Logarithmic resolution of singularities

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The ultimate goal

- For simplicity, we will always take $k = \mathbf{C}$ and one can safely work with complex analytic varieties instead of algebraic k-varieties (using complex topology instead of the étale one).
- Resolution of singularities associates to an integral variety Z a modification (i.e. a proper birational map) $Z_{\mathrm{res}} \to Z$ with a smooth source.
- Modern resolution is canonical and, moreover, functorial w.r.t. smooth morphisms $Z' \to Z$ in the sense that $Z'_{\mathrm{res}} = Z' \times_Z Z_{\mathrm{res}}$. This both simplifies the arguments and has stronger applications (e.g. equivariant resolution).
- A joint project of D. Abramovich, M.T. and J. Włodarczyk aims to resolve morphisms $f\colon Z \to B$ by modifications $Z' \to Z$ and $B' \to B$ and a log smooth morphism $f_{\mathrm{res}}\colon Z' \to B'$ compatible with f (log smoothness is the best one can hope for here).
- Naturally, we want this to be functorial w.r.t. log smooth morphisms.

The state of the art

- So far, we have constructed a new algorithm for desingularization of (log) varieties over a field k of characteristic zero, which is functorial w.r.t. all log smooth morphisms.
- The paper "Principalization of ideals on toroidal orbifolds" by D. Abramovich, M. Temkin and J. Włodarczyk is available at http://www.math.huji.ac.il/∼temkin/papers/LogPrincipalization.pdf
- We expect that (essentially) the same algorithm applies to morphisms, including functorial semistable reduction algorithms, but details are to be worked out.

Plan

1. Classical resolution
   - General framework
   - Induction on dimension
2. Logarithmic geometry
3. Logarithmic algorithms
4. Conclusion

Principalization

- We describe the classical method based on ideas of Hironaka, from his foundational 1964 paper and developed further by himself, Giraud, Bierstone and Milman, Villamayor, Włodarczyk, Kollár, and others. All constructions are local but glue due to the functoriality.
- The first step is to (locally) embed Z into a manifold $X = X_0$ with a boundary $E = E_0 = \emptyset$ (i.e. X is smooth and E is an snc divisor) and to reduce desingularization of Z to principalization of the ideal $I_Z \subset \mathcal{O}_X$ of Z. One iteratively blows up smooth centers V_i sitting over Z and having simple nc with E_i, so that $X_{i+1} = Bl_{V_i}(X_i)$ and $E_{i+1} = f_i^{-1}(V_i) \cup f_i^{-1}(E_i)$ are a manifold with a boundary. The aim is to get $I_n = I_Z\mathcal{O}_{X_n}$ invertible and monomial (i.e. supported on E_n).
- In fact, the last non-empty strict transform $Z_i \subset X_i$ of Z equals to V_i. So, it is smooth, transversal to E_i and desingularizes Z.

Local charts

- (Étale) locally one can always pretend to be working at $O \in X = \mathbf{A}^n$ with coordinates $t_1, \ldots, t_n$ such that $E = V(m_1 \cdot \cdots \cdot m_s) = \cup_{i=1}^{s} E_i$ for $m_j = t_{i_j}$. We call $m_1, \ldots, m_s$ exceptional or monomial coordinates (they are unique up to units), other t_i's are regular coordinates (they are non-canonical).
- V has snc with E means that $V = V(t_1, \ldots, t_r)$ for appropriate coordinates $t_1, \ldots, t_n$.
- $X' = Bl_V(X)$ is covered by r charts. The (new) coordinates on X'_i are $t'_k = \frac{t_k}{t_i}$ for $1 \leq k \leq r$ and $k \neq i$ and $t'_k = t_k$ otherwise. Also $E' = V(t_i) \cup V(t_{i_1} \ldots t_{i_s})$.
- Functions (and ideals) transform via

$$f(t_1, \ldots, t_n) = f(t'_1 t_i'^{a_1}, \ldots, t'_n t_i'^{a_n}) = f'(t'_1, \ldots, t'_n)$$

with $a_i \in \{0, 1\}$.

Example: the pinch points

- The main counterexample to various overoptimistic expectations is Whitney umbrella Z with $I_Z = I = (x^2 - y^2 z)$ and the pinch point O at the origin (it is the only non normal crossings point of Z).
- Blowing up the singular line $V = (x, y)$ resolves Z, but not blowing up of (x, y, z). E.g. in the z-chart we get $I' = (y'^2(x'^2 - y'^2 z')) = I_{Z'} I^2_{E'}$, hence Z' also has a pinch point. In particular, one cannot always blow up the worst singularity and one cannot make Z' nc by modifying the non-nc locus only.
- The actual algorithm blows up the pinch point twice (!) to get enough history (or evidence that this does not work), and then blows up the singular line.

