On local properties of non-Archimedean analytic spaces II.

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Plan:

1. Introduction.

2. Analytic spaces.

3. Graded rings and schemes.

4. Reduction of affinoid algebras and spaces.

5. Category $bir_{\widetilde{k}}$ and reduction of germs.

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1. Introduction.

Let k be a non-Archimedean field whose valuation may be trivial. Our aim is to generalize the notions of reduction of strictly analytic spaces and germes to the general case.

This technique enables us to prove that the embedding functor *st-k-An*$\longrightarrow$ *k-An* is fully faithfull. One obtains also criterions

1. for an analytic space X to be good at a point $x \in X$;
2. for a morphism $Y \longrightarrow X$ to be closed at a point $y \in Y$.

Which generalize analogous criterions for strictly analytic spaces

2. On k-analytic spaces.

Def: Given the algebra $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ of convergent power serieses over k with radii $r_1, \ldots, r_n$, any its quotient algebra $\mathcal{A}$ provided with the residue topology is called an affinoid algebra. It is called strictly affinoid if all radii are taken from $\sqrt{|k^*|}$.

Example: For any $r \notin \sqrt{|k^*|}$ the algebra $K_r = k\{r^{-1}T, rT^{-1}\}$ is a field.

Def: The spectrum $\mathcal{M}(\mathcal{A})$ of a Banach k-algebra $\mathcal{A}$ is the set of all bounded multiplicative seminorms on $\mathcal{A}$.

$x \in \mathcal{M}(\mathcal{A}) \Leftrightarrow \chi_x : \mathcal{A} \longrightarrow \mathcal{H}(x)$, where χ_x is bounded, $\mathcal{H}(x)$ is a non-Archimedean field and $\mathcal{H}(x) = \widehat{Q(\chi_x(\mathcal{A}))}$.

Analytic spaces are glued from spectra of affinoid algebras. An analytic space is good if any its point has an affinoid neighborhood.

Fact: $Y \subset X$ are affinoid, $y \in Y$ is a point $\Rightarrow$ $\mathcal{H}_X(y) = \mathcal{H}_Y(y)$.

(The proof uses Gerritzen-Grauert theorem.)

Corollary: For any analytic space X and a point $x \in X$ the morphism $\mathcal{M}(\mathcal{H}(x)) \xrightarrow{i_x} X$ is well defined.

3. Graded rings and schemes.

Def: Given an abelian group G a G-graded ring A is a ring A with G-graduation $A = \oplus_{g \in G} A_g$.

It seems that the following principle holds: Any definition and proof from algebraic geometry can be generalized to G-graded rings (shemes etc.) if one replaces the words "element of an algebra" by the words "homogenous element of an algebra".

Example: Graded field is a graded ring where any non-zero homogenous element is invertible, it is equivalent to non-existence of non-zero proper graded ideals. If K is a graded field and $g \in G$ is of infinite order over $|K|$, then $K[g^{-1}T, gT^{-1}]$ is a graded field too. If K is a trivially graded field, then $K[G]$ is a graded field.

Def: Given a graded ring A its spectrum $X = \text{Spec}_G(A)$ consists of the set of its prime graded ideals provided with natural topology and of the sheaf $\mathcal{O}_X$ of graded rings.

Def: A graded scheme is a graded locally ringed space locally isomorphic to spectra of graded rings.

The notions of open and closed embeddings, of finite mophisms and of morphisms of finite type are defined in the usual way and satisfy standard properties (local on the base etc.)

Notions of separated and closed morphisms are defined using valuative criterions. (Graded valuation ring $\mathcal{O}$ is an integral local graded ring whose graded quotient field K is such that for any homogenous element $f \in K$ either f or f^{-1} belongs to $\mathcal{O}$.)

4. Reduction of k-affinoid algebras and spaces.

Def: Given a Banach k-algebra $\mathcal{D}$, the spectral norm defines its filtration by positive numbers, the associated $\mathbf{R}_+$-graded algebra is called its reduction and denoted $\widetilde{\mathcal{D}}$.

Fact: A bounded homomorphism of affinoid algebras $\mathcal{A} \longrightarrow \mathcal{B}$ is finite iff its reduction $\widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{B}}$ is finite.

Construction: Given a point x of an affinoid space $X = \mathcal{M}(\mathcal{A})$ the character $\chi_x : \mathcal{A} \longrightarrow \mathcal{H}(x)$ induces a homomorphism $\widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{H}(x)}$. Thus we get a reduction map $\pi_X : X \longrightarrow \widetilde{X} \stackrel{def}{=} \mathrm{Spec}_{\mathbf{R}_+}(\widetilde{\mathcal{A}})$.

