Linear orders, cyclic orders and hereditary G-compactness

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Outline

- The Lascar graph, Lascar distance and G-compactness.
- Cyclic orders, an example of a non-G-compact theory.
- Hereditary G-compactness and linear orders (unstable NIP implies not hereditarily G-compact[†]).

[†] modulo an open conjecture

Lascar graph

The *Lascar graph* consists of the following:

- vertices: small (also infinite!) tuples in $\mathfrak{C}$,
- edges: a, b are connected by an edge if for some $M \preceq \mathfrak{C}$ we have $a \equiv_M b$.

Then

- *Lascar distance* $d_{\mathrm{L}}(a, b)$ = length of the shortest path ($\in \mathbf{N} \cup \{\infty\}$).
- A *Lascar strong type* = a connected component of the Lascar graph.

Remark

If $d_{\mathrm{L}}(a, b) < \infty$, then $a \equiv b$ (and even $a \equiv_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)} b$).

- A theory is *G-compact* if all the Lascar strong types have finite diameter.

Fact

The theory T is G-compact if and only if
all the Lascar strong types have uniformly *bounded diameter.*

Examples

Example

Let T be the theory of dense linear orders without endpoints,
and take any small tuples $a \neq b \subseteq \mathfrak{C}$ such that $a \equiv b$. Then $d_{\mathrm{L}}(a, b) = 1$:
we can just take M that lies to the left of all elements of a, b and by q.e., $a \equiv_M b$.

Example

If $\operatorname{acl}(\emptyset)$ is a model, then T is G-compact (with diameters uniformly bounded by 1).

Example

If T is stable or, more generally, simple, then T is G-compact
(or even more generally, if T has NTP_2 and $\emptyset$ is an extension base).

Cyclic orders

Definition

A ternary relation $C(x, y, z)$ is a (strict, partial) cyclic order if it satisfies:

1. cyclicity: if $C(x, y, z)$, then $C(z, x, y)$,
2. asymmetry: if $C(x, y, z)$, then $\neg C(z, y, x)$,
3. transitivity: if $C(x, y, z)$ and $C(y, z, t)$, then $C(x, y, t)$.

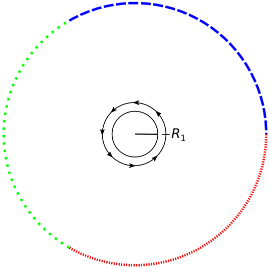
Example

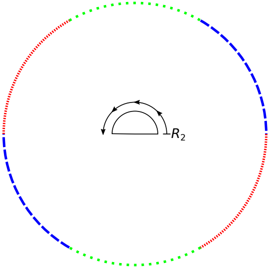
On the unit circle, $C(x, y, z)$ if y lies on the arc from x to z in the counterclockwise direction.

Example

If $(P, <)$ is a partially ordered set, then it has natural cyclic order structure, given by $C(x, y, z)$ if $x < y < z$, $y < z < x$, or $z < x < y$.

Cyclic order with a twist

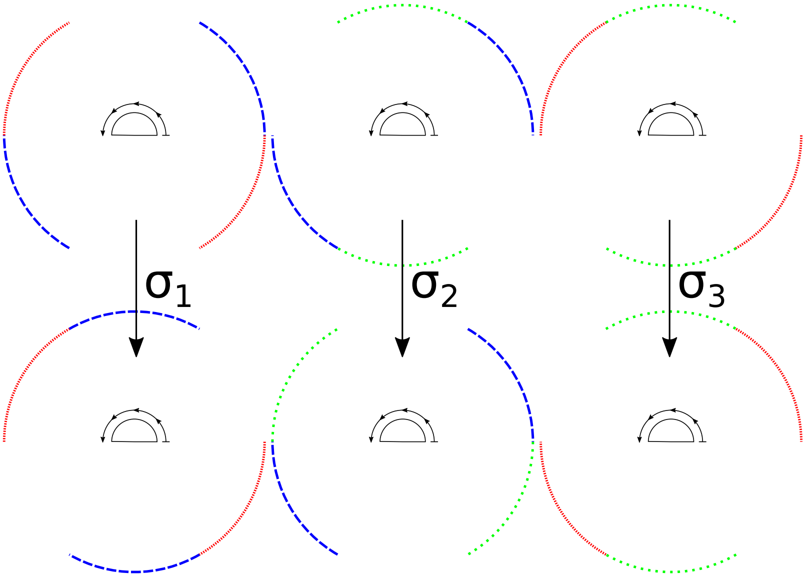




- Consider the structure $M_n = (\mathbf{T}, C, R_n)$, where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the unit circle, C is the cyclic order on $\mathbf{T}$, and R_n is the rotation by $1/n$.
- Let $\mathfrak{C}_n \succeq M_n$ be the monster model.
- For $j = 1, 2, 3$, let M_n^j be the submodel of M_n generated by $[(j-1)/3n, j/3n) + \mathbf{Z}$.
- Note that $M_n = \bigcup_j M_n^j$.
- Furthermore, one can show that $M_n^j \preceq M_n$ and $M_n^{j_1} \cup M_n^{j_2} \preceq M_n$.

(This example is due to Casanovas, Lascar, Pillay and Ziegler.)

R_n-orbits stay together



- For $a \in M_n^1$, $\sigma_3 \circ \sigma_2 \circ \sigma_1(a)$ is $R_n(a)$.
- Each σ_j is a partial elementary map, fixing $M_n^{j-1} \preceq \mathfrak{C}_n$.
- Thus $d_{\mathsf{L}}(a, R_n(a)) \leq 3$.
- It follows that $d_{\mathsf{L}}(a, R_n^k(a)) \leq 3|k|$ for all k.

