Desingularization of quasi-excellent schemes of characteristic zero

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Conventiones

- All geometric spaces (mainly schemes) in this talk are of characteristic zero and all (formal) schemes are noetherian.
- The regular locus X_{reg} consists of points at which X is regular, its complement is the singular locus X_{sing}.
- A sequence of morphisms will often be denoted as $X' \dashrightarrow X$.
- References: a survey "Absolute desingularization in characteristic zero" at arXiv:[1001.5433] and the references given there. Also, [BMT] stands for a joint paper with E.Bierstone and P.Milman "**Q**-universal desingularization" at arXiv:[0905.3580]. These papers and the slides of the lecture are also at my homepage at math.huji.ac.il/∼temkin.

Hironaka's theorem

Theorem (Hironaka, 1964)

Assume that k is a local noetherian ring containing $\mathbf{Q}$ whose completion homomorphism $\phi : k \to \widehat{k}$ is regular (i.e. the geometric fibers of ϕ are regular) and X is integral of finite type over k and with a closed subscheme Z.

Non-embedded desingularization: there exists a sequence of blow ups $X' = X_n \dashrightarrow X_0 = X$ with regular X' and such that the centers of blow ups are regular and lie over X_{sing}.

Embedded desingularization: if X is regular then there exists a sequence of blow ups $X' \dashrightarrow X$ such that the centers are regular, have normal crossings with the exceptional divisors, and $Z' = Z \times_X X'$ is an exceptional divisor (with some multiplicities), in particular, Z' is monomial.

Some comments

- Hironaka proved stronger results, in particular, he considered also non-reduced X and proved that one can choose centers $V_i \subset X_i$ such that X_i is normally flat along V_i.
- The proof is not constructive, and the obtained desingularization is not algorithmic or canonical. In particular, it is not clear how to glue local desingularizations, and this is one of the main reasons why the proof is so difficult.
- The proof cannot be run entirely in the category of algebraic varieties, since formal completions are involved.
- We will mainly discuss the non-embedded desingularization for simplicity of the exposition.

Quasi-excellent schemes

Encouraged by Hironaka's work, Grothendieck wrote the following in [EGA IV_2,§7.8,§7.9], 1965.

- Definition: a scheme k is quasi-excellent or qe (the word was introduced later) if: (N) any integral f.t. k-scheme X has open X_{reg}, and (G) $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}}_{X,x}$ is regular for any $x \in X$.
- Theorem: if any integral f.t. k-scheme X possesses a weak desingularization then k is qe.
- Conjecture/hope: any integral qe scheme possesses a desingularization.
- Remark: Hironaka's method reduces desingularization to the case of $X = \mathrm{Spec}(A)$ for a complete local A. In particular, it implies desingularization of qe schemes of characteristic zero.

Some comments

- The world of qe schemes is the right place to study desingularization: at least qe schemes possess weak local uniformization (Gabber 2007, unpublished) and most experts expect Grothendieck's conjecture to be true.
- Grothendieck's remark is a puzzle (although it is repeated in wikipedia's article on resolution of singularities). No supporting proof was published (to the best of my knowledge).
- Nevertheless, people used the remark as a proved result, in particular, to resolve spectra of affinoid rings in rigid geometry (e.g. $\mathrm{Spec}(\mathbf{Q}_p\{T_1, \ldots, T_n\}/I)$).

Ubiquity of qe schemes

- Almost any "reasonable" noetherian ring or scheme is qe.
- Grothendieck (1965): Y is essentially f.t. over a qe X then Y is qe.
- Gabber (2007): qe rings are preserved under formal completion (very difficult, uses weak local uniformization).
- Fields and number rings are qe. Quasi-excellence of a ring is preserved by passing to a f.g. algebra, formal or convergent power series rings.
- Local rings of varieties, formal varieties, complex analytic spaces, non-archimedean analytic spaces are qe.

The reasons to study desingularization of qe schemes

What is the motivation to study desingularization of qe schemes (beyond the natural wish to clarify Grothendieck's claim and to prove results in largest possible generality)?

- Desingularization is of algebraic nature: we will see that strong enough desingularization of qe schemes implies desingularization of all other geometric objects of characteristic zero, including qe formal schemes, complex analytic spaces, and non-archimedean analytic spaces. Note that desingularization of formal varieties is new.
- Our understanding of the variety case improved a bit in this research ([BMT]).

Functorial desingularization

Definition

A family $\mathcal{F}(X) : X' \dashrightarrow X$ of desingularizations is functorial (w.r.t. smooth morphisms) if for any smooth $f : Y \to X$ the blow up sequence $\mathcal{F}(Y)$ is obtained from the pullback sequence $\mathcal{F}(X) \times_X Y$ by omitting empty blow ups.

