The $PSU(3)$ invariant of the Poincaré homology sphere.

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Abstract

Using the R-matrix formulation of the sl_3 invariant of links, we compute the coloured sl_3 generalised Jones polynomial for the trefoil. From this, the $PSU(3)$ invariant of the Poincaré homology sphere is obtained. This takes complex number values at roots of unity. The result obtained is formally an infinite sum, independent of the order of the root of unity, which at roots of unity reduces to a finite sum. This form enables the derivation of the $PSU(3)$ analogue of the Ohtsuki series for the Poincaré homology sphere, which it was shown by Thang Le could be extracted from the $PSU(N)$ invariants of any rational homology sphere.

Key words: Ohtsuki series, coloured Jones polynomial, quantum groups, trefoil knot, quantum invariants, PSU(3)
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1 Introduction

Suppose that M is a compact oriented 3–manifold without boundary. For any Lie algebra, g, and integral level, k, there is defined an invariant, $Z_{k+\check{c}_g}(M,L)$, of embeddings of links L in M, known as the Witten-Reshetikhin-Turaev invariant (see [16], [10], [15]). It is known that for links in S^3, $Z_K(S^3,L)$ is a polynomial in $q=\exp\frac{2\pi i}{K}$, namely the generalised Jones polynomial of the link L.

Now assume that M is a rational homology sphere, with $H=|H_1(M,\mathbf{Z})|$. In the normalisation for which the invariant for S^3 is 1, denote the invariant

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for the pair $(M, \emptyset)$, as an algebraic function of q at K^{th} roots of unity, by $Z_K(M)$. For a rational homology sphere M, $G = PSU(N)$ and odd prime K, $Z_K(M) \in \mathbf{Z}[q]$ (see [1], [7], [8] and [14]), so that for some $a_{m,K}(M) \in \mathbf{Z}$, one has

$$Z_K(M) = \sum_{m=0}^{\infty} a_{m,K}(M)(q-1)^m .$$

Although the $a_{m,K}$ are not uniquely determined, it is known from [5] that there exist rational numbers $\lambda_m(M)$ such that,

$$a_{m,K}(M) \equiv \lambda_m(M)$$

as elements of $\mathbf{Z}_K$ for all sufficiently large primes K. For integer homology spheres, $\lambda_0(M) = 1$ and $\lambda_1(M) = N(N^2-1)\lambda(M)$ where $\lambda(M)$ denotes the $(SU(2)$–$)$Casson-Walker invariant of M in Casson's normalisation. As a result, one may define a formal power series

$$Z_\infty(M) = \sum_{m=0}^{\infty} \lambda_m (q-1)^m ,$$

with rational coefficients, which is an invariant of rational homology 3–spheres, M, known as the *perturbative invariant* of M. This is expected to be the asymptotic expansion of the trivial connection contribution to $Z_K(M)$.

In this paper we use the R-matrix presentation of link and manifold invariants from [10], to compute the $PSU(3)$ invariant for the Poincaré homology sphere. Various calculations for the Lie group $PSU(3)$ have been carried out elsewhere; we mention [9], [12] and [13] specifically for $PSU(3)$ computations while [3] and [4] obtained more general results on $PSU(N)$ invariants. However the current paper gives, to our knowledge, the first explicit calculation of the coloured Jones polynomial of the trefoil, expressed in a form from which the $PSU(3)$ perturbative invariant of the PHS can be directly obtained.

An outline of the present paper is as follows. In §2, the basic theory associated with the quantum group $U_q sl_3$ will be summarised. In §3, we follow [10] and [5] to give a description of the sl_3 coloured Jones polynomial of a link, and of the $PSU(3)$ invariant of 3-manifolds. This is applied in the last section to compute the coloured Jones polynomial of the trefoil and hence to to demonstrate how the coefficients in the $PSU(3)$ perturbative invariant of the Poincaré homology sphere, may be computed. This is demonstrated for the coefficient of h, which is verified to be in agreement with the appropriate multiple of the Casson invariant, while the results of a computer computation of a few more terms are given.

2 R-matrix and representation theory for $U_q sl_3$

Let $q = v^2 = e^h$ be a formal parameter. Define q-numbers, q-factorials and q-binomial coefficients according to

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}\,, \qquad [n]! = \prod_{i=1}^{n} [i]\,, \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}\,.$$

The dependence on q will be omitted from the notation.

2.1 *The quantum group $U_q sl(3)$*

The quantum group $A = U_q sl(3)$ is defined by generators H_i, X_i and Y_i for $i = 1, 2$ with relations

$$\begin{aligned}
&[H_i, X_j] = A_{i,j} X_j\,, \quad [H_1, H_2] = 0 \\
&[H_i, Y_j] = -A_{i,j} Y_j\,, \quad [X_i, Y_j] = \delta_{i,j} \tfrac{e^{hH_i/2} - e^{-hH_i/2}}{v - v^{-1}}\,, \\
&X_i^2 X_j - [2] X_i X_j X_i + X_j X_i^2 = 0\,, \quad \text{for } |i-j| = 1\,, \\
&Y_i^2 Y_j - [2]\, Y_i Y_j Y_i + Y_j Y_i^2 = 0\,, \quad \text{for } |i-j| = 1\,,
\end{aligned}$$

where $A_{i,j}$ denotes the matrix elements of the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The comultiplication $\Delta\colon A \longrightarrow A \otimes A$ is given by

$$\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i\,, \\
\Delta(X_i) &= X_i \otimes e^{hH_i/4} + e^{-hH_i/4} \otimes X_i\,, \\
\Delta(Y_i) &= Y_i \otimes e^{hH_i/4} + e^{-hH_i/4} \otimes Y_i\,,
\end{aligned}$$

with antipode $S\colon A \longrightarrow A$ being an antihomomorphism acting on the generators by