The role of the boundary

The profits:

- Have canonical monomial coordinates (coming from the history of earlier blow ups). This decreases choices and makes the construction much more canonical.
- Boundary gradually accumulates I: if $I = I^{\mathrm{mon}} I^{\mathrm{pure}}$, where $I^{\mathrm{mon}} = (m_1^{n_1} \cdot \dots \cdot m_s^{n_s})$ and I^{pure} is purely non-monomial, then $I = I^{\mathrm{mon}}$ and $I^{\mathrm{pure}} = (1)$ in the end of the principalization.

The price to pay:

- Must treat E and monomial coordinates with a special care. E.g. any center is either transversal to E_i or lies in it. (Another side of the coin.)

Remark

Many technical complications of the classical algorithm are due to a bad separation of regular and exceptional coordinates (e.g. in the notion of order).

Order reduction

- The main invariant of the algorithm is $d = \mathrm{ord}(I^{\mathrm{pure}})$, where $\mathrm{ord}(J) = \min_{f\in J}\mathrm{ord}(f)$. For example, $\mathrm{ord}(x^2 - yz^2)$ is 2 at any point of the z-axis and $\mathrm{ord}_O(x^5 + y^7, y^2z^3) = 5$.
- One works with marked (or weighted) ideals (I, d) where $d \geq 1$, only uses $X' = Bl_V(X)$ with $V \subseteq (I, d)_{\mathrm{sing}} := \{x \in X | \mathrm{ord}_x(I) \geq d\}$, and updates I by $I' = (I\mathcal{O}_{X'})I_{E'}^{-d}$. E.g. $(x^2 - yz^2, 2)' = (x'^2 - y'^2z', 2)$ on the z-chart.
- Order reduction finds a sequence $X_n \to \cdots \to X$ of such (I, d)-admissible blow ups so that $(I_n, d)_{\mathrm{sing}} = \emptyset$. Its existence implies principalization (e.g. by taking $d = 1$).

Remark

The so-called max order case when $d = \mathrm{ord}(I^{\mathrm{pure}})$ is the main one. It implies the general one relatively easily (and characteristic free). Have to consider the general case due to a bad (inductive) karma.

Maximal contact

- The miracle enabling induction on dimension is that in the maximal order case, order reduction of (I, d) is equivalent to that of $(C(I)|_H, d!)$, i.e. a blow up sequence reduces the order of (I, d) iff it reduces the order of $(C(I)|_H, d!)$. Here $C(I)$ is a coefficient ideal and H is a hypersurface of maximal contact.
- The Main Example: if $I = (t^d + a_2 t^{d-2} + \cdots + a_d)$ with $t = t_1$ and $a_i(t_2, \ldots, t_n)$, then $H = V(t)$ and $C(I) = (a_2^{d!/2}, \ldots, a_d^{d!/d})$.

Remark

(i) Why coefficient ideal? Because, unlike $C(I)|_H$, the stupid restriction $I|_H = (a_d)|_H$ looses a lot of information.
(ii) Each coefficient a_i has natural weight i.
(iii) No problem to have $a_1 = 0$ in characteristic zero.

Derivations

The main tool for a natural description of the algorithm are the derivation ideals $D^n(I) = D(\dots D(I)\dots)$, where $D(I)$ is generated by the elements of I and their derivatives. Derivation decreases order precisely by one and gives a conceptual way to define all basic ingredients (excluding the monomial ones):

(1) $\mathrm{ord}_x(I)$ is the minimal d such that $D^d(I_x) = \mathcal{O}_x$.

(2) Maximal contact is any $H = V(t)$, where t is any regular coordinate in $D^{d-1}(I_x)$ (in particular, H is smooth and $E|_H$ is a boundary).

(3) The coefficient ideal $C(I)$ is just $\sum_{i=0}^{d-1}(D^i(I))^{d!/(d-i)}$.

Remark

The only serious difficulty in proving canonicity of the algorithm is to show independence of the choice of a maximal contact.

Complications

(1) If E is not transversal to H then $E|_H$ makes no sense for us, hence we loose the control on the snc with E. Solution: iteratively reduce the order of I along the locus where the multiplicity s of the old boundary is maximal (i.e. work with $I + I^d_{E(s)}$). Thus, the resulting principalization algorithm consists of $2n$ embedded loops and it improves the lexicographic invariant $(d_1, s_1; d_2, s_2; \dots; d_n, s_n)$.

(2) The module of logarithmic derivations $D^{\log}$ is spanned by $m_j\partial_{m_j}$ and ∂_{t_i} for regular t_i's. These are the derivations preserving E (i..e taking I_E to itself). For almost all needs it is easier and more conceptual to use $D^{\log}$, but it does not compute the order. This is why one has to use the usual derivations.

