# Some computations of Ohtsuki series

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**Abstract.** We present some computational data on Ohtsuki series for a two parameter family of integer homology spheres obtained by surgery around what we call '2–strand knots', closures of the simplest rational tangles. This data allows us to make certain conjectures about the growth rate of the coefficients in Ohtsuki series generally, based on which we introduce an invariant which we call the *slope* $\sigma(M)$ of a manifold $M$ (not to be confused with slopes in hyperbolic geometry). For Seifert fibred manifolds, $M$, the conjectures are known to hold while $\pi^2\sigma(M) \in \mathbf{Q}$; furthermore if $M$ is also an integer homology sphere, $\pi^2\sigma(M) \in \mathbf{Z}$. Assuming the conjectures, the numerical data enables us to give an example of a $\mathbf{Z}$HS for which $\pi^2\sigma(M) \notin \mathbf{Z}$. This paper is based on the first author's M.Sc. thesis.

**Keywords:** Ohtsuki series, Seifert fibred manifold, rational tangle, quantum invariants

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## 1. Introduction

Suppose that $M$ is a compact oriented 3–manifold without boundary. The $sl_2$ Witten-Reshetikhin-Turaev invariant (see (Wi), (RT)) is a complex number invariant $Z_K(M,L)$ of embeddings of links $L$ in $M$, dependent on an integer $K$. It is known that for links in $S^3$, $Z_K(S^3,L)$ is a polynomial in $q=\exp\frac{2\pi i}{K}$ and $q^{-1}$, namely the generalised Jones polynomial of the link $L$.

Now assume that $M$ is a rational homology sphere with $H=|H^1(M,\mathbf{Z})|$. In this paper we will only consider the case of invariants where the link $L$ is empty. From its algebraic definition, e.g. via quantum groups, $Z_K(M,\emptyset)$ can be written as a rational (or polynomial) function of $q$ at $K^{\rm th}$ roots of unity. In the normalization for which $Z_K(S^3,\emptyset)=1$, denote the invariant for the pair $(M,\emptyset)$, by $Z_K(M)$. For odd prime $K$, $Z_K(M)\in\mathbf{Z}[q]$ (see (M1), (M2)), so that for some $a_{m,K}(M)\in\mathbf{Z}$, one



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has

$$Z_K(M) = \sum_{m=0}^{\infty} a_{m,K}(M)(q-1)^m \,. \tag{1}$$

Although the $a_{m,K}$ are not uniquely determined from this relation since $\frac{q^K-1}{q-1}=0$, however they are determined modulo $K$ for $m \le K-2$ when $K$ is prime. It is known from (O1) and (O2) that there exist rational numbers $\lambda_m(M)$ such that,

$$a_{m,K}(M) \equiv \left(\frac{H}{K}\right)\lambda_m(M) \tag{2}$$

as elements of $\mathbf{Z}/K\mathbf{Z}$ for all sufficiently primes $K \ge 2m+3$. The formal power series

$$Z_\infty(M) = \sum_{m=0}^{\infty} \lambda_m (q-1)^m \,,$$

is known as the *Ohtsuki series* of $M$ and by (R3) $Z_\infty(M) \in \mathbf{Z}[\frac{1}{2H}][[h]]$ where $q = 1+h$. For integer homology spheres ($H=1$) we have $Z_\infty(M) \in \mathbf{Z}[[h]]$, while $\lambda_0(M)=1$ and $\lambda_1(M) = 6\lambda(M)$ where $\lambda(M)$ denotes the Casson invariant of $M$. In general $Z_\infty$ is expected to be the asymptotic expansion of the trivial connection contribution to $Z_K(M)$ in Chern-Simons theory.

Relatively few computations of $Z_\infty$ have been carried out. For Lens spaces, Jeffrey's closed formula (Je) for $Z_K$ gives a formula for $Z_\infty$,

$$Z_\infty(L(P,Q)) = q^{\pm 3s(Q,P)} \frac{q^{\frac{1}{2P}} - q^{-\frac{1}{2P}}}{q^{1/2} - q^{-1/2}} \,,$$

where $s(Q,P)$ is a modified Dedekind sum. For Seifert fibred manifolds, $Z_K$ can be written as the asymptotic expansion around $q=1$ of a holomorphic function of $q$ expressed as a complex (contour) integral (LR). Thus for a Seifert fibred manifold which is an $\mathbf{Z}$HS,

$$Z_\infty\left(\Sigma\left(\frac{Q_1}{P_1},\ldots,\frac{Q_N}{P_N}\right)\right) = c.q^{-\frac{\phi}{4}} \int e^{\frac{iKy^2}{8\pi P}} \frac{\prod_{j=1}^N \sinh \frac{y}{2P_j}}{(\sinh \frac{y}{2})^{N-2}}\, dy \,,$$

where $P = \prod P_j$, $c$ is a constant (dependent on $P$) and $\phi$ is a rational number,

$$\phi = 3\mathrm{sign}P + \sum_{j=1}^{N}\left(12 s(Q_j,P_j) - \frac{Q_j}{P_j}\right) \,.$$

For special cases some other formulae are known, for example for $\pm 1$ surgery around the trefoil, that is for the Poincaré homology sphere

$\Sigma(2,3,5)$ and for $\Sigma(2,3,7)$, (see (LZ), (Le))

$$q(q-1)Z_\infty(\Sigma(2,3,5))(q) = 1 - 1/2 \sum_{n=1}^{\infty} \chi(n) q^{(n^2-1)/120}$$

$$(q-1)Z_\infty(\Sigma(2,3,7))(q) = \sum_{n=0}^{\infty} q^{-\frac{1}{2}n(n+1)}(q^{n+1}-1)\dots(q^{2n+1}-1)$$

where the odd function $\chi : \mathbf{Z}/60\mathbf{Z} \longrightarrow \{-1,0,1\}$ takes value $+1$ precisely at 1, 11, 19 and 29 (and nowhere else).