- Main improvement of Hironaka's results for varieties was in constructing algorithmic functorial desingularization of varieties (Bierstone-Milman, Villamayor, Włodarczyk, and many other experts).
- Proofs are much simpler since local solutions glue.
- One works within the category of varieties earning simplicity and sacrificing, for example, formal varieties.

Main results

Theorem (T)

Reduced qe schemes over $\mathbf{Q}$ admit desingularization $\mathcal{F}$ functorial w.r.t. regular morphisms, such that $\mathcal{F}(X) : X' \dashrightarrow X$ is a sequence of blow ups with regular centers.

Normal flatness condition is not achieved so far.

Theorem (T)

Schemes Z embedded into regular qe schemes X over $\mathbf{Q}$ admit desingularization $\mathcal{E}(X, Z) : X' \dashrightarrow X$ that blows up regular centers, monomializes Z and is functorial w.r.t. morphisms $(\overline{X}, \overline{Z}) \to (X, Z)$ s.t. $\overline{X} \to X$ is regular and $\overline{Z} = Z \times_X \overline{X}$.

In the current version, the final boundary is snc but the centers may not have normal crossings with the boundaries.

Desingularization of other geometric spaces

Theorem (T)

(i) $\mathcal{F}$ and $\mathcal{E}$ extend to quasi-compact qe formal schemes and compact analytic spaces of characteristic zero.
(ii) For non-compact spaces, $\mathcal{F}$ and $\mathcal{E}$ can be defined as infinite hypersequences labeled by a well ordered set such that the composition is finite over compact subspaces.

Proof.

Let us outline how $\mathcal{F}(X)$ is constructed for a complex analytic space X. Cover X by closed subspaces X_i of polydiscs. Then $A_i = \mathcal{O}_X(X_i)$ is qe, hence analytification of $\mathcal{F}(\mathrm{Spec}(A_i))$ is a desingularization $\mathcal{F}(X_i)$ of X_i. The construction globalizes because the gluing homomorphisms $\mathcal{O}_X(X_i) \to \mathcal{O}_X(X_i \cap X_j)$ are regular (but not smooth!). □

Applications

- So far, the main case used in applications is desingularization of formal varieties.
- In particular, it was used for study of motivic integration, log canonical thresholds, desingularization of meromorphic connections, and motivic Donaldson-Thomas invariants.
- To study the latter, Kontsevich-Soibelman also conjectured that any morphism between smooth proper formal varieties over $\mathbf{C}[[t]]$ which is generically an isomorphism or bimeromorphic (resp. a rig-isomorphism) factors into a composition of blow ups and downs with smooth centers (resp. lying in the closed fiber). This weak factorization conjecture is expected to follow from desingularization of formal varieties (joint project with D. Abramovich).

The method

Let us describe the main blocks of our method for constructing $\mathcal{F}$ (while $\mathcal{E}$ is built similarly but more technically).

- Step 0. Input: a desingularization of varieties $\mathcal{F}_{\mathrm{Var}}$ which is functorial w.r.t. all regular morphisms.
- Step 1. Localization: reduce the general problem to desingularizing X's such that X_{sing} is a variety (or to desingularizing rig-regular formal varieties $\mathfrak{X}$ – the completion of X along X_{sing}).
- This stage is close in spirit to Grothendieck's claim. It is weaker, but I can prove it and it is still very useful.
- Step 2. Algebraization: realize $\mathfrak{X}$ above as a completion of a variety X, and show that $\mathcal{F}_{\mathrm{Var}}(X)$ induces $\mathcal{F}_{\mathrm{Var}}(\mathfrak{X})$ which is independent of the choice of the algebraization $\widehat{X} \widetilde{\to} \mathfrak{X}$.

Choice of $\mathcal{F}_{\mathrm{Var}}$

- [BMT]: the desingularization algorithm $\mathcal{F}_{\mathrm{Var}}$ of Bierstone-Milman is functorial w.r.t. all regular morphisms, not necessarily of finite type (probably this is true for other methods, but we do not know).
- The main obstacle: all embedded methods replace ideals $\mathcal{I} \subset \mathcal{O}_X$ with other ideals obtained by use of $\mathrm{Der}_{X/k}$. This depends on k, as $\mathrm{Der}_{X/k}\mathcal{I}$ can differ from $\mathrm{Der}_{X/k_0}\mathcal{I}$ for $k_0 \subset k$. Thus, infinite localizations, such as $\mathrm{Spec}(\mathbf{Q}(x)[y]) \hookrightarrow \mathrm{Spec}(\mathbf{Q}[x,y])$, may cause troubles.
- Solution: work with absolute derivations $\mathrm{Der}_{X/\mathbf{Q}}$; this makes various algorithms "absolute", i.e. induced from their restriction onto $\mathbf{Q}$-varieties. (We proved that this does not affect $\mathcal{F}_{\mathrm{Var}}$, so it is already absolute.)

