Logarithmic Geometry and Resolution of Singularities

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in honor of Luc Illusie

- I was lucky, my first project, where logarithmic structures of Fontaine-Illusie were seriously used, was... a joint project with Luc Illusie.
- We worked out (and improved a bit) Gabber's l' strengthening of de Jong's altered resolution.
- The intuition of and the confidence in log geometry I got during this project was very helpful for recent advances with the classical Hironaka's resolution.
- In a joint project with Dan Abramovich and Jarek Włodarczyk we extended the classical canonical/functorial resolution to morphisms (functorial semistable reduction) and obtained a much faster and simpler dream resolution algorithm.
- Quite ironically, the dream algorithm does not use log geometry at all... But it has a log variant developed by Ming Hao Quek.

Plan

The formulation and history

Definition

A $\underline{\mathrm{char}(X)\text{-alteration}}$ of a dominant morphism $f\colon Y \to X$ of integral (log) schemes is $f'\colon Y' \to X'$ with compatible morphisms $Y' \to Y$, $X' \to X$ proper, generically finite of rank not divisible by any l invertible on X.

Theorem (T17, altered resolution of morphisms)

Given a finite type $f\colon Y \to X$ between integral fs log schemes with generically trivial log structures and X of finite type over a qe surface, there is a log smooth $\mathrm{char}(X)$-alteration $f'\colon Y' \to X'$.

- Altered resolution was discovered by de Jong in 95: $\dim(X) \leq 1$ (a point or a trait), also there is an equivariant version.
- de Jong-Abramovich 96: $\mathrm{char}(X) = 0$, X is a point.
- Gabber (announce) $\sim$05: l'-alteration for a single l, $\dim(X) \leq 1$.
- Illusie-T 14: Gabber's program and l'-alteration for any X.

The method

- The proof runs by a direct induction on (relative) dimension: fiber a variety or a morphism by curves $X_d \stackrel{f_d}{\to} X_{d-1} \to \cdots \stackrel{f_1}{\to} X_0$ and iteratively resolve f_i and hence (the corresponding pullback of) X_i.
- The role of log geometry is crystal clear: a relative curve f_i can only be made log smooth (or semistable). The proof is based on properness of $\overline{\mathcal{M}}_{g,n}$ and semistable reduction of Deligne-Mumford (which was the first relative resolution result discovered).
- Control on the rank is done by quotients preserving log smoothness (by so-called toroidal/very tame actions).

Observation

The classical context operated with regular schemes and log structures given by snc divisors, but everything works even easier for log regular log schemes, and this generality becomes critical when considering actions and making them toroidal (so-called torification).

Monoidal democracy and semistable reduction

Principle

Once log structures are used, the right smoothness context is that of all fs log smooth/regular schemes/morphisms. All fs monoids are equal :) If needed, they can be improved combinatorially by a separate routine.

Theorem (Adiprasito-Liu-T 18, semistable reduction for morphisms)

In the altered resolution theorem one can achieve in addition that Y' and X' are regular and the log structures are given by snc divisors.

- This is the best possible resolution of morphisms, locally given by $x_1 = y_1 \dots y_{n_1}, \dots, x_r = y_{n_{r-1}+1} \dots y_{n_r}$.
- It is deduced from [T17] by hard combinatorial methods – a relative version of the main result of [KKMS] on lattice polytops.

The rest of the talk is about a joint project with D. Abramovich and J. Włodarczyk on resolution of singularities over a field k with $\mathrm{char}(k) = 0$: morphisms, log varieties and a bit on a dream algorithm.

References:

- Logarithmic resolution:
  - [ATW17] "Principalization of ideals on logarithmic orbifolds", JEMS **22**, 2020.
  - [ATW20] "Relative desingularization and principalization of ideals".
- Dream algorithms:
  - [ATW19] "Functorial embedded resolution via weighted blowings up".
  - [Quek20] "Logarithmic resolution via weighted toroidal blowings up".

Motivation

- De Jong's method is not canonical, and even the smooth locus of $f\colon Y \to X$ can be modified.
- Main goals of the new project were:
  - Resolve morphisms so that the log smooth locus is preserved, in particular, prove semistable reduction over non-discrete valuation rings.
  - Do this as functorially as possible, e.g. compatible with base extensions (or base changes).
  - Clarify the role of log geometry in the classical resolution.
- The only hope was to use Hironaka's embedded resolution methods with log smooth ambient varieties (or morphisms) instead of the smooth ambient varieties.
- A signature of log geometry in Hironaka's approach and the monoidal democracy principle indicated that this might be possible.