Such computations have been obtained using three approaches. All use a surgery presentation of the manifold via a link.

- One can write a state sum for $Z_K$ in which a state is a labeling of the regions and components of a diagram of the link (see (KL)) and the local weights are quantum dimensions, $\theta$-nets and quantum $6j$-symbols on the different elements of the diagram (components/regions, edges, crossings, respectively). One can use the method of *recombination* here to rewrite the state sum obtained in a possibly simpler way. To extract a formula for $Z_\infty$ now requires some manipulations of the form of the sums involved in $Z_K$ which only work for a small class of manifolds. This was used in (L) and (LZ).
- Using conformal field theory, one can write $Z_K$ as a sum of products of $S$ and $T$ matrix elements which take a particular simple form for special manifolds. This was used to obtain $Z_\infty$ in (Je), (LR).
- Using the formulation of $J_L(\rho_1,\dots,\rho_c)$, the coloured Jones polynomial of a link $L$ coloured by representations $\rho_i$ on its $c$ components, as the image of the link (or rather of a 1-tangle whose closure is the link) under a representation of the category of tangles, one can write $J_L$ in terms of the universal $R$-matrix, as the scalar value taken by an element of $U_q sl_2^{\otimes c}$ in the representations $\rho_i$. From this universal invariant of the link an expression for $Z_\infty(M)$ can be obtained as an infinite sum with the property that coefficients of $(q-1)^n$ (for any $n$) arise only form a finite number of terms in the sum. This was the method used by Le, (Le).

To obtain closed forms for $Z_\infty$ using the first two approaches, relies on the manifold being obtainable from a particularly simple link, since in the end the combinatorial formulae for $Z_K$ depend on quantum $6j$-symbols (or what is the same thing, $R$-matrix entries) which enter at each crossing in a link diagram presentation. These have complicated formulae even for $U_q sl_2$ and so only for those manifolds for which the quantum $6j$ symbols can be manipulated to cancel can a closed form be expected sufficient to find $Z_\infty$. This occurs for Seifert fibred manifolds.

In this paper we will consider a special (two parameter) class of manifolds (see §2) for which the state sum formula is relatively simple, although still complicated enough that a closed formula for $Z_\infty$ is unknown. In (Ja), computer computations were carried out for $Z_\infty$ on this family; we outline the method and some peculiarities relating to numerical precision, in §3. In §4, some of the results are given graphically along with conjectures that they support. Finally, in §5, some possible generalizations are suggested.

## 2. A two-parameter family of manifolds

### 2.1. Definition of the manifolds

Let $K_{(\binom{S}{T})}$ be the knot obtained by connecting two tangles each of which consists of two strands simply braided, with $S$ and $T$ crossings respectively, the two tangles being connected as shown in Figure 1.

Negative $S$ and/or $T$ result in crossings of opposite orientation. Here, $K_{(\binom{S}{T})}$ will be a knot (and not a link) when $S$ and $T$ are not both odd. Note that for different parities of $S$ and $T$, the relative orientations of the different parts of $K_{(\binom{S}{T})}$ will be different. Replacing $(S,T)$ by $(-T,-S)$ doesn't change the knot, while reversing the signs of both $S$ and $T$ (or interchanging $S$ and $T$) changes the knot to its mirror image. Some special cases are, $S=0$ (unknot), $S=\pm 1$ ($(2, T\mp 1)$ torus knot) along with the examples in the table in table below, where the knots have been given with their Conway names.

Table .

| $S$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 4 | 6 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 4 |
| $K_{(\binom{S}{T})}$ | $4_1$ | $5_2$ | $6_1$ | $7_2$ | $8_1$ | $9_2$ | $10_1$ | $7_3$ | $9_3$ | $8_3$ | $9_4$ | $10_3$ |

In this paper we consider the manifolds $M_{(\binom{S}{T})}$ obtained from $S^3$ by surgery on the *framed* knot $K_{(\binom{S}{T})}$ of Figure 1. The number of twists $U$ in $K_{(\binom{S}{T})}$ is chosen so that the resulting blackboard framed knot will

