15 Measurable Utility and the Measurable Choice Theorem

1 Introduction

Economic models with infinitely many economic agents[1] have appeared in the literature in recent years; particular attention has been paid to models in which the set of all economic agents appears as a non-atomic measure space [A_1, A_2, Ka, Sh, V]. Such models are useful for describing mass phenomena in economics, phenomena in which the individual agent is insignificant, but in which sets of agents can exert considerable influence.

This paper is devoted to various mathematical problems raised by the study of such economic models. The first concerns the existence of utility functions for the agents, that are measurable as functions of the agent. Let T be a σ-finite measure[2] space; intuitively, T is the space of agents. Let Ω denote the non-negative orthant of E^n; intuitively, Ω is the set of all bundles containing various (non-negative) quantities of n specified commodities. It is well-known [D] that a preference order $\succ$ on Ω obeying appropriate conditions can be represented by a utility function $v(x)$ (for $x \in \Omega$) that is continuous on Ω. Now if for each agent t in T there is a preference order $\succ_t$, then each such order can be represented by a utility function $v_t(x)$. Under which conditions can this function be chosen so as to be measurable as a function of t, or indeed simultaneously as a function of x and t? A similar question may be asked when the set of outcomes on which the preference orders are defined is more general than Ω, or even when it may vary with the agent t.

The second question concerns "measurable choice." Consider the following:

VON NEUMANN'S THEOREM *Let* T *and* X *be copies of the unit interval, and let* G *be a Borel subset of* $\mathrm{T} \times \mathrm{X}$ *whose projection on* T *is all of* T. *Then there is a measurable function* g *from* T *to* X, *such that* $(t, g(t)) \in \mathrm{G}$ *for almost all* t *in* T.

This proposition, proved in [N], is of fundamental importance in economic investigations of the kind described above (see [A_2, A_3]). However, it is only applicable to these investigations if one assumes that as a measurable space, the space T of agents is essentially the same as the unit

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1. The word "agent" here is used to mean an economic decision-making unit. The comparable word for a game is "player," for a market—"trader."

2. On T, only the notion "measure 0" is of significance in the sequel; the measure itself plays no role.

interval with the Borel sets. This restriction is not as strong as it may originally seem; any Borel subset of any euclidean space (or, indeed, of any complete separable metric space) satisfies it. However, it *would* be desirable to assume as little as possible about the space T, and in particular to avoid any assumptions that explicitly specify its measurable structure.

The third problem concerns the preservation of measurability under projection. It is well known that if T and X are copies of the unit interval, H a Borel subset of $T \times X$, then the projection of H on T is Lebesgue measurable (though it need not be Borel). Here, too, it would be desirable to prove generalizations that assume as little as possible about the space T.

Although there is no immediately apparent conceptual connection between the problems we have raised, they are closely connected. The measurable choice principle and the projection principle are needed in the proofs of our propositions regarding measurable utility (in addition to their uses elsewhere in mathematical economics). We therefore start with the statement and proof of the measurable choice principle in its most general form, continue with the projection principle, and then discuss the question of measurable utility.

We would like to thank Gerard Debreu for very stimulating correspondence and discussions on the subjects of this paper, Joram Lindenstrauss for discussions on the measurable choice principle and in particular for providing the counterexample at the end of section 2, and Shizuo Kakutani for discussions and references on the subject of Complementary Analytic Choice (see the end of section 4).

2 The Measurable Choice Theorem

We shall need the concept of *standard measurable space*. This is a measurable space that is isomorphic to a cartesian product of $\{0, 1\}$ denumerably many times. (The space $\{0, 1\}$ consists of the points 0 and 1 only, and all subsets are measurable. An *isomorphism* between two measurable spaces is a one-one function from one onto the other, that takes measurable sets onto measurable sets in both directions.) The unit interval, any euclidean space, and in fact any non-denumerable Borel subset of any separable complete metric space, with the usual Borel σ-field, is standard. The class of standard spaces thus comprises just about all spaces that are liable to appear as range spaces in economic applications of the measurable choice theorem (in most such applications the range space is simply a euclidean space).

MEASURABLE CHOICE THEOREM *Let* (T, μ) *be a* σ*-finite measure space, let* X *be a standard measurable space, and let* G *be a measurable subset of* $T \times X$ *whose projection on* T *is all of* T. *Then there is a measurable function* g *from* T *to* X, *such that* $(t, g(t)) \in \mathrm{G}$ *for almost all* t *in* T.