Classical resolution

- Resolution of singularities associates to an integral variety Z a modification (i.e. proper birational) $Z_{\mathrm{res}} \to Z$ with a smooth Z_{res}.
- Hironaka 1964 (the Fields medal work): a resolution exists.
- Hironaka, Giraud 70ies: simplifications, maximal contact.
- Villamayor, Bierstone-Milman 80ies-90ies: algorithmic and canonical resolution.
- Włodarczyk 2005: smooth-functoriality, i.e. $Z'_{\mathrm{res}} = Z' \times_Z Z_{\mathrm{res}}$ for any smooth $Z' \to Z$. This both simplifies the arguments and has stronger applications (e.g. equivariant resolution).

Relative and logarithmic resolution

- [ATW17] The classical algorithm has a logarithmic analogue associating to each generically log smooth log variety X a modification $X_{\rm res} \to X$ with a log smooth log DM stack $X_{\rm res}$. It is functorial w.r.t. log smooth morphisms $Y \to X$.
- [ATW20] The same logarithmic resolution algorithm applies to a morphism $f\colon X \to B$ of log schemes: it constructs $X_{\rm res} \to X$ with a log smooth $X_{\rm res} \to B$, but can fail when $\dim(B) > 1$.
- The new ingredient: there exists a modification $h\colon B' \to B$ s.t. the algorithm does not fail for the base change $f'\colon X' \to B'$. Moreover, $X''_{\rm res} \to X'_{\rm res}$ is compatible with further base changes $B'' \to B'$.
- In the current version h is not canonical, so resolution of morphisms is only relatively functorial.
- Work in progress: h can be chosen canonically.

Embedded resolution

- All canonical methods before [ATW17] construct essentially the same algorithm built on Hironaka's framework. Everything is done locally and glues due to the functoriality.
- The resolution is embedded: one (locally) embeds X into a manifold (i.e. a smooth variety) M. To the pair (M, X) one associates a modification of manifolds $f : M_{\mathrm{res}} \to M$ and $X_{\mathrm{res}} \hookrightarrow X \times_M M_{\mathrm{res}}$ is a certain transform of X under f.
- Functorial embedded resolution implies functorial non-embedded one because an embedding $X \hookrightarrow M$ with minimal $\dim(M)$ is unique (étale) locally.

Main choices

The following choices are done in the classical resolution:

(1) Class of modifications: the algorithm iteratively blows up submanifolds $V \subset M$. Notation: $f_i\colon M_{i+1} = \mathrm{Bl}_{V_i}(M_i) \to M_i$.

(2) Transforms: one pullbacks X and subtracts a multiple of the exceptional divisor: $X_{i+1} = f_i^{-1}(X_i) - dE_{f_i}$.

(3) Choice of centers: the order $d = d_1$ of $I = I_X$ at $x \in M$ is a (very crude) primary invariant.

(4) The history: to avoid loops the algorithm encodes history in the iterated exceptional sncd E. The number $s(x)$ of its components at x is another primary invariant.

(5) Induction: one iteratively restricts to hypersurfaces of maximal contact, getting induction on $n = \dim(M)$. The actual invariant, whose maximal locus is blown up, is closer to $(d_1, s_1, d_2, s_2, \dots, d_n)$ with the lex order.

History and a dream algorithm

The classical algorithm has a subtle inductive structure and encodes history of the process in the boundary. With our choices a no-history algorithm does not exist:

Example (No progress.)

Let $\phi = x^2 - yzt$ and $X = V(\phi)$ in $M = \mathbb{A}^4$. Then $V = 0$ is the only smooth S_3-equivariant subscheme containing 0 in $X_{\rm sing}$, but $M' = \mathrm{Bl}_V(M)$ has charts with $X' = f^{-1}(X) - 2E$ having the same singularity, e.g. in M'_y we have
$\phi = (x'y')^2 - y'(y'z')(y't') = y'^2(x'^2 - y'z't')$.

A similar computation shows that blowing up the pinch point of Whitney umbrella $V(x^2 - y^2z)$ yields a pinch point again.

Using weighted blow ups we have constructed in [ATW19] a dream algorithm which just iteratively blows up the maximal invariant locus, so that the invariant drops. No history is needed there.