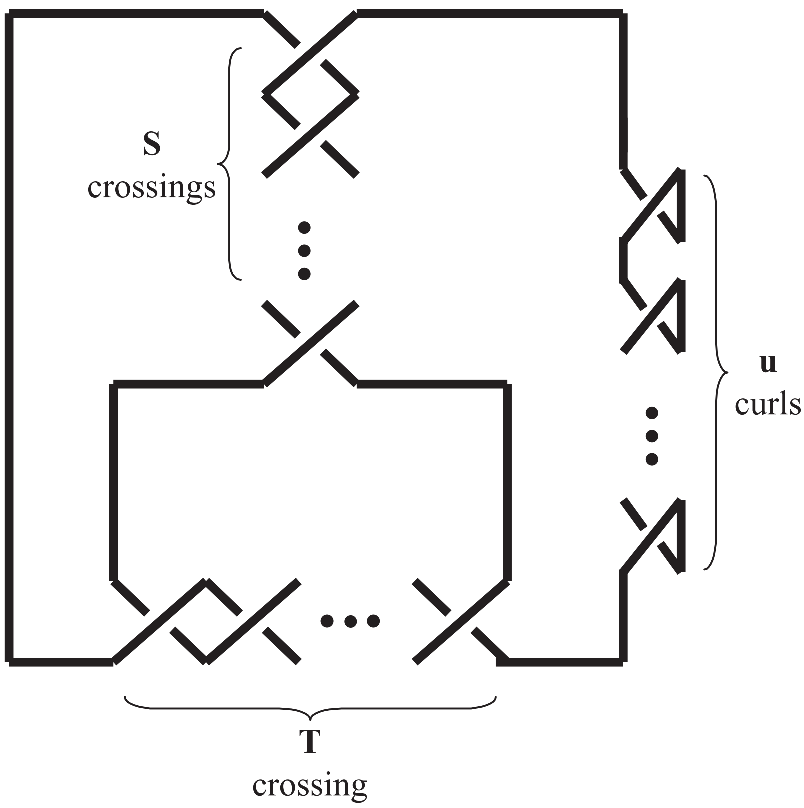


*Figure 1.* Knot family $K_{(\binom{S}{T})}$

have framing $\pm 1$, that is, so that $M_{(\binom{S}{T})}$ will be an integer homology sphere. This requires $U = (-1)^T S - (-1)^S T \pm 1$, where the signs depend on the parities of $S$ and $T$ (since the relative orientations of different sections of the knot depend upon the parities of $S$ and $T$), so in fact for each $S$ and $T$ we obtain two manifolds $M^{\pm}_{(\binom{S}{T})}$.

## 2.2. Calculation of $Z_K$

The Kauffman-Lins state sum formulation (see (KL)) of the WRT invariant $Z_K(M^{\pm}_{(\binom{S}{T})})$ uses a sum over states which are colorings of the one component of $K_{(\binom{S}{T})}$ and of the regions into which the knot diagram of $K_{(\binom{S}{T})}$ in Figure 1 cuts the plane, by colors from the set $\{0, 1, \ldots, K-2\}$. The only admissible colorings are those which satisfy the constraint that for each edge of the knot diagram, the triple

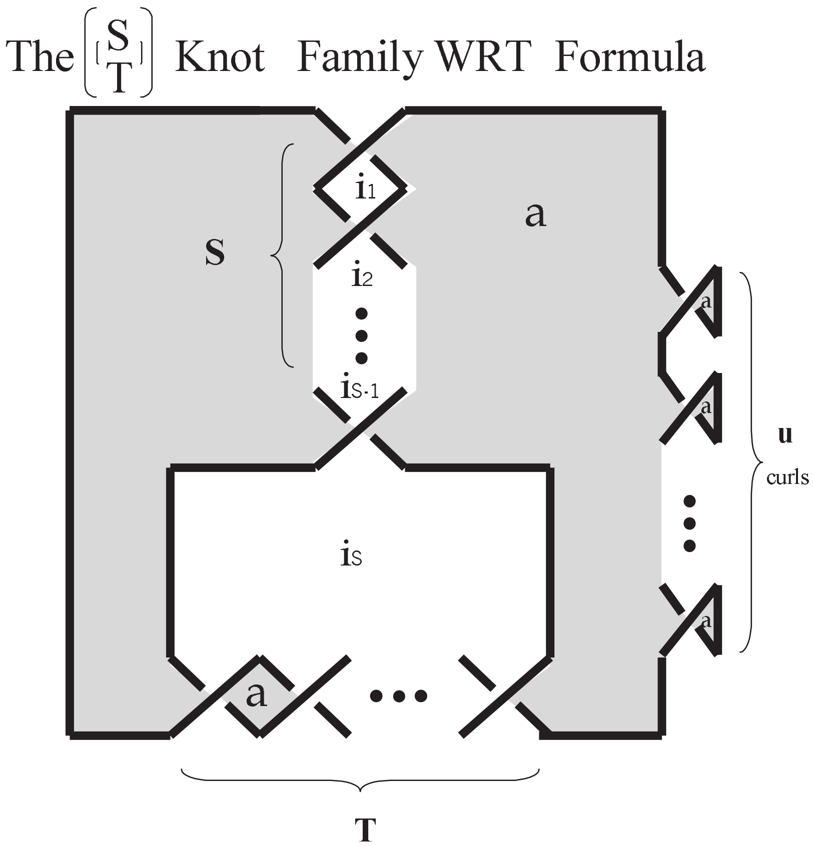


*Figure 2.* Allowed states for $K_{(( \frac{S}{T} ))}$

of colors coming from the component and the two regions on either side of the edge, satisfies the Clebsch-Gordan constraint. The possible colorings in our case are shown in Figure 2. This would give a formula for $Z_K(M^{\pm}_{(( \frac{S}{T} ))})$ as a sum over $S+1$ indices in which the summand contains $S-1$ nontrivial quantum $6j$ symbols.

Using recombination, each crossing can be written as a sum over the label on the internal edge, of the evaluation of an 'H' type graph with two trivalent vertices (Figure 3(1)). This leads to the configuration in Figure 3(2) which can be simplified using Figure 3(3) to a network whose size is independent of $S$ and $T$, although with a non-standard weighting. It has just four trivalent vertices (that is, it is a tetrahedral

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A triple $\{a,b,c\}$ is said to satify the Clebsch-Gordan constraint if they satisfy the triangle inequality $|a-b| \le c \le a+b$ while $a+b+c$ is even and at most $2K-4$.