What is the boundary?

To proceed let us try to understand what the boundary really is.

- Unlike the embedded scheme Z, I think it is wrong to view E as a subscheme of X (though it is determined by it). This is hinted at by functoriality: we consider blow ups $(X', E') \to (X, E)$ which do not take E' to E: one has that $f^{-1}(E) \hookrightarrow E'$ instead of $E' \hookrightarrow f^{-1}(E)$.
- The boundary is also determined by the sheaf of monomials $\mathcal{M}_X = \mathcal{M}_X(\log\ E) = \mathcal{O}^{\times}_{X \setminus E} \cap \mathcal{O}_X \subset (\mathcal{O}_X, \cdot)$ consisting of elements invertible outside of E. This gives the right functoriality: $f^*(\mathcal{M}_X(\log\ E)) \to \mathcal{M}_{X'}(\log\ E')$.
- In fact, the sheaf of monomials $\mathcal{M}_X(\log\ E)$ is precisely what we need from E!
- Locally $\mathcal{M}_X = \mathcal{O}^{\times}_X \times \mathbf{N}^s$ but this splitting (called a monoidal chart) is non-canonical: it is given by fixing exceptional coordinates $m_1, \ldots, m_s$.

Logarithmic varieties

Definition

A logarithmic variety $(X, \mathcal{M}_X)$ consists of a variety X with a sheaf of monoids $\mathcal{M}_X$ and a homomorphism $\alpha_X \colon \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$ such that $\mathcal{M}_X^\times = \alpha_X^{-1}(\mathcal{O}_X^\times)$. A morphism is a compatible pair $f \colon X' \to X$ and $f^*\mathcal{M}_X \to \mathcal{M}_{X'}$.

- The example covering our needs is $(X, \mathcal{M}_X(\log D))$ for a divisor D. Morphisms are $f \colon X' \to X$ s.t. $f^{-1}(D) \hookrightarrow D'$.
- Many constructions extend to log geometry, e.g. $\Omega_{(X,M_X)}$ is generated by Ω_X and elements δm for $m \in M_X$ subject to relations $d\alpha(m) = \alpha(m)\delta m$ (i.e. δm is the log differential of m).
- One also defines log smooth morphisms. As in the classical case, they have locally free sheaves of relative differentials of expected rank.

Toroidal varieties

- Log smooth varieties are just toroidal ones: étale (analytically or formally) locally it suffices to work with the chart $X = \mathrm{Spec}\, \mathbf{C}[M][t_1, \dots, t_l]$ at its origin O, where M is the monoid of integral points in a rational polyhedral cone. The log structure is induced by M and $\Omega_{(X,M)}$ is freely generated by dt_i and δm_i, where $\{m_i\}$ is any basis of M^{gp}. The classical notation is (X, U) or (X, D) with $D = \cup_{m \in M} V(m)$ and $U = X \setminus D$.
- In other words, $\mathcal{O}_{X,x} = \mathbf{C}[[M]][[t_1, \dots, t_l]]$. We view t_i as regular coordinates and all elements of M as monomial coordinates (in particular, M does not have to be free anymore and there is no canonical base of M^{gp}).

Toroidal morphisms

Log smooth morphisms of toroidal varieties are just toroidal morphisms, i.e. they are (étale-locally) modelled on toric maps and formally-locally look as

$$\mathbf{C}[[M]][[t_1,\ldots,t_r]] \hookrightarrow \mathbf{C}[[N]][[t_1,\ldots,t_n]].$$

Example

(i) Semistable maps with appropriate log structures. For example, $\mathrm{Spec}\ \mathbf{C}[x,y] \to \mathrm{Spec}\ \mathbf{C}[\pi]$ given by $\pi = x^a y^b$ is log smooth for the log structures given by $x^{\mathbf{N}} \times y^{\mathbf{N}}$ and $\pi^{\mathbf{N}}$. The relative differentials are spanned by $\delta x = -\frac{b}{a}\delta y$.
(ii) Kummer log-étale covers are obtained when $N \subset \frac{1}{d}M$ and $r = n$. Relative differentials vanish. Finite but usually non-flat, e.g. $\mathrm{Spec}\ \mathbf{C}[x,y] \to \mathrm{Spec}\ \mathbf{C}[x^2, xy, y^2]$ with the log structures of monomials in x, y.

Some remarks

Remark

Toroidal morphisms are log smooth maps of log smooth varieties. In a sense, log geometry extends both to the non-smooth case (and **Z**-schemes).

Remark

The most interesting feature of the new algorithm is functoriality w.r.t. Kummer log-étale covers, e.g. obtained by extracting roots of the monomial coordinates in the classical setting, or obtained by extracting roots of π is in the semistable reduction case. This is out of reach (and unnatural) for the classical algorithms.