R_n-orbits are large

Proposition

If $a, b \in M_n$ are adjacent in the Lascar graph, then $d(a, b) < 2/n$. It follows that in general, $d(a, b) < 2d_L(a, b)/n$ (where d is the metric from $\mathbf{R}$*).*

Corollary

For any n, if $k = \lfloor n/2 \rfloor$, then $d_L(a, R_n^k(a)) = \Theta(n)$.

Proof.

We have $d_L(a, R_n^k(a)) \leq 3k \leq 3n/2$. On the other hand, since $0 \leq k \leq n/2$,

$$d_L(a, R_n^k(a)) > d(a, R_n^k(a)) \cdot n/2 = (k/n) \cdot n/2 = k/2 \geq n/4 - 1$$

$\square$

A non-G-compact theory

Corollary

For any n, if $k = \lfloor n/2 \rfloor$, then $d_{\mathrm{L}}(a, R_n^k(a)) = \Theta(n)$.

Corollary

The theory of the structure $M = (M_n)_{n \in \mathbf{N}}$ is not G-compact.

Proof.

We can find a sequence $a = (a_n)_{n \in \mathbf{N}}$ of elements of M such that the diameter of the Lascar strong type of a_n is $\Omega(n)$. It follows that the diameters of the Lascar strong types are not uniformly bounded, so $\mathrm{Th}(M)$ is not G-compact.
(In fact, the Lascar strong type of a has infinite diameter.) $\square$

Hereditary G-compactness

Definition

We say that a theory T is hereditarily G-compact if for every $M \models T$ and every N interpreted in M (possibly with parameters), the theory $\mathrm{Th}(N)$ is G-compact.

Remark

If T is hereditarily G-compact and G is a group definable in a model of T, then $G^{00}_{\emptyset} = G^{000}_{\emptyset}$.

Example

Every stable theory, and, more generally, every simple theory is hereditarily G-compact.

Question

Is the converse true, i.e. is a hereditarily G-compact theory necessarily simple?

DLO is not hereditarily G-compact

Example

Consider the theory of dense linear orders without endpoints, and the model $(\mathbf{R}, <)$.
The structure M_n is clearly interpretable in $\mathbf{R}$ with parameters $\{0, 1/n, \ldots, (n-1)/n, 1\}$.
Thus, the structure $(M_n)_n$ is interpretable in $\mathbf{R}$ with parameters in $\mathbf{Q} \cap [0, 1]$.
Thus, dense linear orders are not hereditarily G-compact.

Generalised cyclic order with a twist

Definition

Let $(P, <)$ be a poset, and let n be a positive integer. Then $C_n(P)$ is the structure $(P \times \{1, \ldots, n\}, C, R_n)$, where C is the natural cyclic ordering, and R_n given by $R_n(p, i) = (p, i + 1)$ (where $(p, n + 1) = (p, 1)$).

- The structure M_n defined before is essentially $C_n([0, 1))$.
- In any case, P interprets each $C_n(P)$ (without parameters), and hence also $(C_n(P))_{n \in \mathbf{N}}$.

Properties of $C_n(P)$

Definition

We call a poset P *three-splitting* if the initial embeddings of P and $P \oplus P$ into $P \oplus P \oplus P$ are elementary.

- (Is this equivalent to saying that the embedding of P in $P \oplus P$ is elementary?)

Example

$\mathbf{Q}$ and $\mathbf{Z}$ are three-splitting (e.g. by quantifier elimination). The same is true for any models of their theories.

Lemma

If P is three-splitting, then for any $p \in P$, then the diameter of the Lascar strong type of $(p, i) \in C_n(P)$ is $\Omega(n)$.

Discrete linear orders are not hereditarily G-compact

Example

$\mathbf{Q}$ and $\mathbf{Z}$ are three-splitting (e.g. by quantifier elimination). The same is true for any models of their theories.

Lemma

If P is three-splitting, then for any $p \in P$, then the diameter of the Lascar strong type of $(p, i) \in C_n(P)$ is $\Theta(n)$.

Example

A discrete linear order without endpoints is not hereditarily G-compact.

Example

A discrete linear order with both endpoints is not hereditarily G-compact.

Extracting a dense and discrete linear order

Lemma

Suppose L is an $\aleph_0$-saturated infinite linear order.
Then there is some infinite $D \subseteq L$, definable in pure order language,
such that D is either dense or discrete.

Sketch of the proof.

If L contains arbitrarily long finite intervals, by $\aleph_0$-saturation, it contains an infinite discrete interval.
If not, the set $L' \subseteq L$ of all elements without a successor is densely ordered. $\square$

Corollary

An infinite linear order is never hereditarily G-compact.

A conjecture on unstable NIP theories

Conjecture

Suppose T is unstable NIP. Then T interprets an infinite linear order.

- It is known that such T $\bigvee$-interprets an infinite linear orders (Simon).
- It follows that it is true for ω*-categorical* T.
- It is also known to hold if T is unstable „weakly VC-minimal" (Guingona, Laskowski).

Corollary

If the conjecture holds, then every unstable NIP theory is not hereditarily G-compact.

Corollary

If the conjecture holds, then every unstable NIP theory is not hereditarily G-compact.

- What about theories with IP?
- In general, the reduction to linear orders is probably not enough.
  E.g. atomless Boolean algebras do not interpret an infinite linear order (I don't know if they are hereditarily G-compact).
- On the other hand, it seems like all examples of non-G-compactness are essentially NIP.