תכונות

(א)

$$S[a,b]\ni f$$
$$\Downarrow$$
$$\overline{\int_a^b} f = \underline{\int_a^b} f = I_{[a,b]}(f)$$

הוכחה (ל)

(i) $\underline{\int_a^b} f \equiv \sup\limits_{\substack{g\in S[a,b]\\ g\le f}} I_{[a,b]}(g) \ge I_{[a,b]}(f)$ ($f\in S[a,b]$, $f\le f$)

(ii) $f,g\in S[a,b],\ g\le f \overset{\text{מונוט' } I}{\Longrightarrow} I_{[a,b]}(g)\le I_{[a,b]}(f)$

$\Rightarrow \underline{\int_a^b} f \equiv \sup\limits_{\substack{g\in S[a,b]\\ g\le f}} I_{[a,b]}(g) \le I_{[a,b]}(f)$

$\underline{\int_a^b} f = I_{[a,b]}(f) \Leftarrow$ (i), (ii)

$\square$

(ב)

$$f\in B[a,b],\ a<c<b$$
$$\Downarrow$$
$$\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$$
$$\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f$$

הוכחה (ל)

$(\le)$

$g\le f$, $g\in S[a,b]$ $\Rightarrow$ $g|_{[a,c]}\le f|_{[a,c]}$, $g|_{[c,b]}\le f|_{[c,b]}$ (פונקציות מדרגות) $\Rightarrow$ $I_{[a,c]}(g)\le \underline{\int_a^c} f$, $I_{[c,b]}(g)\le \underline{\int_c^b} f$ $\Rightarrow$

$I_{[a,b]}(g) \overset{\text{אדיטיביות של } I}{=} I_{[a,c]}(g)+I_{[c,b]}(g)\le \underline{\int_a^c} f+\underline{\int_c^b} f$

$$\Downarrow$$

לכן: $\underline{\int_a^b} f \equiv \sup\limits_{\substack{g\le f\\ g\in S[a,b]}} I_{[a,b]}(g) \le \underline{\int_a^c} f+\underline{\int_c^b} f$

$(\ge)$ לכל $\varepsilon>0$

$\underline{\int_a^c} f \equiv \sup\limits_{\substack{h\le f\\ h\in S[a,c]}} I_{[a,c]}(h) \Rightarrow \exists h\in S[a,c],\ h\le f,\ I_{[a,c]}(h) > \underline{\int_a^c} f-\varepsilon$

$\underline{\int_c^b} f \equiv \sup\limits_{\substack{k\le f\\ k\in S[c,b]}} I_{[c,b]}(k) \Rightarrow \exists k\in S[c,b],\ k\le f,\ I_{[c,b]}(k) > \underline{\int_c^b} f-\varepsilon$

נגדיר $g:[a,b]\to\mathbb{R}$ ע"י $g(x)=\begin{cases}h(x) & x\le c\\ k(x) & x>c\end{cases}$ $\Rightarrow$ $g\le f$, $g\in S[a,b]$

$I_{[a,b]}(g) = I_{[a,c]}(h)+I_{[c,b]}(k) > \underline{\int_a^c} f + \underline{\int_c^b} f - 2\varepsilon$

$\underline{\int_a^b} f \ge I_{[a,b]}(g) > \underline{\int_a^c} f+\underline{\int_c^b} f-2\varepsilon \overset{\forall\varepsilon>0}{\Longrightarrow} \underline{\int_a^b} f \ge \underline{\int_a^c} f+\underline{\int_c^b} f$ $\square$

(ג)

$$f,g\in B[a,b]$$
$$\Downarrow$$
$$\overline{\int_a^b}(f+g)\le \overline{\int_a^b} f+\overline{\int_a^b} g$$
$$\underline{\int_a^b}(f+g)\ge \underline{\int_a^b} f+\underline{\int_a^b} g$$

הוכחה (ל)

$h,k\in S[a,b]$, $h\le f$, $k\le g$ $\Rightarrow$ $h+k\in S[a,b]$, $h+k\le f+g$ $\Rightarrow \underline{\int_a^b}(f+g)\ge I_{[a,b]}(h+k) = I_{[a,b]}(h)+I_{[a,b]}(k)$

$$\Downarrow$$

$\underline{\int_a^b}(f+g) \ge \sup\limits_{\substack{h,k\in S[a,b]\\ h\le f\\ k\le g}} I_{[a,b]}(h)+I_{[a,b]}(k) = \underline{\int_a^b} f+\underline{\int_a^b} g$