$$S(H_i) = -H_i\,, \qquad S(X_i) = -v X_i\,, \qquad S(Y_i) = -v^{-1} Y_i\,.$$

It is convenient to introduce $E_i = X_i e^{-hH_i/4}$ and $F_i = Y_i e^{hH_i/4}$. This then represents A by algebra generators H_i, E_i and F_i with $i = 1, 2$ and the following relations

$$
\begin{aligned}
[H_i, E_j] &= A_{i,j} E_j \ , \quad [H_1, H_2] = 0 \\
[H_i, F_j] &= -A_{i,j} F_j \ , \ \ [E_i, F_j] \ = \delta_{i,j} \tfrac{e^{hH_i/2} - e^{-hH_i/2}}{1-q} \ , \\
E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 &= \ 0 \ , \quad \text{for } |i-j| = 1 \ , \\
F_i^2 F_j - [2] \, F_i F_j F_i \ + F_j F_i^2 &= \ 0 \ , \quad \text{for } |i-j| = 1 \ ,
\end{aligned}
$$

The comultiplication is now given by

$$
\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i \ , \\
\Delta(E_i) &= E_i \otimes 1 + e^{-hH_i/2} \otimes E_i \ , \\
\Delta(F_i) &= F_i \otimes e^{hH_i/2} + 1 \otimes F_i \ ,
\end{aligned}
$$

while $S(H_i) = -H_i$, $S(E_i) = -e^{hH_i/2} E_i$ and $S(F_i) = -F_i e^{-hH_i/2}$.

There is an adjoint action defined by $\mathrm{ad} = (L \otimes R)(\mathrm{Id} \otimes S)\Delta$, under which E_i acts according to

$$
(\mathrm{ad} E_i) x = E_i x - e^{-hH_i/2} x e^{hH_i/2} E_i \ .
$$

The opposite comultiplication $\Delta' = P \circ \Delta$ induces an adjoint action defined by $\mathrm{ad}' = (L \otimes R)(\mathrm{Id} \otimes S')\Delta'$ under which F_i acts by

$$
(\mathrm{ad}' F_i) x = F_i x - e^{hH_i/2} x e^{-hH_i/2} F_i \ .
$$

Using these actions we define, for $1 \leq i < j \leq 3$, E_{ij} and F_{ij} according to $E_{ii+1} = E_i$, $F_{ii+1} = F_i$ while

$$
\begin{aligned}
E_{13} &= (\mathrm{ad} E_1) E_2 = E_1 E_2 - v E_2 E_1 \ , \\
F_{13} &= (\mathrm{ad}' F_1) F_2 = \ F_1 F_2 - v F_2 F_1 \ .
\end{aligned}
$$

These new elements interact with H_i, E_i and F_i according to

$$
\begin{aligned}
[H_i, E_{13}] &= E_{13} \ , & [H_i, F_{13}] &= -F_{13} \ , \\
E_1 E_{13} &= v^{-1} E_{13} E_1 \ , & F_1 F_{13} &= v^{-1} F_{13} F_1 \ , \\
E_2 E_{13} &= v E_{13} E_2 \ , & F_2 F_{13} &= v F_{13} F_2 \ , \\
[E_1, F_{13}] &= v F_2 e^{-hH_1/2} \ , & [E_2, F_{13}] &= -q F_1 e^{hH_2/2} \ , \\
[E_{13}, F_1] &= -q e^{hH_1/2} E_2 \ , & [E_{13}, F_2] &= v e^{-hH_2/2} E_1 \ , \\
[E_{13}, F_{13}] &= \tfrac{q}{q^{-1}-1} (e^{h(H_1+H_2)/2} - e^{-h(H_1+H_2)/2}) \ .
\end{aligned}
$$

There are three natural copies of $U_q sl_2$ inside $U_q sl_3$, namely those generated by the triples (H_1, E_1, F_1), (H_2, E_2, F_2) and $(H_1 + H_2, E_{13}, -q^{-1}F_{13})$.

There is a Poincaré-Birkhoff-Witt theorem [11] which allows the subalgebra generated by E_i for $i = 1, 2$, to be expressed as a vector space with basis elements of the form $E_2^a E_{13}^b E_1^c$, with $a, b, c \in \mathbf{Z}^+$. Similarly for the subalgebra generated by F_1 and F_2.

2.2 Finite dimensional $U_q sl(3)$-modules

The diagonal matrices $\begin{pmatrix} e^{2\pi i x_1} & & \\ & e^{2\pi i x_2} & \\ & & e^{2\pi i x_3} \end{pmatrix}$ with $x_1 + x_2 + x_3 = 0$, form a maximal torus in SL_3. The roots are $x_i - x_j$ with standard root basis $\{\alpha_1, \alpha_2\}$ with $\alpha_i = x_i - x_{i+1}$. The standard inner product then has $(\alpha_i, \alpha_j) = A_{ij}$. Dual to the standard root basis we have the standard weight basis $\{\lambda_1, \lambda_2\}$ with $(\alpha_i, \lambda_j) = \delta_{ij}$. In our case, $\lambda_i = \sum_j B_{ij}\alpha_j$, where (B_{ij}) is the inverse matrix to (A_{ij}),

$$\lambda_1 = {}^1\!/\!_3(2\alpha_1 + \alpha_2)\,, \qquad \lambda_2 = {}^1\!/\!_3(\alpha_1 + 2\alpha_2)\,.$$

The positive roots are $x_i - x_j$ for $i < j$, and half the sum of the positive roots is $\rho = x_1 - x_3 = \lambda_1 + \lambda_2 = \alpha_1 + \alpha_2$, also being the longest root.

The Cartan subalgebra of A is generated by H_1 and H_2.

