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ASPHERICITY OF ALTERNATING KNOTS[1]

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1. Introduction

A space is said to be *aspheric* if all its homotopy groups of order higher than 1 vanish. A knot K (of multiplicity $\geqq 1$) imbedded in the 3-sphere S^3 is said to be aspheric if $S^3 - K$ is aspheric. The object of this paper is to prove that all alternating knots are aspheric. Two applications of this result are given at the end of the paper.

The basic tool used in the proof is an "addition" theorem for aspheric spaces due to J. H. C. Whitehead [4]. It may be stated as follows:

THEOREM 1.1. *Let $(X; A, B)$ be a triangulable triad, where A and B are connected and $A \cap B$ has a finite number of components C_i. Suppose that*

1. *A, B, and all the C_i are aspheric.*
2. *For each of the C_i, the injections*

$$i_1^*: \pi_1(C_i) \to \pi_1(A) \textit{ and}$$

$$i_2^*: \pi_1(C_i) \to \pi_1(B) \textit{ are isomorphisms into.}$$

Then X is aspheric.

The proof is given in [4].

2. Asphericity of the product of aspheric knots

DEFINITION 2.1. Let K be a knot in S^3, Y a polygonal 2-sphere on S^3, U_1 and U_2 the closures of the two components into which S^3 is divided by Y. Suppose that Y meets K at precisely two points, z_1 and z_2, and that H is a polygonal line segment on Y whose end points are z_1 and z_2. Suppose further that the portions into which K is divided by z_1 and z_2 are called K_1' and K_2', and that we set $K_1' \cup H = K_1$, $K_2' \cup H = K_2$. Then each K_i must lie entirely in one or the other of the U_i, and in fact meets Y only along H; furthermore, each K_i is a knot. If K_1 and K_2 lie in distinct U_i, then K is called a *product* of K_1 and K_2.

The object of this section is to prove that a product of two aspheric knots is aspheric. In the lemmas that follow, we will use the notation introduced in 2.1, and will moreover assume the asphericity of K_1 and K_2. We will omit the proofs of some lemmas; these will offer no difficulty to the reader.

LEMMA 2.2. *$Y - H$ is a deformation retract of both $U_1 - H$ and $U_2 - H$.*

LEMMA 2.3. *$U_k - K_k$ is aspheric, $k = 1, 2$.*

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[1] Except for the applications given in Section 10, this paper forms the substance of a doctoral thesis presented at the Massachusetts Institute of Technology in February, 1955. I should like to express my gratitude to Professor George W. Whitehead not only for his supervision of my thesis, but also for his guidance and teaching throughout my stay at M.I.T.

PROOF. We confine ourselves to the case $k = 1$. We have that $(U_2 - H) \cup (U_1 - K_1) = S^3 - K_1$ is aspheric. But, by 2.2, $Y - H$ is a deformation retract of $U_2 - H$, and hence $(Y - H) \cup (U_1 - K_1) = U_1 - K_1$ is a deformation retract of $S^3 - K_1$. Hence it has the same homotopy type as $S^3 - K_1$, and in particular it is aspheric.

COROLLARY 2.4. *$U_k - K$ is aspheric, $k = 1, 2$.*

LEMMA 2.5. *$Y - K$ has the homotopy type of a 1-sphere; in particular, it is aspheric.*

PROOF. $Y - K = Y - (z_1 \cup z_2)$, and thus is a 2-sphere from which two points have been removed. The result follows at once.

LEMMA 2.6. *If K has one component, then the injection*

$$j^*: H_1(Y - K) \to H_1(U_k - K)$$

is an isomorphism into, $k = 1, 2$.

PROOF. Surround $U_k \cap K$ by a small closed tubular neighborhood M in U_k. Then $\dot{M}$ is[2] a deformation retract of $M - K$, and it follows that $H_1(\dot{M}) = H_1(M - K) = Z$ (Z denotes an infinite cyclic group). If we apply the Mayer-Vietoris theorem to the triad $(U_k; M, \mathrm{Cl}\,(U_k - M))$, we obtain[3] the exact sequence

$$\begin{array}{ccccccc} 0 = H_2(U_k) \to & H_1(\dot{M}) & \xrightarrow{j_1^*} & H_1(M) & \oplus\, H_1\,(\mathrm{Cl}\,(U_k - M)) \to H_1(U_k) = 0. \\ & \| & & \| & \\ & Z & & 0 & \end{array}$$

Hence j_1^* is an isomorphism onto, and $H_1\,(\mathrm{Cl}\,(U_k - M)) = Z$. Since $\mathrm{Cl}\,(U_k - M)$ is a deformation retract of $U_k - K$, it follows that $H_1(U_k - K) = Z$. Now by 2.4, $H_1(Y - K) = Z$. Let R be the component of $Y \cap \dot{M}$ that is closer to z_1, c a singular cycle of $Y - K$ whose image is R and which belongs to a non-zero member of $H_1(Y - K)$. Since R is a deformation retract of $\dot{M}$, and since c is patently not zero-homologous in R, it follows that c is not zero-homologous in $\dot{M}$ either. Hence, since j_1^* is an isomorphism, c is not zero-homologous in $\mathrm{Cl}\,(U_k - M)$ either. Now the injection from $H_1(\mathrm{Cl}\,(U_k - M))$ into $H_1(U_k - K)$ is the inverse of the homomorphism induced by the retraction; it is therefore an isomorphism, and c is not zero-homologous in $U_k - K$ either. Hence Image $j^* \neq 0$. Hence Kernel j^* does not exhaust the whole group. If Kernel $j^* \neq 0$, then Image j^* is a finite group, i.e. Z has non-trivial finite subgroups, an absurdity. Hence Kernel $j^* = 0$, q.e.d.

LEMMA 2.7. *If K has one component, then the injection*

$$i^*: \pi_1(Y - K) \to \pi_1(U_k - K)$$

is an isomorphism into, $k = 1, 2$.

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[2] A dot over a symbol denotes the set of boundary points.

[3] The symbol Cl denotes the closure.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(Y - K) & \xrightarrow{i^*} & \pi_1(U_k - K) \\ \downarrow \rho & & \downarrow \rho' \\ H_1(Y - K) & \xrightarrow{j^*} & H_1(U_k - K) \end{array}$$

where ρ and ρ' are the natural homomorphisms from homotopy groups to homology groups. By 2.5 and 2.6, ρ and j^* are isomorphisms into. Hence $\rho' \circ i^* = j^* \circ \rho$ is an isomorphism into as well. It follows that i^* cannot have positive kernel, q.e.d.

LEMMA 2.8. *i^* is an isomorphism into, whether K has one component or more.*

PROOF. Let K' be the component of K that intersects Y. If we let $i_3{}^*$ and $i_4{}^*$ be the injections

$$i_3{}^*:\pi_1(Y - K) \to \pi_1(U_k - K')$$

and

$$i_4{}^*:\pi_1(U_k - K) \to \pi_1(U_k - K'),$$

then we have $i_3{}^* = i_4{}^* \circ i^*$. By 2.7, $i_3{}^*$ has kernel zero. Hence i^* must have kernel zero, q.e.d.