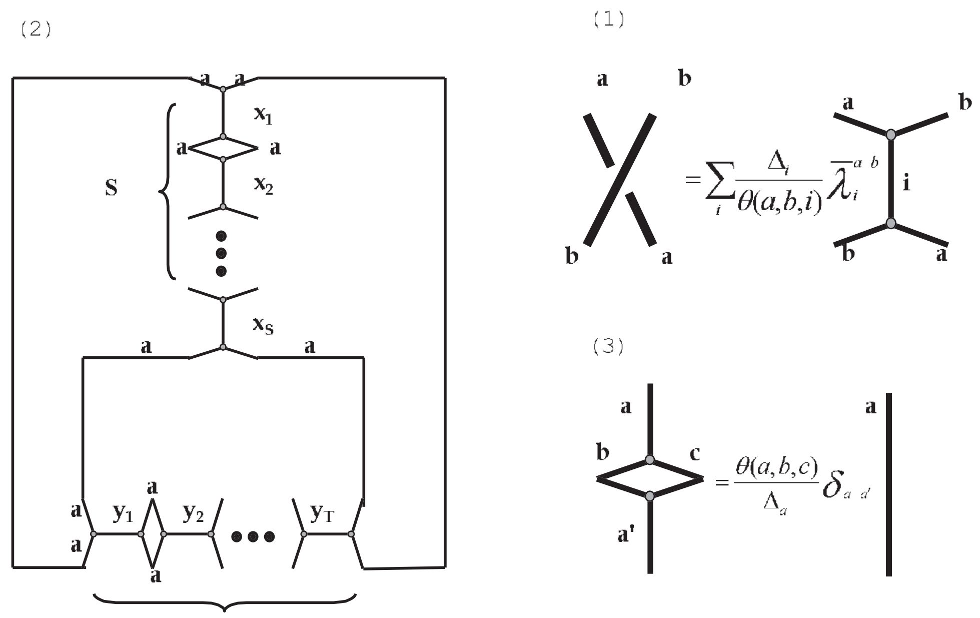


*Figure 3.* Recombination

network) and the resulting formula for the WRT invariant is

$$Z_K(M^{\pm}_{((\frac{S}{T}))}) = G_{\pm}^{-1} \sum_{a,x,y} \Delta_x \Delta_y (\overline{\lambda}_x^{a\ a})^S (\lambda_y^{a\ a})^T (\lambda_0^{a\ a})^U \begin{Bmatrix} a & x & a \\ a & y & a \end{Bmatrix} \quad (3)$$

where the sum is over colors $a$, $x$ and $y$ for which $\{a, a, x\}$ and $\{a, a, y\}$ are admissible triples, while $G_{\pm}$ are the Gauss sums $\sum_a \Delta_a^2 (\lambda_0^{a\ a})^{\pm 1}$ obtained by evaluation on the unknot with framing $\pm 1$. We have used notation,

$$\begin{aligned} \Delta_n &= (-1)^n \frac{A^{2(n+1)} - q^{-2(n+1)}}{q^2 - q^{-2}} \\ \lambda_c^{a\ b} &= (-1)^{(a+b+c)/2} A^{(a(a+2)+b(b+2)-c(c+2))/2} \end{aligned}$$

where $A$ is a fourth root of $q$.

## 2.3. Properties of Ohtsuki series

The Ohtsuki series $\sum_{m=0}^{\infty} \lambda_m (q-1)^m$ will depend on $S$ and $T$ (and $U$, or equivalently on the sign of the surgery). To include this dependence clearly in the notation, we will denote the coefficients by $\lambda_m^{\pm}(S,T)$.

**Theorem** *For fixed parity and signs for $S$ and $T$, $\lambda_m$ is a polynomial in $S$ and $T$ of degree $2m$ with rational coefficients.*

**Proof.** It is known that $M \longrightarrow \lambda_m(M)$ is a finite type invariant of manifolds of order $m$. On the other hand, the map $K \longrightarrow \lambda_m(S^3_K)$ (doing $+1$ surgery on the knot) gives a Vassiliev invariant of order $2m$. (This follows for example from the Melvin-Morton-Rozansky conjecture (B-NG) and Le (Le).)

However if $|S|+|T| > 2m$, then let us pick any $2m+1$ of the crossings in the knot diagram of $K_{(\binom{S}{T})}$ in Figure 1, say $s$ of these crossings will be from the ones that were counted by $S$ and $t$ from the crossings that were counted by $T$, where $s+t = 2m+1$. One can consider the $2^{2m+1}$ knots obtained by variously altering the orientations of these crossings in all possible ways. Since $\lambda_m$ is Vassiliev of order $2m$, the alternating sum of the values of $\lambda_m$ on these knots vanishes. When the orientation of a crossing (say of type $S$) in $K_{(\binom{S}{T})}$ is flipped, the resulting knot is of the same type, by with $S$ changed to $S - 2\mathrm{sign}S$. Without loss of generality, $S$ and $T$ are positive. Then the result of changing the orientations of $i$ crossings of type $S$ and $j$ crossings of type $T$, is $K_{(\binom{S-2i}{T-2j})}$. There are $\binom{s}{i}\binom{t}{j}$ ways in which to pick the subsets of crossings to flip, so that

$$\sum_{i=0}^{s}\sum_{j=0}^{t}(-1)^{i+j}\binom{s}{i}\binom{t}{j}\lambda_m^{\pm}(S-2i, T-2j) = 0\,, \qquad (4)$$

for all $S$ and $T$ positive and all $s$, $t$ with $s+t = 2m+1$. The left hand side of (4) is the order $s+t$ finite difference partial derivative of $\lambda_m^{\pm}$, $s$ times in the $S$ direction and $t$ times in the $T$ direction. Observe also that all the terms in (4) involve $S$ and $T$ of fixed parities. This condition is sufficient to guarantee that (for fixed parities) $\lambda_m^{\pm}(S,T)$ is expressible as a polynomial in $S$ and $T$ of order at most $2m$, as required. Since $\lambda_m \in \mathbf{Z}$, these polynomials must have rational coefficients, in fact $2^{2m}((2m)!!)^2\lambda_m(S,T) \in \mathbf{Z}[S,T]$ where $n!! \equiv \prod_{r=1}^{n} r!$. □