For generic values of q (that is, away from roots of unity), any finite-dimensional irreducible representation of $sl(3)$ has a deformation which is an irreducible representation of A. Such a finite-dimensional irreducible A-module, V_λ, has a highest weight vector u, for which $E_1 u = E_2 u = 0$, and is spanned as a linear space by $F_2^m F_{13}^n F_1^r u$ with m, n and r non-negative integers. The action of H_i on v is as scalars, $H_i u = a_i u$, where $\lambda = a_1\lambda_1 + a_2\lambda_2$ is the highest weight of the module (a_1 and a_2 are arbitrary non-negative integers). With respect to the λ-basis for weights, the actions of F_1, F_{13} and F_2 change the weight by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, respectively.

It will sometimes be convenient to use the basis $\{\alpha_1, \alpha_2\}$ for weights with respect to which λ has coordinates $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} {}^2\!/\!_3 a_1 + {}^1\!/\!_3 a_2 \\ {}^1\!/\!_3 a_1 + {}^2\!/\!_3 a_2 \end{pmatrix}$. Coordinates of weights will now be elements of ${}^1\!/\!_3\mathbf{Z} \times {}^1\!/\!_3\mathbf{Z}$ whose sum of coordinates is integral. The actions of F_1, F_{13} and F_2 change the weight in the α basis by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ respectively. Indeed the weights occurring in V_λ form a hexagonal array

$$\left\{ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \middle| c_i \in b_i + \mathbf{Z}, -b_2 \le c_1 \le b_1, -b_1 \le c_2 \le b_2, -b_1 \le c_1 - c_2 \le b_2 \right\}$$

with vertices

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_2 - b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} -b_2 \\ b_1 - b_2 \end{pmatrix}, \begin{pmatrix} -b_2 \\ -b_1 \end{pmatrix}, \begin{pmatrix} b_2 - b_1 \\ -b_1 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_1 - b_2 \end{pmatrix}.$$

The dimension of the weight space with weight $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is $1 + \min(b_1 - c_1, b_2 - c_2, b_2 + c_1, b_1 + c_2, b_1 + c_1 - c_2, b_2 + c_2 - c_1)$.

The q-Gel'fand-Tsetlin bases for irreducible modules are indexed by tableaux

$$\begin{pmatrix} m_{1,3} & & m_{2,3} & & m_{3,3} \\ & m_{1,2} & & m_{2,2} & \\ & & m_{1,1} & & \end{pmatrix}, \quad \text{where } m_{i,j+1} \le m_{i,j} \le m_{i+1,j+1}\,,$$

where the top row is fixed for a given module. In our case, $m_{1,3} = a_1 + a_2$, $m_{2,3} = a_2$ and $m_{3,3} = 0$. The dimension of this module is $\frac{1}{2}(a_1+1)(a_2+1)(a_1+a_2+2)$.

2.3 Universal R-matrix for A

Rosso [11] gave a multiplicative formula for the universal R-matrix for the quantum group $U_q sl(N+1)$ (see also [2] and [6] where general formulae for $U_q g$ are given),

$$R = \prod_{k=1}^{n} e((1-q^{-1})^2(-q)^{1-l(\beta(k))} E_{\beta(k)} \otimes F_{\beta(k)}; q^{-1}) \cdot \exp\left(\frac{h}{2} t_0\right),$$

where $n = \frac{1}{2}N(N+1)$ is the number of positive roots, $\beta(1), \ldots, \beta(n)$ enumerates the positive roots in an order generated by a choice of minimal representation of the longest element of the Weyl group and $t_0 = \sum_{i,j} B_{ij} H_i \otimes H_j$. Here we use the quantum factorial

$$e(u; q) = \sum_{m=0}^{\infty} \frac{u^m}{(1-q)\ldots(1-q^m)}.$$

In our case $N = 2$, $n = 3$ and the order $\beta(k)$ is $x_2 - x_3$, $x_1 - x_3$ and $x_1 - x_2$. These elements have lengths 1, 2 and 1, respectively, while $E_{x_i - x_j} = E_{ij}$ and

$F_{x_i - x_j} = F_{ij}$, so that

$$R = e((1-q^{-1})^2 E_2 \otimes F_2; q^{-1}) e((1-q^{-1})^2(-q^{-1}) E_{13} \otimes F_{13}; q^{-1})$$
$$e((1-q^{-1})^2 E_1 \otimes F_1; q^{-1}) \cdot e^{\frac{h}{6}(2H_1 \otimes H_1 + 2H_2 \otimes H_2 + H_1 \otimes H_2 + H_2 \otimes H_1)} .$$

This can be written as a sum over non-negative integers p, s, m, n and r

$$R = \sum_{p,s,m,n,r} \frac{(h/6)^{p+s}}{p!s!} v^{\frac{1}{2}m(m-3)+\frac{1}{2}n(n-7)+\frac{1}{2}r(r-3)} (-1)^n \frac{(v-v^{-1})^{m+n+r}}{[m]![n]![r]!}$$
$$E_2^m E_{13}^n E_1^r (2H_1 + H_2)^p (H_1 + 2H_2)^s \otimes F_2^m F_{13}^n F_1^r H_1^p H_2^s .$$

As in any quasi-triangular Hopf algebra, $R^{-1} = (\mathrm{Id} \otimes S^{-1})R$. Using the fact that $S^{-1}(H_i) = -H_i$, $S^{-1}(E_i) = -E_i e^{hH_i/2}$, $S^{-1}(F_i) = -e^{-hH_i/2} F_i$ and $S^{-1}(F_{13}) = -qe^{-h(H_1+H_2)/2}(F_1F_2 - v^{-1}F_2F_1)$, we obtain

$$R^{-1} = \sum_{p,s,m,n,r} \frac{(-h/6)^{p+s}}{p!s!} v^{-\frac{1}{2}[m(m+1)+n(n+1)+r(r+1)]-rn+m(r-n)} \frac{(v^{-1}-v)^{m+n+r}}{[m]![n]![r]!}$$
$$(-1)^n E_2^m E_{13}^n E_1^r (2H_1 + H_2)^p (H_1 + 2H_2)^s$$
$$\otimes H_1^p H_2^s v^{-(r+n)H_1-(n+m)H_2} F_1^r (F_1F_2 - v^{-1}F_2F_1)^n F_2^m .$$

Finally, the element $K \in A$ which will be placed at suitably oriented local maxima/minima in tangle diagrams (see §3) is $K = q^{H_1+H_2}$.