The boundary

- After a blow up $f\colon M' \to M$ each point $x \in E = V(t)$ has a god given coordinate t (unique up to a unit) coming from the history of the resolution. One only uses coordinate systems which include t.
- Inductively, for a sequence $f_i\colon M_{i+1} \to M_i$ we set $E_{i+1} = f_i^{-1}(E_i) \cup E_{f_i}$ and call it the accumulated boundary of M.
- We always work with coordinates $t_1, \dots, t_n$ s.t. $V_i = V(t_{i_1}, \dots, t_{i_l})$ and $E_i = (t_{n-r+1} \dots t_n)$. So, E_i is an snc (simple normal crossings) divisor and V_i has simple normal crossings with E_i (lies in few components and is transversal to others).
- We call the boundary coordinates exceptional or monomial and denote them $m_1, \dots, m_r$. So, $(t_1, \dots, t_n) = (t_1, \dots, t_{n-r}, m_1, \dots, m_r)$.

The role of the boundary

Good news:

- Using canonical monomial coordinates decreases choices, makes the construction more canonical, helps to avoid loops.
- Boundary can accumulate parts of $I = I_X$: we set $I = I^{\mathrm{mon}} I^{\mathrm{pure}}$, where $I^{\mathrm{mon}} = (m_1^{l_1} \dots m_r^{l_r})$ and I^{pure} is purely non-monomial.

Bad news/another side of the same coin:

- Must treat E and monomial coordinates with a special care.
- Less possibilities for coordinates, centers must have snc with E.

Remark

Many technical complications of the classical algorithm are due to a bad separation of regular and exceptional coordinates because both are used to define the order.

Principalization

- All algorithms operate algebraically with $I = I_X$ and solve the following principalization problem: find a sequence of submanifold blow ups $(M_n, E_n) \to \cdots \to (M, E)$ such that $I_n = I_X \mathcal{O}_{X_n}$ is invertible and monomial (i.e. supported on E_n).
- Magic: the last non-empty strict transform $X_l \subset M_l$ of X equals to V_l. So, it is smooth and transversal to E_l.
- Thus, principalization yields resolution $X_l \to X$ and even resolves the boundary $E_l|_{X_l}$ (a strong smell of a log geometry).
- A great profit: working with ideals provides a lot of flexibility.

Order reduction

- The main invariant of the algorithm is $d = \mathrm{ord}(I^{\mathrm{pure}})$, where $\mathrm{ord}(J) = \min_{f\in J} \mathrm{ord}(f)$. For example, $\mathrm{ord}(x^2 - yz^2)$ is 2 at any point of the z-axis and $\mathrm{ord}_O(x^5 + y^7, x^3z^3) = 5$.
- One works with marked (or weighted) ideals (I, d) where $d \geq 1$, only uses $M' = Bl_V(M)$ with $V \subseteq (I, d)_{\mathrm{sing}} := \{u \in M | \mathrm{ord}_u(I) \geq d\}$, and updates I by $I' = (I\mathcal{O}_{M'})I_{E'}^{-d}$. E.g., as we have computed earlier $(x^2 - yzt, 2)' = (x'^2 - y'z't', 2)$ on the y-chart.
- Order reduction finds a sequence $M_n \to \cdots \to M$ of such (I, d)-admissible blow ups so that $(I_n, d)_{\mathrm{sing}} = \emptyset$. Its existence implies principalization just by taking $d = 1$.

Remark

The so-called max order case when $d = \mathrm{ord}(I^{\mathrm{pure}})$ is the main one. It implies the general one relatively easily (and characteristic free). One has to consider the general case due to a bad (inductive) karma.

Maximal contact

- The miracle enabling induction on dimension is that in the maximal order case, order reduction of (I, d) is equivalent to that of $(C(I)|_H, d!)$, i.e. a blow up sequence reduces the order of (I, d) iff it reduces the order of $(C(I)|_H, d!)$. Here $C(I)$ is a coefficient ideal and H is a hypersurface of maximal contact.
- The Main Example: if $I = (t^d + a_2 t^{d-2} + \cdots + a_d)$ with $t = t_1$ and $a_i(t_2, \ldots, t_n)$, then $H = V(t)$ and $C(I) = (a_2^{d!/2}, \ldots, a_d^{d!/d})$.

Remark

(i) Why coefficient ideal? Because, unlike $C(I)|_H$, the stupid restriction $I|_H = (a_d)|_H$ looses a lot of information.
(ii) Each coefficient a_i has natural weight i.
(iii) No problem to have $a_1 = 0$ in characteristic zero (enough $d \in k^\times$).