"Almost all" means that the exceptional set is measurable and of μ-measure 0.

The theorem is false without the requirement that X be standard, as will be shown below by means of an example due to J. Lindenstrauss.

The theorem is also false without the σ-finiteness restriction on T. Indeed, let T be $[0, 1]$ with the Borel sets, and let μ be such that every non-denumerable set has measure ∞. If the theorem were true in this case, then it would follow that g can be chosen so that $(t, g(t)) \in \mathrm{G}$ for *all* t in T, and this is known to be false.

The proof of the measurable choice theorem proceeds by reducing the general case to von Neumann's case in a number of stages. The details of the proof are as follows:

Because of the σ-finiteness condition, we may find a measure μ' equivalent to μ such that $\mu'(\mathrm{T}) = 1$. Since the concepts of "almost all" w.r.t. μ and w.r.t. μ' are equivalent, we may replace μ by μ'. This justifies us in assuming from now on that $\mu(\mathrm{T}) = 1$.

By analogy with topological usage, define a measurable space Y to be *separable* if there is a denumerable family Q of measurable subsets such that the family of all measurable subsets of Y is the smallest σ-ring including Q. The space Y is called *regular* if for all x, $y \in \mathrm{Y}$, there is a measurable set containing x but not y.

The first stage of the proof consists of noting that the theorem is true when both T and X are standard. Indeed this case is covered by von Neumann's theorem (see section 1).

In the second stage of the proof we will assume that the domain T is separable and regular and that the range space X is standard. It is well known that every separable regular measurable space is isomorphic to a subset of a standard space. See, for example, [M], Theorem 2.1 (Mackey's terminology is somewhat different). Therefore T may be included in a standard space U; define a measure ν on U by $\nu(\mathrm{S}) = \mu(\mathrm{S} \cap \mathrm{T})$, and note that $\nu(\mathrm{U}) = 1$.

Let H be a measurable subset of $\mathrm{U} \times \mathrm{X}$ such that $\mathrm{G} = \mathrm{H} \cap (\mathrm{T} \times \mathrm{X})$; it is easy to prove that there is such an H. Let $\pi\mathrm{G}$ and $\pi\mathrm{H}$ be the projections of G and H on U. Of course $\pi\mathrm{H}$ need not be measurable in U, i.e. it need not be in the σ-field associated with U; however, since it is analytic, it must have equal ν-outer and ν-inner measures. To prove this, assume, as one clearly can w.l.o.g., that ν has no atoms; next, apply an isomorphism that takes U to $[0, 1]$ and ν to Lebesgue measure; and finally, recall that every analytic subset of $[0, 1]$ is Lebesgue measurable.

Let ν^+ and ν^- denote the ν-outer and ν-inner measures respectively, and let S^+ be a measurable subset of U such that $S^+ \supset \pi H$ and $\nu(S^+) = \nu^+(\pi H)$. Then

$$\nu^+(\pi H) = \nu(S^+) = \mu(S^+ \cap T) \geqq \mu(\pi H \cap T) = \mu(\pi G) = 1.$$

Hence $\nu^+(\pi H) = 1$, so also $\nu^-(\pi H) = 1$. Hence there is a measurable subset S^- of U such that $S^- \subset \pi H$ and $\nu(S^-) = 1$. Then $H \cup [(U \backslash S) \times X]$ is a measurable subset of $U \times X$ whose projection on U is all of U. So we may apply the first stage to it, and obtain a function h from U to X such that $(t, h(t)) \in H$ except possibly when $t \in U \backslash S^-$. If we now define g to be the restriction of h to T, then g satisfies the demands of the theorem. This completes the proof of the second stage.

In the third stage, we drop the requirement that T be regular; so now T is separable, and X standard. Let us define two points in T to be *equivalent* if every measurable set containing one contains the other. This stage may then be reduced to the previous one by constructing the identification space in which equivalent points are identified. We omit details.