3 Definition of the $PSU(3)$ manifold invariant

3.1 The generalised Jones polynomial

We will use the formulation of the generalised Jones polynomial [10]. Suppose L is an oriented link in S^3 and A is a ribbon Hopf algebra. That is, A is a quasi-triangular Hopf algebra with universal R-matrix, $R \in A \otimes A$ satisfying the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. The square of the antipode is then given by

$$S^2(a) = uau^{-1} , \qquad \text{for all } a \in A ,$$

where $u = m(S \otimes \mathrm{Id})R_{21}$ and $uS(u)$ is central. A is also equipped with a ribbon element v such that

$$v^2 = uS(u) , \qquad S(v) = v , \qquad \epsilon(v) = 1 , \Delta(v) = (R_{12}R_{21})^{-1}(v \otimes v) .$$

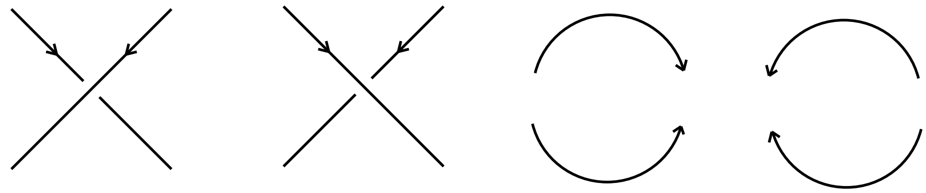

Fig. 1. Generators of the category of tangles

Let $K = v^{-1}u$ so that $S(K) = K^{-1}$.

The *generalised coloured Jones polynomial* of L is defined when each component L_i is coloured by a representation V_i of A and will be denoted by $J_L(V_1, \ldots, V_c)$. We will assume for simplicity that V_i are irreducible. Indeed, according to [10], there is a functor, $\mathcal{F}$, from the category of ribbon tangles to the category of vector spaces, under which links (closed tangles) map to scalars, namely J_L. This functor is defined by its images on the generators shown in Figure 1.

The two orientations of crossing, in which the strands are labelled with representations V and W as shown, have as images the maps $V \otimes W \longrightarrow W \otimes V$ given by $P \circ R$ and $R^{-1} \circ P$, respectively, where P is the permutation of the factors. The four cup and cap sections transform to maps

$$\begin{aligned} V^* \otimes V &\longrightarrow \mathbf{C} \\ (x, y) &\longmapsto x(y) \end{aligned} \qquad \begin{aligned} V \otimes V^* &\longrightarrow \mathbf{C} \\ (y, x) &\longmapsto x(Ky) \end{aligned}$$

and

$$\begin{aligned} \mathbf{C} &\longrightarrow V \otimes V^* \\ 1 &\longmapsto \textstyle\sum_i e_i \otimes e^i \end{aligned} \qquad \begin{aligned} \mathbf{C} &\longrightarrow V^* \otimes V \\ 1 &\longmapsto \textstyle\sum_i K^{-1}e^i \otimes e_i \end{aligned}$$

respectively. Here $\{e^i\}$ is the dual basis for V^* to the basis $\{e_i\}$ for V.

Suppose that L has one component and T is a 1-tangle representation of L, that is $T \in \mathrm{Morph}(a, a)$, where a is the object consisting of one downward oriented point, such that its closure is ambient isotopic to L. Then the above prescription can be used to generate a map $V \longrightarrow V$, for any representation V (the colour of the one open strand). This map commutes with the action of A, so that if A is irreducible it is given by multiplication by a scalar, namely

$$J'_L(V) = \frac{J_L(V)}{J_U(V)},$$

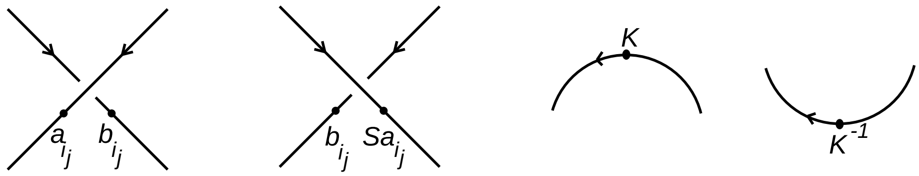

Fig. 2. Decorations on 1-tangle elements

where U is the unknot with framing zero. Indeed $J_U(V)$ is precisely the quantum dimension of the representation V. In this case, the above functorial prescription can be realised from T in the following way. Suppose T is presented as a tangle diagram in generic position, in which the crossings are between downward oriented strands. In the neighbourhood of each crossing decorate the crossing stands by elements of A according to Figure 2, and at local maxima or minima oriented leftwards decorate the strand by $K \in A$. Here $R = \sum_i \alpha_i \otimes \beta_i$ where $\alpha_i, \beta_i \in A$ and i lies in an indexing set. For a tangle with several crossings, i_j denotes an element of the indexing set assigned to the j^{th} crossing.

Then $\mathcal{F}(T)$ is the algebra element read off the diagram by tracing the tangle strand according to its orientation, and writing down the decorations in the form of a product from left to right, and then summing over all labels i_j.

3.2 *PSU*(3) 3-*manifold invariant*

As in [5], we will find it convenient to index irreducible A-modules as $V_{\mu-\rho}$ where $\mu = \lambda + \rho \in \Lambda_{++}$. Thus if $\mu = k_1\lambda_1 + k_2\lambda_2 = c_1\alpha_1 + c_2\alpha_2$, then $k_i = a_i + 1$ and $c_i = b_i + 1$. The dimension of V_λ is $\frac{1}{2}k_1k_2(k_1 + k_2)$ while the quantum dimension (Jones polynomial of the unknot with zero framing) is $[\mu] = [k_1][k_2][k_1 + k_2]/[2]$.