Derivations

The main tool for a choice-free description of the algorithm is the derivation ideals $D(I) = D^1(I)$ generated by the elements of I and their derivations, and its iterations $D^n(I) = D(D^{n-1}(I))$. Note that $\mathrm{ord}_x(I) = \mathrm{ord}_x(D(I)) + 1$ for $x \in V(I)$. The derivation provides a conceptual way to define all basic ingredients excluding the monomial ones:

(1) $\mathrm{ord}_x(I)$ is the minimal d such that $D^d(I_x) = \mathcal{O}_x$.

(2) Maximal contact is any $H = V(t)$, where t is a regular coordinate in $D^{d-1}(I_x)$ (in particular, H is smooth).

(3) The coefficient ideal $C(I)$ is just $\sum_{i=0}^{d-1} (D^i(I))^{d!/(d-i)}$.

Remark

The only serious difficulty in proving canonicity of the algorithm is to show independence of the choice of a maximal contact.

Log derivations

The module of logarithmic derivations $D^{\log}$ is spanned by $m_j\partial_{m_j}$ and ∂_{t_i} for regular t_i's. These are the derivations preserving E (i.e. taking I_E to itself). For almost all needs it is easier and more conceptual to use $D^{\log}$, but it does not compute the order. This is why one has to use the usual derivations and runs into two complications as follows.

Choice of the maximal contact

(1) If E is not transversal to H then $E|_H$ makes no sense for us, hence we loose the control on the choice of centers having snc with E.

Solution: new boundary is transversal to H (and any center lying in it), so first iteratively reduce the order of I along the locus where the multiplicity s of the old boundary is maximal (practically, work with $I + I^d_{E(s)}$). Thus, our primary invariant is (d, s^{old}) ordered lexicographically.

Monomial contribution to the order

(2) When $\mathrm{ord}(I) \geq d$ but $\mathrm{ord}(I^{\mathrm{pure}}) < d$ we cannot proceed by looking only at I^{pure}. This happens because I^{mon} contributes to the order and causes that $(I, d)_{\mathrm{sing}} \neq \emptyset = (I^{\mathrm{pure}}, d)_{\mathrm{sing}}$.

Solution:
1. Reduce $e = \mathrm{ord}(I^{\mathrm{pure}})$ only along the locus where $\mathrm{ord}(I^{\mathrm{mon}}) \geq d - e$. Practically, we resolve the so-called companion ideal, which is the weighted sum of (I^{pure}, e) and $(I^{\mathrm{mon}}, d - e)$.
2. Once $e = 0$ (i.e. $I^{\mathrm{pure}} = (1)$), apply a purely combinatorial step to I^{mon}.

What is the boundary?

To proceed let us try to understand what the boundary really is.

- Unlike the embedded scheme X, I think it is wrong to view the boundary as a subscheme E of M (though it is determined by E). This is hinted at by functoriality: we consider blow ups $M' \to M$ which do not take E' to E, i.e. $(M', E') \to (M, E)$ is not a morphism of pairs.
- But $(M', E') \to (M, E)$ is a morphism of log schemes, once we view the boundary as the associated log structure $\mathcal{M}_M = \mathcal{M}_M(\log\ E) = \mathcal{O}^{\times}_{M \setminus E} \cap \mathcal{O}_M \subset (\mathcal{O}_M, \cdot)$ given by elements invertible outside of E.
- Furthermore, the sheaf of monomials $\mathcal{M}_M(\log\ E)$ is precisely what we need from E for principalization via factoring out I^{mon}!

Logarithmic parameters

- Étale locally log smooth log varieties are toroidal varieties, i.e. of the form $T = \mathrm{Spec}\, k[M][t_1, \dots, t_l]$ for a sharp fs M.
- We view t_i as regular coordinates and all elements of M as monomial coordinates at the origin of T.
- $\Omega_{(T,M)}$ is freely generated by dt_i and δm_j, where $\{m_j\}$ is any basis of M^{gp}.
- Monomial democracy: M does not have to be free anymore and there is no canonical base of M^{gp}, all monomials are equal :)

Remark

The most interesting feature of the new algorithm is functoriality w.r.t. Kummer log-étale covers, e.g. obtained by extracting roots of the monomial coordinates in the classical setting, or obtained by extracting roots of a uniformizer π in the semistable reduction case. This is out of reach (and unnatural) for the classical algorithms.