THEOREM 2.9. *Under the conditions described at the beginning of this section, K is aspheric.*

PROOF. This is a consequence of 1.1, 2.4, 2.5, and 2.8; we must set $X = S^3 - K$, $A = U_1 - K$, and $B = U_2 - K$.

3. The projection of a knot

We may take S^3 to be the union of two closed tetrahedra, T_1 and T_2, whose boundaries are identified; thus we may set $\dot{T}_1 = \dot{T}_2 = S^2$. At the same time it will be convenient to single out one point $s \in S^3$ not lying on K, and to place a cartesian co-ordinate system on $S^3 - s = R^3$ so that

(3.1) One of the faces of S^2 lies in the $x - y$ plane R^2 of R^3; this face will be called F.

(3.2) The projection $p:R^3 \to R^2$ given by $p((x, y, z)) = (x, y)$ takes K into Int F.

(3.3) p is a regular projection (see [3]).

(3.4) If $p(a) = p(b)$, $a, b \in K$, $a \neq b$, then a lies *inside* a straight segment of K.

The images under p of the points described in 3.4 are called *double points*, and the segments of $P = p(K)$ connecting double points are called *sides* of P. It will be assumed from now on that P is connected.

As is well known, P divides S^2 into two classes of regions, called "black" and "white" for convenience, in such a way so that each side is always the common boundary of a black and a white region; these regions are all connected and simply connected proper open subsets of the 2-sphere, and are therefore open disks. Pick an orientation ω on S^2, and call this the "positive" orientation; ω induces

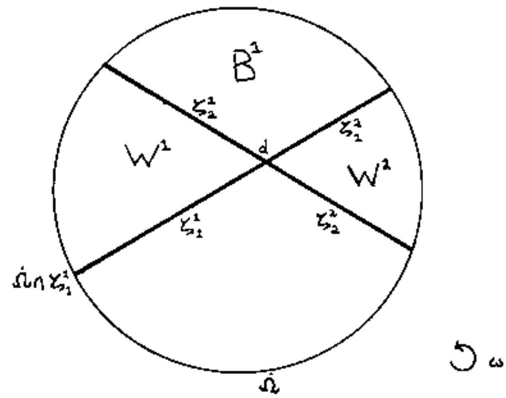

FIG. 1

"positive" orientations on each of the black and the white regions. We will denote the white regions by W, the black regions by B, and the double points by d.

We can now construct a function[4] $\Delta = \Delta_K$ from the set D of all double points d of P to the set consisting of 1 and -1, as follows: Let the two white regions on whose boundary d lies be W^1 and W^2, and let $p^{-1}(d) = x_1 \cup x_2$, where x_1 has the larger z-co-ordinate. Let Ω be a small circular neighborhood of d. The portion of P lying within Ω is the projection of two disjoint segments of K, ξ_1 and ξ_2, where $x_i \in \xi_i$, $i = 1, 2$. Suppose we divide $P \cap \Omega$ into four segments, ζ_j^i, $i, j = 1, 2$, having only d in common, such that $\zeta_1^i \cup \zeta_2^i$ lies in W^i, and $\zeta_j^1 \cup \zeta_j^2 = p(\xi_j)$. (See Figure 1.)

DEFINITION 3.5. If the orientation induced on W^1 by ζ_1^1, oriented from $\dot{\Omega} \cap \zeta_1^1$ to d, is ω, then $\Delta(d) = 1$. If it is $-\omega$, then $\Delta(d) = -1$.

Note that this definition is completely unambiguous. For if ζ_1^1, oriented as indicated, induces ω on W^1, then it induces $-\omega$ on the black region B^1 on whose boundary ζ_1^1 lies. But ζ_2^2 also lies on the boundary of B^1; hence ζ_2^2, oriented from d to $\cap \zeta_2^2$, induces $-\omega$ on B^1, and therefore induces ω on W^2. But then ζ_1^2, oriented from $\dot{\Omega} \cap \zeta_1^2$ to d, must also induce ω on W^2.

It is a well-known fact (see [3]), that once the white and black regions are determined, a knot is determined to within equivalence by its projection P and the function Δ. An alternating knot is characterized by the property that Δ is a constant and that P is connected (we have been assuming the latter all along). It will be no loss of generality to assume that $\Delta = 1$ always for the alternating knots that will be considered in this paper.

4. The graph of a knot

Let us suppose that the white region W has n sides, and let Λ be a closed circular disk with n equally spaced points δ_i on its boundary, it being understood

[4] Δ coincides with Reidemeister's Incidence number η for the black regions on which d lies, in those cases when $\eta \neq 0$ (see [3]).

that δ_i and δ_{i+1} are adjacent. The arcs into which $\dot{\Lambda}$ is divided by the δ_i are called the *sides* of Λ. There is a mapping

$$\sigma : \Lambda \to (W)$$

that takes each side of Λ onto a side of P and each of the δ_i onto a double point of P, and that is one-one except possibly on the δ_i; i.e., it is the extension to Λ of a homeomorphism from Int Λ onto W.

If $n > 2$, then the order of the δ_i on $\dot{\Lambda}$ induces an orientation on $\dot{\Lambda}$ and hence on Λ; we stipulate that this orientation shall go into ω under σ. We further stipulate that the *radii* of Λ, i.e. the straight lines η_i connecting the center θ of Λ to the δ_i, go into polygonal segments under σ. We set $\sigma(\eta_i) = g_i$, $\bigcup_i g_i = A$. The union of all the A for all the white spaces W of P will be called the *graph* of K, and will be denoted by G or G_K. The *vertices* of G are the $\sigma(\theta)$, which will also be called the *centers* of the white spaces W of P; in each white space, there is one and only one center. Note that $P \cap G = D$, though none of the points of D are vertices of G. The points of D will also be called double points of G. Note that as a graph, P is determined by G; and it then follows from the remark at the end of the previous section that

REMARK 4.1. *A knot is determined to within equivalence by its graph G and the values of Δ on the double points of G.*

5. Products of knot projections

DEFINITION 5.1. Let Y_1 be a polygonal 1-sphere on F, V_1 and V_2 the closures of the two components into which S^2 is divided by Y_1. Suppose that the intersection of P with Y_1 consists of precisely two points, z_1 and z_2, which are not double points of P; we set $P \cap V_1 = P_1'$, $P \cap V_2 = P_2'$. Let H_1 be one of the segments into which Y_1 is divided by z_1 and z_2, and set $P_1' \cup H_1 = P_1$, $P_2' \cup H_1 = P_2$. If P_1 and P_2 lie in distinct V_i, then P is called a *product* of P_1 and P_2, and (P, Δ_K) is called a *product* of $(P_1, \Delta_K \mid D \cap P_1)$ and $(P_2, \Delta_K \mid D \cap P_2)$. (The product is denoted by a dot between the factors.)

THEOREM 5.2. *Under the conditions of* 5.1, *there are knots K_1 and K_2 such that $P_i = p(K_i)$, $p \mid K_i$ satisfy* 3.2, 3.3, *and* 3.4, *and* $\Delta_{K_i} = \Delta_K \mid D \cap P_i$.