Main results

Ignoring an orbifold aspect, our main result is:

Theorem (Log principalization)

Given a toroidal variety X with an ideal $I \subset \mathcal{O}_X$ there exists a sequence of admissible blowings up of toroidal varieties $X_n \to \cdots \to X$ such that the ideal $I\mathcal{O}_{X_n}$ is monomial. This sequence is compatible with log smooth morphisms $X' \to X$.

As in the classical situation this implies

Theorem (Log resolution)

For any integral logarithmic variety Z there exists a modification $Z_{\mathrm{res}} \to Z$ such that Z_{res} is log smooth. This is functorial w.r.t. log smooth morphisms $Z' \to Z$.

The method

In brief, we want to log-adjust all parts of the classical algorithm. The main adjustment is to only use log derivations:

(1) $\mathrm{logord}_x(I)$ is the minimal d such that $(D^{\log})^d(I_x) = \mathcal{O}_x$.
(2) Maximal contact is any $H = V(t)$, where t is any regular coordinate in $(D^{\log})^{d-1}(I_x)$ (in particular, H is smooth and $E|_H$ is a boundary).
(3) The coefficient ideal $C(I)$ is just $\sum_{i=0}^{d-1}((D^{\log})^i(I))^{d!/(d-i)}$.
(4) In addition, J is (I, d)-admissible if $I \subseteq J^d$ and, for appropriate coordinates, $J = (t_1, \ldots, t_l, m_1, \ldots, m_r)$ for any set of monomials. Then $X' = Bl_J(X)$ is toroidal and the d-transform $I' = (I\mathcal{O}_{X'})(J\mathcal{O}_{X'})^{-d}$ is defined.

Infinite log order

- Note that $\mathrm{logord}(t_i) = 1$ but $\mathrm{logord}(m) = \infty$. This is the main novelty that allows functoriality w.r.t. extracting roots of monomials (Kummer covers).
- As a price we have to do something special when $\mathrm{logord}(I) = \infty$, but this is simple: just start with blowing up the minimal monomial ideal I_{mon} containing I. For example, if $I = (\sum_{i \in \mathbf{N}^r} m_i t^i)$ then $I_{\mathrm{mon}} = (m_i)$. This single toroidal blow up makes logord finite. (This result is due to Kollár.)
- Our algorithm is simpler, in particular, it does not have to separate the old boundary (max contact is given by a regular coordinate!).
- In a sense, we completely separate dealing with regular coordinates via log order and dealing with monomials via combinatorics (i.e. toroidal blow ups).
- The invariant is just $(d_n, \ldots, d_1)$ with $d_i \in \mathbf{N} \cup \{\infty\}$.

Orbifolds

- Is all this so elementary? Where is the cheating?
- Well. Our algorithm does not distinguish monomials and their roots. In fact, we view this as a serious advantage (log smooth functoriality). As another side of the coin, to achieve correct weights and admissibility, the algorithm often insists to use Kummer monomials $m^{1/d}$.
- This can be by-passed by working on log-étale Kummer covers, which is ok due to the strong functoriality we prove. The Kummer-local description remains the same as we saw. However, in order to describe the algorithm via modifications of X we have to use orbifolds and non-representable modifications $X' \to X$ that we call Kummer blow ups.
- This is ok for applications, because we can remove the stacky structure afterwards by a separate torification algorithm. Though the latter is only compatible w.r.t. smooth morphisms.

An example

Example

(i) Take $X = \mathrm{Spec}\, \mathbf{C}[t, m]$ and $I = (t^2 - m^2)$. Then $\mathrm{logord}_O(I) = 2$, $H = V(t)$, $C(I) = (m^2, 2)$, the order reduction of $C(I)|_H$ blows up $(m^2)^{1/2} = (m)$, and the order reduction of I blows up (t, m). Just as in the classical case.
(ii) If $I = (t^2 - m)$ then $\mathrm{logord}_O(I) = 2$, $H = V(t)$, $C(I) = (m, 2)$, the order reduction of $C(I)|_H$ blows up $(m^{1/2})$, and the order reduction of I blows up $(t, m^{1/2})$. This is a non-representable Kummer blow up whose coarse moduli space $Bl_{(t^2,m)}(X)$ is not toroidal.

Remark

More generally, the weighted blow up of $((t_1, d_1), \ldots, (t_r, d_r))$ in $\mathbf{A}^n$ is the coarse space of a non-representable modification with a smooth source. So, Kummer blow ups might be useful for other classical problems in birational geometry.

Thanks to the organizers!

Happy birthday Askold, and "ad mea ve'esrim".