Main results

Ignoring an orbifold aspect, our main result is:

Theorem (Log principalization)

Given a toroidal variety T with an ideal $I \subset \mathcal{O}_T$ there exists a sequence of admissible blowings up of toroidal varieties $T_n \to \cdots \to T$ such that the ideal $I\mathcal{O}_{T_n}$ is monomial. This sequence is compatible with log smooth morphisms $T' \to T$.

As in the classical situation this implies

Theorem (Log resolution)

For any integral logarithmic variety X there exists a modification $X_{\mathrm{res}} \to X$ such that X_{res} is log smooth. This is functorial w.r.t. log smooth morphisms $X' \to X$.

The method

In brief, we want to log-adjust all parts of the classical algorithm. The main adjustment is to only use log derivations:

(1) $\mathrm{logord}_x(I)$ is the minimal d such that $(D^{\log})^d(I_x) = \mathcal{O}_x$.

(2) Maximal contact is any $H = V(t)$, where t is any regular coordinate in $(D^{\log})^{d-1}(I_x)$ (in particular, H is toroidal).

(3) The coefficient ideal $C(I)$ is just $\sum_{i=0}^{d-1}((D^{\log})^i(I))^{d!/(d-i)}$.

(4) In addition, J is (I,d)-admissible if $I \subseteq J^d$ and, for appropriate coordinates, $J = (t_1, \ldots, t_l, m_1, \ldots, m_r)$ for any set of monomials. Then $X' = Bl_J(X)$ is toroidal and the d-transform $I' = (I\mathcal{O}_{X'})(J\mathcal{O}_{X'})^{-d}$ is defined. Note that J is submonomial – a monomial ideal on the log submanifold $V(t_1, \ldots, t_l)$.

Infinite log order

- Note that $\mathrm{logord}(t_i) = 1$ but $\mathrm{logord}(m) = \infty$. This is the main novelty that allows functoriality w.r.t. extracting roots of monomials (Kummer covers).
- As a price we have to do something special when $\mathrm{logord}(I) = \infty$, but this is simple: just start with blowing up the minimal monomial ideal I_{mon} containing I. For example, if $I = (\sum_{i\in\mathbb{N}^r} m_i t^i)$ then $I_{\mathrm{mon}} = (m_i)$. The single toroidal blow up makes logord finite! (This result is due to Kollár.)
- Our algorithm is simpler, in particular, it avoids both complications (max contact is given by a regular coordinate!).
- In a sense, we completely separate dealing with regular coordinates via log order and dealing with monomials via combinatorics (i.e. toroidal blow ups).
- The invariant is just $(d_1, \ldots, d_n)$ with $d_i \in \mathbb{N}$, $d_n \in \{0, \infty\}$.

Orbifolds

- Is all this so elementary? Where is the cheating?
- Well. Our algorithm does not distinguish monomials and their roots. In fact, we view this as a serious advantage (log smooth functoriality). As another side of the coin, to achieve correct weights and admissibility, the algorithm often insists to use Kummer monomials $m^{1/d}$.
- This can be by-passed by working on log-étale Kummer covers, which is ok due to the log smooth functoriality we prove. The Kummer-local description remains the same as we saw. However, in order to describe the algorithm via modifications of T we have to use orbifolds and non-representable modifications $T' \to T$ that we call Kummer blow ups.
- This is ok for applications, because we can remove the stacky structure afterwards by a separate torification algorithm. Though the latter is only compatible w.r.t. smooth morphisms.

An example

Example

(i) Take $T = \mathrm{Spec}\ \mathbb{C}[t, m]$ and $I = (t^2 - m^2)$. Then $\mathrm{logord}_O(I) = 2$, $H = V(t)$, $C(I)|_H = (m^2, 2)$, the order reduction of $C(I)|_H$ blows up $(m^2)^{1/2} = (m)$, and the order reduction of I blows up (t, m). Just as in the classical case.
(ii) If $I = (t^2 - m)$ then $\mathrm{logord}_O(I) = 2$, $H = V(t)$, $C(I)|_H = (m, 2)$, the order reduction of $C(I)|_H$ blows up $(m^{1/2})$, and the order reduction of I blows up $(t, m^{1/2})$. This is a non-representable Kummer blow up whose coarse moduli space $Bl_{(t^2,m)}(T)$ is not toroidal.

Remark

More generally, the weighted blow up of $((t_1, d_1), \ldots, (t_r, d_r))$ in $\mathbb{A}^n$ is the coarse space of a non-representable modification with a smooth source. They are used in the dream algorithm of [ATW19] and should be useful for other classical problems in birational geometry.