REMARK. The restriction $Y_1 \subset F$ in 5.1 is inessential, but makes the proof of 5.2 easier. In actuality, it would be sufficient to assume $Y_1 \subset S^2$.

PROOF. By the Jordan Theorem, one of the V_i, say V_1, must lie in F. Suppose the z-co-ordinates of K are strictly bounded from above by a_2 and from below by a_1. Let $U_1 = V_1 \times [a_1, a_2]$, where $[a_1, a_2]$ is the closed interval from a_1 to a_2. Let H be an arbitrary segment on $\dot{U}_1 = Y$ such that $p(H) = H_1$ without double points. Then 2.1 is satisfied, $p \mid K_i = P_i$, and $p \mid K_i$ satisfy 3.2. Furthermore, H_1 contains no double points of Image p, and it follows that H_1 contains no double points of Image $p \mid K_i$; thus $p \mid K_i$ satisfy 3.3 and 3.4 also. Lastly, if we take the region containing points of S^2 not in F to be black, then a point in a white region of P_i will also lie in a white region of P, and thus if d is a double point of P_i, we obtain without difficulty that $\Delta_{K_i}(d) = \Delta_K(d)$. This completes the proof.

6. The index of the graph

Denote by λ_i the arcs of Λ joining δ_i and δ_{i+1}, and set $l_i = \sigma(\lambda_i)$. If there are only two δ_i, denote by λ_0 that arc which when oriented from δ_0 to δ_1, induces an orientation on Int Λ that goes into ω under σ. Denote the regions into which G divides S^2 by M_k; then we have the following lemma:

LEMMA 6.1. *Each M_k contains one and only one black region B_k, and B_k is the only black region that M_k meets.*

PROOF. (a) We first prove that there is at least one black region that meets M_k. Let d be an arbitrary double point lying on $\dot{M}_k$. Let Ω be a disk with center d of radius so small so that $\Omega \cap G$ consists of two straight line segments meeting at d, and $\Omega \cap P$ consists of four straight line segments meeting at d. $\Omega \cap G$ divides Ω into two regions, R_1 and R_2, each of which contains two segments of $\Omega \cap P$. Each point on P is a boundary point of a black region, and hence for each

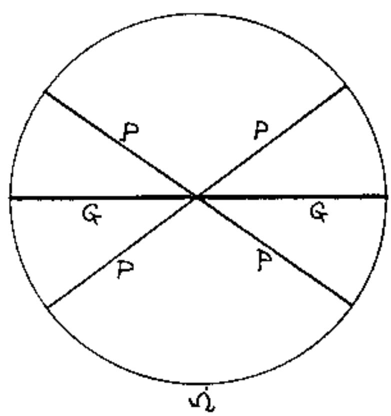

FIG. 2

of the two regions R_1 and R_2, there is a black region meeting it. Since M_k must meet at least one of R_1 or R_2 (as d is a boundary point of M_k), it follows that M_k must meet a black region. This region will be called B_k. (See Figure 2.)

(b) We next prove that $M_k \supset B_k$. For, if not, then M_k must have a boundary point in B_k. But $\dot{M}_k \subset G$, and it follows that G must meet a black region, which is impossible, as G is contained in the union of the closures of the white regions.

(c) Finally, we show that each M_k contains only one black region. Let m denote the number of M_k's, w the number of white regions, b the number of black regions, $o(D)$ the number of double points. In the graph P, each vertex $d \in D$ lies on precisely four edges, and each edge has precisely two vertices $d \in D$. Hence the number of edges of P is $2o(D)$. Since the number of faces of P is given by $b + w$, Euler's formula yields

$$o(D) - 2o(D) + (b + w) = 2,$$

or

$$b = 2 + o(D) - w.$$

On the other hand, in the graph G the vertices are the centers of the white spaces, so that there are precisely w vertices. What is more, there is one and only one double point on each edge of G; thus the number of edges of G is $o(D)$. It follows that

$$w - o(D) + m = 2,$$

or

$$m = 2 + o(D) - w = b.$$

We already know that each M_k contains at least one black region. Since there are no more black regions than M_k's, each M_k can contain only one black region, q.e.d.

Let x be the center of the white space W.

DEFINITION 6.2. Given k, we define the index $I_k(x)$ of M_k on x by

(1) If x is not a boundary point of M_k, $I_k(x) = 0$.

(2) If x is a boundary point of M_k, we let $I_k(x) + 1$ be the number of distinct l_i bounding W that lie in M_k.

Thus if only one of the l_i lies in M_k, then $I_k(x) = 0$. The index measures the multiplicity with which a region M_k meets a vertex x. The index will also be denoted by $I_k(x; G)$, in order to show explicitly its dependence on G.

DEFINITION 6.3. The index of a vertex x of G is given by

$$I(x; G) = \sum_k I_k(x; G);$$

the index of the entire graph G is given by

$$I[G] = \sum_x I(x; G).$$

The proof of the asphericity of non-trivial alternating knots is accomplished by induction on the index $I[G]$. In this induction, the proof that the theorem holds for $I[G] = 0$ is more difficult than the inductive step. We will deal with the inductive step first. It is sufficient to prove that every alternating knot K whose graph G has index larger than zero is the product of two alternating knots K_1 and K_2 each of whose graphs has index less than $I[G]$. For then by the induction hypothesis, K_1 and K_2 are both aspheric; and from 2.9 it then follows that K is aspheric as well.

LEMMA 6.4. *Let K be an alternating knot whose projection P has a graph G of positive index. Let Δ_0 denote a function identically equal to* 1; *we may assume without loss of generality that $\Delta_K = \Delta_0$. Then there are two knot projections P_1 and P_2, with graphs G_1 and G_2 respectively, such that*

$$(1) \qquad (P, \Delta_K) = (P_1, \Delta_0)\cdot(P_2, \Delta_0)$$

$$(2) \qquad \max (I[G_1], I[G_2]) < I[G].$$

PROOF. We note that since the index of K is larger than zero, there must be at least one vertex x of G, and one M_k, such that $I_k(x; G) > 0$. Let W be the white region in which x lies, l and l' two sides of W lying in M_k, λ and λ' the inverse images under σ of l and l' respectively, β and β' arbitrary points lying in

λ and λ' respectively. Connect β and β' to θ by straight lines α and α', and set $\sigma(\alpha) = A$, $\sigma(\alpha') = A'$. $A \cup A'$ is a simple curve starting at $\sigma(\beta)$, ending at $\sigma(\beta')$, containing x, and lying entirely within W. By 6.1 and 6.2, both $\sigma(\beta)$ and $\sigma(\beta')$ must lie on $\dot{B}_k$. They may therefore be connected by a simple polygonal curve A'' lying entirely within B_k. Then $A \cup A' \cup A''$ is a polygonal 1-sphere on F intersecting P at only two points, namely $\sigma(\beta)$ and $\sigma(\beta')$, which are not double points of P. Hence $A \cup A' \cup A''$ satisfies the conditions for Y_1 stated in 5.1.

