Strong type spaces as quotients of Polish groups

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- (We have a blanket assumption that the theory we are working in is countable.)
- The goal: understanding strong type spaces.
- The idea: the Galois groups, strong type spaces, quotients of type-definable groups all behave like quotients of compact Polish groups.
- We have shown that, in a very strong sense (especially under NIP hypotheses), they *are* quotients of compact Polish groups.
- This observation (and the related theory) can be used to recover essentially all known theorems about cardinality and the so-called Borel cardinality of strong type spaces and quotients of type-definable groups.

Strong types, connected components

Definition

Let $X \subseteq \mathfrak{C}$ be a type-definable set. An equivalence relation E on X is *invariant* if it is $\mathrm{Aut}(\mathfrak{C})$-invariant, and it is *bounded* if it has a small number of classes.

Definition

A *strong type* is a bounded invariant equivalence relation which refines $\equiv$.

Definition

A *strong type space* is the quotient X/E, where E is a strong type on X.

Definition

Given a ($\emptyset$-)type-definable group G, the connected component $G^{00}_{\emptyset}$ is the smallest ($\emptyset$-)type-definable subgroup of G of small index.

Logic topology

Definition

Given a ($\emptyset$-)type-definable set X and a bounded invariant equivalence relation E on X, a set $A \subseteq X/E$ is closed in the *logic topology* if its preimage in X is type-definable.

Fact

The logic topology is compact (because X is type-definable), it is Hausdorff $\iff$ E is type-definable.

- These quotients also have a well-defined "Borel cardinality".
- In particular, we have the logic topology on $G/G^{00}_{\emptyset}$ (because the coset equivalence relation is bounded and invariant).

Toy examples

- Consider a type-definable group G and its connected component $G^{00}_{\emptyset}$.
- Then $G/G^{00}_{\emptyset}$ is a compact Polish group (with the logic topology).
- For any $H \leq G$, $G^{00}_{\emptyset} \leq H$, the quotient G/H and $(G/G^{00})/(H/G^{00}_{\emptyset})$ are essentially the same (topologically, descriptive-set-theoretically).
- Likewise, $\mathrm{Gal}_{\mathrm{KP}}(T)$ is a compact Polish group.
- For any complete $\emptyset$-type p and strong type E coarser than $\equiv_{\mathrm{KP}}$ on $X = p(\mathfrak{C})$, then $\mathrm{Gal}_{\mathrm{KP}}(T)$ acts transitively on X/E.
- For any $a \models p$, X/E and $\mathrm{Gal}_{\mathrm{KP}}(T)/\mathrm{Stab}_{\mathrm{Gal}_{\mathrm{KP}}(T)}([a]_E)$ are essentially the same.
- But this only works when H contains $G^{00}_{\emptyset}$, or when E is coarser than $\equiv_{\mathrm{KP}}$ (because $\mathrm{Gal}(T)$ is not Hausdorff, so not Polish)...

Towards an application: a trichotomy in Polish groups

Fact (Useful Fact)

Suppose G is a compact Polish group, while $H \leq G$ is analytic.
Then exactly one of the following holds:

- *H is open and $[G : H]$ is finite,*
- *H is closed and $[G : H] = 2^{\aleph_0}$,*
- *H is not closed, $[G : H] = 2^{\aleph_0}$ and G/H is not smooth (in the sense of Borel cardinality).*

In particular, G/H is smooth if and only if H is closed, and $[G : H]$ is finite (and H is open) or $[G : H] = 2^{\aleph_0}$.

- We want to show an analogous fact for strong type spaces/quotients of type-definable group.

An easier trichotomy

Proposition

Let $p \in S(\emptyset)$ and let E be an invariant equivalence relation on $X = p(\mathfrak{C})$, coarser than $\equiv_{\mathrm{KP}}$, analytic. Then we have exactly one of the following:

- *E is relatively definable and X/E is finite,*
- *E is type-definable and $|X/E| = 2^{\aleph_0}$,*
- *E is not type-definable and $|X/E| = 2^{\aleph_0}$ and X/E is not smooth.*

Idea.

We pull X/E up to $\mathrm{Gal}_{\mathrm{KP}}(T)$, apply the Useful Fact, and then push its conclusion back down. □

- We have an analogous conclusion for quotients of type-definable groups by (invariant) analytic subgroups containing $G^{00}_{\emptyset}$.
- But this approach is completely useless for arbitrary bounded invariant equivalence relations on $p(\mathfrak{C})$, quotients by arbitrary bounded invariant subgroups.

Theorem

Let $X = p(\mathfrak{C})$ for some $p \in S(\emptyset)$. Then there is a compact Polish group $\hat{G}$ such that for every strong type E on X, there is a $\hat{H} \leq \hat{G}$ such that:

- *$\hat{H}$ is closed iff E is type-definable,*
- *$\hat{H}$ is open iff E is relatively definable (in X^2),*
- *$\hat{H}$ is analytic if E is analytic (in particular, it has the Baire property),*
- *$\hat{G}/\hat{H} \leq_B X/E$ and $\hat{G}/\hat{H} \sim_B X/E$ if p has NIP.*

Theorem

Given a type-definable G, there is a compact Polish $\hat{G}$ such that for every $H \leq G$ of bounded index, there is a $\hat{H} \leq \hat{G}$ (... analogous conclusion).

