spaces $\phi_{U,V} : \mathcal{M}(A_U) \to \mathcal{M}(A'_V)$ for each $U \in \tau$, $V \in \tau'$ with $f(U) \subseteq V$.
- ϕ is a quasi-isomorphism if ϕ is a homeomorphism and $\phi_{U,V}$ are embeddings of affinoid domains.
- The **category of k-analytic spaces** k-An is obtained from the category with strong morphisms by inverting the family of quasi-isomorphisms (which mainly reduces to refining atlases).

Compatibilities

Any affinoid space $X = \mathrm{Spec}(A)$ can be viewed as an analytic space with the trivial atlas $\tau = \{X\}$.

Fact

- *The functor k-Aff $\to$ k-An is fully faithful.*
- *The functor st-k-An $\to$ k-An is fully faithful.*

The first result reduces to Tate's acyclity theorem. The second one is surprisingly difficult (though it is easy for separated spaces); in particular, it implies that any two strictly analytic structures on an analytic space are equivalent.

Analytic domains

Definition

- A subset $Y \subseteq X$ is an **analytic domain** in (X, A, τ_0) if Y admits a quasi-net $\{V_i\}$ such that each V_i is an affinoid domain in some $U_i \in \tau_0$. Such Y acquires a natural structure of a k-analytic space.
- A covering of Y by analytic domains $Y_i \subseteq Y$ is **admissible** if $\{Y_i\}$ is a quasi-net on Y. By τ (resp. τ^c) we denote the G-topology of (resp. compact) analytic domains with admissible coverings.
- Y is **good** if any its point possesses a neighborhood which is an affinoid domain.

Structure sheaves

Fact

Let (X, A, τ_0) be an analytic space.

- *The correspondence $U \mapsto \mathcal{A}_U$ for $U \in \tau_0$ naturally extends to a τ^c-sheaf $\mathcal{O}_{X_G^c}$ of Banach k-algebras.*
- *$\mathcal{O}_{X_G^c}$ uniquely extends to a τ-sheaf $\mathcal{O}_{X_G}$ of k-algebras and the categories of coherent sheaves are equivalent $\mathrm{Coh}(\mathcal{O}_{X_G}) \widetilde{\to} \mathrm{Coh}(\mathcal{O}_{X_G^c})$. Actually, $\mathcal{O}_{X_G}(U) = \mathrm{Mor}_{k-An}(U, \mathbf{A}_k^1)$.*
- *The restriction $\mathcal{O}_X$ of $\mathcal{O}_{X_G}$ to the usual topology can be pathological, e.g. one may have $\mathcal{O}_{X,x} \widetilde{\to} k$ for non-discrete points x. But if X is good then $\mathcal{O}_X$ behaves well and $\mathrm{Coh}(\mathcal{O}_{X_G}) \widetilde{\to} \mathrm{Coh}(\mathcal{O}_X)$.*

GAGA

Fact

- *Assume $\mathcal{X}$ is a variety over k. For any good k-analytic space Y let $F_{\mathcal{X}}(Y)$ be the set of morphisms of locally ringed spaces $(Y, \mathcal{O}_Y) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then $F_{\mathcal{X}}$ is represented by a good k-analytic space $X = \mathcal{X}^{\mathrm{an}}$ called the* **analytification** *of $\mathcal{X}$.*
- *The construction is functorial, and $f : \mathcal{Y} \to \mathcal{X}$ is separated, proper, finite, immersion, isomorphism, etc., iff f^{an} is so.*
- *If X is proper or k is trivially valued, then analytification of modules induces an equivalence $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}) \widetilde{\to} \mathrm{Coh}(\mathcal{O}_{\mathcal{X}^{\mathrm{an}}})$. In particular, {finite X-schemes} $\Leftrightarrow$ {finite $\mathcal{X}^{\mathrm{an}}$-spaces}.*

Generic fibers of formal schemes

- An **admissible formal scheme** over the ring of integers k° of k is a flat finitely presented k°-scheme $\mathfrak{X}$, i.e. $\mathfrak{X}$ admits a finite union by formal schemes $\mathrm{Spf}(A)$, where $A = k^\circ\{T_1, \dots, T_n\}/(f_1, \dots, f_m)$ and $\pi^n \notin (f_1, \dots, f_m)$.
- **Closed fiber** $\mathfrak{X}_s$ is the reduction of $(\mathfrak{X}, \mathfrak{X}_s/(\pi))$ for $\pi \in k^\circ$ with $0 < |\pi| < 1$.
- **Generic fiber** of $\mathfrak{X} = \mathrm{Spf}(A)$ above is the affinoid space $\mathfrak{X}_\eta = \mathcal{M}(A \otimes_{k^\circ} k)$.
- η takes open immersions to embeddings of affinoid domains and hence extends to a functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$ between the category of k°-admissible formal schemes and the category of compact strictly k-analytic spaces.

Raynaud's theory

- If $f : \mathfrak{Y} \to \mathfrak{X}$ is finite, a closed immersion, an open immersion, etc., then f_η is finite, a closed immersion, embedding of analytic domain, etc. Furthermore, f is separated, proper iff f_η is so.
- Raynaud's theory: η takes the family $\mathcal{FB}$ of blow ups along open ideals to isomorphisms. Moreover, η is isomorphic to the localization by $\mathcal{FB}$, i.e. $k^\circ\text{-}Adm/\mathcal{B} \widetilde{\to} comp\text{-}st\text{-}k\text{-}An$.
- These categories are also equivalent to the categories of qcqs rigid k-analytic spaces and Huber's adic spaces of finite type over (k, k°).

Set-theoretical comparison

The objects of all four categories are related as follows. For simplicity we consider only the set-theoretical level.

- $|\mathfrak{X}_\eta^{\mathrm{ad}}| \widetilde{\to} \operatorname{proj\,lim}_{\mathfrak{X}_i \to \mathfrak{X}} |\mathfrak{X}_i|$ where the limit is taken over all admissible blow ups $\mathfrak{X}_i \to \mathfrak{X}$. Points of the adic generic fiber $\mathfrak{X}_\eta^{\mathrm{ad}}$ are continuous valuations of arbitrary heights.
- $\mathfrak{X}_\eta^{\mathrm{an}} \subseteq \mathfrak{X}_\eta^{\mathrm{ad}}$ consists of valuations of height one. Also, there is a projection $\mathfrak{X}_\eta^{\mathrm{ad}} \to \mathfrak{X}_\eta^{\mathrm{an}}$ and $\mathfrak{X}_\eta^{\mathrm{an}}$ is the maximal Hausdorff quotient of $\mathfrak{X}_\eta^{\mathrm{ad}}$.
- The rigid space $\mathfrak{X}_\eta^{\mathrm{rig}}$ consists of all points of X_η^{an} with $\mathcal{H}(x) \subset k^a$.