16 Linearity of Unrestrictedly Transferable Utilities

We quote from Ref. [1]:

Most of the past work in n-person (game) theory has supposed that, in addition to receiving the payoffs prescribed by the rules of the game, the players are permitted to make additional transfers—side payments in the delicate language of the theory, considerations or bribes in more direct vocabularies. Indeed, a far stronger assumption is made which is generally subsumed under the phrase that utility is 'unrestrictedly transferable.' Of course, it is never utility as such that is transferred, for utility is a derivative concept, but commodities to which utility can indirectly be attached by the players. To make any sense of the elliptic concept of unrestricted transferability and of the mathematics employed, one must suppose that there exists an infinitely divisible, real and desirable commodity (which for all the world behaves like money) such that any reapportionment of it among the players results in increments and decrements of individual utilities which sum to zero according to some specific set of utility scales for the players. This can happen if money exists, provided that each player's utility for money is linear and that the zero and unit of each utility function is so chosen that the conservation of money implies the conservation of utility. When else it can realistically happen is obscure.

It is the purpose of this note to answer the question raised in the last sentence of this quotation. We will show that when $n \geqslant 3$ unrestrictedly transferable utilities imply utilities for each player that are linear in money. When $n = 2$ this implication need not hold.

Let us denote by P the set of possible outcomes p of the game in question, before side payments are made. The utility function of each player i is a function of the outcome p and of the amount of money x that he gets as a result of side payments after the play is over. We know that this utility function is uniquely determined "up to the choice of a zero and a unit," i.e., up to an additive and a positive multiplicative constant. The assumption that utility is "unrestrictedly transferable," as defined in the above quotation, may be formulated as follows: For each player i $(i = 1, \ldots, n)$, it is possible to choose additive and multiplicative constants in such a way so that the resulting utility function $u_i(x,p)$ satisfies the following conditions for each fixed $p \in P$:

u_i is monotonic[1] in x; (1)

$$\sum_{i=1}^{n} x_i = 0 \text{ implies } \sum_{i=1}^{n} u_i(x_i,p) = \sum_{i=1}^{n} u_i(0,p). \tag{2}$$

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1. That the utility functions are monotonic is implied by the use of the word "desirable" to describe money in the cited passage. The monotonicity assumption may be replaced by the apparently weaker assumption that there is at least one player whose utility function is bounded in some finite interval. Only the most wildly pathological functions fail to satisfy this condition.

Mathematically, we are given n real-valued functions $u_1(x,p),\ldots,$ $u_n(x,p)$, where x ranges over the real numbers and p ranges over a set P; it is assumed that for each $p \in P$, the u_i obey (1) and (2). We wish to prove that for $n \geqslant 3$ there are functions $c(p)$ and $k_i(p) (i = 1,\ldots,n)$ defined on P, such that

$$u_i(x,p) = c(p)x + k_i(p); \tag{3}$$

here $c(p)$ is independent of x and i, and $k_i(p)$ is independent of x.

Fix p. For each i, let $f_i(x) = f_i(x,p) = u_i(x,p) - u_i(0,p)$; then $f_i(0) = 0$ and

$$\sum_{i=1}^{n} x_i = 0 \text{ implies } \sum_{i=1}^{n} f_i(x_i) = 0. \tag{4}$$

Let i and j be distinct players and x an arbitrary real number. If we set $x_i = x$, $x_j = -x$, and $x_m = 0$ for $m \neq i$, $m \neq j$ and apply (4), we obtain

$$f_i(x) = -f_j(-x). \tag{5}$$

If $k \neq i$, $k \neq j$, we prove similarly that $f_k(x) = -f_j(-x)$. Hence $f_i(x) = f_k(x)$. Since this holds for arbitrary i and k, it follows that there is an f such that

$$f(0) = 0 \tag{6}$$

and $f_1(x) = f_2(x) = \ldots = f_n(x) = f(x)$ for all x. From (4) we then obtain that