Suppose that q is a K^{th} root of unity. The prescription of [5] defines for a link L with c components and framing $f_i = \pm 1$ on the i^{th} component,

$$\langle L \rangle = \sum_{\mu_1,\ldots,\mu_c} J'_L(\mu_1, \ldots, \mu_c) \prod_{i=1}^{c} [\mu_i]^2 q^{\sum_i f_i \Omega(\mu_i)} ,$$

where J'_L denotes the coloured Jones polynomial coloured by representations $V_{\mu_i+\rho}$, normalised to 1 on the unknot and the sum is over representations μ_i for which k_1 and k_2 are integers congruent modulo 3 with $0 < k_i < K$ and $k_1 + k_2 < K$. As noted in [5], if J'_L is extended using certain symmetry properties, then equivalently (up to a constant which cancels in the computation of Z_K), the sum may be taken over representations for which $c_i \in \{1, \ldots, K-1\}$. Here

$\Omega(\mu)$ is the framing normalisation

$$\begin{aligned}\Omega(\mu) = q^{1/3(a_1^2+a_1a_2+a_2^2)+a_1+a_2} &= q^{b_1^2-b_1b_2+b_2^2+b_1+b_2} \\ = \quad q^{1/3(k_1^2+k_1k_2+k_2^2)-1} &= q^{c_1^2-c_1c_2+c_2^2-1} .\end{aligned}$$

The $PSU(3)$ invariant of a 3-manifold M obtained from S^3 by surgery around a link L is then

$$Z_K(M) = G_+^{-\sigma_+} G_-^{-\sigma_-} \langle L \rangle ,$$

where $\sigma_\pm$ are the numbers of positive/negative eigenvalues of the linking matrix of L and $G_\pm$ are the bracket values for the unknot with framings ± 1. When L is a knot ($c = 1$) with framing $f = \pm 1$, this reduces to

$$Z_K(M) = \frac{\sum_\mu q^{f\Omega(\mu)} [\mu]^2 J_L'(V_{\mu+\rho})}{\sum_\mu q^{f\Omega(\mu)} [\mu]^2} . \tag{1}$$

3.3 The $PSU(3)$ perturbative invariant

Writing the coloured Jones polynomial as a power series in h ($q = e^h$), the coefficients are polynomials in the colour. More precisely, a version of the Melvin–Morton–Rozansky conjecture allows a representation in the form

$$J_L'(V_{\mu+\rho}) = \sum_{i_1,i_2} a_{i_1 i_2}(h) \cdot q^{i_1 k_1 + i_2 k_2} , \tag{2}$$

where $a_{i_1 i_2}$ are power series in h, and the sum is over integers i_1 and i_2.

By translating the summand, one can show, as for one-dimensional Gauss sums, that

$$\sum_{k_1,k_2} q^{i_1 k_1 + i_2 k_2} \cdot q^{f/3(k_1^2+k_1k_2+k_2^2)} = q^{-f(i_1^2 - i_1 i_2 + i_2^2)} \sum_{k_1,k_2} q^{f/3(k_1^2+k_1k_2+k_2^2)} , \tag{3}$$

where the sum is over the parallelogram in c_i's mentioned in the previous section (equivalently, over a fundamental region with k_1 and k_2 congruent modulo 3, up to translations in the k_1 and k_2 directions, say modulo $3K$). Since $[k]^2 = (v - v^{-1})^{-2}(q^k + q^{-k} - 2)$, we can expand $[\mu]^2$ as

$$[k_1]^2 [k_2]^2 [k_1 + k_2]^2 = (v - v^{-1})^{-6} \sum_{a,b} C_{ab} q^{ak_1 + bk_2} ,$$

where f_{ab} take non-zero integer values for precisely 17 lattice points (a,b), as represented by the matrix

$$C = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 2 & -2 & 0 \\ 1 & 2 & -6 & 2 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

in which the lattice coordinates a and b take the values -2, -1, 0, 1 and 2. Evaluating the individual sums obtained from this expansion, using 3, it follows that

$$\sum_{k_1,k_2} [k_1]^2[k_2]^2[k_1+k_2]^2 q^{f/3(k_1^2+k_1k_2+k_2^2)} = -\frac{6[2]fq^{-2f}}{(v-v^{-1})^3} \sum_{k_1,k_2} q^{f/3(k_1^2+k_1k_2+k_2^2)} .$$

Equation (1) now gives

$$Z_K(M) = -\frac{1}{6}[2]fq^{2f}(v-v^{-1})^3 \sum_{i_1,i_2} b_{i_1i_2} q^{-f(i_1^2-i_1i_2+i_2^2)} , \tag{4}$$

where

$$[\mu]^2 J'_L(V_{\mu+\rho}) = \sum_{i_1,i_2} b_{i_1i_2} q^{i_1k_1+i_2k_2} ,$$

and $b_{i_1i_2}$ are Laurent power series in h. A term $q^{i_1k_1+i_2k_2}$ in J'_L contributes $q^{i_1k_1+i_2k_2}[\mu]^2 = (v-v^{-1})^{-6}[2]^{-2}\sum_{a,b} C_{ab} q^{(i_1+a)k_1+(i_2+b)k_2}$ to $J'_L[\mu]^2$ and therefore its contribution to $Z_K(M)$ is, according to (4)

$$\begin{aligned} &\tfrac{f}{6}[2]^{-1}\,(v-v^{-1})^{-3} q^{-f(i_1^2-i_1i_2+i_2^2)} \\ &\quad [6q^{2f} - 2q^f(q^{2i_1-i_2}+q^{i_2-2i_1}+q^{2i_2-i_1}+q^{i_1-2i_2}+q^{i_1+i_2}+q^{-i_1-i_2}) \\ &\quad +2q^{-f}(q^{3i_1}+q^{-3i_1}+q^{3i_2}+q^{-3i_2}+q^{3i_1-3i_2}+q^{3i_2-3i_1}) \\ &\quad -q^{-2f}(q^{4i_1-2i_2}+q^{2i_2-4i_1}+q^{4i_2-2i_1}+q^{2i_1-4i_2}+q^{2i_1+2i_2}+q^{-2i_1-2i_2})] . \end{aligned}$$