In the fourth and final stage, we drop the requirement that T be separable; so now T is an arbitrary measurable space on which there is defined a totally finite measure ν, and X is standard. For any family $\mathscr{R}$ of measurable subsets of T, let $(\mathscr{R})$ denote the σ-field generated by $\mathscr{R}$, i.e. the smallest σ-field that includes $\mathscr{R}$, and let $T_{\mathscr{R}}$ denote the measurable space whose points are the points of T and whose measurable sets are the members of $(\mathscr{R})$. Then we claim that for every set G that is measurable in $T \times X$, there is a *denumerable* family $\mathscr{R}$ of measurable sets in T such that G is measurable in $T_{\mathscr{R}} \times X$. Indeed, let $\mathscr{B}$ be the family of all sets G for which there is such a denumerable family. Then clearly all measurable rectangles are in $\mathscr{B}$; indeed for them, $\mathscr{R}$ can consist of a single set. Furthermore, if $G \in T_{\mathscr{R}} \times X$, then the complement of G is in $T_{\mathscr{R}} \times X$, so $\mathscr{B}$ is closed under the formation of complements. Finally, since the denumerable union of denumerable families is again a denumerable family, it follows that $\mathscr{B}$ is closed under the formation of denumerable unions as well. Hence $\mathscr{B}$ is a σ-field containing all rectangles, so it must contain all sets measurable in the product space.

In particular, if $G \subset T \times X$ satisfies the hypotheses of the measurable choice theorem, we may find a denumerable family $\mathscr{R}$ such that G is measurable in $T_{\mathscr{R}} \times X$. But clearly $T_{\mathscr{R}}$ is separable, so we may apply the third stage to the measure space $(T_{\mathscr{R}}, \mu|(\mathscr{R}))$. Hence there is a function g from T to X, which is measurable w.r.t. the field $(\mathscr{R})$, and such that

$(t, g(t)) \in G$ except possibly for t in a set S such that $S \in \mathscr{R}$ and $\mu(S) = 0$. But if g is measurable-$\mathscr{R}$, it is a fortiori measurable, and from $S \in \mathscr{R}$ it follows that S is measurable. This completes the proof of the theorem.

The Lindenstrauss counterexample, which shows that X cannot be replaced by an arbitrary measurable space, is as follows: let T be the half-open interval $[0, 1)$ with the Borel sets, and let μ be Lebesgue measure. Let X be the subspace of $[0, 1)$ that one usually uses to show that there are Lebesgue non-measurable sets; that is, the sets $X + r$ are mutually disjoint for rational r, and $\bigcup_r (X + r) = [0, 1)$ (addition is modulo 1). Let D be the subset of $T \times X$ defined by

$$D = \{(t, x) : t = x\}.$$

D is a sort of diagonal of $T \times X$. It's easy to prove that D is measurable; one builds a finite number of small rectangles whose union covers D, lets the rectangles get smaller, and the intersection is D. Note that the projection of D on X exhausts X, but the projection on T does not exhaust T; in fact, the projection is precisely X, which may be considered a subset of T.

Let

$$G = \bigcup_r (D + (r, 0)).$$

Clearly $D + (r, 0)$ is measurable, and hence G is measurable. The projection of $D + (r, 0)$ on T is $X + r$, and hence the projection of G exhausts T. Since the $X + r$ are disjoint, it follows that for each t there is precisely one point $g(t) \in X$ such that $(t, g(t)) \in G$. Clearly it is sufficient to prove that there is no measurable function that differs from g on a set of Lebesgue measure 0 only.

We first prove a lemma: If S is a measurable subset of T such that $S + r = S$ for all rational r, then $\mu(S) = 0$ or $\mu(S) = 1$. To prove this, suppose $\mu(S) > 0$. Then we can find a point t_0 at which S has density 1; hence for every ε and every sufficiently small interval $[s, t)$ around t_0, we will have $\mu(S \cap [s, t)) > (1 - \varepsilon)(t - s)$. In particular, if m is a sufficiently large positive integer, we can find an interval $[s, t)$ such that $t - s = 1/m$ and $\mu(S \cap [s, t)) > (1 - \varepsilon)/m$. Then

$$\mu(S) = \mu(S \cap [0, 1)) = \mu\left(S \cap \sum_{j=1}^{m} ([s, t) + j/m)\right)$$

$$= \sum_{j=1}^{m} \mu(S \cap ([s, t) + j/m))$$

$$= \sum_{j=1}^{m} \mu((S + j/m) \cap ([s,t) + j/m))$$

$$= \sum_{j=1}^{m} \mu((S \cap [s,t)) + j/m)$$

$$= \sum_{j=1}^{m} \mu(S \cap [s,t)) > m(1-\varepsilon)/m = 1-\varepsilon.$$

Since ε may be chosen arbitrarily small, the lemma follows.