Furthermore, we claim that P_1 and P_2 lie in distinct V_i, according to the terminology of 5.1. Indeed, $\dot{W}$ is also a simple closed curve, and divides S^2 into two disconnected regions, V_3 and V_4. If P_1 and P_2 both lie in V_1, say, then it follows that $\dot{W}$ lies entirely within V_1. This implies that Int W and $\dot{V}_1$ are disjoint. But $\dot{V}_1 = A \cup A' \cup A''$, and it follows that Int W is disjoint from $A \cup A'$, which is absurd. By 5.1 it now follows that

$$\begin{aligned}(P, \Delta_K) &= (P_1, \Delta_0 \mid D \cap P_1)\cdot(P_2, \Delta_0 \mid D \cap P_2)\\ &= (P_1, \Delta_0)\cdot(P_2, \Delta_0).\end{aligned}$$

The first part of our theorem is thus proved.

Let us now examine the relationship of the graphs G_i to the graph G. For definiteness, we look at G_1. Except for the white region W, all of the white re-

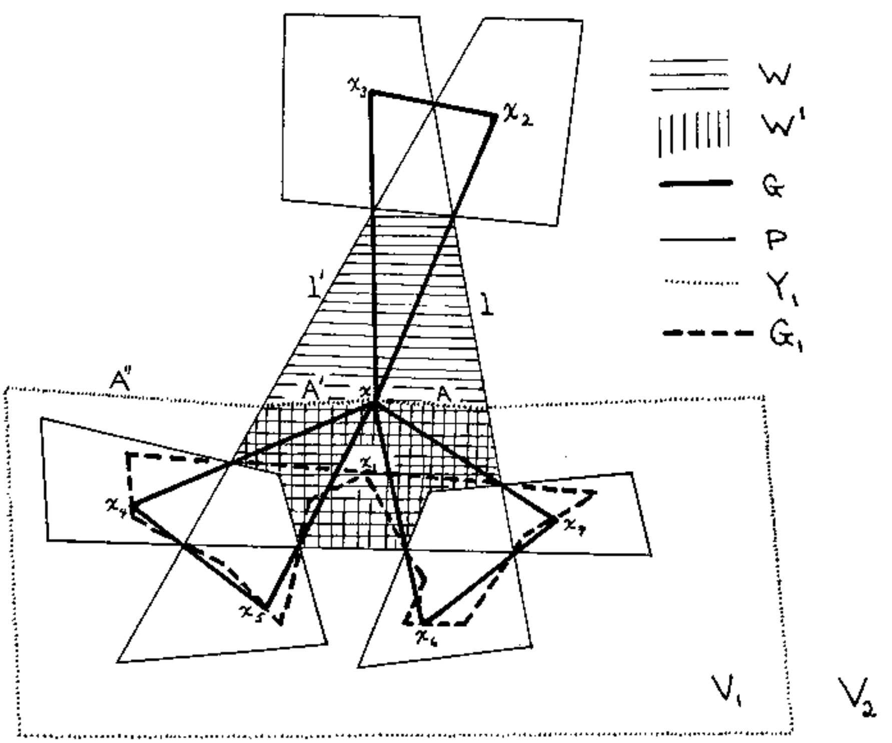

FIG. 3

gions of P_1 are either removed entirely or remain unchanged versions of the white regions of P; of course, those white regions of P lying on the other side of Y_1 from P_1 are not represented at all among the white regions of P_1. Except for B_k, all the black regions of P_1 are also unchanged versions of the black regions of P. The change that must be made in B_k in order to convert it from a black region of P to a black region B_k^1 of P_1 is a very simple one; we simply take B_k^1 to be $B_k \cup V_2$, where V_2 is the region of S^2 containing P_2. Similarly W^1 is defined to be $W \cap V_1$. (See Figure 3.)

Let x_1 be the center of W^1, and let M_k^1 be the region of $S^2 - G_1$ containing B_k^1. Then $I_k(x_1 ; G_1) < I_k(x; G)$. This may be seen as follows: Each complete side of W^1 not intersecting Y_1 is also a complete side of W. If such a side lies in M_k^1, then it lies on $\dot{B}_k^1$; but $B_k^1 = B_k \cup V_2$, and since a side of W^1 not intersecting Y_1 cannot lie on $\dot{V}_2$, it follows that it must lie on $\dot{B}_k$. Hence it lies in $\dot{M}_k$. Hence the number of distinct sides of W^1 lying in M_k^1 and not intersecting Y_1 does not exceed the number of distinct sides of W lying in M_k and not intersecting Y_1. Furthermore, there are exactly two sides of W lying in M_k and intersecting Y_1, namely l and l', while there is only one side of W^1 lying in M_k^1 and intersecting Y_1, namely the side containing $A \cup A'$, since all the rest of Y_1 lies strictly inside B_k. Hence $I_k(x; G)$ exceeds $I_k(x_1 ; G_1)$ by at least 1.

Next, let us consider a region of $S^2 - G_1$ which is not M_k^1; we call it M_s^1. M_s^1 cannot intersect Y_1 and so lies entirely in the interior of V_1. It is therefore equal to some component M_s of $S^2 - G$. M_s^1 intersects precisely as many sides of white regions of P_1 as M_s does; and each such complete side of a white region of P_1 is also a complete side of a white region of P, as none of them intersect Y_1. Hence for each center x_i ($i \neq 1$) associated with P_1, we may write

$$I_s(x_i ; G_1) = I_s(x_i ; G);$$

and for $i = 1$, we may write

$$I_s(x_1 ; G_1) = I_s(x; G).$$

Summing up, we obtain

$$\begin{aligned} I[G_1] &= \textstyle\sum_i \sum_s I_s(x_i ; G_1) \\ &= \textstyle\sum_s I_s(x_1 ; G_1) + \sum_{i \neq 1} \sum_s I_s(x_i ; G_1) \\ &= I_k(x_1 ; G_1) + \textstyle\sum_{s \neq k} I_s(x_1 ; G_1) + \sum_{i \neq 1} \sum_s I_s(x_i ; G_1) \\ &< I_k(x; G) + \textstyle\sum_{s \neq k} I_s(x; G) + \sum_{i \neq 1} \sum_s I_s(x_i ; G); \end{aligned}$$

we must keep in mind that s runs over the indices of only those regions of $S^2 - G$ that lie within V_1, and i runs over the indices of only those centers that lie within V_1. If we now let t run over the indices of *all* regions of $S^2 - G$, and allow j to run over the indices of *all* centers, then we obtain that the last expression above is

$$\leqq I_k(x; G) + \textstyle\sum_{t \neq k} I_t(x; G) + \sum_{j \neq 1} \sum_t I_t(x_j ; G)$$

$$= \sum_t I_t(x; G) + \sum_{j\neq 1} \sum_t I_t(x_j ; G)$$
$$= I[G].$$

Hence $I[G_1] < I[G]$, and our theorem is proved.

THEOREM 6.5. *Let K be an alternating knot whose graph G has positive index. Then K is the product of two alternating knots whose graphs have lower index than G.*

REMARK. This theorem completes the inductive step, leaving us to prove asphericity in the case in which the index is zero.

PROOF. This is an immediate consequence of 6.4 and 5.2.