It can be directly verified that this represents a power series in h (with only non-negative powers of h) $\sum_{n=0}^{\infty} p_n(i_1,i_2)h^n$ where the first three terms are given by

$$p_0 = 1 , \qquad p_1 = -4f\Delta , \qquad p_2 = 5\Delta^2 + 2\Delta ,$$

where $\Delta = i_1^2 - i_1 i_2 + i_2^2$. More generally, p_n is a polynomial in i_1 and i_2 of degree at most $2n$. Thus we have a linear map from terms in $J'_L(V_{\rho+\mu})$ to terms in $Z_r(M)$, over the ring of formal power series in h,

$$\begin{aligned} \tau: \qquad B &\longrightarrow \mathbf{Q}[[h]] \\ q^{i_1 k_1 + i_2 k_2} &\longmapsto \sum_{n=0}^{\infty} p_n(i_1, i_2) h^n \,, \end{aligned}$$

where B is the ring generated by $q^{i_1 k_1 + i_2 k_2}$ over $\mathbf{Q}[[h]]$. The coefficient p_n itself can be extended linearly to a map $B \longrightarrow \mathbf{Q}[[h]]$, and can be expressed as the formal evaluation at $k_1 = k_2 = 0$ of a differential operator of order $2n$ (in $h^{-1}\frac{\partial}{\partial k_1}$ and $h^{-1}\frac{\partial}{\partial k_2}$). The following lemma is an immediate consequence, and will be used in the next section to compute the first coefficients in the perturbative $PSU(3)$ invariant of the Poincaré homology sphere.

Lemma *If $x \in B$ is a product of r terms ($r \geq 2n$) of the form $q^{a_1 k_1 + a_2 k_2 + b} - q^{a'_1 k_1 + a'_2 k_2 + b'}$ (with a_i, a'_i, b and b' integers), then $p_n(x)$ is divisible by h^{r-2n} and hence $\tau(x)$ is divisible by $h^{\left[\frac{r+1}{2}\right]}$.*

For $K \in \mathbf{N}$, J'_L reduces to a finite combination of terms as in (2), that is the sum over i_1 and i_2 becomes a finite sum. Then $Z_K(M) = \tau(J'_L)$. On the other hand, one can always write J'_L in the form of an infinite sum of form (2), the only dependence on K being via the variable q, which is such that at a root of unity, all but a finite number of terms vanish. Thus, so long as certain formal convergence conditions are satisfied by J'_L (which it can be seen from the R-matrix description of J'_L are always satisfied), $\tau(J'_L)$, where J'_L is considered as an infinite series (2), will be a well-defined power series in $\mathbf{Q}[[h]]$. This is $Z_\infty(M)$, the $PSU(3)$ perturbative invariant of M.

4 Evaluations on the trefoil

4.1 Diagrammatric evaluation

We use the 1-tangle realisation, T, of the trefoil knot shown in Figure 3. The prescription given in the last section gives

$$\mathcal{F}(T) = \sum_{i_1 i_2 i_3} \beta_{i_3} \alpha_{i_2} \beta_{i_1} K \alpha_{i_3} \beta_{i_2} \alpha_{i_1} \,.$$

Suppose that V is an irreducible representation of $U_q sl_3$. Then $\mathcal{F}(T)$ will act as multiplication by $J'_{L'}(V)$ on V, and this action may therefore be determined by

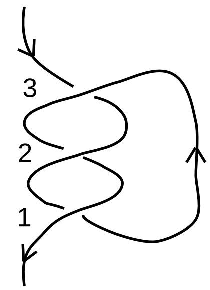

Fig. 3. Trefoil

calculating the action on the highest weight vector u. Here L' is the trefoil with framing three. Using the formula for the universal R-matrix given in the last section, we see that α_{i_1} is a product of H_i's with $E_2^{m_1}E_{13}^{n_1}E_1^{r_1}$, which annihilates u unless $m_1 = n_1 = r_1 = 0$. Similarly from β_{i_3}, we see that $m_3 = n_3 = r_3 = 0$. Thus we obtain

$$J'\ _{L'}(V)u = \sum_{\substack{p_1,s_1,p_2,s_2,p_3,s_3\\ m,n,r}} \frac{\left(\frac{h}{6}\right)^{p_1+s_1+p_2+s_2+p_3+s_3}}{p_1!s_1!p_2!s_2!p_3!s_3!}(-1)^n \frac{(v-v^{-1})^{m+n+r}}{[m]![n]![r]!}$$
$$v^{\frac{1}{2}[m(m-3)+n(n-7)+r(r-3)]} H_1^{p_3}H_2^{s_3}E_2^mE_{13}^nE_1^r(2H_1+H_2)^{p_1}(H_1+2H_2)^{s_2}H_1^{p_1}H_2^{s_1}$$
$$K(2H_1+H_2)^{p_2}(H_1+2H_2)^{s_2}F_2^mF_{13}^nF_1^rH_1^{p_2}H_2^{s_2}(2H_1+H_2)^{p_1}(H_1+2H_2)^{s_1}u\,.$$

Performing the sum over the p_i's and s_i's gives

$$J'_{L'}(V)u = \sum_{m,n,r}(-1)^n\frac{(v-v^{-1})^{m+n+r}}{[m]![n]![r]!}E_2^mE_{13}^nE_1^nF_2^mF_{13}^nF_1^ru$$
$$q^{a_1^2+a_1a_2+a_2^2+a_1+a_2}v^{\frac{1}{2}[m(m-7)+n(n-15)+r(r-7)]-3ma_2-3n(a_1+a_2)-3ra_1}\,.$$