From this lemma we deduce the following corollary:

If S is a not necessarily measurable subset of $[0,1)$ such that $S + r = S$ for all rational r, then the outer measure of S is either 0 or 1. Indeed, let S' be a measurable set containing S, such that $\mu(S') = \mu^{+}(S)$, where μ^{+} denotes outer measure. (For example, take S' to be the intersection of a sequence $\{S_i\}$ of measurable sets containing S, such that the measure of S_i differs from $\mu^{+}(S)$ by $< 1/i$). Then $S' + r$ is measurable for all r, and includes $S + r = S$. Therefore $S'' = \cap (S' + r)$ is measurable, and includes S; since $S'' \subset S'$, it follows that $\mu(S'') = \mu^{+}(S)$. Now $S'' + r = S''$ for all rational r, hence $\mu(S'') = 0$ or $\mu(S'') = 1$, and the corollary is proved.

Obviously the outer measure of X cannot vanish; let Y be a measurable subset of T such that $Y \supset X$ and $\mu(Y) = \mu^{+}(X) > 0$. Using density considerations as in the proof of the lemma, we can find two disjoint intervals $I_1 = [s_1, t_1)$ and $I_2 = [s_2, t_2)$ such that $\mu(Y \cap I_1) > 0$, $\mu(Y \cap I_2) > 0$. Then also $\mu^{+}(X \cap I_1) > 0$, $\mu^{+}(X \cap I_2) > 0$. Let

$$X_1 = \bigcup_r ((X \cap I_1) + r)$$

$$X_2 = \bigcup_r ((X \cap I_2) + r).$$

Then $X_1 + r = X_1$ and $X_2 + r = X_2$ for all rational r. But $X_1 \supset X \cap I_1$, and hence $\mu^{+}(X_1) \geqq \mu^{+}(X \cap I_1) > 0$; similarly $\mu^{+}(X_2) > 0$. So from the corollary it follows that $\mu^{+}(X_1) = \mu^{+}(X_2) = 1$, and since $X_1 \cap X_2 = \varnothing$, it follows that $\mu^{-}(X_1) = \mu^{-}(X_2) = 0$, where μ^{-} denotes the inner measure. But $X_1 = g^{-1}(I_1)$, and it follows that there is no measurable function g' such that g differs from g' on a set of measure 0. This completes the example.

3 The Projection Theorem

PROJECTION THEOREM *Let* (T, μ) *be a* σ*-finite measure space, let* X *be a standard measurable space, and let* H *be a measurable subset of* $T \times X$.

Then the projection of H *on* T *differs from a measurable set by a set of measure* 0.

The theorem is false without the requirement that X be standard; this may be seen at once by taking H = D in the Lindenstrauss example (see section 2).

The proof proceeds by reducing the general case to the case in which T is standard, in much the same way that the measurable choice theorem is proved. The case in which T is standard is well-known (see for example Kuratowski [Ku]), as we said in the introduction. We will omit the details of the proof.

4 Measurable Utility

Let X be a topological space. A *continuous preference order* (cpo) on X is a relation $\succsim$ on X that is transitive, reflexive, and complete[3] and such that for each y in X, the sets $\{x : x \succsim y\}$ and $\{x : y \succsim x\}$ are closed. Given such a relation, we will define $x \sim y$ if $x \succsim y$ and $y \succsim x$, and $x \succ y$ if $x \succsim y$ but not $x \sim y$. A function v from X to the reals is said to *represent* the cpo $\succsim$ if it is continuous and

$$x \succsim y \Longleftrightarrow v(x) \geqq v(y).$$

Rather than stating our most general result at once, we will first state and prove a special case. As in section 1, Ω will denote the non-negative orthant of a Euclidean space. For the purposes of this paper, a measurable space is an abstract space together with a σ-field[4] of subsets.

PROPOSITION 1 *Let* T *be an arbitrary measurable space. For each t in* T, *let $\succsim_t$ be a continuous preference order on Ω; assume that for all x, y in Ω the set $\{t : x \succ_t y\}$ is measurable. Then each cpo $\succsim_t$ can be represented by a function v_t in such a way so that $v_t(x)$ is simultaneously measurable[5] in t and x.*

Proof First assume that all the preference orders are non-trivial, i.e. that for each t there are x and y in Ω such that $x \succ_t y$. Let R be the set of rational points in Ω, Γ' the set of rational numbers in the open unit interval $(0, 1)$, and $\Gamma = \Gamma' \cup \{0, 1\}$. Let $\{r_1, r_2, \ldots\}$ and $\{\gamma_1, \gamma_2, \ldots\}$ be arbitrary but fixed enumerations of R and Γ' respectively. For each t define a function v_t from R to Γ inductively as follows:

3. i.e. for all x, y, either $x \succsim y$ or $y \succsim x$ or both.
4. non-empty family closed under denumerable unions and complementation.
5. i.e. measurable on the product space $\mathrm{T} \times \Omega$, when Ω is invested with the Borel structure.