7. The canonical form of a knot

In accordance with 6.5, we assume from now on that the knot K possesses a graph G of zero index. Let H be a "tube" around G of small radius; that is, H is the set of all points in R^3 at a distance $\leqq \varepsilon$ from G, where ε is some number sufficiently small so as not to obscure the "structure" of G; it will become clear in the course of the sequel how small ε must be chosen. We may assume without loss of generality that in the neighborhood of a double point, G consists of one straight line segment; if this is not already so, it can be achieved by slight distortions in the mappings σ.

Let us now take an arbitrary double point d and consider a small neighborhood of d in G. For convenience, we take d to be the origin of co-ordinates, and the small neighborhood of d in G to be the section of the x-axis extending from $x = -1$ to $x = 1$. The surface $\dot{H}$ corresponding to this section of G may be characterized by the equations

$$-1 \leqq x \leqq 1$$
$$y = \varepsilon \cos \theta$$
$$z = \varepsilon \sin \theta,$$

where θ is the parameter denoting the angle which the line from the point (x, y, z) on $\dot{H}$ to the point $(x, 0, 0)$ makes with the line from $(x, 0, 0)$ to $(x, -1, 0)$, and we fix $\theta = \frac{1}{2}\pi$ at $(x, 0, \varepsilon)$.

We now define a knot K' as follows: Except in the neighborhood of a double point, K' is given by $\dot{H} \cap R^2$. In the neighborhood of d, i.e., in the portion of $\dot{H}$ characterized by the above equations, K' has two components, the first having the equation

$$\theta = \tfrac{1}{2}\pi x + \tfrac{1}{2}\pi,$$

and the second having the equation

$$\theta = \tfrac{1}{2}\pi x - \tfrac{1}{2}\pi.$$

K' is defined in a similar fashion for the other double points.

It is important to note that the definition of K' does not depend on which side of d is chosen to be along the negative x-axis and which side along the positive

x-axis. This is easily seen if we choose a new co-ordinate system in which the former positive x-axis becomes the negative x-axis, and the former negative x-axis becomes the positive x-axis, while the z-axis remains unchanged (the direction of the z-axis is not at our disposal, since Δ, which is to remain invariant, was defined in terms of the direction of the z-axis). If we agree that the "usual" orientation induced by the co-ordinate system (i.e., the orientation induced by ((1, 0), (0, 1), (0, 0)) on the triangle bounded by those points) shall agree with ω, then this new co-ordinate system, which we will denote by (x', y', z'), is simply a rotation of the old one by π; we have

$$x' = -x$$
$$y' = -y$$
$$z' = z.$$

θ' is defined to be 0 at the point $x' = 0$, $y' = -\varepsilon$, $z' = 0$, and $\frac{1}{2}\pi$ at the point $x' = 0$, $y' = 0$, $z' = \varepsilon$. That is to say, θ' is 0 at $x = 0$, $y = \varepsilon$, $z = 0$, and $\frac{1}{2}\pi$ at $x = 0$, $y = 0$, $z = \varepsilon$. The equations of K' near d in terms of the new co-ordinate system are

$$\theta' = \tfrac{1}{2}\pi x' + \tfrac{1}{2}\pi$$

and $$\theta' = \tfrac{1}{2}\pi x' - \tfrac{1}{2}\pi \qquad \text{respectively.}$$

In terms of the old co-ordinate system, these are simply

$$-\theta + \pi = \tfrac{1}{2}\pi(-x) + \tfrac{1}{2}\pi$$

and $$-\theta + \pi = \tfrac{1}{2}\pi(-x) - \tfrac{1}{2}\pi \qquad \text{respectively,}$$

i.e., $$\theta = \tfrac{1}{2}\pi x + \tfrac{1}{2}\pi$$

and $$\theta = \tfrac{1}{2}\pi x - \tfrac{1}{2}\pi \qquad \text{respectively,}$$

which are the same equations as those we obtained before. Summing up, we have proved

THEOREM 7.1. *K' is invariantly defined in terms of the direction of the z-axis and of ω.*

THEOREM 7.2. *G is the graph of K'.*

PROOF. Let $P' = p(K')$. If we declare the regions of $S^2 - P'$ containing vertices of G to be white regions, and the others to be black regions, then no side of P' is the common boundary of two white regions. One way of seeing this is by noting that a side that is the common boundary of two white regions has to have, at any given point, *two* half-edges of G within a distance of ε from it, each of the two half-edges radiating from a different vertex. It is clear that if ε is sufficiently small, this is impossible except in the neighborhood of double points.

Similarly, by construction it is apparent that every point t of P' lies within ε of some half-edge γ of G. The side of P' containing t therefore bounds the white region determined by the vertex of G lying in γ. Thus each side of P' bounds one, but not two, white regions. Hence it must bound one white region and one black region, and our division of $S^2 - P'$ into white and black regions turns out to be in accordance with the rules.

Finally, it is immediate by construction that the double points of P' are the same as the double points of G. If we fix attention on a given white region W, we see that from its center, a vertex of G, there radiate half-edges to all the double points of P' lying on $\dot{W}$. But these are precisely the double points of P' lying on $\dot{W}$. Hence $G \cap \dot{W}$ may be considered that portion of the graph of P' that lies in W, and it follows that the graph of P' is indeed G.

THEOREM 7.3. *$\Delta_{K'}$ is a constant.*

REMARK. Because ω is taken to be the usual orientation on R^2, $\Delta_{K'}$ is in fact identically 1. But this fact is unimportant. If it had turned out to be -1, we would have either altered the definition of K', or fixed attention on alternating knots whose Δ is identically -1.

PROOF. This is a consequence of the fact that K' is defined near d without reference to the particular d under consideration.

THEOREM 7.4. *K' is equivalent to K.*

REMARK. In view f this theorem, we will henceforth take K to be equal to K'.

PROOF. This is a consequence of 7.3, 7.2, and 4.1.

It should be noted that K has now ceased to be polygonal; strictly speaking, this is against the rules. What we should now do is approximate K' by a polygonal K. However, such a procedure is clearly possible, does not alter the proof materially, and merely complicates matters. We will therefore refrain from carrying out this portion of the proof in detail.

8. Application of Whitehead's theorem

We are now ready to proceed with the proof of asphericity. Set $X = S^3 - K$, $A = H - K$, $B = \mathrm{Cl}\,(S^3 - H) - K$. The proof of asphericity will be accomplished by the application of 1.1 to the triad $(X; A, B)$. Several of the lemmas of this section are constructions which can be easily performed in view of the detailed geometrical situation described in the previous section. The proofs of these lemmas will be omitted.

Denote the plane $z = \varepsilon$ by Z_1, the plane $z = -\varepsilon$ by Z_2. Define the graph G' as follows: Except in the neighborhood of double points, $G' = (Z_1 \cap \dot{H}) \cup (Z_2 \cap \dot{H})$. In the neighborhood of a double point, we define G' as consisting of two components, which are given by the following equations (we adopt the notation of the previous section):

$$\theta = \tfrac{1}{2}\pi x + \pi$$

and
$$\theta = \tfrac{1}{2}\pi x.$$

As in the previous section, G' is invariantly defined.