$\square$

דוגמא (לא צריך שוויון)

$f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$ $\qquad$ $g(x)=\begin{cases}1 & x\notin\mathbb{Q}\\ 0 & x\in\mathbb{Q}\end{cases}$

$\underline{\int_a^b}(f+g)=\overline{\int_a^b}(f+g)=b-a \Leftarrow (f+g)(x)=1\ \forall x \Leftarrow$

$\underline{\int_a^b} f=\underline{\int_a^b} g=0$, $\overline{\int_a^b} f=\overline{\int_a^b} g=b-a$ $\Leftarrow$ $\begin{cases}\underline{S}_T(f)=\underline{S}_T(g)=0\\ \overline{S}_T(f)=\overline{S}_T(g)=b-a\\ \text{לכל חלוקה } T\end{cases}$

כי בכל קטע $\left.\begin{matrix}\inf=0\\ \sup=1\end{matrix}\right\}$ לפונק' f,g

$$\Downarrow$$

$\underline{\int_a^b}(f+g) > \underline{\int_a^b} f+\underline{\int_a^b} g$

$\overline{\int_a^b}(f+g) < \overline{\int_a^b} f+\overline{\int_a^b} g$

(ד) $f\in\mathcal{B}[a,b],\ k\in\mathbb{R}$

$$\Downarrow$$

$$\overline{\int_a^b} k\cdot f=\begin{cases}k\overline{\int_a^b}f & (k\ge 0)\\ k\underline{\int_a^b}f & (k<0)\end{cases}$$

$$\underline{\int_a^b} k\cdot f=\begin{cases}k\underline{\int_a^b}f & (k\ge 0)\\ k\overline{\int_a^b}f & (k<0)\end{cases}$$

הוכחה (ד)

$$\begin{matrix}g\in\mathcal{S}[a,b]\\ g\le k\cdot f\end{matrix}\iff\begin{matrix}\frac1k\cdot g\in\mathcal{S}[a,b]\\ \frac1k\cdot g\le f\end{matrix}\Rightarrow I_{[a,b]}\left(\tfrac1k\cdot g\right)\le\underline{\int_a^b}f\quad(k>0)$$

$$I_{[a,b]}\left(\tfrac1k\cdot g\right)=\tfrac1k I_{[a,b]}(g)$$

$$\Rightarrow I_{[a,b]}(g)\le k\underline{\int_a^b}f$$

לכן $\underline{\int_a^b}k\cdot f\equiv\sup_{\substack{g\in\mathcal{S}[a,b]\\ g\le k\cdot f}} I_{[a,b]}(g)\le k\cdot\underline{\int_a^b}f$

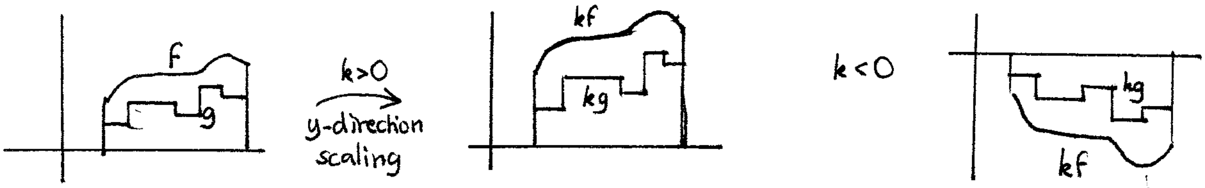
$$\underline{\int_a^b} f\le\frac1k\underline{\int_a^b}k\cdot f\quad:\begin{cases}f\rightsquigarrow k\cdot f\\ k\rightsquigarrow\frac1k\end{cases}\text{ במקום}$$

$$\underline{\int_a^b}kf\ge k\cdot\underline{\int_a^b}f\Leftarrow$$

$$\underline{\int_a^b}k\cdot f=k\cdot\underline{\int_a^b}f\Leftarrow$$

□

$k<0$: תרגיל!