Definition of $v_t(r_1)$

if $r_j \succsim_t r_1$ for all j, define $v_t(r_1) = 0$;
if $r_1 \succsim_t r_j$ for all j, define $v_t(r_1) = 1$;
otherwise, define $v_t(r_1) = \gamma_1$.

Definition of $v_t(r_n)$

if $r_j \succsim_t r_n$ for all j, define $v_t(r_n) = 0$;
if $r_n \succsim_t r_j$ for all j, define $v_t(r_n) = 1$;
if $r_n \sim_t r_j$ for some $j < n$, define $v_t(r_n) = v_t(r_j)$ for that j;
otherwise, define $v_t(r_n)$ to be the first γ_i such that

$$\max\{v_t(r_j) : 1 \leqq j < n \quad \text{and} \quad r_n \succ_t r_j\} < \gamma_i$$
$$< \min\{v_t(r_j) : 1 \leqq j < n \quad \text{and} \quad r_j \succ_t r_n\}.$$

First note that for each r_j in R, $v_t(r_j)$ is a measurable function of t. The proof is by induction on j, and the reader will have no difficulty in supplying it. Next, note that for each t, v_t is continuous on R and can be extended to a continuous function on Ω in a unique fashion; the extension will also be called v. Now if x is allowed to range over Ω, and α is an arbitrary member of $[0, 1]$, then $v_t(x) \leqq \alpha$ if and only if for each positive integer k there is a positive integer j such that $\|x - r_j\| < 1/k$ and $v_t(r_j) < \alpha + (1/k)$. Together with the measurability of $v_t(r_j)$ established above, this completes the proof of simultaneous measurability when none of the preference orders are trivial.

For the general case, note that triviality of the preference order on Ω is equivalent to triviality of its restriction to R; from this it easily follows that the set of all t with trivial preference orders is measurable. If we then let v_t vanish identically in x for all t with trivial orders, and define it as above for the other t, we again get simultaneous measurability.

MEASURABLE UTILITY THEOREM *Let* T *be a* σ*-finite measure space. For each* t *in* T, *let* $\varphi(t)$ *be a connected subset of* E^n *and let* $\succsim_t$ *be a continuous preference order on* $\varphi(t)$. *Assume that the set*

$$\{(x, y, t) : x \in \varphi(t), x \succsim_t y\}$$

is a subset of $E^n \times E^n \times T$ *that is measurable in the product structure, when* E^n *is invested with the Borel structure. Then almost every cpo* $\succsim_t$ *can be represented by a function* v_t *in such a way so that* $v_t(x)$ *is simultaneously measurable in* t *and* x.

Proof The proof is a variant of that of the previous proposition. Let G denote the graph of φ. From the assumptions of the theorem it follows that G is a Borel subset of $E^n \times E^n \times T$, since it is the intersection of the displayed expression with the set $\{(x, y, t) : x = y\}$. Obviously $\varphi(t)$ is separable for each t. What is needed is some way of choosing a denumerable dense sequence in each of the $\varphi(t)$ in a "measurable" way. More precisely, we want a sequence of measurable functions $\mathbf{r}_1, \mathbf{r}_2, \ldots$ from T to E^n such that for almost all $t \in T$, $\mathbf{r}_i(t) \in \varphi(t)$ for each i, and $\{\mathbf{r}_1(t), \mathbf{r}_2(t), \ldots\}$ is dense in $\varphi(t)$.