## 3. Computer calculations of Ohtsuki series

Using (3), $Z_K(M^{\pm}_{((\frac{S}{T}))})$ can be computed for all the $K-1$ different $K^{\text{th}}$ roots of unity, $q = q_s = \exp\frac{2\pi i s}{K}$ $(1 \le s < K)$ for fixed $S$, $T$ and prime $K$. From these values, the coefficients $a_{m,K}(M^{\pm}_{((\frac{S}{T}))}) \in \mathbf{Z}$ can be found in (1) where we assume that they vanish past the $(K-2)^{\text{th}}$. Namely, one solves the following linear system of size $K-1$,

$$Z_K|_{q=q_s} = \sum_{m=0}^{K-2} a_{m,K}.(q_s - 1)^m \ , \tag{5}$$

where the dependence on the manifold has been omitted from the notation. The integrality of the solutions provides a check on the errors in the computations of the individual $WRT$ invariants at different roots of unity.

There a number of computational challenges here.

### 3.1. Growth of number of states

As was mentioned in the previous section, the sum obtained from naively applying, say the Kauffman-Lins prescription, to the knot diagram Figure 1 leads to a sum over a number of states which grows exponentially with $S$, since there would be a summation over $S$ indices $i_j$, and the non-trivial quantum $6j$-symbols effectively introduce a further $S-1$ summations, where the summand is now a product (or quotient of products) of quantum numbers and factorials. This problem is obviated with the used of recombination; thus (3) only involves a sum over 4 indices ($a$, $x$, $y$ and an additional index from the single quantum $6j$-symbol) so that there are the order of $K^3$ states to be summed over (or $K^4$ with a summand which is a quotient of products of quantum numbers and factorials), and this is independent of $S$ and $T$.

### 3.2. Numerical precision and summand size

Since $q$-factorials can be numerically very large, the individual terms in (3) can be many orders of magnitude larger than their sum and this necessitates using very high precision arithmetic.

We give the example of $K = 97$ for the manifold $M^{+}_{((\frac{3}{2}))}$ obtained by surgery around the knot $5_2$ (obtained from $S = 3$, $T = 2$, $U = 6$).

For $s = 1$, $[90]! \approx 10^{114}$ and the order of summands in $Z_K$ for $s = 2$ is $\sim 10^{26}$, while their sum ($Z_{97}$) is $\sim 10^6$. To obtain an accuracy of $d$ significant figures in the result then requires $20 + d + e$ figures in the summands, where the number of states is of order $10^e$. See Figure 4 which shows a plot of the numerical sizes of $Z_K$ and the individual terms in the sum for $Z_K$ (more precisely, the sum of absolute values of the real part of terms in the sum for $Z_K$) on a log scale, against $s$.

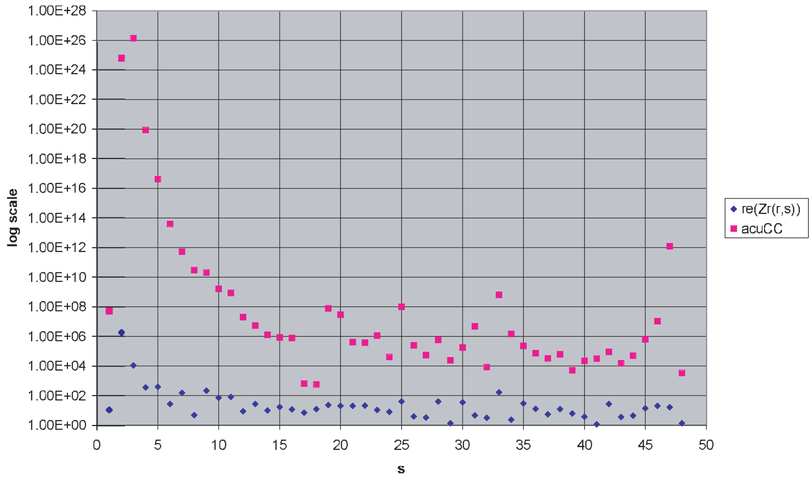


*Figure 4.* Relative sizes of $Z_K$ and its summands against $s$

Because of the very large numerical values of the quantum factorials, despite the smaller sizes of terms in the sums ($10^{100}$ versus $10^{26}$ in the above example), it is necessary to be able to work with numbers with large exponent (say 300) though maybe only 30 significant figures. It is very costly in time to use higher precision than absolutely necessary, so some special routines were written to easily manipulate sum numbers with large exponents, but (comparitively) not so high precision. For example, calculating $Z_{97}$ for all $s$ using 30-digits precision took 596 seconds on a Pentium 600MHz computer but it takes 1740 seconds to compute it for only one $s$ using 80-digits precision!

Because of the large discrepancy in orders of magnitude of the individual terms and the total sum, which as can be see from Figure 4 only occurs for some values of $s$ (near 0 and $K$), it is necessary for those values of $s$ to perform the computations with specially higher precision. Therefore two different routines were written, and used on different values of $s$. For details, see (Ja).