Fact:
(i) The reduction map $\pi_X : X \longrightarrow \widetilde{X}$ is surjective.
(ii) Any point $\widetilde{x} \in \widetilde{X}_{gen}$ has a unique preimage.
(iii) The set $\phi_X^{-1}(\widetilde{X}_{gen})$ is the Shilov boundary of X.

Remark: Using reduction of affinoid spaces, the notion of formal affinoid covering due to Bosch may be generalized straightforwardly. However an analog of Raynaud's theory of formal schemes is lacking.

5. Category $bir_{\widetilde{k}}$ and reduction of germs.

Let $\widetilde{k}$ be a G-graded field.

Def: For a graded field $K/\widetilde{k}$, define its Zariski-Riemann space as $\mathbf{P}_K = \{\nu | \widetilde{k} \subset \mathcal{O}_\nu \subset K\}$, its topology basis consists of affine sets $\mathbf{P}_K\{f_1, \ldots, f_n\} = \{\nu | f_1, \ldots, f_n \in \mathcal{O}_\nu\}$.

Def: $bir_{\widetilde{k}}$ is the category of maps $X \xrightarrow{f_X} \mathbf{P}_K$, where f_X is a local homeomorphism and X is a connected quasi-compact and quasi-separated topological space. A morphism is a pair, an embedding $i : K \hookrightarrow L$ and a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f_X} & \mathbf{P}_L \\ h \downarrow & & \downarrow i^{\#} \\ X & \xrightarrow[f_Y]{} & \mathbf{P}_K \end{array}$$

Def: (i,h) is separated (resp. proper) if $Y \longrightarrow \mathbf{P}_L \times_{\mathbf{P}_K} X$ is injective (resp. bijective).

Given a subgroup H of G, such that $H \supset |\widetilde{k}^*|$, we say that an object (X, K) of $bir_{\widetilde{k}}$ is H-strict if it can be defined using only elements of K_H. Equivalently, (X, K) is H-strict iff $X = X_H \times_{\mathbf{P}_{K_H}} \mathbf{P}_K$ for some object (X_H, K_H). Analogously one defines H-strict morphisms.

Fact: The embedding functor *st-H-*$bir_{\widetilde{k}} \longrightarrow bir_{\widetilde{k}}$ is fully faithfull.

The reduction functor $Red : Germs \longrightarrow bir_{\widetilde{k}}$ ($X_x \mapsto \widetilde{X}_x$) is defined as follows. Given a point x of an affinoid space $X = \mathcal{M}(\mathcal{A})$ set $\widetilde{X}_x = \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\{\widetilde{\chi}_x(\widetilde{\mathcal{A}})\}$. One checks that $\widetilde{X}_x$ depends only on the germ X_x. In general case one covers X_x by affinoid germs X_x^i and glues $\widetilde{X}_x$ from $\widetilde{X}_x^i$.

Fact: $Y \subset X$ is an analytic subdomain, $x \in Y$ $\Rightarrow \widetilde{Y}_x \longrightarrow \widetilde{X}_x$ is an open embedding.

Fact: $x \in Y, Z \subset X \Rightarrow \widetilde{Y_x \cap Z_x} \widetilde{\longrightarrow} \widetilde{Y}_x \cap \widetilde{Z}_x$ and $\widetilde{Y_x \cup Z_x} \widetilde{\longrightarrow} \widetilde{Y}_x \cup \widetilde{Z}_x$; if $X = \mathcal{M}(\mathcal{A})$ and $f \in \mathcal{A}$, then $\widetilde{X_x\{f\}} \widetilde{\longrightarrow} \widetilde{X}_x\{\widetilde{f}\}$.

Th: $\{$equiv. classes of subdomains $Y_x \subset X_x\}$ $\Leftrightarrow \{$open quasi-comp. subsets of $\widetilde{X}_x\}$.

6. Main results and applications.

Th:
(i) A germ X_x is good iff $\widetilde{X}_x$ is affine.
(ii) A morphism $Y_y \longrightarrow X_x$ is closed (resp. separated) iff its reduction $\widetilde{Y}_y \longrightarrow \widetilde{X}_x$ is proper (resp. separated).
(iii) A germ X_x is strictly analytic iff $\widetilde{X}_x$ is $|k^*|$-strict.

Corollary: The embedding functor *st-k-An*$\longrightarrow$ *k-An* is fully faithfull.

Corollary: Properness of a morphism of analytic spaces is a G-local on the base property.