Let $\{R_1, R_2, \ldots\}$ be a denumerable basis for the topology of E^n. For every positive integer k, define $Q_k = \{t \in T : R_k \cap \varphi(t) \neq \varnothing\}$. One has $Q_k = \mathrm{proj}_T[(T \times R_k) \cap G]$. Since G is Borel, it follows from the projection theorem that there is a measurable set P_k that differs from Q_k by a set of measure 0. Define a sequence of set-valued functions $\{\psi_k\}$ by

$$\psi_k(t) = \begin{cases} R_k \cap \varphi(t) & \text{for } t \in P_k \\ \varphi(t) & \text{otherwise.} \end{cases}$$

Under this definition it may happen that $\psi_k(t)$ is empty for t in a certain measurable set of measure 0. Define

$$\varphi_k(t) = \begin{cases} \psi_k(t) & \text{when } \psi_k(t) \neq \varnothing \\ \varphi(t) & \text{otherwise.} \end{cases}$$

Clearly the graph of φ_k is a Borel set, and hence by the measurable choice theorem we can find functions $\mathbf{q}_n(t)$ such that q_k differs from a measurable function on a set of measure 0 and $\mathbf{q}_k(t) \in \varphi_k(t)$ for all t. From this it is easy to construct a sequence of measurable functions $\{\mathbf{r}_1, \mathbf{r}_2, \ldots\}$ such that for almost all t, $\{\mathbf{r}_1(t), \mathbf{r}_2(t), \ldots\}$ is a dense subsequence[6] of $\varphi(t)$.

The reader will have no difficulty in reconstructing the remainder of the proof from that of the previous proposition.

The theorem will still hold if the assumption that $\varphi(t)$ is connected is replaced by one of two assumptions: either that $\varphi(t)$ is independent of t, or that $\varphi(t)$ has a finite number of components for each t. But it seems difficult to dispense with it altogether. To appreciate the difficulties, let us specialize to the case in which T is the unit interval with the Borel structure and Lebesgue measure. Let X be a standard measurable space. Recall that a subset G of a Euclidean space is called *analytic* (A) if it is the projection of a Borel set in a higher dimensional space; complementary analytic (CA) if it is the complement of an A set; PCA if it is the projection of a CA set; and CPCA if it is the complement of a PCA set.

6. We are grateful to Professor G. Debreu for communicating to us a method used by C. Castaing in an unpublished paper dated December 1966 (see also [C]), which considerably simplified the construction of the $\mathbf{r}_k$.

The measurable choice theorem still holds if one assumes that the G in the statement of that theorem need only be analytic rather than Borel; this is easily seen. But we do not know whether the measurable choice theorem holds if G is assumed to be *complementary analytic*. Let us call this hypothesis (i.e. that it does hold) the "complementary analytic choice hypothesis," or CA choice for short. Then the *measurable utility theorem without any connectedness assumption on $\varphi(t)$ would follow from CA choice.*

We do not know whether CA choice is true or false; unfortunately, though, it seems for all practical purposes to be unusable. The discussion of this question brings us to the roots of set theory. Consider the statement:

1) CA choice is false.

The statement (1) *is consistent with the axioms of set theory.* To demonstrate this, consider the following statements:

2) There is a subset of the line that is both PCA and CPCA and that is not Lebesgue measurable.

3) There is a (point-valued) function with a CA graph that is not Lebesgue measurable.

First we establish that 1), 2), and 3) are all equivalent. The equivalence of 2) and 3) is known; see [Si, p. 56]. Moreover, 3) obviously implies 1), since we may take the set-valued function needed to contradict CA choice to be precisely the point-valued function of 3). To complete the equivalence proof, one need only show that "not 3)" implies CA choice. Again referring to page 56 of [Si], we see that if F is a set-valued function with a CA graph, then there is a point-valued function f with a CA graph such that $f(t) \in F(t)$ for all t. But then from "not 3)" it follows that f is Lebesgue measurable, implying CA choice. Thus 1), 2), and 3) are all equivalent.

For the sake of simplicity, we will consistently refer to 2) in the sequel, which is the best known of the above three equivalent statements. Gödel proved [G] that 2) is consistent with set theory; so 2) also is, as asserted above. But it also raises some new questions. In the same paper, Gödel proved that 2) *and* the Continuum Hypothesis are simultaneously consistent with set theory. This leads one to wonder whether Paul Cohen's recent results or methods can be used to prove not only the independence of the Continuum Hypothesis, but also that of 2). If 2) is independent, then at least we would be sure that the existence of an appropriate utility in the disconnected case is consistent. Specifically, we would like to know the answers to the following two questions:

(i) Is 2) independent of set theory?

(ii) Is 2) independent of set theory, when the Continuum Hypothesis is added as an axiom of set theory?

Of course there may be a simple direct proof of the theorem we want, without using CA choice. In view of the above considerations, though, it would be surprising if there were a simple counterexample, because that would yield a proof of 2).

Note

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