The result of these two factors is to practically limit the size of $K$ (though not the complexity of the knot, that is $S$ and $T$, since the difficulty in calculation is practically independent of $S$ and $T$). For $M^+_{(\binom{3}{2})}$ these techniques allow computation for all primes $K$ less then 137 with a total running time less than 1.5 days (using a 586 1GHz computer).

Knowing $a_{m,K}$ for primes $K = K_i \geq m + 2$ determines $\lambda_m$ modulo $\prod K_i$, according to their definition in (2) (recall that here $H = 1$). From Ohtsuki's theorem, it is known that this class, as an integer in $[-\frac{1}{2}\prod K_i, \frac{1}{2}\prod K_i]$, stabilizes as the number of primes used increases. This gives an algorithm for obtaining (the first few terms in) Ohtsuki series. The practical limitation on $K$ above gives a limitation on how many terms can be calculated. The full implementation of this algorithm for our manifolds $M^{\pm}_{(\binom{S}{T})}$ can be found at `http://www.ma.huji.ac.il/~ruthel/nori/index.html`.

## 4. Results and conjectures

### 4.1. Individual Ohtsuki series

For small $K$ (up to 100), it turns out that the number of states is $\approx K^{2.9}$ (rather than $K^3$) while the running time is $\approx K^{4.2}$ (rather than $K^4$) due to the increase in computational time of each summand and initialization procedures, for the larger values of $K$. For $M^+_{(\binom{3}{2})}$ discussed above ($(S,T,U) = (-2,-3,6)$), $K < 137$ was sufficient to compute the first 21 coefficients of $Z_\infty$,

$$
\begin{aligned}
&Z_\infty(M^+_{(\binom{3}{2})}) \\
&= 1 + 12h + 258h^2 + 7756h^3 + 300055h^4 + 14192892h^5 \\
&\ +793556722h^6 + 51201783488h^7 + 3744412949224h^8 \\
&\ +306062634843942h^9 + 27651533457983745h^{10} \\
&\ +2736207255879667844h^{11} + 294306807889008940143h^{12} \\
&\ +34188707473104409330168h^{13} \\
&\ +4265845139103716469762268h^{14} \\
&\ +568978507509845435699024672h^{15} \\
&\ +80787229265313530505892175542h^{16} \\
&\ +12165972894589961487357113418955h^{17} \\
&\ +1936811327962748352514940775515283h^{18} \\
&\ +325007156713501796302801741846095206h^{19} \\
&\ +57334985329655520887251821186176103843h^{20} \\
&\ +10607981215487793536113323249915379712259h^{21} \\
&\ +2053956644731187123340443541756436810603354h^{22} + \cdots
\end{aligned}
$$

where $h = q - 1$.

### 4.2. Polynomiality of coefficients

We already know that for fixed $m$, the dependence on $S$ and $T$ (of fixed parity) of $\lambda^{\pm}_m(S,T)$ is polynomial of degree $2m$. Since for $S = 0$ (or $T = 0$), $K_{(\binom{S}{T})}$ is the unknot for which $Z_\infty = 1$, thus $\lambda^{\pm}_m(S,T)$ is divisible by $S$ (for $S$ even) and by $T$ (for $T$ even) and by $ST$ (for $S$, $T$ both even).

Since the WRT invariants of the mirror image manifold $\bar{M}$ of $M$ are given by $Z_K(\bar{M})(q) = Z_K(M)(q^{-1})$ at all roots of unity, thus $Z_\infty(\bar{M})$ can be obtained from $Z_\infty(M)$ by replacing $q$ by $q^{-1}$, that is by substituting $\frac{1}{1+h} - 1 = -\frac{h}{1+h}$ for $h$. This gives (complicated) relations between $\lambda^{\pm}_m(S,T)$ and $\lambda^{\mp}_m(T,S)$, while $\lambda^{\pm}_m(-T,-S) = \lambda^{\pm}_m(S,T)$. Hence it is only necessary to compute the polynomials $\lambda^+_m(S,T)$ for the two parity combinations, (even, even), (even, odd). Knowing the degrees of the polynomials, numerical results for a large enough number of pairs $(S,T)$ are sufficient to determine these polynomials. The results for the first few coefficients are, for $S$ and $T$ both even,

$$
\begin{aligned}
\lambda^+_0 &= 1 \\
\lambda^+_1 &= -\tfrac{3}{2}ST \\
\lambda^+_2 &= \tfrac{3}{4}ST(5ST + T - S + 3)
\end{aligned}
$$

and for $S$ even, $T$ odd,

$$\begin{aligned}
\lambda_0^+ &= 1 \\
\lambda_1^+ &= \tfrac{3}{4}S(S+2T) \\
\lambda_2^+ &= \tfrac{25}{32}S^4+\tfrac{25}{8}S^3T+\tfrac{15}{4}S^2T^2-\tfrac{1}{2}S^3-\tfrac{3}{2}S^2T-\tfrac{3}{4}ST^2-\tfrac{1}{2}S^2+\tfrac{1}{4}ST-\tfrac{1}{4}S
\end{aligned}$$

Observe, for example, that $\lambda_1$ is divisible by 6, and indeed the Casson invariant is

$$\lambda(M^+_{(\binom{S}{T})}) = \frac{1}{6}\lambda_1^+(S,T) = \begin{cases} -\frac{1}{4}ST & \text{for } S \text{ and } T \text{ both even} \\ \frac{1}{8}S(S+2T) & \text{for } S \text{ even and } T \text{ odd} \end{cases}$$

## 4.3. Slopes

For fixed $S$ and $T$, the dependence on $m$ of $\lambda_m^{\pm}(S,T)$ is more interesting. See Figure 5 for graphs of the ratios $\frac{\lambda_n^+(S,T)}{\lambda_{n-1}^+(S,T)}$ for $S=-2$ and several odd values of $T$. From this figure there is an obvious conjecture.