Main idea of localization

- Resolving $X = X_0$ by a blow up sequence $\mathcal{F}(X) = (\ldots \dashrightarrow X_i \dashrightarrow X_0)$, we can run induction by codimension and localize w.r.t. X_0 rather than current X_i.
- This reduces the problem to finding a resolution $\mathcal{F}_{\mathrm{Var}}$ of schemes X' with f.t. morphism $f : X' \to X$ s.t. X is local with closed point $x \in X$ and $X'_{\mathrm{sing}} \subseteq f^{-1}(x)$. In particular, X'_{sing} is a variety.
- $\mathcal{F}$ is built uniformly for all schemes, so it differs from $\mathcal{F}_{\mathrm{Var}}$ already on varieties.
- The method is very robust, characteristic free and applies to all versions of desingularization (embedded, functorial, etc.).
- In particular, non-functorial desingularization of general qe scheme follows from Hironaka's theorem (but the reduction is, probably, not what Grothendieck meant).

The inductive scheme

- Set $\mathcal{F}^0(X) = \mathrm{Id}_X$ and built inductively a blow up sequence $\mathcal{F}^d(X) : X' \dashrightarrow X$ that resolves X over an open set U whose complement Z is of codimension at least $d+1$.
- For illustration, assume that $Z = z$ is a point. Each center $V_i \subset X_i$ may be singular over z, so insert $\mathcal{F}^d(V_i)$ into $\mathcal{F}^d(X)$ before blowing up V_i. In the end apply the blow up induced from $\mathcal{F}_{\mathrm{Var}}(X'_z)$, where $X'_z = \mathrm{Spec}(\mathcal{O}_{X,z}) \times_X X'$.
- In general, do this simultaneously for all codimension $d+1$ points of Z, and extend the inserted blow ups by taking Zariski closure of the centers.
- The sequence $\mathcal{F}^d(X)$ stabilizes by noetherian induction (even when $\dim(X) = \infty$!), so set $\mathcal{F} = \lim \mathcal{F}^d$.
- It is useful to have qe pathologies in mind. The main induction is by codimension. (How else can one treat $\dim(X) = \infty$ or even $\dim(X_{\mathrm{sing}}) = \infty$?)

Completion

- If X_{sing} is a variety then to construct $\mathcal{F}_{\mathrm{Var}}(X)$ is equivalent to construct $\mathcal{F}_{\mathrm{Var}}(\mathfrak{X})$, where $\mathfrak{X}$ is the completion of X along X_{sing}.
- Indeed, $\mathfrak{X}$ is a rig-regular formal variety (i.e. its "generic fiber is regular"), so $\mathcal{F}_{\mathrm{Var}}(\mathfrak{X})$ blows up only open ideals (or the subschemes of the closed fiber of $\mathfrak{X}$), and those algebraize to ideals on X (or subschemes of X_{sing}).
- We use that X is qe, so X and $\mathfrak{X}$ are related by regular "morphisms". In particular, their regular loci match.
- We also use that $\mathfrak{X}$ is qe. This is much simpler than Gabber's results, and was proved by Valabrega in 1970ies.

Resolution of formal varieties: a challenge

- Modern algorithms (e.g. the algorithm of Bierstone-Milman) do not imply analytic desingularization in a formal way, but can be rephrased in the analytic contexts almost without changes. The same should be done for formal varieties.
- I expect that one can reformulate the method of Bierstone-Milman (and, probably, other methods) for <u>all</u> formal varieties at cost of working with continuous derivatives $\widehat{\mathrm{Der}}_{\mathfrak{X}/k}$. Moreover, one can obtain an absolute algorithm (compatible with all regular morphisms) by taking absolute continuous derivatives $\widehat{\mathrm{Der}}_{\mathfrak{X}/\mathbf{Q}}$.
- In the case of success, one would obtain strongest forms of embedded and non-embedded desingularization (since the localization stage can "chew" any input).

Resolution of formal varieties by algebraization

- Algebraization of an affine rig-smooth formal variety $\mathfrak{X}$ over $k[[t]]$ is possible by results of Elkik, so $\mathfrak{X} \widetilde{\to} \widehat{X}$ for a variety X. Fortunately, this case suffices to construct $\mathcal{F}$ and $\mathcal{E}$.
- The main problem is that the choice of X and even its ground field is absolutely non-unique. So, functoriality requires a subtle study. It is done by descent to the ground field $\mathbf{Q}$ (to by-pass non-uniqueness of the ground field) and using Elkik's result that $\mathfrak{X}$ is determined up to an isomorphism by $\mathfrak{X} \otimes_{k[[t]]} k[[t]]/(t^n)$ for $n \gg 0$.
- Fortunately, this suffice for desingularization but we lose: (a) independence of characteristic (rig-regular versus rig-smooth), (b) normal flatness condition (involves not rig-regular completions), and profit: (c) additional troubles in the embedded case (currently, no result is proved to algebraize rig-regular divisors on $\mathfrak{X}$).