(ה) $f\in\mathcal{B}[a,b]$

$$\Downarrow$$

$$\underline{\int_a^b}f\le\overline{\int_a^b}f$$

הוכחה

$$\begin{matrix}g,h\in\mathcal{S}[a,b]\\ g\le f,\ h\ge f\end{matrix}\Rightarrow g\le h\ \underset{\text{מונוטוניות}}{\overset{I_{[a,b]}}{\Rightarrow}}\ I_{[a,b]}(g)\le I_{[a,b]}(h)$$

$$\underset{\sup_g}{\Rightarrow}\ \sup_{\substack{g\in\mathcal{S}[a,b]\\ g\le f}}I_{[a,b]}(g)\le I_{[a,b]}(h)$$

$$\sup_{\substack{g\in\mathcal{S}[a,b]\\ g\le f}}I_{[a,b]}(g)=\underline{\int_a^b}f$$

$$\underset{\inf_h}{\Rightarrow}\quad\underline{\int_a^b}f\le\inf_{\substack{h\in\mathcal{S}[a,b]\\ h\ge f}}I_{[a,b]}(h)\equiv\overline{\int_a^b}f$$

□

(ו) $f\in\mathcal{B}[a,b],\ f\ge 0$

$$\Downarrow$$

$$\underline{\int_a^b}f,\ \overline{\int_a^b}f\ge 0$$

הוכחה

$\underline{\int}$: $\mathcal{S}[a,b]\ni 0\le f\ \Rightarrow\ 0=I_{[a,b]}(0)\le\underline{\int_a^b}f$

$\overline{\int}$: $\overset{(ה)}{\Rightarrow}\ \overline{\int_a^b}f\ge\underline{\int_a^b}f\ge 0$

□

(ז) $f\in\mathcal{B}[a,b]$

$g(x)=f(x+c)$

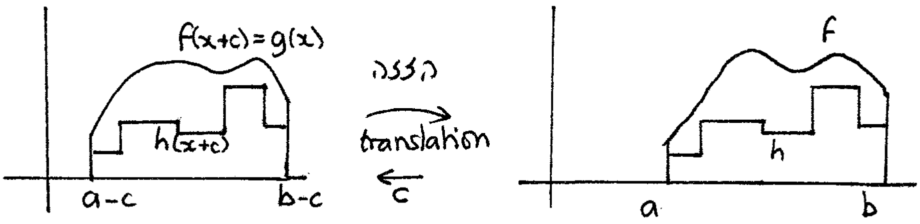
$$\Downarrow$$

$g\in\mathcal{B}[a-c,b-c]$

$$\underline{\int_{a-c}^{b-c}}g=\underline{\int_a^b}f$$

$$\overline{\int_{a-c}^{b-c}}g=\overline{\int_a^b}f$$

$\underline{\int}$:



תרגיל!

$f\in\mathcal{B}[a,b]$

$g(x)=f(\lambda x),\ \lambda>0$

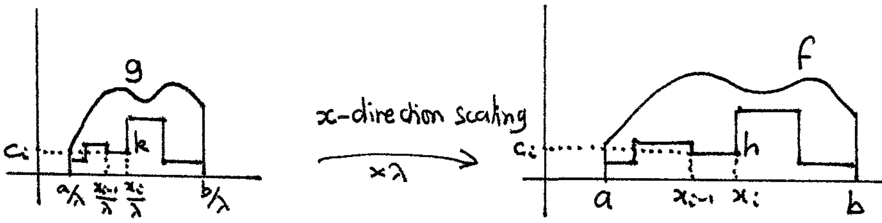
$$\Downarrow$$

$g\in\mathcal{B}[a/\lambda,b/\lambda]$

$$\underline{\int_{a/\lambda}^{b/\lambda}}g=1/\lambda\underline{\int_a^b}f$$

$$\overline{\int_{a/\lambda}^{b/\lambda}}g=1/\lambda\overline{\int_a^b}f$$

$\underline{\int}$:



$$k\in\mathcal{S}[a/\lambda,b/\lambda],\ k\le g\ \underset{k(x)\equiv h(\lambda x)}{\iff}\ h\in\mathcal{S}[a,b],\ h\le f$$

$$\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}-\frac{x_{i-1}}{\lambda}\right)c_i\equiv I_{[a/\lambda,b/\lambda]}(k)=1/\lambda\cdot I_{[a,b]}(h)\equiv 1/\lambda\sum_{i=1}^{n}(x_i-x_{i-1})c_i$$

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<u>הגדרה</u> $f\in\mathcal{B}[a,b]$ <u>נקראת פונקציה אינטגרבילית</u> אם"ם $\underline{\int_a^b} f=\overline{\int_a^b} f$ ונסמן ב- $f\in\mathcal{R}[a,b]$
(לפי רימן) Riemann integrable
גם כתוב $\int_a^b f\equiv\overline{\int_a^b} f=\underline{\int_a^b} f$.

<u>משפט</u> $f\in\mathcal{R}[a,b]$ אם"ם $0=\inf_T(\overline{S}_T(f)-\underline{S}_T(f))$.