An ~~easier~~ trichotomy

Corollary

Let $p \in S(\emptyset)$ and let E be an invariant equivalence relation on $X = p(\mathfrak{C})$, ~~coarser than $\equiv_{\mathrm{KP}}$~~ bounded, analytic.
Then we have exactly one of the following:

- *E is relatively definable and X/E is finite,*
- *E is type-definable and $|X/E| = 2^{\aleph_0}$,*
- *E is not type-definable and $|X/E| = 2^{\aleph_0}$ and X/E is not smooth.*

Idea.

We pull X/E up to ~~$\mathrm{Gal}_{\mathrm{KP}}(T)$~~ $\hat{G}$, apply the Useful Fact, and then push its conclusion back down. □

A trichotomy in type-definable groups

Corollary

Let G be a type-definable group and let $H \leq G$ be invariant of small index, analytic. Then we have exactly one of the following:

- *H is relatively definable and $[G : H]$ is finite,*
- *H is type-definable and $[G : H] = 2^{\aleph_0}$,*
- *H is not type-definable and $[G : H] = 2^{\aleph_0}$ and G/H is not smooth.*

Idea.

We pull G/H up to $\hat{G}$, apply the Useful Fact, and push its conclusion back down. □

- This implies that for an analytic H, $[G : H]$ is finite, $2^{\aleph_0}$ or unbounded.
- This is *not* true if H is arbitrary (there are "Vitali"-like counterexamples).

Rosenthal compacta

Fact (Rosenthal, Bourgain, Fremlin and Talagrand)

Let X be a compact Polish space, and let $A \subseteq C(X)$ be bounded in the $\sup$ norm. The following are equivalent:

- *$\overline{A}$ (the pointwise closure in X^X) consists of Borel functions.*
- *$\overline{A}$ has the Fréchet-Urysohn property (for any $B \subseteq \overline{A}$, $\overline{B}$ = the limits of sequences in B).*
- *A contains no "independent sequence" ($\leftrightsquigarrow$ NIP).*
- *A contains no "ℓ^1-sequence".*

Definition

Given such A, we say that $\overline{A}$ with the pointwise convergence topology (or any space homeomorphic to it) is a *Rosenthal compact*.

The Ellis semigroup, the Ellis group and its canonical Hausdorff quotient

Definition

Given a group G of homeomorphisms of a compact Hausdorff space X, the Ellis semigroup $EL = E(G, X)$ is the pointwise closure of G in X^X (with composition as the semigroup operation).

- The Ellis semigroup is a compact left topological semigroup.

Definition

We say that the action of G on X is *tame* if $E(G, X)$ is Rosenthal.

- Such EL always contains so-called ‘Ellis groups” $u\mathcal{M}$, which come equipped with a compact semitopological group structure (not Hausdorff).
- $u\mathcal{M}$ has a canonical (compact) Hausdorff group quotient $u\mathcal{M}/H(u\mathcal{M})$.

Construction of $\hat{G}$

- Recall that we want to express X/E as $\hat{G}/\hat{H}$, where $X = p(\mathfrak{C})$ for $p \in S(\emptyset)$.
- We choose a countable *ambitious* (i.e. homogeneous in a weak sense) model M which realises p.
- We the action of $\mathrm{Aut}(M)$ on $S_m(M)$.
- (Assume NIP for simplicity.)
- Because of NIP, this action is tame,
  i.e. the Ellis group $EL = E(\mathrm{Aut}(M), S_m(M))$ is Rosenthal.
- This implies that $\hat{G} = u\mathcal{M}/H(u\mathcal{M})$ is a compact Polish group
  (as a countably tight compact Hausdorff group).
- (Without NIP we have to work a bit more to obtain $\hat{G}$.)

Some more ideas from the proof

We show that we have the following commutative diagram:

$$\begin{array}{ccccc} EL & & \twoheadrightarrow & & \hat{G} = u\mathcal{M}/H(u\mathcal{M}) \\ \downarrow\!\!\!\downarrow & & & & \downarrow\!\!\!\downarrow \\ S_m(M) & \twoheadrightarrow & X_M & \twoheadrightarrow & X/E \end{array}$$

- $X_M = \{\mathrm{tp}(a/M) \mid a \models p\} = \{\mathrm{tp}(a/M) \mid a \in X\}$
- The map $EL \to S_m(M)$ is just evaluation at $\mathrm{tp}(m/M)$.
- The map $EL \to \hat{G}$ is a certain natural epimorphism (given by $f \mapsto ufuH(u\mathcal{M})$, *not* continuous!).
- The map $\hat{G} \to X/E$ factors through an orbit map $\mathrm{Gal}(T) \to X/E$ via an epimorphism $\hat{G} \to \mathrm{Gal}(T)$.
- It follows that $\hat{G}$ acts on X/E, and $\hat{H}$ is just the stabiliser of $[a]_E$ for some $a \in X(M)$.
- Then we work (a lot) more to show that this $\hat{G}$ and $\hat{H}$ have all the required properties.

Concluding remarks

- There is a weaker variant of the trichotomy which applies in the case when the domain is not $p(\mathfrak{C})$ (i.e. we have smoothness $\iff$ type-definability).
- We can also consider a $Y \subsetneq p(\mathfrak{C})$ type-definable with parameters, and the theorem essentially applies in this case (under reasonable assumptions).
- The group $\hat{G}$ can be chosen in a sort-of natural way (independently of p), but there seems to be no "canonical" choice (we need to choose an appropriate countable model M).
- I have given a general (abstract) framework in which these sorts of results can be proved.