**Conjecture** *If $M$ is a rational homology sphere, then the ratio $\frac{\lambda_n(M)}{\lambda_{n-1}(M)}$ is asymptotically linear in $n$, for large $n$.*

When this conjecture holds, set

$$\sigma(M) = \lim_{n\longrightarrow\infty} \frac{\lambda_n(M)}{n\lambda_{n-1}(M)} \ ;$$

this will be called the *slope* of $M$ (not to be confused with slopes in hyperbolic geometry).

It is known from (LR) that the slope exists for Seifert fibred manifolds and is

$$\sigma\left(\Sigma\left(\frac{Q_1}{P_1},\ldots,\frac{Q_N}{P_N}\right)\right) = \pm\frac{\prod P_i}{\pi^2} = \pm\left(\pi^2\sum_{i=1}^{N}\frac{Q_i}{P_i}\right)^{-1} .$$

As a corollary $\pi^2\sigma(M)\in\mathbf{Q}$ for such manifolds, while if in addition the manifold is an integer homology sphere then $\pi^2\sigma(M)\in\mathbf{Z}$. Since the WRT invariant (and therefore also the Ohtsuki series) is multiplicative under connect sum,

$$|\sigma(M\#N)| = \max(|\sigma(M)|,|\sigma(N)|)$$

when $|\sigma(M)|\neq|\sigma(N)|$, while under mirror image, the sign of the slope is reversed ($\sigma(\overline{M}) = -\sigma(M)$).

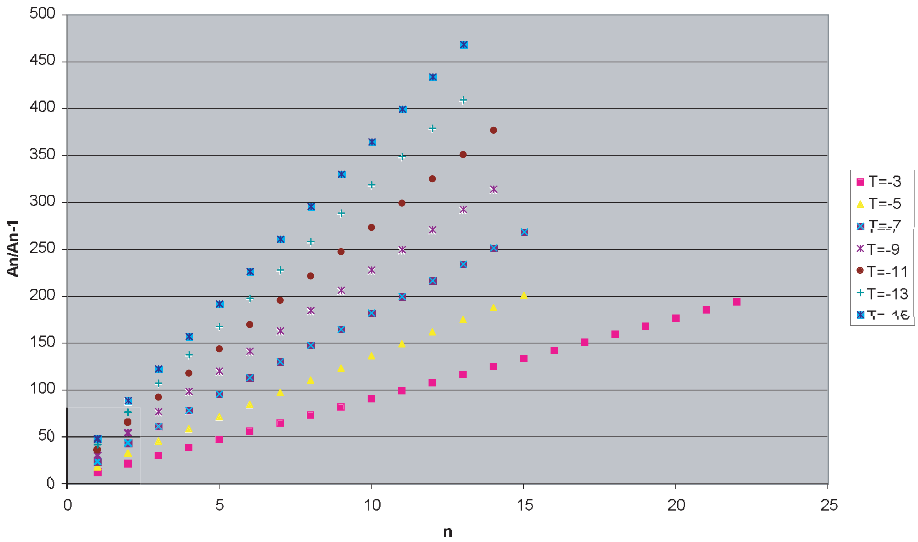


*Figure 5.* $\lambda^+_n(S,T)/\lambda^+_{n-1}(S,T)$ for $S=-2$ against $n$

Assuming that slopes exist for our family of manifolds, from the polynomiality of the terms $\lambda^{\pm}_m(S,T)$, one can make a further conjecture.

**Conjecture** *The slope for the manifolds $M^{\pm}_{((\frac{S}{T}))}$ is polynomial in $S$ and $T$ of degree at most* 2*.*

Observe that this conjecture **does not** follow from the previous conjecture and the theorem giving $\lambda^{\pm}_m(S,T)$ as a polynomial of degree $2m$ in $S$ and $T$, since we have no control on the growth rate of the coefficients in such polynomials with $m$ (it is conceivable that the lower order coefficients may grow more rapidly that the leading one). This conjecture is backed by numerical data. See for example Figure 6 which gives the slopes for $S=-3$ plotted against $T$ (even); and Figure 7 which gives slopes for $S$ and $T$ even, plotted against $T$ for various values of $S$.

Again assuming this conjecture, we can theoretically determine what its form should be. For the case of $S$ odd and $T$ even, we know that the knot for $S=\pm1$ is the $(2,T\mp1)$ torus knot, surgery around which gives for $M^{\delta}_{((\frac{S}{T}))}$ the Seifert fibred manifold on three fibers $\Sigma(2,T-S,2T-2S-\delta)$, and hence its slope is $\frac{2}{\pi^2}(T-S)((2T-2S)\delta-1)$. Hence for general (odd) $S$ and even $T$, the quadratic giving the slope must still have coefficient $\frac{4}{\pi^2}$ for $T^2$.