A change of framing by 1 contributes a factor $q^{1/3(a_1^2+a_1a_2+a_2^2)+a_1+a_2}$, as can be verified by computing the action of $\sum_i \beta_i K\alpha_i$, coming from a 1-tangle representation of the unknot with framing 1. Thus the coloured Jones polynomial of the trefoil with zero framing, L, is given by

$$\frac{J_L(V)}{J_U(V)}u = \sum_{m,n,r}(-1)^n\frac{(v-v^{-1})^{m+n+r}}{[m]![n]![r]!}E_2^mE_{13}^nE_1^nF_2^mF_{13}^nF_1^ru$$
$$v^{\frac{1}{2}[m(m-7)+n(n-15)+r(r-7)]-3ma_2-3n(a_1+a_2)-3ra_1-4a_1-4a_2}\,.$$

4.2 Quantum group calculation

We shall calculate the action of $E_2^mE_{13}^nE_1^rF_2^mF_{13}^nF_1^r$ on the highest weight vector u in the irreducible A-module, V_λ with highest weight $\lambda = a_1\lambda_1 + a_2\lambda_2$. This action will be multiplication by a scalar.

Let $u_{m,n,r} = F_2^m F_{13}^n F_1^r u$. By the Poincaré-Birkhoff-Witt theorem [11], a basis for the subalgebra of A generated by F_1 and F_2 is $\{F_2^m F_{13}^n F_1^r\}_{m,n,r}$, and therefore the vectors $u_{m,n,r}$ span V_λ.

Since $[E_1, F_{13}] = v^2 e^{-hH_1/2} F_2$, we can inductively show that

$$E_1^s F_{13} = F_{13} E_1^s + v^{s+1}[s] e^{-hH_1/2} F_2 E_1^{s-1} \ .$$

Again using induction, one obtains

$$E_1^s F_{13}^n = \sum_{j=0}^{\min n,s} v^{j(j+s-n+1)} \begin{bmatrix} n \\ j \end{bmatrix} \frac{[s]!}{[s-j]!} e^{-hjH_1/2} F_2^j F_{13}^{n-j} E_1^{s-j} \ .$$

Since $\{F_1^p u\}_{p=0}^{a_1}$ is a $U_q sl_2$-module, under the action of the subalgebra generated by H_1, E_1 and F_1, thus

$$E_1^t F_1^r u = v^t \frac{[r]![a_1 - r + t]!}{[r-t]![a_1 - r]!} F_1^{r-t} u \ . \tag{5}$$

Applying (4.2) to the vector $F_1^r u$ and using with $t = s - j$, we find that $E_1^s u_{m,n,r} = F_2^m (E_1^s F_{13}^n) F_1^r u$ is given by

$$E_1^s u_{m,n,r} = \sum_{j=\max 0,s-r}^{\min n,s} v^{j(j-s-a_1+2r)+s} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ s-j \end{bmatrix} \frac{[s]![a_1 - r + s - j]!}{[a_1 - r]!} u_{m+j,n-j,r-s+j} \ . \tag{6}$$

Next observe that $[E_{13}, F_2] = v e^{-hH_2/2} E_1$, so that

$$\begin{aligned} E_{13} u_{m,n,r} = E_{13} F_2 u_{m-1,n,r} = F_2 \ E_{13} u_{m-1,n,r} &+ v^{2m+n-a_1-a_2+r+1}[n] u_{m,n-1,r} \\ &+ v^{2m+n-a_2-r+1}[a_1 - r + 1][r] u_{m-1,n,r-1} \end{aligned}$$

using (6) with $s = 1$ to reduce $E_1 v_{m-1,n,r}$. Iterating this equation leads to

$$\begin{aligned} E_{13} u_{m,n,r} = F_2^m E_{13} u_{0,n,r} &+ v^{m+n-a_1-a_2+r+2}[m][n] u_{m,n-1,r} \\ &+ v^{m+n-a_2-r+2}[m][a_1 - r + 1][r] u_{m-1,n,r-1} \ . \end{aligned} \tag{7}$$

Since $[E_{13}, F_{13}] = \frac{q^2}{1-q}(v^{H_1+H_2} - v^{-H_1-H_2})$, one can prove inductively that

$$E_{13} u_{0,n,r} = F_{13}^n E_{13} u_{0,0,r} - v^3 [a_1 + a_2 - n - r + 1][n] u_{0,n-1,r} \ .$$

The first term vanishes, since $E_{13}F_1^r u = 0$. Substituting this into (7) gives

$$E_{13}u_{m,n,r} = -v^{m+3}[a_1+a_2-m-n-r+1][n]u_{m,n-1,r} \\ +v^{m+n-a_2-r+2}[m][a_1-r+1][r]u_{m-1,n,r-1}\,.$$

Applying this equation recursively leads to

$$E_{13}^k\ u_{m,n,r} = \sum_{i=\max 0,k-n}^{\min k,r} (-1)^{k-i} v^{i(n-a_2-r+i-k-1)+k(m+3)} \begin{bmatrix} k \\ i \end{bmatrix} \\ \frac{[m]![a_1-r+i]![r]![n]![a_1+a_2-m-n-r+k]!}{[m-i]![a_1-r]![r-i]![n-k+i]![a_1+a_2-m-n-r+i]!} u_{m-i,n-k+i,r-i}\,.$$

The particular case of interest for us is

$$E_{13}^n u_{m+j,n-j,j} = (-1)^{n-j} v^{-j(j+a_2+1)+n(m+j+3)}[n]! \\ \frac{[m+j]![a_1]![a_1+a_2-m-j]!}{[m]![a_1-j]![a_1+a_2-m-n]!} u_{m,0,0}\,.$$