<u>הוכחה</u> $(\Rightarrow)$ נניח ש- $\inf_T(\overline{S}_T(f)-\underline{S}_T(f))=0$.

נניח ש $\varepsilon>0$. לכן קיימת חלוקה T כך ש- $\overline{S}_T(f)-\underline{S}_T(f)<\varepsilon$

$$\overline{\int_a^b} f\le\underline{\int_a^b} f \overset{\forall\varepsilon>0}{\Longleftarrow} \underbrace{\overline{\int_a^b} f\le}_{\text{הגדרה של }\overline{\int}} \overline{S}_T(f)\underbrace{<}_{\text{תכונה של }T}\underline{S}_T(f)+\varepsilon\underbrace{\le}_{\text{הגדרה של }\underline{\int}}\underline{\int_a^b} f+\varepsilon \quad\Leftarrow$$

$$\Downarrow\ \overline{\int}\ge\underline{\int}$$

$$\overline{\int_a^b} f=\underline{\int_a^b} f$$

$$\Downarrow$$

$$f\in\mathcal{R}[a,b]$$

$(\Leftarrow)$ נניח ש- $f\in\mathcal{R}[a,b]$, $\varepsilon>0$.

$$\underset{T,U\ \text{חלוקות}}{\inf}(\overline{S}_T(f)-\underline{S}_U(f))=0\Leftarrow\inf_T\overline{S}_T(f)=\sup_U\underline{S}_U(f)\overset{\text{הגדרות}}{\Longleftarrow}\overline{\int_a^b} f=\underline{\int_a^b} f$$

$$\Downarrow$$

קיימות חלוקות U,T כך ש- $\overline{S}_T(f)-\underline{S}_U(f)<\varepsilon$

$$\Downarrow\ W=T\cup U$$

$$\overline{S}_W(f)-\underline{S}_W(f)\underbrace{\le}_{\substack{\overline{S}_W(f)\le\overline{S}_T(f)\\ \underline{S}_W(f)\ge\underline{S}_U(f)}}\overline{S}_T(f)-\underline{S}_U(f)<\varepsilon$$

לכן: $\inf_W(\overline{S}_W(f)-\underline{S}_W(f))=0$ $\square$

<u>משפט</u> $\mathcal{S}[a,b]\subsetneqq\mathcal{R}[a,b]\subsetneqq\mathcal{B}[a,b]$

<u>הוכחה</u> * $\mathcal{R}[a,b]\subseteq\mathcal{B}[a,b]$ לפי הגדרה
* $\mathcal{S}[a,b]\subseteq\mathcal{R}[a,b]$ מתכונה א' דף 14 א'

<u>דוגמא</u> $f\in\mathcal{B}[a,b]$, $f\notin\mathcal{R}[a,b]$

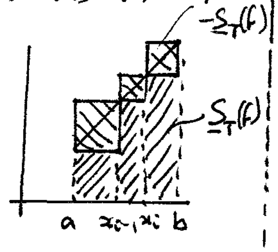
$$f(x)=\begin{cases}0 & x\in\mathbb{Q}\\ 1 & x\notin\mathbb{Q}\end{cases}$$

$\underline{S}_T(f)=0,\ \overline{S}_T(f)=b-a\quad\forall T$

<u>דוגמא</u> $f\in\mathcal{R}[a,b]$, $f\notin\mathcal{S}[a,b]$

$f(x)=x$

$$\overline{S}_T(f)-\underline{S}_T(f)=\sum_{i=1}^n(x_i-x_{i-1})^2\Leftarrow\begin{cases}\overline{S}_T(f)=\sum_{i=1}^n(x_i-x_{i-1})x_i\\ \underline{S}_T(f)=\sum_{i=1}^n(x_i-x_{i-1})x_{i-1}\end{cases}$$



נבחר חלוקה T כך ש-
$x_i-x_{i-1}=h\quad\forall i$
כ"א $x_i=a+ih\quad 0\le i\le n$
וגם $b=a+nh\Leftarrow h=\frac{b-a}{n}$

$$T=(a<a+h<a+2h<\cdots<\underset{b}{\underset{\shortparallel}{a+nh}})$$

$$\sum_{i=1}^n(x_i-x_{i-1})^2=nh^2=\frac{(b-a)^2}{n}\xrightarrow[n\to\infty]{}0$$

$$\inf_T(\overline{S}_T(f)-\underline{S}_T(f))=0\Rightarrow f\in\mathcal{R}[a,b]$$