Define a map $r_1 : G' \to G$ as follows: Except in the neighborhood of a double point, r_1 is simply the projection p. In the neighborhood of a double point, let

$q = (x, y, z)$ be a point of G'. Then $r_1(q) = (x, 0, 0)$. It is clear that r_1 is a covering mapping.

LEMMA 8.1. *There are deformation retractions*[5] $r_2: A \cap B \to G'$ *and* $r_3: A \to G$ *such that* $r_1 \circ r_2 = r_3 \mid A \cap B$.

THEOREM 8.2. *A is aspheric.*

PROOF. By 8.1, A has the homotopy type of G. But G is a graph, and hence its universal covering space, a tree, has the homotopy type of a point. Hence G is aspheric, and the theorem follows.

THEOREM 8.3. *The components of $A \cap B$ are aspheric.*

PROOF. By 8.1, $A \cap B$ has the homotopy type of G'. The proof is now similar to that of 8.2.

LEMMA 8.4. *B has the homotopy type of* $\mathrm{Cl}\,(S^3 - H)$.

LEMMA 8.5. *There is a deformation retraction* $r_4: H - G \to \dot{H}$.

LEMMA 8.6. *There is a deformation retraction* $r_5: S^3 - G \to \mathrm{Cl}\,(S^3 - H)$.

PROOF. Define r_5 by $r_5 \mid H - G = r_4$, $r_5 \mid S^3 - H =$ identity.

LEMMA 8.7. *$T_1 - G$ and $T_2 - G$ have the homotopy types of T_1 and T_2.*

THEOREM 8.8. *B is aspheric.*

PROOF. We note first that by 8.4 and 8.6, B has the homotopy type of $S^3 - G$. Next, we apply 1.1 to the triad $(S^3 - G; T_1 - G, T_2 - G)$. Since T_1 and T_2 are clearly aspheric, it follows from 8.7 that $T_1 - G$ and $T_2 - G$ are aspheric as well. Furthermore,

$$(T_1 - G) \cap (T_2 - G) = S^2 - G = \bigcup_s M_s .$$

Each of the M_s is an open disk, and hence is first of all aspheric, and secondly has a fundamental group equal to zero. It follows that the injections from each of the components of $S^2 - G$ into $T_1 - G$ and $T_2 - G$ must be isomorphisms into. Hence all the conditions of 1.1 are satisfied and it follows that $S^3 - G$ is aspheric, q.e.d.

THEOREM 8.9. *For each of the components C_i of $A \cap B$, the injection*

$$i_1^*: \pi_1(C_i) \to \pi_1(A)$$

is an isomorphism into.

PROOF. We first assume that $A \cap B$ has only one component (the case of the non-orientable Reidemeister surface). Let $i_1: A \cap B \to A$ be the inclusion mapping. Then by 8.1, the diagram

$$\begin{array}{ccc} A \cap B & \xrightarrow{r_2} & G' \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ A & \xrightarrow{r_3} & G \end{array}$$

is commutative. It follows that for the homomorphisms induced by these mappings on the fundamental groups, we have $r_3^* \circ i_1^* = r_1^* \circ r_2^*$. r_1 is a covering

[5] All deformation retractions will be onto.

mapping, and hence r_1^* has kernel zero; r_2 is a deformation retraction, and hence r_2^* is an isomorphism onto. It follows that the right side of the above equation must have kernel zero; Therefore i_1^* has kernel zero also.

The proof is the same when $A \cap B$ has two components, C_1 and C_2 (the case of the orientable Reidemeister surface); we simply replace $A \cap B$ by C_1 or C_2.

9. The injection i_2^*

It remains to prove that $i_2^*:\pi_1(C_i) \rightarrow \pi_1(B)$ is an isomorphism into. We confine ourselves to the case in which $A \cap B$ is connected; the proof in the other case is entirely analogous.

Let W^j be a white region, $d_i = d_i^j$ one of its double points. Let $g_i^j:I \rightarrow g_i$ be a path starting at the center x^j of W^j and ending at the double point d_i. Suppose $\tilde{d}_i$ to be a point of G' lying over d_i. Then there is a unique path $\tilde{g}_i^j:I \rightarrow G'$ starting at $\tilde{d}_i$ such that $r_1 \circ \tilde{g}_i^j = g_i^j$. Up to the present we have indexed the double points d_i, and consequently also the $\tilde{d}_i$, in the positive orientation around a certain white region W^j. We now abandon this notation and simply let the $\tilde{d}_i$ range over all the points of G' lying over double points of G. We thus obtain $2o(D)$ distinct $\tilde{d}_i$, each of which has a different index. We then have that with each $\tilde{d}_i$ there are associated two paths, $\tilde{g}_i^{j_1}$ and $\tilde{g}_i^{j_2}$, corresponding to the two white regions on whose boundary $\tilde{d}_i$ lies. We set

$$f_i = \tilde{g}_i^{j_1} \cdot (\tilde{g}_i^{j_2})^{-1}$$

(the choice of which path comes first is arbitrary). Call the points $r_1^{-1}(x^j)$ vertices of G'. Then f_i is a path in G' starting at one vertex, ending at another, and passing through precisely one $\tilde{d}_i$.

The construction of the fundamental group of G' may be accomplished by means of a well-known process. We first pick a maximal tree T' in G', and a base point y_0; y_0 is a vertex of G', and every other vertex of G' also lies in T'. For each vertex y_j of G' there is a unique edge-path in T' starting at y_0 and ending at y_j. Hence it follows that there is a path $h_j:I \rightarrow T'$ starting at y_0 and ending at y_j and unique up to homotopy. h_j may be considered to be the unique product of successive f_i's or their inverses; indeed, with each edge of the unique edge-path we may associate an f_i, and these will be the f_i's entering into h_j. Let d_{ij} be an edge of G' not lying in T', starting at y_i and ending at y_j. Let h_{ij} be a path whose image is d_{ij}, starting at y_i and ending at y_j. Then the loop $h_i \cdot h_{ij} \cdot h_j^{-1}$ represents a member of $\pi_1(G')$; it is well known that the set of all these members is a set of free generators for $\pi_1(G')$. It is clear that h_{ij} may be taken to be a single f_i (or its inverse); in fact, that f_i whose end points are y_i and y_j is the h_{ij} we want. It follows that there is a set of products of f_i's and their inverses that represent a set of free generators of $\pi_1(G')$. Summing up, we have proved

LEMMA 9.1. *Each member of $\pi_1(G')$ may be represented by a product of f_i's or their inverses.*

COROLLARY 9.2. *Each member of $\pi_1(G')$ may be represented by a product of f_i's or their inverses in which no f_i is preceded or followed by its inverse.*

Remark. In point of fact, it is not too difficult to see that with this restriction the product becomes unique (i.e., the order of the f_i is unique; the multiplications may of course be performed in any order in which one chooses, leading to products that are in general not equal but homotopic). We do not need this fact, but in any case we will call this product the *canonical* product corresponding to the member of $\pi_1(G')$ in question.

Note that the double points of G' all lie in R^2, and in fact in a black region.

Lemma 9.3. *In the canonical product, no two successive factors have their double points lying in the same M_k.*

Proof. By 9.2, the two factors, say f_1 and f_2^{-1}, can only have one end point in common. Call this end point y_0. We adopt the notation of Section 7.