Combining with (6) for $s=r$ and noting that $E_2^m F_2^m u = v^m \frac{[m]![a_2]!}{[a_2-m]!}u$, we obtain

$$E_2^m E_{13}^n E_1^r F_2^m F_{13}^n F_1^r u = \sum_{j=0}^{\min n,r} (-1)^{n-j} v^{j(r+n-a_1-a_2-1)+r+m+n(m+3)} \\ \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ j \end{bmatrix} [n]![r]! \frac{[m+j]![a_1]![a_2]![a_1+a_2-m-j]!}{[a_1-r]![a_2-m]![a_1+a_2-m-n]!} u\,.$$

4.3 PSU(3) invariant of the Poincaré HS

Combining the results of the last two subsections gives the coloured Jones polynomial of the trefoil as

$$\frac{J_L(V)}{J_U(V)} = \sum_{m,n,r} \sum_{j=0}^{\min n,r} (v-v^{-1})^{m+n+r} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ j \end{bmatrix} \frac{[m+j]![a_1]![a_2]![a_1+a_2-m-j]!}{[m]![a_1-r]![a_2-m]![a_1+a_2-m-n]!} \\ (-1)^j v^{\frac{1}{2}[m(m-5)+n(n-9)+r(r-5)]-3ma_2-(3n+j+4)(a_1+a_2)-3ra_1+j(r+n-1)+mn}\,.$$

Observe that the dependence of the summand on the colour variables a_1 and a_2 is via a factor

$$(v-v^{-1})^{m+n+r} v^{-(3n+3r+j+4)a_1-(3m+3n+j+4)a_2}[a_1]\ldots[a_1-r+1] \\ \cdot[a_2]\ldots[a_2-m+1]\cdot[a_1+a_2-m-j]\ldots[a_1+a_2-m-n+1]\,,$$

which is $(v-v^{-1})^j$ times a product of $m+r+n-j$ differences of the form $q^{l(a_1,a_2)}-q^{l'(a_1,a_2)}$ where l and l' are affine linear functions of the colours a_1 and a_2. By the lemma in §3.3, the contribution to $Z_r(M)$ coming from this term will be divisible by $h^{[(m+r+n+j+1)/2]}$. Thus, in order to compute the coefficient of any power of h in $\tau(J'_L)$, only a finite number of terms (m,n,r,j) will be involved. The resulting series is $Z_\infty(M)$, the perturbative $PSU(3)$ invariant of M.

In particular, those terms contributing to h^0 and h^1 terms in $Z_\infty(M)$ will come from $m+n+r+j\leq 2$. The restriction $j\leq n,r$ ensures that in these cases $j=0$. The values of (m,n,r) and the associated terms in J'_L are listed below. We have transformed a_i into $k_i=a_i+1$, while the last column lists the contribution (up to order h) to $Z_\infty(M)$.

$$
\begin{array}{llr}
(0,0,0)\colon q^{4-2k_1-4k_2} & \longrightarrow & 1+(4-16f)h\\
(1,0,0)\colon q^{4-2k_1-3k_2}-q^{5-2k_1-4k_2} & \longrightarrow & (-1+20f)h\\
(0,1,0)\colon q^{4-3k_1-3k_2}-q^{6-4k_1-4k_2} & \longrightarrow & (-2+28f)h\\
(0,0,1)\colon q^{4-3k_1-2k_2}-q^{5-4k_1-2k_2} & \longrightarrow & (-1+20f)h\\
(2,0,0)\colon q^{4-2k_1-4k_2}-(q^5+q^6)q^{-2k_1-5k_2}+q^{7-2k_1-6k_2} & \longrightarrow & -8fh\\
(0,2,0)\colon q^{4-4k_1-4k_2}-(q^6+q^7)q^{-5k_1-5k_2}+q^{9-6k_1-6K_2} & \longrightarrow & -8fh\\
(0,0,2)\colon q^{4-4k_1-2k_2}-(q^5+q^6)q^{-5k_1-2k_2}+q^{7-6k_1-2k_2} & \longrightarrow & -8fh\\
(1,1,0)\colon q^{4-3k_1-4k_2}+q^{8-4k_1-6k_2}-q^{7-4k_1-5k_2}-q^{5-3k_1-5k_2} & \longrightarrow & -4fh\\
(1,0,1)\colon q^{4-3k_1-3k_2}+q^{6-4k_1-4k_2}-q^{5-3k_1-4k_2}-q^{5-4k_1-3k_2} & \longrightarrow & 4fh\\
(0,1,1)\colon q^{4-4k_1-3k_2}+q^{7-6k_1-4k_2}-q^{5-5k_1-3k_2}-q^{6-5k_1-4k_2} & \longrightarrow & -4fh
\end{array}
$$

Hence $Z_\infty = 1+24fh+O(h^2)$, in agreement with [5] which predicts that the coefficient of h in the $PSU(N)$ perturbative invariant of an integer homology sphere is $N(N^2-1)\lambda(M)$, where $\lambda(M)$ is the Casson invariant. Using Mathematica [17], the above algorithm can easily be implemented to give the first few terms for both positive and negative integer surgery on the trefoil (that is, for the Seifert fibred manifolds $\Sigma(2,3,5)$ and $\Sigma(2,3,7)$),

$$
\begin{aligned}
Z_\infty(M^+) &= 1+24h+432h^2+7920h^3+161424h^4+3718483.2h^5+\cdots,\\
Z_\infty(M^-) &= 1-24h+576h^2-15264h^3+455904h^4-15324595.2h^5+\cdots.
\end{aligned}
$$

Changing variable to $x = q - 1$, we get series

$$Z_\infty(M^+) = 1 + 24x + 420x^2 + 7496x^3 + 149934x^4 + 3409140x^5 + 97500090x^6 + 2508211080x^7 + \cdots ,$$

$$Z_\infty(M^-) = 1 - 24x + 588x^2 - 15848x^3 + 479334x^4 - 16263600x^5 + 615044648x^6 - 25720830360x^7 + \cdots ,$$

which we can expect to have integer coefficients as for the ($SU(2)$) Ohtsuki series for integer homology spheres.

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