Functorial desingularization of quasi-excellent schemes over $\mathbf{Q}$

M. Temkin

Introduction

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- Desingularize certain formal varieties using algebraization.
- Desingularize quasi-excellent schemes using decompletion and induction on codimension.
- If the problem is solved for quasi-excellent schemes functorially then one automatically obtains desingularization of other geometric spaces, including formal schemes (new), complex analytic spaces and non-archimedean analytic spaces (e.g. Berkovich or rigid).

Plan

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2. Definitions and main results
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3. The method
   (a) Full functoriality of varieties
   (b) Formal varieties and algebraization
   (c) Quasi-excellent schemes and decompletion
   (d) Other categories

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- Such f is a desingularization if it is an isomorphism over the regular locus X_{reg}.
- f is strong if it comes equipped with a fixed factorization into a composition of blow ups along regular centers.
- A family of desingularizations $\mathcal{F}(X) : X' \to X$ is functorial with respect to a family of morphism S if for any $g : Y \to X$ in S one has that $\mathcal{F}(Y) = \mathcal{F}(X) \times_X Y$ (up to eliminating of blow ups along empty centers).

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(iii) Functoriality is defined as earlier.

Known results

- Hironaka 1964: strong but non-functorial embedded and non-embedded desingularizations are possible for schemes of finite type over a local ring *k* (of *cha* $= 0$) such that the morphism $\mathrm{Spec}(\widehat{k}) \to \mathrm{Spec}(k)$ is regular (i.e. (is flat) and has (geometrically) regular fibers).

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- Bierstone-Milman-Temkin 2009 (preprint): functoriality w.r.t. all regular morphisms between varieties.

Definition and motivation

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- Definition: A noetherian scheme X is qe (or quasi-excellent) if: (G) the homomorphism $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}}_{X,x}$ is regular for any $x \in X$, and (N) any scheme X' of f.t. over X has open X'_{reg}.

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The modern conjecture/hope is that qe schemes admit strong desingularization functorial in regular morphisms.

Examples of qe schemes and rings

- Quasi-excellence of a ring is preserved by: a quotient, a localization, passing to a polynomial ring and a formal (resp. convergent) power series ring. (The latter is very difficult; proved by Gabber few years ago.)

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- In particular, all geometric objects we discussed are glued from local objects controlled by qe rings along regular gluing homomorphisms (e.g. formal localization).

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From now on we will discuss non-embedded desingularization because it is technically easier.

Absolute desingularization of varieties

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- The main obstacle: all embedded methods replace ideals $\mathcal{I} \subset \mathcal{O}_X$ with some derivative ideals obtained by applying $\mathrm{Der}_{X/k}$. This depends on k, as $\mathrm{Der}_{X/k}\mathcal{I}$ can differ from $\mathrm{Der}_{X/k_0}\mathcal{I}$ for $k_0 \subset k$.

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- Solution: work with absolute derivations $\mathrm{Der}_{X/\mathbf{Q}}$. (We proved that for the algorithm of Bierstone-Milman the absolute version is the old one.)

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- Quasi-excellence implies that a desingularization $X' \to X$ gives rise after the completion to a desingularization $\mathfrak{X}' \to \mathfrak{X}$.
- Functoriality w.r.t. regular morphisms allows to glue these local desingularizations to a global one.

Non-uniqueness of algebraization

- If we algebraize $\mathfrak{X}$ with the ground field morphism $i : \mathfrak{X} \to \mathrm{Spec}(k)$ fixed then different algebraizations X and X' are not too different because they admit a common étale cover (locally over the closed fiber). It follows from functoriality that they give rise to the same desingularization of $\mathfrak{X}$.

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- Unfortunately, i has a lot of deformations (unlike the case of usual varieties). If the ground field $k \subset \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ is not fixed then it is a real headache to compare two algebraizations.
- Idea: by Elkik's results a sufficiently thick closed fiber $X_n = (\mathfrak{X}_s, \mathcal{O}_{\mathfrak{X}}/T^n\mathcal{O}_{\mathfrak{X}})$ determines $\mathfrak{X}$. A technical trick shows that in our case the desingularization $\mathcal{F}(X)$ actually depends only on the thick neighborhoods $X_n \hookrightarrow X$ of X_{sing}.

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- I expect that one can reformulate these methods for <u>all</u> formal varieties at cost of working with continuous derivatives $\widehat{\mathrm{Der}}_{\mathfrak{X}/k}$. Moreover, one can obtain an absolute algorithm (compatible with all regular morphisms) by taking absolute continuous derivatives $\widehat{\mathrm{Der}}_{\mathfrak{X}/\mathbf{Q}}$.

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- Benefits: simpler argument, strong embedded desingularization.

Schemes with small singular locus

- Consider the category of pairs (X, D) where X is a reduced qe scheme and D is a Cartier divisor that is isomorphic to a variety and contains X_{sing}. Morphisms are regular morphisms $f : X' \to X$ such that $f^{-1}(D) = D'$.

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- We obtain desingularization of a scheme X whose singular locus is a variety. The desingularization $\mathcal{F}_{\mathrm{Var}}(X, D)$ (and its functoriality) depends (at least a priori) on the choice of D.

A construction of $\mathcal{F}$

- The general desingularization $\mathcal{F}(X) : X' \to X$ of qe schemes is constructed from $\mathcal{F}_{\mathrm{Var}}(X, D)$. For functoriality reasons, $\mathcal{F} \neq \mathcal{F}_{\mathrm{Var}}$ even on varieties.

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- The construction runs by induction on codimension in X. At d-th stage we have a blow up functor $\mathcal{F}_n : X_m \to \dots \to X$ which is a strong desingularization over the points of codimension at most d.
- Let $Z = \{z_1, \dots, z_n\} \subset X$ be the "bad" points of codimension $d + 1$. We will produce $\mathcal{F}_{d+1}$ by <u>inserting</u> few blow ups with centers over the closure of Z.

Construction of $\mathcal{F}$: continuation

- For the localization $X' = X_Z$ at Z we solve this as follows. First insert the blow up along Z into $\mathcal{F}_d(X_Z)$ as the first blow up. Next desingularize all centers V_i of the blow ups by inserting $\mathcal{F}_d(V_i)$ before blowing V_i up. Finally, set $Z_m = Z \times_{X'} X'_m$ and add $\mathcal{F}_{\mathrm{Var}}(X'_m, Z_m)$ in the end of the sequence.

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This scheme of induction is very robust and applies to any variant of desingularization (e.g. various versions of embedded desingularization). So, as soon as the case of formal varieties is settled, the remaining part is nearly automatic.

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- If $\mathfrak{X}' = \mathrm{Spf}(B)$ is an open subscheme in $\mathfrak{X}_i$ then $\mathcal{F}(\mathfrak{X}')$ is compatible with $\mathcal{F}(\mathfrak{X}_i)$ because $A \to B$ is a regular homomorphism. It follows that local desingularizations $\mathcal{F}(\mathfrak{X}_i)$ glue to a global desingularization $\mathcal{F}(\mathfrak{X})$.

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- Functoriality of $\mathcal{F}(\mathfrak{X})$ in regular morphisms is checked similarly.