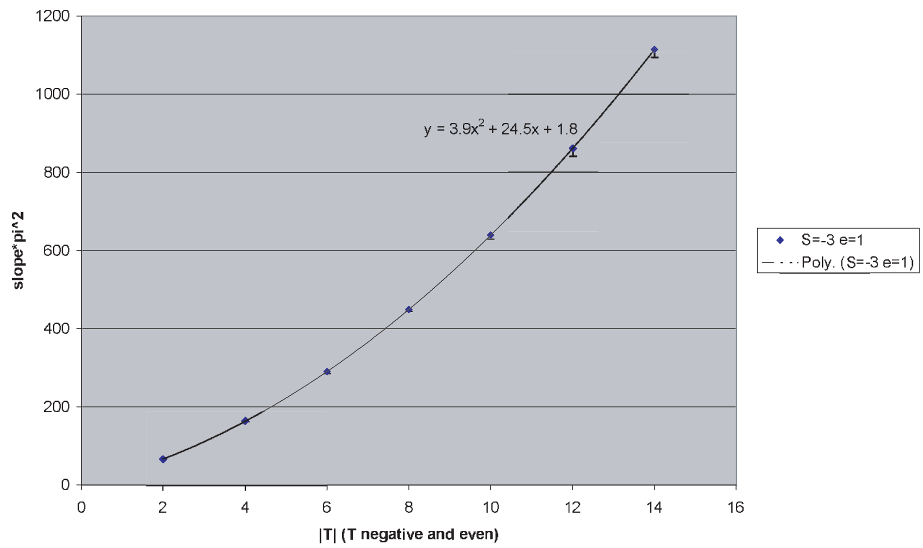


*Figure 6.* Graph of slopes for $S = -3$ and $T$ even

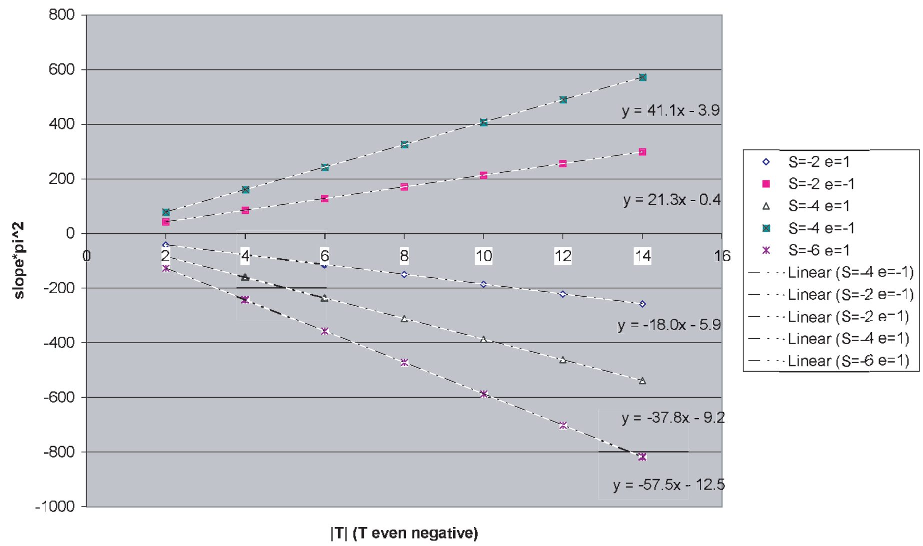


*Figure 7.* Graph of slopes for $S$ and $T$ even, against $T$

For the case of $S$ and $T$ both even, we have no similar data from special cases, since although we know that for $S = 0$, the manifold obtained is $S^3$ and $Z_\infty = 1$, there is no well-defined slope!

Finally we remark that the numerical data and precision is sufficient to demonstrate (assuming the conjectures) the existence of integer homology spheres for which $\pi^2\sigma(M) \notin \mathbf{Z}$, for example $M^+_{(\binom{3}{2})}$.

## 5. Conclusions

We have presented numerical data on Ohtsuki series for a 2-parameter family of integer homology 3-spheres (which includes some hyperbolic manifolds), which led to the conjectural introduction of a 'new' invariant of manifolds, which we called the *slope*. For Seifert fibred manifolds this is known to exist. The numerical data indicates that the slope (assuming it exists) is quadratic in the two parameters of the manifold.

It may be noted that the computer program may be used to compute the Ohtsuki series for surgery around an arbitrary knot, though the complexity of the calculation is likely to be prohibitively high for more general manifolds. However, for an arbitrary knot diagram, adding a 2-strand (that is, replacing any disc which contains within it just two non-intersecting strands in the knot diagram, by a braid $\sigma_1^S$) results in a manifold after surgery whose Ohtsuki series has a computational complexity essentially independent of $S$. The coefficients in the Ohtsuki series will be polynomial in $S$ by the same argument as used in this paper, and one may suppose that the slope will be quadratic in $S$.

In future work it is hoped to extend the calculations to manifolds obtained by surgery on closures of arbitrary rational tangles, since then the recombination technique used in this paper can be applied to reduce the computation of $Z_K$ to a small (fixed) number of quantum $6j$ symbols, after which the complexity is essentially independent of the numbers of twists in the diagram (apart from its depth). We also remark the there is some hope of obtaining proofs of formulae at least for leading coefficients of $\lambda^\pm_m$, using the direct presentation of the coloured Jones polynomial via universal $R$-matrices (see Le's approach to Ohtsuki series (Le)).

It does not seem that Witten-Chern-Simons theory sheds any light on the slope, or its physical meaning. In fact a somewhat 'orthogonal' way of viewing the Ohtsuki series of manifold invariants is via an asymptotic expansion of the ratio of adjacent coefficents, in order words not only the leading coefficent (that of $n$, which we have called the slope) but also other coeffients may be of interest.

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