We may assume without loss of generality that y_0 lies above R^2, that for $i = 1$ and 2, f_i starts at y_0, and that the portion of the image of f_i that is higher than R^2 lies over the half plane $x \leqq 0$. We fix attention on the white region W whose center is $r_1(y_0)$.

The point $\tilde{d}_i$ has the co-ordinates (0, 1, 0). We may connect it by a straight line segment to the point e_i of P that has the co-ordinates $(-1, 1, 0)$. This straight line segment does not intersect G (in the neighborhood of $\tilde{d}_i$ it is parallel to G); hence e_i lies in the M_k in which $\tilde{d}_i$ lies. Let us orient the section $\overline{d_ie_i}$ of P given by

$$y = -\sin \tfrac{1}{2}\pi x$$

$$0 \geqq x \geqq -1$$

in the positive direction from d_i to e_i. This orientation induces the positive orientation ω on the cell E_i bounded by $\overline{d_ie_i}$, by the line segment

$$x = -1$$

$$0 \leqq y \leqq 1,$$

and by the line segment

$$y = 0$$

$$0 \geqq x \geqq -1.$$

Since E_i has no interior points lying on P, it must lie either completely within or completely outside of W. On the other hand, some of the boundary points of E_i are interior points of W, e.g. the point $(-1, 0, 0)$. Hence E_i must lie completely within W. Since the positive orientation from d_i to e_i along $\overline{d_ie_i}$ induces the positive orientation on E_i, it follows that it must also induce the positive orientation on W. (See Figure 4.)

Now let us assume that e_1 and e_2 lie in the same l_j, say l_0. Since at least one of the end points of the l_j on which e_i lies must be d_i, it follows that both d_1 and d_2 are end points of l_0. But by 9.1, f_1 and f_2 are not the same, so that d_1 and d_2 must be distinct. Hence the two end points of l_0 are d_1 and d_2.

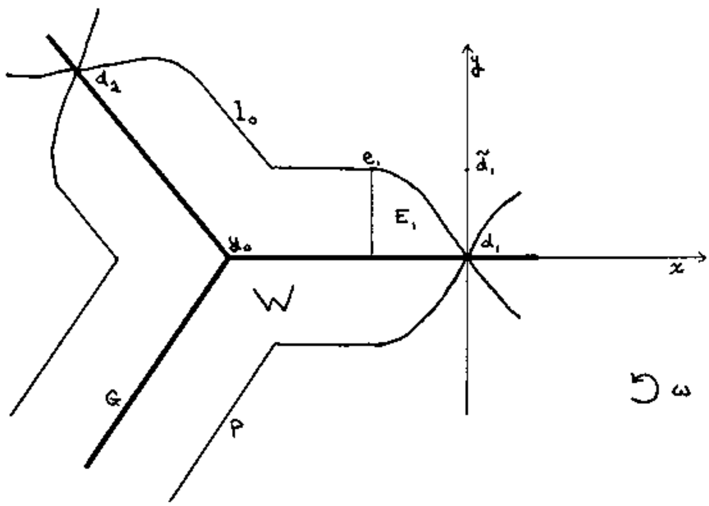

FIG. 4

The segment $\overline{d_1e_1}$ is simply a part of l_0. We know that the positive orientation on W induces an orientation on $\overline{d_1e_1}$ that goes from d_1 to e_1. Hence it induces an orientation on l_0 that goes from d_1 to d_2. But by the same reasoning, the positive orientation on W must induce an orientation on l_0 that goes from d_2 to d_1. Thus the assumption that e_1 and e_2 lie in the same l_j has led to a contradiction. Hence it follows from the assumption that $I(G) = 0$ that e_1 and e_2 lie in distinct M_k. But since e_i and $\tilde{d}_i$ lie in the same M_k, it follows that $\tilde{d}_1$ and $\tilde{d}_2$ lie in distinct M_k, q.e.d.

We now construct a covering space for $S^3 - G$. Set $S_i = T_i - G$, $i = 1, 2$. We construct a class $\mathfrak{S}_1$ of copies S_1^0, $S_1^{i_1i_2}$, $\cdots$, $S_1^{i_1i_2\cdots i_t}$, $\cdots$ of S_1, where the i_j each range from 1 to the number m of distinct M_k, and t may be any even positive integer, with the sole restriction that in any specific copy of S_1, we have $i_j \neq i_{j+1}$, $j = 1, \cdots, t - 1$. Similarly, we construct a class $\mathfrak{S}_2$ of copies $S_2^{i_1}$, $S_2^{i_1i_2i_3}$, $\cdots$, $S_2^{i_1i_2\cdots i_t}$, $\cdots$ of S_2, with precisely the same stipulations as above, except that t is restricted to be an odd positive integer. Note only that the copy indexed by 0 is omitted in this case. Set $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2$. Since each member of $\mathfrak{S}$ is a copy of one of the original S_i, there exist homeomorphisms

$$\phi_j^{(t)}: S_j \rightarrow S_j^{(t)},$$

(t) being an abbreviation for the expression $i_1i_2 \cdots i_t$, or, if $t = 0$, for the expression 0. Define $M_k^{(t)}$ for $k = 1, \cdots, m$ by

$$M_k^{(t)} = \phi_j^{(t)}(M_k).$$

Note that the subscript j is always uniquely defined by the superscript (t); in fact it is 1 if t is even and is 2 if t is odd. We will call this function of t by the name $q(t)$.

Now take the union of all the members of $\mathfrak{S}$, and identify M_k^0 with M_k^k and $M_k^{(t)}$ with $M_k^{(t)k}$ in the natural manner if $k \neq i_t$. Call the resulting identification space $\tilde{B}$. Define $\theta:\tilde{B} \to S^3 - G$ by $\theta \mid S_j^{(t)} = (\phi_j^{(t)})^{-1}$. Then θ is well-defined, and we have

LEMMA 9.4. *$\tilde{B}$ is a covering space of $S^3 - G$ with θ as covering mapping.*

REMARK. In fact, $\tilde{B}$ is the universal covering space of $S^3 - G$. We do not need this fact.

PROOF. θ is certainly a local homeomorphism on Int $S_i^{(t)}$, since there are no identifications there. Let $[s]$ be an abbreviation for $j_1 j_2 \cdots j_s$, or for 0 if $s = 0$. It is sufficient to prove that each $M_k^{(t)}$ is identified with precisely one other copy of M_k, which we will call $M_k^{[s]}$, and that $q(t) \neq q(s)$. We distinguish several cases, with which we will deal one by one.

(a) $(t) = 0$. In this case we set $[s] = k$. No other identifications are made.

(b) $(t) = k$. In this case we set $[s] = 0$. Although we have a term i_1 in the superscript (t), no other identifications are made because i_1 is the last term of the superscript and $i_1 = k$.

(c) (t) has more than one component, and for its last term i_t we have $i_t = k$. In this case we set $[s] = i_1 \cdots i_{t-1}$. Then $M_k^{[s]}$ must be identified with $M_k^{(t)}$ because $i_{t-1} \neq k$, since we must have $i_{t-1} \neq i_t$. What is more, no other identifications are made because $i_t = k$.