$$\sum_{i=1}^{n} x_i = 0 \text{ implies } \sum_{i=1}^{n} f(x_i) = 0 \tag{7}$$

and from (5) that

$$f(x) = -f(-x). \tag{8}$$

Now for arbitrary x and y, let $x_1 = x$, $x_2 = y$, $x_3 = -x-y$, and $x_m = 0$ for $m > 3$. From (7) and (8) it follows that $f(x+y) = -f(-x-y) = f(x) + f(y)$. This result, together with the assumption that the u_i (and therefore also f) are monotonic, implies the linearity of f according to a known result. Hence f_i, which is the same as f, is linear in x; in other words, $u_i(x,p) - u_i(0,p) = f_i(x,p) = c(p)x$. Setting $k_i(p) = u_i(0,p)$, we obtain (3).

We remark that to obtain (3) it is not necessary to assume that (2) holds for all members $(x_1,\ldots,x_n)$ of euclidean n-space. Let us assume that $\sum_{i=1}^{n} u_i(x_i,p) = \sum_{i=1}^{n} u_i(0,p)$ in an open and connected subset V of the hyperplane $\sum_{i=1}^{n} x_i = 0$. Then we can deduce (3) for all x in the projection $\pi_i(V)$ of V on the x_i-axis.[2]

The proof is a slight adaptation of the above proof. Fix p. Let $(x_1^o, \ldots, x_n^o) \in V$, and set $f_i(x) = u_i(x + x_i^o, p) - u_i(x_i^o, p)$. The proof now goes through as before, with x restricted to a small neighborhood of 0; we obtain (3), with x restricted to a small open interval around x_i^o. But all of $\pi_i(V)$ may be covered by such intervals, and in the places where the intervals overlap, we must get the same coefficients $c(p)$ and $k_i(p)$. A simple topological argument now leads to the desired result.

If we are willing to add the assumption that the u_i are continuous in x, then we can extend our result to the case in which V is the closure of an open connected set or lies between such a set and its closure.

The importance of these remarks lies in the fact that, in most practical cases, not all possible combinations of side payments actually come into question. For example, a player i would not be willing to accept a total side payment x which would yield him a utility $u_i(x,p)$ smaller than what he can guarantee himself by his own efforts. Thus each player has a certain minimum b_i below which he will not accept side payments. In the case in which side payments are limited by this factor and by this factor only, V is the simplex formed by the intersection of the "corner" $x_1 \geqslant b_1, \ldots, x_n \geqslant b_n$ with the hyperplane $\sum_{i=1}^{n} x_i = 0$. Although V itself is not open, in this particular case the linearity of the u_i on V follows from their linearity on the interior of V, and the continuity of the u_i is not needed.

It is of interest to ask under what conditions the marginal utilities $c(p)$ are independent of the outcomes p. One condition that ensures this is

$$u_i(0,p_1) = u_i(x,p_2) \text{ implies } u_i(y,p_1) = u_i(x+y,p_2). \tag{9}$$

Intuitively, (9) says that the difference between outcomes p_1 and p_2 can be replaced in a natural way by the monetary value x. Another case in which we know that $c(p)$ is independent of p is that in which P is the cartesian product of sets $P_1, \ldots, P_n$, and the utility $u_i(x,p)$ depends only on the i-th coordinate of p.

To construct a counterexample for $n = 2$, it is sufficient to construct u_1 and u_2 so that f_1 and f_2 are nonlinear and $f_1(x) = -f_2(-x)$. A simple example can be obtained by setting $u_1(x,p) = u_2(x,p) = x^3 + k_i(p)$, k_i being arbitrary. To obtain a more "natural" example, we distinguish between "pure" outcomes and "mixed" outcomes, i.e., probability combinations of pure outcomes. For pure outcomes p, let $k_1(p)$ and $k_2(p)$

2. This holds true even if V depends on p.

take integral values only, and set

$$u_i(x,p) = 2\pi(x + k_i(p)) + \sin 2\pi x, \quad i = 1,2. \tag{10}$$

The utilities for mixed outcomes can easily be calculated from (10). This example is more "natural" because it satisfies (9).

Reference

1. Luce, R. D., and H. Raiffa, *Games and Decisions*, John Wiley (1957), p. 168.