(d) For the last term i_t of (t), we have $i_t \neq k$. In this case we set

$$[s] = i_1 \cdots i_t k.$$

Since $i_t \neq k$, no other identifications are made.

Note that in each case $q(t) \neq q(s)$.

LEMMA 9.5. *A compact subset of $\tilde{B}$ can meet only finitely many members of S.*

PROOF. Let Q be a compact subset of $\tilde{B}$. Let $N_i^{(t)}$ be the union of $S_i^{(t)}$ and the interiors of all the members of $\mathfrak{S}$ that meet $S_i^{(t)}$. Then $N_i^{(t)}$ is an open set and the set of all $N_i^{(t)}$ is an open covering of $\tilde{B}$. Hence a finite number of $N_i^{(t)}$ suffice to cover Q. But each $N_i^{(t)}$ meets only $m + 1$ members of $\mathfrak{S}$. Hence Q may be covered by a finite number of members of $\mathfrak{S}$. Since each member of $\mathfrak{S}$ meets only a finite number of other members of $\mathfrak{S}$, our lemma is proved.

LEMMA 9.6. *The injection $i_5^*:\pi_1(G') \to \pi_1(S^3 - G)$ is an isomorphism into.*

PROOF. Let η be a non-trivial member of $\pi_1(G')$; then the canonical product representing η has more than one factor. It is sufficient to prove that $i_5^*(\eta)$ is not the identity.

Let F be the canonical product representing η, and let $i_5:G' \to S^3 - G$ be the inclusion mapping. Then $i_5 \circ F$ is a path in $S^3 - G$. If $i_5^*(\eta)$ is the identity, then $i_5 \circ F$ is zero-homotopic in $S^3 - G$. Hence it may be lifted into a loop in $\tilde{B}$. That is to say, there is a map $F':I \to \tilde{B}$ such that $F'(0) = F'(1)$ and $\theta \circ F' = i_5 \circ F$.

Since I is compact, $F'(I)$ must be compact as well. Hence by 9.5, the number of components of (t) in the $S_i^{(t)}$ that $F'(I)$ meets is bounded. Let s be the maximum number of components of a superscript that actually occurs in a member of S that $F'(I)$ meets. That is, we have that $F'(I)$ meets an $S_i^{(s)}$, but it is not true that $F'(I)$ meets any $S_j^{[t]}$ with $t > s$.

Set $F' = \prod_j f'_{i_j}$, where we are stipulating that $\theta \circ f'_{i_j} = f_{i_j}$. Suppose that $f'_{i_0}(I)$ meets $S_i^{(s)}$, and that indeed $f'_{i_0}(1) \in S_i^{(s)}$. Let the next factor of F' be f'_{i_1}. $(\theta \circ f'_{i_j})(I) = f_{i_j}(I)$ certainly crosses some M_k; hence $f'_{i_j}(I)$ must cross an $M_k^{(s)}$ for $j = 0$ and 1. However, f'_{i_0} and f'_{i_1} cannot cross the same $M_k^{(s)}$, for otherwise f_{i_0} and f_{i_1} would cross the same M_k, a violation of 9.3. Thus for either $j = 0$ or $j = 1$, we must have that f'_{i_j} crosses an $M_k^{(s)}$ with $k \neq i_s$. But this $M_k^{(s)}$ is identified with $M_k^{(s)k}$, so that $f'_{i_j}(I)$ meets $M_k^{(s+1)}$ (with $i_{s+1} = k$), and therefore also meets $S_{q(s+1)}^{(s+1)}$, contrary to the assumption that $F'(I)$ meets no $S_j^{[t]}$ with $t > s$. Thus we have shown that for a non-trivial η, $i_5^*(\eta)$ cannot be the identity, so that i_5^* must be an isomorphism into.

Let $i_6: B \to S^3 - G$ be the inclusion mapping.

Lemma 9.7. *The injection $i_6^*: \pi_1(B) \to \pi_1(S^3 - G)$ is an isomorphism onto.*

Proof. Let $i_7: B \to \mathrm{Cl}(B)$ be the inclusion mapping (the closure is to be taken in S^3). Then the injection i_7^* is an isomorphism onto. Hence it is sufficient to prove that the injection

$$i_8^*: \pi_1(\mathrm{Cl}(B)) \to \pi_1(S^3 - G)$$

is an isomorphism onto. From $r_5 \circ i_8 =$ identity, it follows that $r_5^* \circ i_8^* =$ identity. But since r_5 is a deformation retraction, r_5^* is an isomorphism onto, and so i_8^* must be an isomorphism onto as well.

Theorem 9.8. *The injection*

$$i_2^*: \pi_1(A \cap B) \to \pi_1(B)$$

is an isomorphism into.

Proof. Let $i_9: G' \to A \cap B$ be the inclusion mapping. Then the diagram

$$\begin{tikzcd} A \cap B \arrow[d, "i_2"'] & G' \arrow[l, "i_9"'] \arrow[d, "i_5"] \\ B \arrow[r, "i_6"'] & S^3 - G \end{tikzcd}$$

is commutative. It follows that the diagram

$$\begin{tikzcd} \pi_1(A \cap B) \arrow[r, "r_2^*"] \arrow[d, "i_2^*"'] & \pi_1(G') \arrow[d, "i_5^*"] \\ \pi_1(B) \arrow[r, "i_6^*"'] & \pi_1(S^3 - G) \end{tikzcd}$$

is commutative as well, since $r_2^* = (i_9^*)^{-1}$ (this is a consequence of the fact that $r_2 \circ i_9$ is the identity and r_2 is a deformation retraction). What is more, both r_2^* and i_6^* are isomorphisms onto (r_2 is a deformation retraction; i_6^* by 9.7). Since i_3^* is an isomorphism into (by 9.6), i_2^* is an isomorphism into as well.

THEOREM 9.9. *Alternating knots are aspheric.*

PROOF. Follows from 1.1, 8.2, 8.3, 8.8, 8.9, and 9.8.

10. Applications

DEFINITION 10.1. A knot K is said to be *separable* if it possesses a disconnected projection.

THEOREM 10.2. *No alternating knot is separable.*[6]

PROOF. For, if it were, it would not be aspheric, contrary to 9.9.

DEFINITION 10.3. Let K be a knot of multiplicity 1. K is called *A-unknotted* if $\pi_1(S^3 - K)$ is infinite cyclic. K is called *D-unknotted* if there exists a map $f: E \to S^3$, where E is a triangulated 2-cell, such that f is a simplicial non-degenerated map of E into some rectilinear subdivision of S^3, f maps $\dot{E}$ topologically onto K, and $f(E - \dot{E}) \subset S^3 - K$.

LEMMA 10.4. *If K is aspheric and D-unknotted, then K is A-unknotted.*

The proof is given in [2].

LEMMA 10.5. *If K is alternating and A-unknotted, then K is unknotted.*

The proof is given in [1].

THEOREM 10.6. *If K is alternating and D-unknotted, then K is unknotted.* (This is Dehn's Lemma for alternating knots.)

PROOF. Follows from 9.9, 10.4, and 10.5.

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[6] This result has been obtained independently by R. H. Crowell [1].